

MULTIPLICATIVE RENORMALIZATION METHOD  
FOR ORTHOGONAL POLYNOMIALS

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# Abstract

To study the orthogonal polynomials, Asai, Kubo and Kuo recently have developed the multiplicative renormalization method. Motivated by infinite dimensional white noise analysis, it is an alternative to the computational part of the classical Gram-Schmidt process to find the orthogonal polynomials for a given measure. Instead of finding the orthogonal polynomials recursively as described in the Gram-Schmidt process, one analyzes different types of generating functions systematically in order to obtain polynomials after power series expansion. This work also produces the Jacobi-Szegö parameters easily and paves the way for the study of one-mode interacting Fock spaces related to these parameters. They have verified the classical measures and their corresponding orthogonal polynomials. In this thesis, we take this to the next level in order to classify measures with certain generating functions which leads to some new measures. We will begin with a description of the mathematical background of the method and we will re-derive all measures with generating functions of the exponential type as did Meixner and Morris in different settings. Next we shall derive all measures with generating functions of the fractional type, which will yield new measures. We will also show the relation between consecutive Gegenbauer measures and neighbouring Jacobi polynomials. Finally we will demonstrate the uniqueness of the uniform measure with a certain type of generating function.

# Chapter 1

## Introduction

This dissertation concerns generating functions for orthogonal polynomials. Given an arbitrary probability measure  $\mu$  on the set  $\mathbb{R}$  of real numbers, it has always been a challenge to find an associated collection of orthogonal polynomials. For example, let  $\mu$  be the Gaussian measure on the real line with mean 0 and variance  $\sigma^2$ . Equivalently,

$$d\mu(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx.$$

It is well known that the Hermite polynomials

$$H_n(x; \sigma^2) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^k k! (n-2k)!} (-\sigma^2)^k x^{n-2k}, \quad n = 0, 1, 2, \dots,$$

form an orthogonal basis for the space  $L^2(\mu)$ . Furthermore, we have a generating function

$$e^{tx - \frac{\sigma^2 t^2}{2}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x; \sigma^2). \quad (1.0.1)$$

Theoretically, one would derive the Hermite polynomials from the Gaussian measure  $\mu$  by employing the Gram-Schmidt orthogonalization process to the system  $\{1, x, x^2, \dots, x^n, \dots\}$ . However, this method is rather impractical because it does not give the explicit form of the  $n$ th Hermite polynomial.

On the other hand, if one knows how to derive the generating function  $\psi(t, x) = e^{tx - \frac{\sigma^2 t^2}{2}}$  from the measure  $\mu$ , then (s)he may obtain the Hermite polynomials by expanding the generating function  $\psi$  as a power series in  $t$ . Therefore in general, the underlying problem is to find out how to derive the generating function from the probability measure  $\mu$ . In their recent series of papers, N. Asai, I. Kubo and H-H. Kuo [10],[12],[11],[13],[14], developed a new method for deriving generating

functions. The idea arises from the multiplicative renormalization method introduced by T. Hida [17] in 1975, which enabled to treat white noise as an infinite dimensional generalized function. For example, consider the function

$$\varphi(t, x) = e^{tx}, \quad (1.0.2)$$

where we regard  $x$  as a random variable with Gaussian distribution. Then we may take the expectation in order to obtain

$$E_\mu \varphi(t, \cdot) = e^{\frac{\sigma^2 t^2}{2}}.$$

In white noise analysis, the *multiplicative renormalization* of the function  $\varphi(t, x)$  is defined by the equation

$$\psi(t, x) := \frac{\varphi(t, x)}{E_\mu \varphi(t, \cdot)}.$$

But in our case,  $\frac{\varphi(t, x)}{E_\mu \varphi(t, \cdot)} = e^{tx - \frac{\sigma^2 t^2}{2}}$ , which is the generating function in equation (1.0.1); hence we have derived the generating function by the multiplicative renormalization of the function in equation (1.0.2) with respect to the Gaussian measure  $\mu$ . Now,  $\psi$  has power series expansion

$$\psi(t, x) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n.$$

Expanding  $\psi(t, x)$  in the following way gives us the  $n$ th orthogonal polynomial explicitly:

$$\begin{aligned} \psi(t, x) &= e^{tx} e^{-\frac{\sigma^2 t^2}{2}} \\ &= \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} t^n \right) \left( \sum_{n=0}^{\infty} \frac{(-\sigma^2)^n}{2^n n!} t^{2n} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2k}}{(n-2k)!} \frac{(-\sigma^2)^k}{2^k k!} \right) t^n. \end{aligned}$$

It follows immediately that  $P_n(x)$  is exactly the  $n$ th Hermite polynomial. Hence we have derived the Hermite polynomials from the Gaussian measure  $\mu$  by using

the functions  $\varphi(t, x) = e^{tx}$  and  $\psi(t, x) = e^{tx - \frac{\sigma^2 t^2}{2}}$ . Furthermore, for any  $t$  and  $s$ , we have that

$$E_\mu \psi(t, \cdot) \psi(s, \cdot) = e^{\sigma^2 ts};$$

that is,  $E_\mu \psi(t, \cdot) \psi(s, \cdot)$  is a function of  $ts$ , which implies that the Hermite polynomials are orthogonal with respect to the Gaussian measure  $\mu$ .

The original motivation for AKK papers, is the relation between the concept of interacting Fock spaces and orthogonal polynomials. In chapter 10, we will explain more about interacting Fock spaces, specifically the one mode interacting Fock space. However, in this thesis, we will focus more on the generating functions of orthogonal polynomials of different types. This approach has been very systematic and fruitful as the reader will observe as (s)he goes on. We not only covered many classical orthogonal polynomials, but also introduced some new measures and find some new generating functions for existing measures.

Chapter 2 consists of the literature and the AKK method with the utmost generality. Some of the key concepts in orthogonal polynomials will be touched and the most general theorems of AKK method will be proved. We will also give most general calculation methods for Jacobi-Szegő parameters. In chapter 3, we will study the interesting case which covers most of the classical orthogonal polynomials with respect to the given classical measures, namely the case with the generating function of the form  $h(\rho(t)x)$ . In chapters 4, 5 and 6 we will change our direction, fix some particular functions, namely  $h(x) = e^x$ ,  $h(x) = \frac{1}{1-x}$  and  $h(x) = \frac{1}{(1-x)^2}$ , and then come up with all possible measures and orthogonal polynomials corresponding to this particular generating functions. Chapter 5 contains the most beautiful fruit of this research, where we found some new measures. I hope these new measures will be very useful in mathematical physics and practical applications wherever

Wigner Distributions are used. In chapter 7, we will prove the existence of four Jacobi-type measures corresponding to the case  $h(x) = \frac{1}{(1-x)^c}$ , with  $c > 0$ . We also derive some beautiful integral formulas. In the next chapter, we will prove that in fact these are the only possible ones for an arbitrary  $c$ , along with the trivial delta measures for general  $c$ . In Chapter 9, we will enter the world of  $q$ -orthogonal polynomials. This time we will fix the  $\rho(t) = t$  and come up with the corresponding measures and polynomials. Chapter 10 of this dissertation is devoted to the theory of operators on interacting Fock spaces.

## Chapter 2

# Orthogonal Polynomials and AKK Method

### 2.1 An Overview of Orthogonal Polynomials

Since the introduction of orthogonal polynomials by Murhpy [22] in 1835 and the development of measure theory, it has always been a question for mathematicians to find the corresponding orthogonal polynomials for a given measure  $\mu$ . There has been enormous results published for this problem. Although the Gram-Schmidt process works theoretically, it is impractical for most of the cases to find the  $n$ th orthogonal polynomial explicitly.

Another approach is to use the generating function. By a generating function for the sequence of orrhthogonal polynomials  $\{P_n(x)\}_{n=0}^{\infty}$  in  $L^2(\mu)$ , we mean a function  $\phi(t, x)$  with a power series expansion in  $t$  about 0

$$\phi(t, x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n. \quad (2.1.1)$$

So if we know the generating function, we can write the power series and then calculate the  $P_n(x)$  explicitly for each  $n$ . There have been many papers published in this direction. In this method one start with a particular form of generating function and then classify the corresponding measures and the orthogonal polynomials. Probably the most famous paper in this direction is the paper by Meixner [20], in which he classifies all of the measures with an exponential generating function. More pricesely he classified all the measures and corresponding orthogonal polynomials with the generating function of the form

$$f(t)e^{g(t)x} = \sum_{n=0}^{\infty} a_n P_n(x) t^n. \quad (2.1.2)$$

Many others found “Meixner Class” by using different methods. Among them was Morris[21] who used the Natural Exponential Families. This method was later improved by Hassairi[16] to give a meaning to the concept of  $\mu - 2$  orthogonality.

## 2.2 AKK Method

In this section we will define our workframe and discuss the most general theory. Let  $\mu$  be a probability measure on  $\mathbb{R}$  having finite moments of all orders, that is

$$\int_{\mathbb{R}} |x|^n d\mu(x) < \infty, \quad n \geq 0,$$

such that the linear span of  $\{x^n\}_{n=0}^{\infty}$  is dense in  $L^2(\mu)$ . Assume the probability measure  $\mu$  takes on exactly  $m$  distinct values at points  $x_1, \dots, x_m$ . Now observe that  $E_{\mu}[Q_m(x)^2] = 0$  where  $Q_m(x) = \prod_{i=1}^m (x - x_i)$ . Hence this case is uninteresting to us because the corresponding Jacobi-Szegő parameters do not define an interacting Fock space (Chapter 10). So from now on we will not consider these degenerate cases and we will always assume that  $\mu$  takes infinitely many distinct values.

By the Gram-Schmidt orthogonalization process, there exists a sequence  $\{P_n(x)\}_{n=0}^{\infty}$  of orthogonal polynomials such that  $P_n(x)$  is a polynomial of degree  $n$  with the leading coefficient of 1 for  $n \geq 0$ . More precisely  $P_0(x) = 1$  and for  $n \geq 1$ :

$$P_n(x) = x^n - \sum_{i=0}^{n-1} \frac{\int_{\mathbb{R}} x^n P_i(x) d\mu(x)}{\int_{\mathbb{R}} P_i(x)^2 d\mu(x)} P_i(x). \quad (2.2.1)$$

One can also prove the existence of the following 3-term recursion formula:

$$(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_{n-1}P_{n-1}(x), \quad n = 0, 1, 2, \dots \quad (2.2.2)$$

where  $P_{-1}(x) = \omega_{-1} = 0$  and  $\omega_n > 0$ . To see this, observe that for the  $(n+1)$ th degree polynomial  $xP_n(x)$ , we have some coefficients  $\{c_i\}_{i=0}^{n+1} \in \mathbb{R}$  such that  $xP_n(x) = \sum_{i=0}^{n+1} c_i P_i(x)$ . Now multiplying both sides by  $P_i(x)$ , where  $0 \leq i \leq n$ , and taking the expectation one obtains  $E_{\mu}[xP_n(x)P_i(x)] = c_i E_{\mu}[P_i(x)^2]$ . But for

$i \leq n - 2$ , we have  $E_\mu[xP_n(x)P_i(x)] = E_\mu[P_n(x)xP_i(x)] = 0$ , which gives  $c_i = 0$ .

One can also observe that  $w_n = \frac{E_\mu(P_{n+1}(x)^2)}{E_\mu(P_n(x)^2)} > 0$ .

Before proceeding any further, we should define the key concepts behind our work.

**Definition 2.1.** *A pre-generating function is a function with power series expansion in  $t$  of the form*

$$\varphi(t, x) = \sum_{n=0}^{\infty} g_n(x)t^n.$$

where  $g_n$  is a polynomial of degree  $n$  for each  $n \geq 0$  and  $\limsup_{n \rightarrow \infty} \|g_n\|_{L^2(\mu)}^{1/n} < \infty$ .

Moreover, if the sequence  $\{g_n\}_{n=0}^{\infty}$  is orthogonal in  $L^2(\mu)$  then we drop the prefix "pre" and call it a generating function for the measure  $\mu$ .

Observe that the second condition gives the analyticity of the  $E_\mu[\varphi(t, x)]$  on some neighbourhood of  $t = 0$ . To prove this result, observe that for

$$|t| < R = \liminf_{n \rightarrow \infty} \|g_n\|_{L^2(\mu)}^{-1/n} > 0,$$

the series

$$|\varphi(t, x)| = \left| \sum_{n=0}^{\infty} g_n(x)t^n \right| \leq \sum_{n=0}^{\infty} \|g_n\|_{L^2(\mu)} |t|^n$$

converges. Moreover  $E_\mu[\varphi(0, \cdot)] = g_0(x) \neq 0$ . So there exists  $0 < r < R$  for which if we have  $|t| < r$  then  $E_\mu[\varphi(t, \cdot)] \neq 0$ .

**Definition 2.2.** *The multiplicative renormalization of a pre-generating function  $\varphi(t, x)$  is defined to be the function*

$$\psi(t, x) = \frac{\varphi(t, x)}{E_\mu[\varphi(t, x)]} = \sum_{n=0}^{\infty} A_n(x)t^n.$$

One can prove that the multiplicative renormalization of a pre-generating function is also a pre-generating function with the additional property  $E_\mu[\psi(t, x)] = 1$ .

This part is almost obvious since one has

$$E_\mu[\psi(t, x)] = E_\mu\left[\frac{\varphi(t, x)}{E_\mu[\varphi(t, x)]}\right] = \frac{1}{E_\mu[\varphi(t, x)]} E_\mu[\varphi(t, x)] = 1.$$

Now by the analyticity of  $E_\mu[\varphi(t, \cdot)]$ , we know  $\frac{1}{E_\mu[\varphi(t, x)]}$  has power series expansion around  $t = 0$ .

The following theorem is the backbone of the AKK-Method

**Theorem 2.3.** *Let  $\psi(t, x)$  be the multiplicative renormalization of a pre-generating function  $\varphi(t, x)$  as given in Definition 2.1 and 2.2. Then the polynomials  $A_n$  are orthogonal if and only if  $E_\mu[\psi(t, \cdot)\psi(s, \cdot)]$  is a function of  $ts$ . Hence  $\varphi(t, x)$  is a generating function for the orthogonal polynomials for the measure  $\mu$ .*

*Proof.* Suppose the polynomials  $A_n$  are orthogonal. Then one has

$$E_\mu[\psi(t, \cdot)\psi(s, \cdot)] = \sum_{n,m=0}^{\infty} E_\mu[Q_n Q_m] t^n s^m = \sum_{n=0}^{\infty} E_\mu Q_n^2 (ts)^n.$$

The converse can be proved easily by differentiating with respect to  $s$ ,  $m$  times and setting  $s = 0$ . □

In order to find the orthogonal polynomials for a  $\mu$ , one can start with a certain form of pre-generating function, then calculate its multiplicative renormalization and finally calculate  $E_\mu[\psi(t, x)\psi(s, x)]$  to see if it is a function of  $ts$ . If it is a function of  $ts$  then one simply expands it as a power series in  $t$  about 0, whose existence is guaranteed by the Definition 2.1, to find the orthogonal polynomials. One might need to divide each polynomial by the leading coefficient to get the monic  $P_n(x)$ .

To find the Jacobi-Szegő parameters, multiply the equation (2.2.2) by  $P_n(x)$  and take the expectation to get

$$\alpha_n = \frac{E_\mu[xP_n^2]}{E_\mu[P_n^2]}.$$

Here we just used the orthogonality of  $P_n(x)_{n=0}^\infty$ , namely  $E_\mu[P_n P_m] = 0$  for  $n \neq m$  and  $E_\mu[xP_n P_m] = 0$  for  $|n - m| \neq 1$ . Similarly multiply the equation (2.2.2) by  $P_{n-1}(x)$  and take the expectation to get

$$\omega_{n-1} = \frac{E_\mu[xP_n P_{n-1}]}{E_\mu[P_{n-1}^2]}.$$

Observing that

$$E_\mu[xP_n P_{n-1}] = E_\mu[P_n(P_n + Q_{n-1} + \cdots + Q_0)] = E_\mu P_n^2$$

gives us the following nice formula

$$\omega_{n-1} = \frac{E_\mu[P_n^2]}{E_\mu[P_{n-1}^2]}.$$

Define  $\lambda_0 = 1$  and for  $n \geq 1$ ,  $\lambda_n = \prod_{i=0}^{n-1} \omega_i$ . Then one gets

$$\lambda_n = E_\mu[P_n^2].$$

This formulas are classical formulas in the theory of orthogonal polynomials. Now the following theorem tells us how to find them directly from generating function

**Theorem 2.4.** *Let  $\varphi(t, x) = \sum_{n=0}^\infty a_n P_n(x) t^n$  be a generating function. Then we have the following equations*

$$\lim_{t \rightarrow 0} \varphi(t, \frac{x}{t}) = \sum_{n=0}^\infty a_n x^n, \quad (2.2.3)$$

$$E_\mu[\varphi(t, \cdot)^2] = \sum_{n=0}^\infty a_n^2 \lambda_n t^{2n}, \quad (2.2.4)$$

$$E_\mu[x\varphi(t, \cdot)^2] = \sum_{n=0}^\infty a_n^2 \lambda_n \alpha_n t^{2n} + 2a_n a_{n-1} \lambda_n t^{2n-1}, \quad (2.2.5)$$

where  $a_{-1} = 0$  for convention. Moreover if  $\mu$  is symmetric about  $y$ -axis then we have  $\alpha_n = 0$ .

*Proof.* First follows from the fact that  $P_n(x) = x^n + \text{lower degree terms}$  and the last two follows from previous discussion.  $\square$

Once we have the generating function for this theorem, we can find the Jacobi-Szegö parameters. Hence, in this method, the real challenge is how to choose the right pre-generating function. The right pre-generating function should enable one to take expectations as discussed above. This forces one to study certain types of pre-generating functions, rather than handling it with this generality.

# Chapter 3

## A Special Generating Function $h(\rho(t)x)$

### 3.1 General Set-up of the Case

In this chapter, we will deal with the pre-generating functions of the form  $h(\rho(t)x)$  where  $h$  and  $\rho$  are functions with power series expansion about 0. Let us define the following functions in line with chapter 2:

$$\theta(t) = \int_{\mathbb{R}} h(tx) d\mu(x), \quad (3.1.1)$$

$$\tilde{\theta}(t, s) = \int_{\mathbb{R}} h(tx)h(sx) d\mu(x). \quad (3.1.2)$$

It is clear that since  $h$  is analytic about 0,  $\theta$  and  $\tilde{\theta}(t, s)$  are also analytic about 0 and setting  $t = 0$  gives  $\theta(0) = h(0) \neq 0$ . Now let us explore some of the properties of  $h$  and  $\rho$  so that we can have a generating function described in the previous chapter. If  $\frac{h(\rho(t)x)}{E_{\mu}[h(\rho(t)x)]}$  is a generating function for the measure  $\mu$ , we must have the following :

$$h(\rho(t)x) = \theta(\rho(t)) \sum_{n=0}^{\infty} a_n P_n(x) t^n, \quad (3.1.3)$$

where  $a_n \neq 0$  and  $P_n(x)$  is a polynomial of degree  $n$  as discussed in chapter 2. Setting  $t = 0$  yields  $\rho(0) = 0$  and  $\theta(0) \neq 0$ . By taking the derivative  $n$  times and setting  $t = 0$  one gets

$$h^{(n)}(0)\rho'(0)^n x^n + \dots = \theta(0)n!a_n P_n(x) \quad (3.1.4)$$

which tells us that  $\rho'(0) \neq 0$  and  $h^{(n)}(0) \neq 0$  for each  $n$ . The above discussion can be summarized in the following theorem.

**Theorem 3.1.** Let  $\frac{h(\rho(t)x)}{E_\mu[h(\rho(t)x)]}$  be a generating function for orthogonal polynomials  $\{P_n(x)\}_{n=0}^\infty$  with respect to the measure  $\mu$ . Assume further that  $h$  and  $\rho$  are analytic about 0. Then we must have  $\rho(0) = 0$ ,  $\rho'(0) \neq 0$  and for every  $n \geq 0$   $h^{(n)}(0) \neq 0$ .

Now the main theorem in Chapter 2 becomes the following :

**Theorem 3.2.** Let  $h$  and  $\rho$  satisfy the conditions in the previous theorem. Then the function

$$\Theta_\rho(t, s) = \frac{\tilde{\theta}(\rho(t), \rho(s))}{\theta(\rho(t))\theta(\rho(s))} \quad (3.1.5)$$

is a function of  $ts$  if and only if the multiplicative renormalization

$$\psi(t, x) = \frac{h(\rho(t)x)}{E_\mu[h(\rho(t)x)]} \quad (3.1.6)$$

is a generating function for  $\mu$ .

Given a measure  $\mu$  as described in chapter 2, one can try to fix either  $h$  or  $\rho$  and try to find the other one which satisfies the theorem above. Since by the first theorem  $\rho$  has fewer constraints, it is more logical to fix  $h$  and find the corresponding  $\rho$ . The work of AKK dealt with this case. However, the other approach is also fruitful, as it will be discussed in chapter 9. Until Chapter 9 however, we will work in the style of AKK, namely we will first fix  $h$  and try to find a corresponding  $\rho(t)$ . If we can find a pair of  $h$  and  $\rho$  such that  $\Theta_\rho(t, s)$  is a function of  $ts$ , then we have our generating function  $\frac{h(\rho(t)x)}{\theta(\rho(t))}$ . One then can successively find orthogonal polynomials by expanding in power series of  $t$  and find the Jacobi-Szegö parameters by using the formulas discussed in Chapter 2. Observe that to find the  $w_n$ 's, all one needs is the expansion of  $\Theta_\rho(t, t) = \Theta_\rho(t^2)$  about  $t = 0$ . This will be very useful for finding the Jacobi-Szegö parameters and hence for interacting Fock spaces. Another useful part of this case is the fact that  $\Theta_\rho(t, t) = \Theta_\rho(t^2) = \int_{\mathbb{R}} (\psi(t, x))^2 d\mu(x) \geq (\int_{\mathbb{R}} \psi(t, x) d\mu(x))^2 = 1$  by the Cauchy-

Schwartz inequality. Hence  $\Theta_\rho(0) \geq 1$ . In the upcoming chapters, this criteria will enable us to differentiate between formal polynomials (with respect to a linear functional) and classical polynomials (with respect to a real measure on  $\mathbb{R}$ ).

## 3.2 Some Special Candidates for $h(x)$

Right now, the real question is how to choose an appropriate  $h$  for which we can find a corresponding  $\rho$ . Since by theorem 3.1  $h^{(n)}(0) \neq 0$ , we can focus on the following two types of functions

$$h(x) = e^x, \tag{3.2.1}$$

$$h(x) = \frac{1}{(1-x)^c}, \quad c > 0. \tag{3.2.2}$$

It is evident that both functions satisfy the necessary condition  $h^{(n)}(0) \neq 0$ . There are also many other candidates such as  $h(x) = -\ln(1-x)$  or  $h(x) = \arctan(x) + \cos(x)$ , but we want to be able to take the integrals such as in (3.1.2). As it is shown in the series of papers by AKK, these two functions cover many classical examples of orthogonal polynomials. In this dissertation, we have only analyzed the above two functions because the other functions appear to be hopeless. Some variant's of these functions also turn out to be fruitful which is not included here but will come as a paper soon.

It is easy to observe that in both cases  $\lim_{t \rightarrow 0} \varphi(t, \frac{x}{t}) = \lim_{t \rightarrow 0} \psi(t, \frac{x}{t})$ . For the first case one finds  $a_n = \frac{\rho'(0)^n}{n!}$  and for the second  $a_n = \binom{c}{n} \rho'(0)^n$

I will close the section by giving an example to demonstrate the power of this method:

**Example 3.3.** Let  $\mu$  be the semicircle distribution given by

$$d\mu(x) = \frac{2}{\pi} \sqrt{1-x^2} dx, \quad |x| < 1.$$

Try the pre-generating function of the type  $h(x) = \frac{1}{1-x}$  or equivalently  $\varphi(t, x) = \frac{1}{1-\rho(t)x}$ . Then we can calculate the expectation, by using the trigonometric substitution,  $E_\mu[\varphi(t, \cdot)] = \frac{2}{1+\sqrt{1-\rho(t)^2}}$ . So we have the multiplicative renormalization  $\psi(t, x) = \frac{1+\sqrt{1-\rho(t)^2}}{2} \frac{1}{1-\rho(t)x}$ . Now one can calculate

$$E_\mu[\psi(t, \cdot)\psi(s, \cdot)] = \frac{1}{2} \left( 1 + \frac{\rho(t)\sqrt{1-\rho(s)^2} - \rho(s)\sqrt{1-\rho(t)^2}}{\rho(t) - \rho(s)} \right).$$

By the main theorem in Chapter 2, the above expression must be a function of  $ts$ . One can observe (in fact prove) for  $\rho(t) = \frac{2t}{t^2+1}$ , we have  $E_\mu[\psi(t, \cdot)\psi(s, \cdot)] = \frac{1}{1-ts}$ , which is clearly a function of  $ts$ . So we have the generating function  $\psi(t, x) = \frac{1}{1-2tx+t^2}$ . Moreover since  $c = 1$  and  $\rho'(0) = 2$  we get  $a_n = 2^n$ . Hence

$$\psi(t, x) = \frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty} 2^n P_n(x) t^n.$$

A simple expansion will yield

$$P_n(x) = x^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \left( \frac{-1}{4x^2} \right)^k.$$

Finally, by observing that this measure is symmetric, one evaluates  $\alpha_n = 0$ . One may also calculate that  $E_\mu[\psi(t, \cdot)^2] = \frac{1}{1-t^2}$  hence  $\lambda_n = \left(\frac{1}{4}\right)^n$  which gives  $\omega_n = \frac{1}{4}$ . One can also observe that  $2^n P_n(x)_{n=0}^{\infty}$  are the classical Chebyshev polynomials of the second kind.

# Chapter 4

## The Exponential Case $h(x) = e^x$

### 4.1 Exponential Type Generating Functions

In this chapter we take the opposite approach of that of Chapter 3. Now we will fix the  $h(x) = e^x$  and find all corresponding measures. This work has been done by Meixner[20] in 1934 but the AKK method will lead to the same results far more easily as will be shown here. The middle K of AKK, Izumi Kubo has published this result as a paper[18].

The theorem in Chapter 3 tells us that in order to have a generating function, we must have

$$\Theta_\rho(t, s) = \frac{\tilde{\theta}(\rho(t), \rho(s))}{\theta(\rho(t))\theta(\rho(s))} \quad (4.1.1)$$

is a function of  $ts$ . This means that we must have

$$\theta(\rho(t))\theta(\rho(s))\Theta_\rho(ts) = \tilde{\theta}(\rho(t), \rho(s)). \quad (4.1.2)$$

Now observe that for  $h(x) = e^x$  we have

$$\tilde{\theta}(t, s) = \int_{\mathbb{R}} e^{tx} e^{sx} d\mu(x) = \int_{\mathbb{R}} e^{(t+s)x} d\mu(x) = \theta(t + s). \quad (4.1.3)$$

So we must satisfy the following functional equation:

$$\theta(\rho(t))\theta(\rho(s))\Theta_\rho(ts) = \theta(\rho(t) + \rho(s)). \quad (4.1.4)$$

We have  $\theta(0) = h(0) = 1$  by Chapter 3. So setting  $s = 0$  one gets  $\Theta_\rho(0) = 1$ . Again from Chapter 3,  $\rho(0) = 0$  and  $\rho'(0) \neq 0$ . Since all the functions involved are analytic, we can take the derivative of both sides with respect to  $s$  twice to get

$$\theta'(\rho(t) + \rho(s))\rho'(s) = \theta(\rho(t))[\theta'(\rho(s))\rho'(s)\Theta_\rho(ts) + \theta(\rho(s))\Theta'_\rho(ts)t], \quad (4.1.5)$$

and

$$\theta''(\rho(t) + \rho(s))\rho'(s)^2 + \theta'(\rho(t) + \rho(s))\rho''(s) = \quad (4.1.6)$$

$$\begin{aligned} & \theta(\rho(t))[\theta''(\rho(s))\rho'(s)^2\Theta_\rho(ts) + \theta'(\rho(s))\rho''(s)\Theta_\rho(ts) \\ & + 2t\theta'(\rho(s))\rho'(s)\Theta'_\rho(ts) + t^2\theta(\rho(s))\Theta''_\rho(ts)]. \end{aligned}$$

Setting  $s = 0$  and dividing both sides by  $\rho'(0) \neq 0$  in the equation (4.1.5) yields

$$\theta(\rho(t))\left(\frac{\Theta'_\rho(0)}{\rho'(0)}t + \theta'(0)\right) = \theta'(\rho(t)). \quad (4.1.7)$$

Now setting  $s = 0$  and dividing both sides by  $\rho'(0)^2$  in the equation (4.1.6) yields

$$\theta''(\rho(t)) + \theta'(\rho(t))\frac{\rho''(0)}{\rho'(0)^2} = \quad (4.1.8)$$

$$\theta(\rho(t))\left[\frac{\Theta''_\rho(0)}{\rho'(0)^2}t^2 + \frac{2\theta'(0)\Theta'_\rho(0)}{\rho'(0)}t + \theta''(0) + \frac{\theta'(0)\rho''(0)}{\rho'(0)^2}\right].$$

Taking the derivative of the equation (4.1.7) with respect to  $t$  one gets

$$\theta''(\rho(t))\rho'(t) = \theta'(\rho(t))\rho'(t)\left[\frac{\Theta'_\rho(0)}{\rho'(0)}t + \theta'(0)\right] + \frac{\Theta'_\rho(0)}{\rho'(0)}\theta(\rho(t)). \quad (4.1.9)$$

Pulling  $\theta''(\rho(t))$  from equation (4.1.8) and  $\theta'(\rho(t))$  from equation (4.1.7) and using in (4.1.9) one gets

$$\theta(\rho(t))\left[\rho'(t)\left[\left(\frac{\Theta''_\rho(0) - (\Theta'_\rho(0))^2}{\rho'(0)^2}\right)t^2 - \frac{\Theta'_\rho(0)\rho''(0)}{\rho'(0)^3}t + \theta''(0) - (\theta'(0))^2\right] - \frac{\Theta'_\rho(0)}{\rho'(0)}\right] = 0. \quad (4.1.10)$$

It is evident that  $\theta(\rho(t)) \neq 0$  identically and if  $\Theta'_\rho(0) = 0$  then by equation (4.1.7) we have  $\theta(t) = e^{\theta'(0)t}$  which gives the trivial point measure  $\delta_{\theta'(0)}$ . So from now on, we assume  $\Theta'_\rho(0) \neq 0$ . Then one finds

$$\rho'(t) = \frac{1}{\left(\frac{\Theta''_\rho(0) - (\Theta'_\rho(0))^2}{\Theta'_\rho(0)\rho'(0)}\right)t^2 - \frac{\rho''(0)}{\rho'(0)^2}t + \frac{(\theta''(0) - (\theta'(0))^2)\rho'(0)}{\Theta'_\rho(0)}}. \quad (4.1.11)$$

Setting  $t = 0$  gives  $\frac{(\theta''(0) - (\theta'(0))^2)\rho'(0)}{\Theta'_\rho(0)} = \frac{1}{\rho'(0)}$ . Hence we arrive at a necessary condition for  $\rho(t)$  :

$$\rho'(t) = \frac{1}{\left(\frac{\Theta''_\rho(0) - (\Theta'_\rho(0))^2}{\Theta'_\rho(0)\rho'(0)}\right)t^2 - \frac{\rho''(0)}{\rho'(0)^2}t + \frac{1}{\rho'(0)}}. \quad (4.1.12)$$

In the equation (4.1.7) to find  $\theta(\rho(t))$  we can multiply both sides by  $\rho'(t)$ . Doing so and taking the integral of both sides we get

$$\theta(\rho(t)) = \exp\left(\int_0^t \rho'(y)\left(\frac{\Theta'_\rho(0)}{\rho'(0)}y + \theta'(0)\right)dy\right). \quad (4.1.13)$$

In a similar fashion, one also gets the equation for  $\Theta'_\rho(ts)$

$$\Theta_\rho(ts) = \exp\left(\int_0^t (\rho'(s+y)\left(\frac{\Theta'_\rho(0)}{\rho'(0)}(s+y) + \theta'(0)\right) - (\rho'(y)\left(\frac{\Theta'_\rho(0)}{\rho'(0)}y + \theta'(0)\right))dy\right). \quad (4.1.14)$$

## 4.2 Measures and Orthogonal Polynomials

If necessary dividing  $\rho(t)$  by  $\rho'(0) \neq 0$ , replacing the measure  $d\mu(x)$  by  $d\mu(\frac{x}{\rho'(0)})$  and arranging the interval where the measure is defined accordingly, without loss of generalization one may assume  $\rho'(0) = 1$ . Then we have

$$\rho'(t) = \frac{1}{\beta t^2 + \gamma t + 1} \quad (4.2.1)$$

$$\theta(\rho(t)) = \exp\left(\int_0^t \rho'(y)(ay + b)dy\right), \quad (4.2.2)$$

where  $\beta = \frac{\Theta''_\rho(0) - (\Theta'_\rho(0))^2}{\Theta'_\rho(0)\rho'(0)}$ ,  $\gamma = \frac{\rho''(0)}{\rho'(0)^2}$ ,  $a = \frac{\Theta'_\rho(0)}{\rho'(0)}$  and  $b = \theta'(0)$ . The equation in (4.2.2) can also be written as

$$\theta(\rho(t)) = \exp\left(\int_0^t \rho'(y)aydy\right) \cdot \exp(b\rho(t)) \quad (4.2.3)$$

If necessary shifting the measure right by  $b$  units, namely replacing  $\mu(x)$  by  $\mu(x+b)$ , we may assume  $b = 0$ . We already know  $a > 0$ . Due to the nature of  $\rho'(t)$  it is essential for us to study the quadratic form  $\beta t^2 + \gamma t + 1$  in order to get a complete classification.

**Case 1:**  $\beta = \gamma = 0$

We have  $\rho'(t) = 1$  which gives  $\rho(t) = t$ . Hence by the equation (4.2.2) we have

$$\theta(t) = \exp\left(a\frac{t^2}{2}\right).$$

So we have the Gaussian measure with mean 0 and variance  $a$ . The corresponding polynomials are the well-known Hermite polynomials.

**Case 2:**  $\beta = 0$  and  $\gamma \neq 0$

We have  $\rho'(t) = \frac{1}{1+\gamma t}$ , which means  $\rho(t) = \frac{\log(1+\gamma t)}{\gamma}$  and  $\theta\left(\frac{\log(1+\gamma t)}{\gamma}\right) = \exp\left(\frac{a}{\gamma}\right)$ . Hence

$$\theta(t) = \exp\left(\frac{a}{\gamma^2}(e^{\gamma t} - 1 - \gamma t)\right).$$

Clearly, we have the Poisson measure with parameter  $a$ . It is also well-know that the corresponding orthogonal polynomials are the Charlier polynomials.

**Case 3:**  $4\beta = \gamma^2 > 0$

We have  $\rho'(t) = \left(\frac{2}{2+\gamma t}\right)^2$  that is equivalent to  $\rho(t) = \frac{2t}{2+\gamma t}$ . Hence

$$\theta\left(\frac{2t}{2+\gamma t}\right) = \exp\left(\frac{4a}{\gamma^2}\left(\log\left(1 + \frac{\gamma}{2}t\right)\right) - \frac{\gamma t}{2+\gamma t}\right)$$

, or equivalently

$$\theta(t) = \exp\left(\frac{-4a}{\gamma^2}\left(\log\left(1 - \frac{\gamma}{2}t\right) + \frac{\gamma}{2}t\right)\right).$$

This gives the gamma distribution with parameter  $a$ . So we have the classical Laguerre polynomials.

**Case 4:**  $4\beta > \gamma^2 > 0$

Using the formula

$$\int \frac{1}{(x+A)^2 + B^2} = \frac{1}{B} \arctan\left(\frac{x+A}{B}\right)$$

we get

$$\rho(t) = \frac{2}{d}\left(\arctan\left(\frac{2\beta t + \gamma}{d}\right) - \arctan\left(\frac{\gamma}{d}\right)\right)$$

where  $d = \sqrt{4\beta - \gamma^2}$ . Moreover the summation formula for tangent, that is

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

, yields

$$\rho(t) = \frac{2}{d} \arctan\left(\frac{dt}{2 + \gamma t}\right).$$

So one gets

$$\theta(\rho(t)) = \exp\left(\frac{a}{\beta} \left(\frac{1}{2} \log(\beta t^2 + \gamma t + 1) - \frac{\gamma}{d} \arctan\left(\frac{dt}{2 + \gamma t}\right)\right)\right)$$

or with the help of some trigonometry equivalently

$$\theta(t) = \exp\left[\frac{a}{\beta} \left(\log\left(\frac{\cos(\arctan(\frac{\gamma}{d}))}{\cos(\frac{d}{2}t + \arctan(\frac{\gamma}{d}))}\right) - \frac{\gamma}{2}t\right)\right].$$

This gives the Levy stochastic area.

**Case 5:**  $4\beta < \gamma^2$

In this case  $\beta t^2 + \gamma t + 1$  will have 2 distinct real roots. Let  $p, q$  be real numbers such that  $p + q = \gamma$  and  $pq = \beta$  and  $p > q$ . Observe  $\frac{-1}{p}$  and  $\frac{-1}{q}$  are the roots of the equation  $\beta t^2 + \gamma t + 1 = 0$ . One can easily find

$$\rho(t) = \frac{1}{p - q} \log\left(\frac{1 + pt}{1 + qt}\right)$$

and

$$\theta(\rho(t)) = \frac{a}{\beta(p - q)} [p \log(1 + qt) - q \log(1 + pt)]$$

by using the regular partial fraction method on the integrals. Playing some with the equation one gets :

$$\theta(t) = \left(1 - \frac{q}{p}\right)^{\frac{a}{\beta}} \left(1 - \frac{q}{p} \exp((p - q)t \frac{-a}{\beta})\right) \exp\left(\frac{-a}{p}t\right).$$

This case gives the negative binomial if  $\beta < 0$  and the degenerate binomial if  $\beta > 0$  and  $\frac{a}{\beta}$  is an integer.

One can check that in all these mentioned cases,  $\Theta_\rho(t, s)$  is indeed a function of  $ts$ . As expected we arrive at the same class as Meixner[20] did in his original 1934 paper. This class has also been studied in the language of natural exponential families and in 1982 Morris[21] classified them as the measures with quadratic variation.

# Chapter 5

## The Fractional Case $h(x) = \frac{1}{1-x}$

### 5.1 Fractional Type

In this chapter, we will discuss the fixed function  $h(x) = \frac{1}{1-x}$ . Like in Chapter 4, in which we discussed the case  $h(x) = e^x$ , we must satisfy the following equation to be satisfied:

$$\Theta_\rho(t, s) = \Theta_\rho(ts) = \frac{\tilde{\theta}(\rho(t), \rho(s))}{\theta(\rho(t))\theta(\rho(s))}. \quad (5.1.1)$$

Now by using the identity  $\frac{1}{(1-tx)(1-sx)} = \frac{t}{t-s} \frac{1}{1-tx} + \frac{s}{s-t} \frac{1}{1-sx}$ , observe that for  $h(x) = \frac{1}{1-x}$  and  $t \neq s$  we have the following :

$$\tilde{\theta}(t, s) = \frac{t\theta(t)}{t-s} + \frac{s\theta(s)}{s-t}. \quad (5.1.2)$$

Here we may understand the relation in the limit sense for  $t = s$ . Hence, we must have

$$\Theta_\rho(ts)(\rho(t) - \rho(s))\theta(\rho(t))\theta(\rho(s)) = \rho(t)\theta(\rho(t)) - \rho(s)\theta(\rho(s)) \quad (5.1.3)$$

where  $\rho(t)$  and  $\theta(t)$  are analytic functions about 0 with  $\theta(0) = 1$ ,  $\rho(0) = 0$ , and  $\rho'(0) \neq 0$ , which follows from chapter 3. Setting  $s = 0$  in equation (5.1.3), we get  $\Theta_\rho(0) = 1$ . Taking derivatives of both sides with respect to  $s$ , we get

$$\begin{aligned} \theta(\rho(t)) \left[ t\Theta'_\rho(ts)(\rho(t) - \rho(s))\theta(\rho(s)) - \Theta_\rho(ts)\rho'(s)\theta(\rho(s)) \right. \\ \left. + \Theta_\rho(ts)(\rho(t) - \rho(s))\theta'(\rho(s))\rho'(s) \right] \\ = -\rho'(s)\theta(\rho(s)) - \theta(\rho(s))\rho(s)\rho'(s). \end{aligned} \quad (5.1.4)$$

Now by setting  $s = 0$  and dividing both sides by  $\rho'(0) \neq 0$ , we get

$$\theta(\rho(t)) = \frac{1}{1 - (at + b)\rho(t)}, \quad (5.1.5)$$

, where  $a = \frac{\Theta'_\rho(0)}{\rho'(0)}$  and  $b = \theta'(0)$ . Putting (5.1.5) in (5.1.3) and making some simplifications, we get

$$(\Theta_\rho(ts) - 1)(\rho(t) - \rho(s)) = a\rho(t)\rho(s)(t - s). \quad (5.1.6)$$

By taking derivatives with respect to  $s$  twice, we get

$$t\Theta'_\rho(ts)(\rho(t) - \rho(s)) - (\Theta_\rho(ts) - 1)\rho'(s) = a\rho(t)(\rho'(s)(t - s) - \rho(s)) \quad (5.1.7)$$

and

$$t^2\Theta''_\rho(ts)(\rho(t) - \rho(s)) - 2t\Theta'_\rho(ts)\rho'(s) - (\Theta_\rho(ts) - 1)\rho''(s) = a\rho(t)(\rho''(s)(t - s) - 2\rho'(s)). \quad (5.1.8)$$

Setting  $s = 0$  and simplifying yields:

$$2\Theta'_\rho(0)\rho'(0)t = \rho(t)[t^2\Theta''_\rho(0) - a\rho''(0)t + 2a\rho'(0)]. \quad (5.1.9)$$

If  $\Theta'_\rho(0) = 0$ , then  $a = 0$ . By equation (5.1.5) one gets  $\theta(t) = \frac{1}{1-bt}$ , which gives the point measure  $\delta_b$ . So, from now on, we may assume  $\Theta'_\rho(0) \neq 0$ , which results in the following final form for  $\rho(t)$

$$\rho(t) = \frac{2t}{\alpha t^2 + 2\beta t + \gamma}, \quad (5.1.10)$$

where  $\alpha = \frac{\Theta''_\rho(0)}{\Theta'_\rho(0)\rho'(0)}$  and  $\beta = \frac{-\rho''(0)}{2\rho'(0)^2}$  and  $\gamma = \frac{2}{\rho'(0)}$ .

Using this result in equation (5.1.6), we get

$$\Theta_\rho(ts) = 1 + \frac{2ats}{\gamma - \alpha ts} \quad (5.1.11)$$

, which is indeed a function of  $ts$ . So by the main theorem in Chapter 1 we have the following generating function for all measures which fall in this category.

$$\psi(t, x) = \frac{h(\rho(t)x)}{\theta(\rho(t))} = \frac{1 - (at + b)\rho(t)}{1 - \rho(t)x} \quad (5.1.12)$$

## 5.2 The Jacobi-Szegö Parameters

Without calculating the explicit measures or polynomials, one can calculate the Jacobi-Szegö parameters as shown below. We can employ the equations (2.2.3), (2.2.4) and (2.2.5) from Chapter 2 to find the  $\alpha_n$  and  $\omega_n$ . Putting  $t = s$  in the equation (5.1.11) we get:

$$1 + \frac{2at^2}{\gamma - \alpha t^2} = \int_{\mathbb{R}} \left( \frac{(1 - (at + b)\rho(t))}{(1 - \rho(t)x)} \right)^2 d\mu(x) = \int_{\mathbb{R}} \psi(t, x)^2 d\mu(x) = \sum_{n=0}^{\infty} a_n^2 \lambda_n t^{2n} \quad (5.2.1)$$

The evaluation of  $\{a_n\}_{n=0}^{\infty}$  in the formula is given in Chapter 3. In our particular case  $a_n = \rho'(0)^n = \left(\frac{2}{\gamma}\right)^n$  for  $n \geq 0$ . Hence:

$$\lambda_n = \frac{a\gamma}{2} \left(\frac{\alpha\gamma}{4}\right)^{n-1}, n \geq 1.$$

So

$$\omega_n = \begin{cases} \frac{a\gamma}{2}, & \text{if } n = 0, \\ \frac{\alpha\gamma}{4}, & \text{if } n \geq 1. \end{cases} \quad (5.2.2)$$

Finding  $\alpha_n$  is a little more involved:

$$\begin{aligned} & \sum_{n=0}^{\infty} \left( a_n^2 \alpha_n \lambda_n t^{2n} + 2a_n a_{n-1} \lambda_n t^{2n-1} \right) \\ &= \int_{\mathbb{R}} x \psi(t, x)^2 d\mu(x) \\ &= (1 - (at + b)\rho(t))^2 \int_{\mathbb{R}} \frac{x}{(1 - \rho(t)x)^2} d\mu(x) \\ &= (1 - (at + b)\rho(t))^2 \frac{1}{\rho(t)} \left[ \int_{\mathbb{R}} \frac{1}{(1 - \rho(t)x)^2} d\mu(x) - \int_{\mathbb{R}} \frac{1}{(1 - \rho(t)x)} d\mu(x) \right] \\ &= (1 - (at + b)\rho(t))^2 \frac{1}{\rho(t)} \left[ \frac{1 + \frac{2at^2}{\gamma - \alpha t^2}}{(1 - (at + b)\rho(t))^2} - \frac{1}{1 - (at + b)\rho(t)} \right] \\ &= b + 2at \frac{\gamma + \beta t}{\gamma - \alpha t^2}. \end{aligned}$$

Hence we get

$$\alpha_n = \begin{cases} b, & \text{if } n = 0, \\ \beta, & \text{if } n \geq 1. \end{cases} \quad (5.2.3)$$

### 5.3 The Orthogonal Polynomials

In this section we will find the formal orthogonal polynomials. By formal we mean with respect to a linear functional not necessarily with respect to a measure on  $\mathbb{R}$ .

We first state a simple identity without a proof:

**Lemma 5.1 (A Simple Identity).** *For a suitable interval for  $z$  about 0 we have the following identity :*

$$\frac{1}{1 + Az + Bz^2 - zx} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-m}{m} \left( \frac{-B}{(x-A)^2} \right)^m \right) ((x-A)z)^n.$$

Let us define  $Q_{-1} = Q_{-2} = 0$  and for  $n \geq 0$

$$Q_n(x) = \left( \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-m}{m} \left( \frac{-\alpha\gamma}{(x-\beta)^2} \right)^m \right) (x-\beta)^n.$$

Now we have our generating function

$$\begin{aligned} \psi(t, x) &= \frac{h(\rho(t)x)}{\theta(\rho(t))} \\ &= \frac{1 - (at + b)\rho(t)}{1 - \rho(t)x} \\ &= \frac{1 + (\beta - b)\left(\frac{2t}{\gamma}\right) + \left(\frac{(\alpha - 2a)\gamma}{4}\right)\left(\frac{2t}{\gamma}\right)^2}{1 - \frac{2t}{\gamma}\left(x - \beta - \frac{\alpha\gamma}{4}\left(\frac{2t}{\gamma}\right)\right)}. \end{aligned}$$

Using the lemma stated above, we get

$$\psi(t, x) = \sum_{n=0}^{\infty} \left(\frac{2}{\gamma}\right)^n P_n(x) t^n,$$

where

$$P_n(x) = Q_n(x) + (\beta - b)Q_{n-1}(x) + \left(\frac{(\alpha - 2a)\gamma}{4}\right)Q_{n-2}(x).$$

The very first few polynomials are given as:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x - b \\ P_2(x) &= x^2 - (\beta + b)x + \frac{2b\beta - a\gamma}{2} \end{aligned}$$

One can also observe the relation between Chebyshev polynomials of the second kind  $U_n(x)$  and  $Q_n(x)$  given as

$$Q_n(x) = \left(\frac{\sqrt{\alpha\gamma}}{2}\right)^n U_n\left(\frac{x-\beta}{\sqrt{\alpha\gamma}}\right)$$

## 5.4 The Corresponding Measures

From equation 5.2.2 we know that  $w_n$  are given as below:

$$\omega_n = \begin{cases} \frac{a\gamma}{2}, & \text{if } n = 0, \\ \frac{\alpha\gamma}{4}, & \text{if } n \geq 1. \end{cases} \quad (5.4.1)$$

To be able to have a real measure on  $\mathbb{R}$ , we must have  $w_n > 0$ . Hence we must have  $a, \alpha, \gamma$  of the same sign and non-zero. Shohat and Tamarkin[23] in the theorem 1.11 of their book give the uniqueness condition as below:

**Lemma 5.2.** *The moment problem has a unique solution if  $\sum_{n=0}^{\infty} \frac{1}{\lambda_n^{\frac{1}{2n}}} = \infty$ .*

This is clearly our case since  $\lim_{n \rightarrow \infty} \lambda_n^{\frac{1}{2n}} = \frac{2}{\sqrt{\alpha\gamma}} \neq 0$ . Now let the measure  $\mu$  is a solution for the equation

$$\theta(\rho(t)) = \int_{\mathbb{R}} \frac{1}{1 - \rho(t)x} d\mu(x) = \frac{1}{1 - (at + b)\rho(t)} \quad (5.4.2)$$

with

$$\rho(t) = \frac{2t}{\alpha t^2 + 2\beta t + \gamma}. \quad (5.4.3)$$

Substituting  $x = \sqrt{\alpha\gamma}y + \beta$ ,  $A = \frac{a}{\alpha}$ ,  $B = \frac{b-\beta}{\sqrt{\alpha\gamma}}$ ,  $t = \sqrt{\frac{\gamma}{\alpha}} \frac{z}{1+\sqrt{1-z^2}}$  and

$dv(y) = d\mu(\sqrt{\alpha\gamma}y + \beta)$  we get that we must solve the following equation which holds for small  $z$ :

$$\int_{\mathbb{R}} \frac{1}{1 - zy} dv(y) = \frac{1}{1 - A - Bz + A\sqrt{1-z^2}} \quad (5.4.4)$$

with the knowledge of the uniqueness of the measure  $v(y)$  and  $A > 0$ . If necessary, by replacing  $dv(y)$  by  $dv(-y)$ , one may assume  $B \geq 0$ . Now we will prove the following theorem

**Theorem 5.3.** *The unique solution given to the above equation is given by the measure*

$$dv(y) = W_0 \frac{1}{(1-py)(1-qy)} \frac{2\sqrt{1-y^2}}{\pi} dy 1_{[-1,1]} + W_1 \delta_{\frac{1}{p}} + W_2 \delta_{\frac{1}{q}}$$

, where  $p = \frac{B(1-A)+A\sqrt{B^2+2A-1}}{A^2+B^2}$  and  $q = \frac{B(1-A)-A\sqrt{B^2+2A-1}}{A^2+B^2}$ . Here  $W_0 = \frac{A}{2(A^2+B^2)}$ ,  $W_1 = \frac{1-A-pB-A\sqrt{1-p^2}}{p(q-p)(A^2+B^2)}$  and  $W_2 = \frac{1-A-qB-A\sqrt{1-q^2}}{q(p-q)(A^2+B^2)}$ .

To prove this theorem we need an integral formula

**Lemma 5.4.** *For  $|t| \leq 1$  or  $t$  with non-zero imaginary part we have:*

$$\int_{-1}^1 \frac{1}{1-tx} \frac{2\sqrt{1-x^2}}{\pi} dx = \frac{2}{1+\sqrt{1-t^2}} = \frac{2(1-\sqrt{1-t^2})}{t^2}.$$

First let us make some simple calculations related to  $p$  and  $q$ :

$$\begin{aligned} pq &= \frac{1-2A}{A^2+B^2} \\ p+q &= \frac{2B(1-A)}{A^2+B^2} \\ p-q &= \frac{2A\sqrt{B^2+2A-1}}{A^2+B^2} \\ \sqrt{1-p^2} &= \frac{|A(1-A)-B\sqrt{B^2+2A-1}|}{A^2+B^2} \\ \sqrt{1-q^2} &= \frac{|A(1-A)+B\sqrt{B^2+2A-1}|}{A^2+B^2} \end{aligned}$$

Now we may calculate:

$$\int_{\mathbb{R}} \frac{1}{1-zy} dv(y) = W_0 \int_{-1}^1 \frac{1}{(1-zy)(1-py)(1-qy)} \frac{2\sqrt{1-y^2}}{\pi} dy + W_1 \frac{1}{1-\frac{z}{p}} + W_2 \frac{1}{1-\frac{z}{q}}$$

Observe that

$$\begin{aligned} \frac{1}{(1-zy)(1-py)(1-qy)} &= \frac{z^2}{(z-p)(z-q)} \frac{1}{(1-zy)} \\ &+ \frac{p^2}{(p-z)(p-q)} \frac{1}{(1-py)} + \frac{q^2}{(q-z)(q-p)} \frac{1}{(1-qy)} \end{aligned}$$

and hence by using the integral lemma above we must have the following equality:

$$\begin{aligned}
W_0 & \left[ \frac{z^2}{(z-p)(z-q)} \frac{2(1-\sqrt{1-z^2})}{z^2} + \frac{p^2}{(p-z)(p-q)} \frac{2(1-\sqrt{1-p^2})}{p^2} \right. \\
& \left. + \frac{q^2}{(q-z)(q-p)} \frac{2(1-\sqrt{1-q^2})}{q^2} \right] + W_1 \frac{p}{p-z} + W_2 \frac{q}{q-z} \\
& = \frac{1}{1-A-Bz+A\sqrt{1-z^2}} = \frac{1-A-Bz-A\sqrt{1-z^2}}{(A^2+B^2)(z-p)(z-q)}
\end{aligned}$$

or equivalently by multiplying all sides by  $(A^2+B^2)(z-p)(z-q)$  and simplifying some we come up with the equation:

$$\begin{aligned}
& -2W_0(A^2+B^2) + 2W_0(A^2+B^2)\sqrt{1-z^2} \\
& + (z-p) \left[ \frac{2W_0(A^2+B^2)(1-\sqrt{1-q^2})}{q-p} + W_2q(A^2+B^2) \right] \\
& + (z-q) \left[ \frac{2W_0(A^2+B^2)(1-\sqrt{1-p^2})}{p-q} + W_1p(A^2+B^2) \right] = -1+2A+Bz+A\sqrt{1-z^2}
\end{aligned}$$

By equating the coefficients of the powers of  $z$ , one can easily calculate

$$\begin{aligned}
W_0 & = \frac{A}{2(A^2+B^2)} \\
W_1 & = \frac{1-A-pB-A\sqrt{1-p^2}}{p(q-p)(A^2+B^2)} \\
& = \frac{|A(1-A)-B\sqrt{B^2+2A-1}| - (A(1-A)-B\sqrt{B^2+2A-1})}{2p(A^2+B^2)\sqrt{B^2+2A-1}} \\
W_2 & = \frac{1-A-qB-A\sqrt{1-q^2}}{q(p-q)(A^2+B^2)} \\
& = -\frac{|A(1-A)+B\sqrt{B^2+2A-1}| - (A(1-A)+B\sqrt{B^2+2A-1})}{2q(A^2+B^2)\sqrt{B^2+2A-1}}
\end{aligned}$$

For  $0 < A < 1$ , we have  $W_2 = 0$ . And again in this case if  $B \leq 1-A$  we have  $W_1 = 0$  and if  $B > 1-A$  then we have a positive real weight  $W_1$ . Now if  $A > 1$  then we have  $B^2+2A-1 > 0$ ,  $p > 0$  and  $q < 0$ . Hence both  $W_1$  and  $W_2$  are positive real numbers. For different cases, the values of  $A$  and  $B$ ,  $W_1$  and  $W_2$  are given in the following table

**TABLE 1**Table of Values of  $A$  and  $B$ 

<i>Cases</i>	$W_1$	$W_2$
$0 < A \leq 1$ and $B \leq 1 - A$	0	0
$0 < A \leq 1$ and $B > 1 - A$	$\frac{B^2 - (1-A)^2}{(AB + (1-A)\sqrt{B^2 + 2A - 1})\sqrt{B^2 + 2A - 1}}$	0
$1 < A$ and $B \leq A - 1$	$\frac{(AB + (A-1)\sqrt{B^2 + 2A - 1})}{(2A - 1)\sqrt{B^2 + 2A - 1}}$	$\frac{A(A-1)}{(B(A-1) + A\sqrt{B^2 + 2A - 1})\sqrt{B^2 + 2A - 1}}$
$1 < A$ and $B > A - 1$	$\frac{(AB + (A-1)\sqrt{B^2 + 2A - 1})}{(2A - 1)\sqrt{B^2 + 2A - 1}}$	$\frac{B}{(B(A-1) + A\sqrt{B^2 + 2A - 1})}$

# Chapter 6

## The Case $h(x) = \frac{1}{(1-x)^2}$

### 6.1 Generating Functions

In this case, like in the previous one, we must satisfy

$$\Theta_\rho(t, s) = \Theta_\rho(ts) = \frac{\tilde{\theta}(\rho(t), \rho(s))}{\theta(\rho(t))\theta(\rho(s))}. \quad (6.1.1)$$

Let us define a new function  $g$ :

$$g(t) = \int_{\mathbb{R}} \frac{1}{1-tx} d\mu(x). \quad (6.1.2)$$

Using the same identity from Chapter 5

$$\frac{1}{1-Az} \frac{1}{1-Bz} = \frac{1}{A-B} \left[ \frac{A}{1-Az} - \frac{B}{1-Bz} \right].$$

Observe that for  $h(x) = \frac{1}{(1-x)^2}$ , we have:

$$\begin{aligned} \tilde{\theta}(\rho(t), \rho(s)) &= \int_{\mathbb{R}} \frac{1}{((1-\rho(t)x)(1-\rho(s)x))^2} d\mu(x) \\ &= \frac{1}{(\rho(t) - \rho(s))^2} \int_{\mathbb{R}} \left( \frac{\rho(t)}{1-\rho(t)x} - \frac{\rho(s)}{1-\rho(s)x} \right)^2 d\mu(x) \\ &= \frac{\left[ \rho(t)^2\theta(\rho(t)) + \rho(s)^2\theta(\rho(s)) - \frac{2\rho(t)\rho(s)}{\rho(t)-\rho(s)}(\rho(t)g(\rho(t)) - \rho(s)g(\rho(s))) \right]}{(\rho(t) - \rho(s))^2}. \end{aligned} \quad (6.1.3)$$

Hence we must have

$$\Theta_\rho(ts)(\rho(t) - \rho(s))^3\theta(\rho(t))\theta(\rho(s)) = \quad (6.1.4)$$

$$\left[ \rho(t)^2\theta(\rho(t)) + \rho(s)^2\theta(\rho(s)) \right] [\rho(t) - \rho(s)] - 2\rho(t)\rho(s) [\rho(t)g(\rho(t)) - \rho(s)g(\rho(s))],$$

where  $\rho(t)$  is an analytic function about 0 with  $\rho(0) = 0$  and  $\rho'(0) \neq 0$ . Moreover, both  $\theta(t) = \int_{\mathbb{R}} \frac{1}{(1-tx)^2} d\mu(x)$  and  $g(t) = \int_{\mathbb{R}} \frac{1}{1-tx} d\mu(x)$  are analytic about 0 and

$\theta(0) = g(0) = 1$ . Setting  $s = 0$  in Equation (6.1.4) we get  $\Theta_\rho(0) = 1$ . Taking derivatives of both sides with respect to  $s$  and setting  $s = 0$  yields

$$g(\rho(t)) = \left[1 - (at + b)\rho(t)\right]\theta(\rho(t)), \quad (6.1.5)$$

where  $\Theta'_\rho(0) = 2ac$  and  $\theta'(0) = 2b$  and  $\rho'(0) = c \neq 0$ . Observe that if  $a = 0$ , we get the trivial measure  $\delta_b$ . So from now on, we will assume  $a \neq 0$ . Putting (6.1.5) in (6.1.4), we get

$$\begin{aligned} \Theta_\rho(ts)(\rho(t) - \rho(s))^3\theta(\rho(t))\theta(\rho(s)) &= [\rho(t)^2\theta(\rho(t) + \rho(s)^2\theta(\rho(s))](\rho(t) - \rho(s)) \\ -2\rho(t)\rho(s)[\rho(t)(1 - (at + b)\rho(t))\theta(\rho(t) - \rho(s)(1 - (as + b)\rho(s))\theta(\rho(s))] & \quad (6.1.6) \end{aligned}$$

Taking derivatives with respect to  $s$  twice and setting  $s = 0$  gives

$$\theta(\rho(t)) = \frac{1}{1 - 2(at + b)\rho(t) + (dt^2 + et + f)\rho(t)^2}, \quad (6.1.7)$$

where  $\Theta''_\rho(0) = 6dc^2$ ,  $\theta''(0) = 6f$  and  $p''(0) = \frac{4abc^2 - 3c^2e}{a}$ . Now taking the derivative of  $\theta(\rho(t))$  twice and setting  $t = 0$  gives  $f = b^2 + \frac{a}{2c}$ . Putting this back in to (6.1.6) we get that we must have

$$\begin{aligned} \Theta_\rho(ts)(\rho(t) - \rho(s))^3 &= [\rho(t)^2(1 - 2(as + b)\rho(s) + (ds^2 + es + b^2 + \frac{a}{2c})\rho(s)^2) \\ &+ \rho(s)^2(1 - 2(at + b)\rho(t) + (dt^2 + et + b^2 + \frac{a}{2c})\rho(t)^2)](\rho(t) - \rho(s)) \\ -2\rho(t)\rho(s)[\rho(t)(1 - (at + b)\rho(t))(1 - 2(as + b)\rho(s) + (ds^2 + es + b^2 + \frac{a}{2c})\rho(s)^2) \\ -\rho(s)(1 - (as + b)\rho(s))(1 - 2(at + b)\rho(t) + (dt^2 + et + b^2 + \frac{a}{2c})\rho(t)^2)]. & \quad (6.1.8) \end{aligned}$$

Taking derivatives with respect to  $s$  three times and setting  $s = 0$  gives:

$$(At^3 + Bt^2 + Ct + D)\rho(t)^2 - 2(Et^2 + Ft + G)\rho(t) + t = 0 \quad (6.1.9)$$

where  $A = \frac{h^{(3)}(0)}{24ac^3}$ ,  $B = \frac{9de-10abd}{4a^2}$ ,  $C = \frac{27e^2c^3+42a^2b^2c^3-66abec^3+9a^3c^2-a^2\rho^{(3)}(0)}{12a^2c^3}$ ,  
 $D = \frac{e}{2ac}$ ,  $E = \frac{d}{a}$ ,  $F = \frac{e-ab}{a}$ ,  $G = \frac{1}{2c}$ .

By taking the 3<sup>rd</sup> derivative of the equation (6.1.9) and setting  $t = 0$ , we get

$$\rho^{(3)}(0) = \frac{c^2(3a^3 + 30a^2b^2c - 42abec + 15ce^2 - 8ad)}{a^2}.$$

This gives  $C = \frac{6a^2b^2c+3a^3-12abce+6ce^2+4ad}{6a^2c}$ .

Solving equation (6.1.9) for  $\rho(t)$  we get

$$\rho(t) = \frac{Q_2 \pm \sqrt{Q_2^2 - tQ_3}}{Q_3} \quad (6.1.10)$$

where  $Q_2 = Et^2 + Ft + G$  and  $Q_3 = At^3 + Bt^2 + Ct + D$ . Now taking the fourth derivative of both sides in equation (6.1.8), setting  $s = 0$  and simplifying the alone  $t$  by using (6.1.9), we get

$$R_4\rho(t)^2 + R_3\rho(t) + R_2 = 0 \quad (6.1.11)$$

where  $R_4, R_3$  and  $R_2$  are polynomials of degree 4, 3 and 2 respectively. (One can observe  $R_3 \neq \frac{2Q_2R_2}{t}$  so this is not a trivial multiple of equation (6.1.9)). One can pull  $\rho(t)^2$  from equation (6.1.9) and using here we get

$$\rho(t) = \frac{tR_4 - R_2}{2Q_2R_4 + R_3} = \frac{S_5}{S_6} \quad (6.1.12)$$

where  $S_6$  and  $S_5$  are polynomials of degree 6 and 5 respectively.

Comparing (6.1.10) with (6.1.12), it is evident that  $Q_2^2 - tQ_3$  must be a perfect square. If  $Q_2^2 - tQ_3 = T_2^2$  then  $(Q_2 - T_2)(Q_2 + T_2) = tQ_3$ , and hence,  $Q_2 \pm T_2 = tT_1$ , where  $T_2, T_1$  are polynomials of degree 2 and 1 respectively. This gives us the following formula for  $\rho(t)$ :

$$\rho(t) = \frac{2t}{\alpha t^2 + \beta t + \gamma}. \quad (6.1.13)$$

Using (6.1.13) in (6.1.9) we get that we must have the following :

$$4(At^3 + Bt^2 + Ct + D)t - 4(Et^2 + Ft + G)(\alpha t^2 + \beta t + \gamma) + (\alpha t^2 + \beta t + \gamma)^2 = 0.$$

So, we must have

$$4A - 4E\alpha + \alpha^2 = 0 \quad (6.1.14)$$

$$4B - 4F\alpha - 4E\beta + 2\alpha\beta = 0 \quad (6.1.15)$$

$$4C - 4G\alpha - 4F\beta - 4E\gamma + 2\alpha\gamma + \beta^2 = 0 \quad (6.1.16)$$

$$4D - 4G\beta - 4F\gamma + 2\beta\gamma = 0 \quad (6.1.17)$$

$$\gamma^2 - 4G\gamma = 0 \quad (6.1.18)$$

By the equation (6.1.18) we get  $\gamma = 4G$  ( $\gamma = 0$  gives contradiction ( $\rho'(0) = \frac{2}{\gamma}$ )).

By using it in the (6.1.17), we get  $\beta = 4F - \frac{D}{G}$  and finally using this in the (6.1.16)

we get  $\alpha = \frac{4F\beta + 4E\gamma - \beta^2 - 4C}{2\gamma - 4G}$ . Using the definition of  $A, \dots, G$  we get

$$\alpha = \frac{16ad + 12abce - 3e^2c - 6a^3 - 12a^2b^2c}{6a^2}, \beta = \frac{3e - 4ab}{a} \text{ and } \gamma = \frac{2}{c}. \quad (6.1.19)$$

Now we will use these relations in (6.1.14) and (6.1.15) to get the following:

$$A = \frac{\alpha(4\frac{d}{a} - \alpha)}{4} \quad (6.1.20)$$

and

$$(2ab - e)(3d - 2a\alpha) = 0$$

Now putting (6.1.13) back in (6.1.8) and using  $c = \frac{2}{\gamma}$  and  $e = \frac{a\beta + 4ab}{3}$  we get

$$(\Theta_\rho(ts) - 1)(\alpha st - \gamma)^3 + 4\alpha(a - d\alpha)t^3s^3 - 12\gamma(a - d\alpha)s^2t^2 + 4a\gamma^2st = \quad (6.1.21)$$

$$4s^3t^3 \frac{X(s+t) + Y + 3atsZ}{3(t-s)^2}$$

where  $X = 12(3d - 2a\alpha)(2b - \beta)$ ,  $Y = 9a^2\gamma - 24d\gamma + 4ab^2 + a\beta^2 + 9a\alpha\gamma - 4ab\beta$

and  $Z = \alpha^2 - 4a\alpha + 4d$ . So we must have

$$\alpha^2 - 4a\alpha + 4d = 0 \quad (6.1.22)$$

$$9a^2\gamma - 24d\gamma + 4ab^2 + a\beta^2 + 9a\alpha\gamma - 4ab\beta = 0 \quad (6.1.23)$$

$$(3d - 2a\alpha)(2b - \beta) = 0 \quad (6.1.24)$$

A simple calculation will give

$$\Theta_\rho(t, s) = \Theta_\rho(ts) = \frac{1 - (3 - 4\frac{a}{\alpha})\frac{\alpha ts}{\gamma}}{(1 - \frac{\alpha ts}{\gamma})^3}. \quad (6.1.25)$$

In the following sections we will talk about two cases shaped around the equation (6.1.24).

## 6.2 Corresponding Measures, Jacobi-Szego parameters and Polynomials

Like in the previous Chapter, after necessary transformations, one can safely assume  $\alpha = \gamma = 1$ ,  $\beta = 0$ . In this case we have two cases by equation

**Case 1**  $\beta = 2b = 0$

In this case using the equations 6.17 and 6.18 one gets  $a = 1$  or  $a = \frac{2}{3}$ . So, one gets  $g(\rho(t)) = 1 + t^2$  and  $g(\rho(t)) = (1 + t^2)(1 - \frac{t^2}{3})$  respectively. As a result one gets the measures  $d\mu(x) = \frac{2\sqrt{1-x^2}}{\pi}$  and  $d\mu(x) = \frac{8(1-x^2)\sqrt{1-x^2}}{3\pi}$  on  $[-1, 1]$ , respectively. Hence the corresponding polynomials are the Gegenbauer polynomials with parameters 1 and 2. Jacobi–Szegő parameters are well-known in the literature.

**Case 2**  $3d = 2a\alpha$

In this case by equations 2.17 and 2.18 one gets  $a = 3/4$  and  $b = \pm\frac{1}{4}$ . Like in the previous case one gets  $g(\rho(t)) = (1 \mp \frac{t}{2})(1 + t^2)$ . Hence the measures are  $d\mu(x) = \frac{2(1-x^2)\sqrt{1-x^2}}{(1\pm x)\pi}$  on  $[-1, 1]$ . These are the Jacobi measures with  $(a, b) = (2, 1)$  or  $(a, b) = (1, 2)$ . Hence the corresponding polynomials are special Jacobi polynomials. Their Jacobi–Szegő parameters are well-known as well.

So we reached the following theorem :

**Theorem 6.1.** *There are only 4 measures along with the trivial delta measure for the case  $c = -2$ . The following table shows the measures in terms of  $d\mu(x) = \frac{2}{\pi}\sqrt{1-x^2}$ , Jacobi-Szegő parameters and corresponding polynomials*

**TABLE 2**The Complete Classification of Generating Functions for Case  $c=-2$ 

Measures	$g(\rho(t))$	Polynomials	$\alpha_n, \omega_n$
$\mu$	$(1 + t^2)$	$P_n(x) - \frac{n-1}{4(n+1)}P_{n-2}(x)$	$0, \frac{1}{4}$
$\frac{4(1-x^2)\mu}{3}$	$(1 + t^2)(1 - \frac{t^2}{3})$	$P_n(x)$	$0, \frac{(n+1)(n+4)}{4(n+2)(n+3)}$
$(1 \mp x)\mu$	$(1 \mp \frac{t}{2})(1 + t^2)$	$P_n(x) \mp \frac{n}{2(n+1)}P_{n-1}(x)$	$\frac{\mp 1}{2(n+1)(n+2)}, \frac{(n+1)(n+3)}{4(n+2)^2}$

# Chapter 7

## Verification of Four Measures for the Case $h(x) = \frac{1}{(1-x)^c}$

In this chapter we will verify 4-measures which works for the case  $\frac{1}{(1-x)^c}$ . In the AKK series, they gave only the beta type distribution with the parameter  $c$ . Let  $\mu$  be the beta-type distribution with parameter  $\beta > -1/2$  given by

$$d\mu(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1/2)} (1 - x^2)^{\beta-1/2} dx, \quad |x| < 1,$$

where  $\Gamma(\cdot)$  is the Gamma function. In the paper by AKK,  $\frac{1}{(1-\rho(t)x)^\beta}$  was used to find a generating function. Then they got

$$E_\mu \varphi(t, \cdot) = \left( \frac{2}{1 + \sqrt{1 - \rho(t)^2}} \right)^\beta.$$

After heavy computation which is not given in that paper, they got

$$\rho(t) = \frac{2t}{t^2 + 1}.$$

Here is that computation

$$E_\mu \varphi(t, \cdot) \varphi(s, \cdot) = \left( \frac{2}{1 + \sqrt{1 - \rho(t)^2}} \right)^\beta \left( \frac{2}{1 + \sqrt{1 - \rho(s)^2}} \right)^\beta \\ \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\beta + 1) \Gamma(\beta)}{\Gamma(\beta + 1 + n) \Gamma(\beta - n)} \left( \frac{\rho(t) \rho(s)}{(1 + \sqrt{1 - \rho(t)^2})(1 + \sqrt{1 - \rho(s)^2})} \right)^n$$

So they found the generating function of

$$\psi(t, x) = \frac{1}{(1 - 2tx + t^2)^\beta}.$$

This generating function gives the classical Gegenbauer Polynomials.

Now in this paper, we are going to prove that  $\frac{1}{(1-\rho(t)x)^{\beta+1}}$  can also be used as a pre-generating function. Calculation shows us that

$$E_\mu \varphi(t, \cdot) = \frac{1}{\sqrt{1 - \rho(t)^2}} \left( \frac{2}{1 + \sqrt{1 - \rho(t)^2}} \right)^\beta.$$

A further calculation shows us

$$E_\mu \varphi(t, \cdot) \varphi(s, \cdot) = \frac{2^{2\beta} ((1 + \sqrt{1 - \rho(t)^2})(1 + \sqrt{1 - \rho(s)^2}) + \rho(t)\rho(s))}{\sqrt{1 - \rho(t)^2} \sqrt{1 - \rho(s)^2}}$$

$$\frac{\left( (1 + \sqrt{1 - \rho(t)^2})(1 + \sqrt{1 - \rho(s)^2}) \right)^\beta}{\left( (1 + \sqrt{1 - \rho(t)^2})(1 + \sqrt{1 - \rho(s)^2}) - \rho(t)\rho(s) \right)^{2\beta+1}}$$

so we must use the same

$$\rho(t) = \frac{2t}{t^2 + 1}.$$

So we get the generating function of

$$\psi(t, x) = \frac{1 - t^2}{(1 - 2tx + t^2)^{\beta+1}}.$$

Polynomials can be done very similarly. (Observe  $a_n = \frac{2^n}{n!} \frac{\Gamma(\beta+1+n)}{\Gamma(\beta+1)}$ ) Now for Szegő-Jacobi parameters observe  $\mu$  is symmetric so  $\alpha_n = 0$ . To find the  $\omega_n$ , one can evaluate

$$E_\mu \psi(t, x)^2 = \frac{1 + t^2}{(1 - t^2)^{2\beta+1}} = \sum_{n=0}^{\infty} \frac{2(\beta + n)\Gamma(2\beta + n)}{\Gamma(2\beta + 1)n!} t^{2n}$$

so by some calculation we get

$$\lambda_n = \frac{2^{1-2n}(\beta + n)\Gamma(2\beta + n)n!\Gamma(\beta + 1)^2}{\Gamma(2\beta + 1)\Gamma(\beta + 1 + n)^2}, n \geq 0.$$

So we get

$$\omega_n = \frac{(n + 1)(n + 2\beta)}{4(\beta + n + 1)(\beta + n)} n \geq 0$$

This is the same as the results in the AKK paper which tells that the polynomials should be same as well. With similar methods one can prove that

$$\psi(t, x) = \frac{1 - t}{(1 - 2tx + t^2)^{\beta+1}}$$

and

$$\psi(t, x) = \frac{1 + t}{(1 - 2tx + t^2)^{\beta+1}}$$

are generating functions for the measures

$$d\mu(x) = \frac{1 - x}{\sqrt{\pi}} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1/2)} (1 - x^2)^{\beta-1/2} dx, \quad |x| < 1,$$

and

$$d\mu(x) = \frac{1 + x}{\sqrt{\pi}} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1/2)} (1 - x^2)^{\beta-1/2} dx, \quad |x| < 1.$$

# Chapter 8

## Complete Classification for General $c$

### 8.1 The Functional Equation for the Generating Functions

Let

$$\int_{\mathbb{R}} \frac{1}{(1 - \rho(t)x)^c} d\mu(x) = f(t) \quad (8.1.1)$$

By the AKK theorem we must have the following:

$$\int_{\mathbb{R}} \frac{1}{(1 - \rho(t)x)^c} \frac{1}{(1 - \rho(s)x)^c} d\mu(x) = f(t)f(s)h(st) = A \quad (8.1.2)$$

We also know to be able to have generating function we must have  $c$  different from negative integers and 0. Moreover we must have  $\rho(0) = 0$  and  $\rho'(0) \neq 0$ . Clearly  $f(0) = h(0) = 1$  Now by taking the derivative of both sides with respect to  $s$  we get:

$$\int_{\mathbb{R}} \frac{1}{(1 - \rho(t)x)^c} \frac{c\rho'(s)x}{(1 - \rho(s)x)^{c+1}} d\mu(x) = \frac{\partial A}{\partial s} \quad (8.1.3)$$

Hence we get

$$\int_{\mathbb{R}} \frac{1}{(1 - \rho(t)x)^c} \frac{1}{(1 - \rho(s)x)^{c+1}} d\mu(x) = \frac{\partial A}{\partial s} \frac{\rho(s)}{c\rho'(s)} + A \quad (8.1.4)$$

Now taking derivative with respect to  $t$  this time gives us

$$\int_{\mathbb{R}} \frac{c\rho'(t)x}{(1 - \rho(t)x)^{c+1}} \frac{1}{(1 - \rho(s)x)^{c+1}} d\mu(x) = \frac{\partial^2 A}{\partial ts} \frac{\rho(s)}{c\rho'(s)} + \frac{\partial A}{\partial t} \quad (8.1.5)$$

Hence we get

$$\int_{\mathbb{R}} \frac{1}{(1 - \rho(t)x)^{c+1}} \frac{1}{(1 - \rho(s)x)^{c+1}} d\mu(x) = \left( \frac{\partial^2 A}{\partial ts} \frac{\rho(s)}{c\rho'(s)} + \frac{\partial A}{\partial t} \right) \frac{\rho(t)}{c\rho'(t)} + \frac{\partial A}{\partial s} \frac{\rho(s)}{c\rho'(s)} + A \quad (8.1.6)$$

On the other hand by using equation (8.1.4) for  $t$  and  $s$  we have:

$$\int_{\mathbb{R}} \frac{1}{(1 - \rho(t)x)^{c+1}} \frac{1}{(1 - \rho(s)x)^{c+1}} d\mu(x) = \frac{\rho(t)\left(\frac{\partial A}{\partial t} \frac{\rho(t)}{c\rho'(t)} + A\right) - \rho(s)\left(\frac{\partial A}{\partial s} \frac{\rho(s)}{c\rho'(s)} + A\right)}{\rho(t) - \rho(s)}. \quad (8.1.7)$$

Now equating the equations 8.1.6 and 8.1.7 we see that we must have

$$c\left(\frac{\partial A}{\partial t} \rho'(s) - \frac{\partial A}{\partial s} \rho'(t)\right) = (\rho(t) - \rho(s)) \frac{\partial^2 A}{\partial ts}. \quad (8.1.8)$$

By definition of  $A$  we evaluate the partial derivative with respect to  $t$

$$\frac{\partial A}{\partial t} = f(s)(f'(t)h(ts) + sh'(st)f(t)), \quad (8.1.9)$$

and with respect to  $ts$

$$\frac{\partial^2 A}{\partial ts} = f'(s)(f'(t)h(ts) + sh'(st)f(t)) + f(s)(f'(t)th'(ts) + h'(st)f(t) + sth''(st)f(t)). \quad (8.1.10)$$

Putting these in equation (8.1.8) we find that we must have:

$$\begin{aligned} c\left[(f(s)(f'(t)h(ts) + sh'(st)f(t))\rho'(s) - f(t)(f'(s)h(ts) + th'(st)f(s))\rho'(t))\right] = \\ (\rho(t) - \rho(s))\left[f'(s)(f'(t)h(ts) + sh'(st)f(t)) + f(s)(f'(t)th'(ts) + h'(st)f(t) \right. \\ \left. + sth''(st)f(t)\right]. \end{aligned} \quad (8.1.11)$$

In this equation setting  $s = 0$  gives

$$f'(t)(c - (at + b)p(t)) = f(t)(ap(t) + c(at + b)p'(t)), \quad (8.1.12)$$

where  $a = \frac{h'(0)}{p'(0)}$  and  $b = \frac{f'(0)}{p'(0)}$ . Using this information in equation (8.1.11) one gets,

$$p(t) = \frac{2t}{t^2 + 1}. \quad (8.1.13)$$

If necessary one would use a transformation as in Chapter 6. Then the equation 8.1.12 becomes

$$f'(t) = 2f(t) \frac{bc + (ac + a)t - bct^2 + (a - ac)t^3}{(1 + t^2)(c - 2bt + (c - 2a)t^2)} \quad (8.1.14)$$

and the equation 8.1.11 becomes

$$\begin{aligned} & (2ac^2 + 2ac^3 - 4abcs + 2ac^2s^2 - 2ac^3s^2 - 4abct - 4a^2st - 4a^2cst + 4ac^2st + 2ac^2t^2 - 2ac^3t^2 \\ & \quad + 4a^2s^2t^2 - 4a^2cs^2t^2 - 2ac^2s^2t^2 + 2ac^3s^2t^2)h(st) \\ & + (-c^2 - c^3 + 2bcs - c^2s^2 + c^3s^2 + 2bct - 4b^2st + 4b^2cst + c^2st + 3c^3st + 4abcs^2t \\ & - 8bc^2s^2t + c^2s^3t - 4ac^2s^3t + c^3s^3t - c^2t^2 + c^3t^2 + 4abcs^2t^2 - 8bc^2s^2t^2 + 4a^2s^2t^2 + 4b^2s^2t^2 \\ & + 4a^2cs^2t^2 + 4b^2cs^2t^2 - c^2s^2t^2 - 8ac^2s^2t^2 + 3c^3s^2t^2 - 2bcs^3t^2 + 4abcs^3t^2 + c^2st^3 - 4ac^2st^3 \\ & \quad + c^3st^3 - 2bcs^2t^3 + 4abcs^2t^3 - 4a^2s^3t^3 + 4a^2cs^3t^3 + c^2s^3t^3 - c^3s^3t^3)h'(st) \\ & + (st(st - 1)(c - 2bt + (c - 2a)t^2)(c - 2bs + (c - 2a)s^2)h''(st) = 0 \quad (8.1.15) \end{aligned}$$

Now this holds for all  $s$  and  $t$ . Differentiating with respect to  $s$  and setting  $s = 0$  gives that we must have:

$$\begin{aligned} -4a^2 - 8ab^2 - 4a^2c + 8ab^2c + 6ac^2 + 4a^2c^2 + 6ac^3 + 4a^2c^3 - 2c^2h''(0) \\ -c^3h''(0) &= 0(1) \\ 4bch''(0) - 16abc^2 &= 0(2) \\ 2ac^2 - 4a^2c^2 + 2ac^3 - 4a^2c^3 + 2ach''(0) - 2c^2h''(0) + c^3h''(0) &= 0(3) \end{aligned}$$

Now from the second condition it is clear that we have to consider two cases

**Case1** :  $b = 0$

From (1) we get that

$$h''(0) = \frac{2a(1+c)(-2a+3c^2+2ac^2)}{c^2(2+c)}. \quad (8.1.16)$$

Recall that  $c \neq 0$  and  $c \neq -2$  Using this in (3) we get

$$a(a-c)(a+ac-c^2)(c-1) = 0. \quad (8.1.17)$$

Recall that  $c \neq 0$ . **Case2** :  $h''(0) = 4ac$

Now from (3) we observe we must have  $a = \frac{2c-1}{2}$  Using this in (1) we get

$$(4b^2 - 1)(c-1)(2c-1) = 0 \quad (8.1.18)$$

## 8.2 Orthogonal Polynomials and Jacobi-Szegö parameters

In this section we will derive the orthogonal polynomials and Jacobi-Szegö parameters. From previous section we get that

$$f(t) = \int_{\mathbb{R}} \frac{(1+t^2)^c}{(1-2tx+t^2)^c} d\mu(x) \quad (8.2.1)$$

and hence  $\psi(t, x) = \frac{1}{f(t)} \frac{(1+t^2)^c}{(1-2tx+t^2)^c}$  is going to be the generating function. To keep matters simple lets define

$$g(t) = \frac{(1+t^2)^c}{f(t)} \quad (8.2.2)$$

then  $\psi(t, x) = \frac{g(t)}{(1-2tx+t^2)^c}$  will be the generating function. The equation

$$f'(t) = 2f(t) \frac{bc + (ac+a)t - bct^2 + (a-ac)t^3}{(1+t^2)(c-2bt+(c-2a)t^2)} \quad (8.2.3)$$

simplifies to

$$g'(t) = -2g(t) \frac{bc + (ac+a-c^2)t}{(c-2bt+(c-2a)t^2)}. \quad (8.2.4)$$

Now we have studied  $c = 1$  which appears in both cases in Chapter 5. So from now on assume  $c \neq 1$ . We have a total of 6 subcases corresponding to the first two cases

**Case 1:**  $b = 0$

Subcase 1: If  $a = 0$ , then

we get  $g(t) = (1 - 2\frac{b}{c}t + t^2)^c$  and  $h(ts) = 1$

Subcase 2: If  $a - c = 0$ , then

we get  $g(t) = 1 - t^2$  and  $h(ts) = \frac{1+ts}{(1-ts)^{2c-1}}$

Subcase 3: If  $ac + a - c^2 = 0$ , then

we get  $g(t) = 1$  and  $h(ts) = \frac{1}{(1-ts)^{2c-1}}(1 + \sum_{n=1}^{\infty} \prod_{i=1}^n \frac{i-c}{i+c} (ts)^n)$

**Case 2:**  $a = c - \frac{1}{2}$

Subcase 4: If  $2c - 1 = 0$ , then

we get  $g(t) = \sqrt{1 - 4bt + t^2}$  and  $h(ts) = 1$

Subcase 5: If  $b = \frac{1}{2}$ , then

we get  $g(t) = 1 - t$  and  $h(ts) = \frac{1}{(1-ts)^{2c-1}}$

Subcase 6: If  $b = \frac{-1}{2}$ , then

we get  $g(t) = 1 + t$  and  $h(ts) = \frac{1}{(1-ts)^{2c-1}}$

Each of this subcases give a sequence of orthogonal polynomials and one can calculate the Jacobi-Szegő parameters easily.

### 8.3 Corresponding Measures

Now lets calculate the measures For Subcases 1 and 4 we clearly have the delta measure at  $\frac{b}{c}$ . For non-trivial measures defined on real line, we must have  $a > 0$ . Hence to have a measure from Subcases 5 and 6 we must have  $c > 1/2$  and for this case it is easy to observe that we have the Jacobi measures with parameters  $(c, c - 1)$  and  $(c - 1, c)$ . For the Subcases 2 and 3 one needs to be careful. Observe that  $\int_{\mathbb{R}} \psi(t, x)\psi(s, x)d\mu(x) = h(ts)$  and hence  $\int_{\mathbb{R}} \psi(t, x)^2d\mu(x) = h(t^2)$ . To have measures we must have  $\lambda_n > 0$ . So the coefficients in the expansion of  $h(t^2)$  must be positive. Hence one gets for Subcase 2 the condition  $c > 1/2$ . But in that case one easily gets the Gegenbauer measure with parameter  $c - 1$ . Similarly one can calculate that for  $h''(0) > 0$  one needs  $c > -1/2$  for the Subcase 3. In this case we have the Gegenbauer measure with parameter  $c$ . Hence we find out that for general

$c$  up to the scaling of the interval we have at most 4 different measures along with the trivial delta measure.

# Chapter 9

## The Case $\rho(t) = t$

### 9.1 The $q$ -orthogonal Polynomials Arising from AKK Method

We are considering the case  $\varphi(t, x) = h(\rho(t)x)$ , where  $h$  and  $\rho$  are analytic around 0. The case when  $h$  is fixed is studied in the Chapters 4, 5, 6 and 8. We have found all the possible cases for  $\rho(t)$  when  $h$  is given. In this Chapter, we will take the opposite approach; that is, we shall fix  $\rho(t)$  and classify all possible functions  $h$ . The case  $\rho(t) = t$  is likely to be the simplest, yet it is still quite difficult. The main theorem in Chapter 3 becomes

$$\int h(tx)h(sx) d\mu(x) = \int h(tx) d\mu(x) \cdot \int h(sx) d\mu(x) \cdot g(st),$$

where  $g$  is a function of  $st$  and  $h^n(0) \neq 0$  for every  $n$ . Setting  $s = 0$  yields  $g(0) = 1$ . If necessary, by multiplying by a constant, without loss of generality, we may assume that  $h(0) = 1$ . Let  $h(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $g(x) = \sum_{n=0}^{\infty} c_n x^n$  and  $\int x^n d\mu(x) = b_n$ . We have  $a_0 = b_0 = c_0 = 1$  and  $a_n = \frac{h^{(n)}(0)}{n!} \neq 0$ . In the right hand side we have

$$\int h(tx) d\mu(x) = \int \sum_{n=0}^{\infty} a_n (tx)^n d\mu(x) = \sum_{n=0}^{\infty} a_n t^n \int x^n d\mu(x) = \sum_{n=0}^{\infty} a_n b_n t^n.$$

Furthermore, the left hand side is

$$\begin{aligned} \int h(tx)h(sx) d\mu(x) &= \int \left( \sum_{n=0}^{\infty} a_n (tx)^n \right) \cdot \left( \sum_{m=0}^{\infty} a_m (sx)^m \right) d\mu(x) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n (a_m a_{n-m} t^m s^{n-m}) \right) b_n. \end{aligned}$$

Now, rearranging the sum in the increasing order of  $s^n$ , we get that the left hand side is equal to

$$\sum_{n=0}^{\infty} \left( \sum_{m=0}^n a_n a_m b_{n+m} t^m \right) s^n$$

and that the right hand side is equal to

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_{n-k} b_{n-k} c_k t^k \right) \cdot \left( \sum_{j=0}^{\infty} a_j b_j t^j \right) \cdot s^n.$$

Therefore for every  $n$  we have that

$$\sum_{m=0}^n a_n a_m b_{n+m} t^m = \left( \sum_{k=0}^n a_{n-k} b_{n-k} c_k t^k \right) \cdot \left( \sum_{j=0}^{\infty} a_j b_j t^j \right).$$

Now let  $a_{-1} = a_{-2} = \dots = b_{-1} = b_{-2} = \dots = 0$ . Then we have that

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{n-k} b_{n-k} c_k t^k \right) \cdot \left( \sum_{j=0}^{\infty} a_j b_j t^j \right) \cdot s^n.$$

Equating powers of  $t$ , we obtain

$$a_n a_m b_{n+m} = \sum_{k=0}^{\infty} a_{n-k} b_{n-k} a_{m-k} b_{m-k} c_k.$$

Now, we know that  $a_0 = b_0 = c_0 = 1$  and  $a_n \neq 0$ .

For  $m = 0$ , we obtain  $a_n a_0 b_n = a_n b_n a_0 b_0 c_0$ .

For  $m = 1$ , we obtain

$$a_1 a_n b_{n+1} = a_n a_1 b_n b_1 + a_{n-1} b_{n-1} c_1. \quad (9.1.1)$$

When  $m = 2$ , we obtain

$$a_2 a_n b_{n+2} = a_2 b_2 a_n b_n + a_1 b_1 c_1 a_{n-1} b_{n-1} + c_2 a_{n-2} b_{n-2}, \quad (9.1.2)$$

and so on. Replacing  $c_n$  by  $a_1^{2n} c_n$  and  $a_n$  by  $a_1^{2n} c_n$ , we may assume that  $a_1 = 1$ .

One needs to study the cases  $b_1 = 0$  and  $b_1 \neq 0$  separately.

**Case 1:**  $b_1 = 0$

By (9.1.1) we have  $a_n b_{n+1} = a_{n-1} b_{n-1} c_1$ . Since  $a_n \neq 0$ , we have that  $c_n \neq 0$  and

$b_i = 0$  when  $i$  is odd. Setting  $n = 2l + 1$ , we obtain  $a_{2l+1}b_{2l+2} = a_{2l}b_{2l}c_1$ .

Define  $r_n := \frac{a_{n-1}c_1}{a_n}$ . Then  $a_n = \frac{(c_1)^n}{r_n \cdots r_2 r_1}$ ,  $r_1 = c_1$  and  $b_{2n} = r_{2n-1}r_{2n-3} \cdots r_1$ ,

$b_2 = c_1 = r_1$ . Using this in equation (9.1.2), we get for  $n = 2l + 2$  that  $a_2a_{2l+2}b_{2l+4} =$

$a_2b_2a_{2l+2}b_{2l+2} + c_2a_{2l}b_{2l}$ , or equivalently  $\frac{r_1}{r_2} \cdot \frac{r_1^2}{r_{2l+2}r_{2l+1}} \cdot r_{2l+3}r_{2l+1} = \frac{r_1}{r_2} \cdot r_1 \cdot \frac{r_1^2}{r_{2l+2}r_{2l+1}} \cdot$   
 $r_{2l+1} + c_2$ . That is,  $r_1^3r_{2l+3} = r_1^4 + c_2r_2r_{2l+2}$ , or

$$r_{2l+1} = r_1 + \frac{c_2r_2}{r_1^3} \cdot r_{2l}, r_0 = 0. \quad (9.1.3)$$

Now using (9.1.3) and the fact that  $b_1 = b_3 = 0$ , we have that

$$a_3a_nb_{n+3} = a_2b_2a_{n-1}b_{n-1}c_1 + a_{n-3}b_{n-3}c_3$$

. Letting  $n = 2l + 3$ , we obtain

$$a_3a_{2l+3}b_{2l+6} = a_2b_2a_{2l+2}b_{2l+2}c_1 + a_{2l}b_{2l}c_3$$

, or equivalently,

$$r_{2l+5} = r_3 + \frac{c_3r_2r_3}{r_1^5}r_{2l+2}$$

. Using equation (9.1.3) one more time, we obtain

$$r_{2l+2} = r_2 + \frac{c_3}{c_2r_1^3}(r_1^4 + c_2r_2^2)r_{2l}. \quad (9.1.4)$$

Let  $A = \frac{c_2r_2}{r_1^3}$  and  $B = \frac{c_3}{c_2r_1^3}(c_1^4 + c_2r_2^2)$ . Then

$$r_{2n+1} = r_1 + Ar_{2n}$$

$$r_{2n+2} = r_2 + Br_{2n},$$

so

$$r_4 = r_2 + Br_2 = r_2(1 + B)$$

$$r_6 = r_2 + Br_4 = r_2 + Br_2(1 + B) = r_2(1 + B + B^2)$$

$$r_8 = r_2 + Br_6 = r_2 + r_2B(1 + B + B^2) = r_2(1 + B + B^2 + B^3).$$

In general, by induction one may show  $r_{2n} = r_2 \cdot \frac{B^n - 1}{B - 1}$ , where we take the limit if  $B = 1$ . Hence,  $r_{2n+1} = r_1 + Ar_2 \cdot \frac{B^n - 1}{B - 1}$ . To summarize,  $A_n = \frac{(r_1)_n}{r_n \cdots r_1}$ ,  $b_{2n} = r_{2n-1} \cdots r_1$ ,  $b_{2n+1} = 0$ , and  $c_1 = r_1, r_2, c_2, c_3$  are free parameters (for now). Moreover,  $r_{2n} = r_2 \cdot \frac{B^n - 1}{B - 1}$  and  $r_1 + Ar_2 \cdot \frac{B^n - 1}{B - 1}$ , where  $A = \frac{c_2 r_2}{r_1^3}$  and  $B = \frac{c_3}{c_2 r_1^5} (c_1^4 + c_2 r_2^2)$ . This case give rise to q-orthogonal polynomials described in Wall [25]. In the literature they are known as Wall polynomials.

**Case 2:**  $b_1 \neq 0$

This case is not done yet in this thesis.

# Chapter 10

## Interacting Fock Spaces

The original motivation for the development of AKK method was to study interacting Fock spaces. In this chapter we will give some preliminary results and definitions related to these spaces.

We have defined the sequence  $\lambda = \{\lambda_n\}_{n=0}^{\infty}$  in Chapter 2 of this thesis. Assume this sequence satisfies a technical condition

$$\inf \lambda_n^{\frac{1}{n}} > 0.$$

For such a sequence  $\lambda$  we define a Hilbert space  $\Gamma_\lambda$  by

$$\Gamma_\lambda = \{(a_0, a_1, \dots, a_n, \dots) | a_n \in \mathbb{C}, \sum_{n=0}^{\infty} \lambda_n a_n^2 < \infty\}$$

with norm  $\|\cdot\|$  given by

$$\|(a_n)\|_\lambda = \left( \sum_{n=0}^{\infty} \lambda_n |a_n|^2 \right)^{\frac{1}{2}} \quad (10.0.1)$$

This Hilbert space is called the *one-mode interacting Fock space* associated with  $\lambda$  (hence  $\mu$ ). Define the following annihilation operator  $A$

$$A\Phi_n = \begin{cases} 0, & \text{if } n = 0, \\ \omega_{n-1}\Phi_{n-1}, & \text{if } n \geq 1. \end{cases} \quad (10.0.2)$$

where number vector  $\Phi_n = (0, 0, \dots, 0, 1, 0, \dots)$  with 1 in the  $n + 1$ st position for  $n \geq 0$ . The first vector  $\Phi_0$  is called the *vacuum vector*. This is a densely defined operator on  $\Gamma_\lambda$ . The adjoint of this operator  $A^*$  is called the creation operator which after an easy check satisfies

$$A^*\Phi_n = \Phi_{n+1}, n \geq 0. \quad (10.0.3)$$

To be complete we need to define the following operators :

$$N\Phi_n = n\Phi_n \quad (10.0.4)$$

$$\alpha_N\Phi_n = \alpha_n\Phi_n \quad (10.0.5)$$

In 1998, L. Accardi and M. Bożejko[2] stated the following result: There exists a unitary isomorphism  $U : \Gamma_\lambda \mapsto L^2(\mu)$  satisfying the following 3 conditions

$$(1) U\Phi_0 = 1,$$

$$(2) UA^*U^*P_n = P_{n+1}$$

$$(3) U(A + A^* + \alpha_N)U^* = X,$$

where  $X$  is the multiplication operator by  $x$  on  $L^2(\mu)$ . This unitary isomorphism  $U$  is canonical in the sense of the last condition. More precisely the multiplication operator  $X$  on  $L^2(\mu)$  corresponds to a linear combination of creation, annihilation and number operators on the  $\Gamma_\lambda$ .

The notion of interacting Fock spaces was introduced in 1992[6] and later axiomatized in 1997[7] with the hopes that the category of interacting Fock spaces for general probability measures could play the role played by Fock space for Gaussian measures. The above result of L. Accardi and M. Bożejko[2] gave the first confirmation. Notion of one-mode interacting Fock space generalized to several variables by Accardi and Nahni. Finally Accardi Kuo and Stan generalized to infinite dimensional. Unlike the one-mode case, in the multi-mode case not all interacting Fock spaces are canonically isomorphic to the spaces of orthogonal polynomials. However, those being so are somewhat characterized. More precisely in the multi-mode case, the interacting Fock spaces which are canonically isomorphic to spaces of orthogonal polynomials are characterized in terms of a sequence of quadratic

commutation relations among finite dimensional matrices. It has also been shown that the quantum decomposition of a vector-valued random variable can be written down as a sum of creation, annihilation and number operators as long as it has finite moments of all orders. Hence, the codification of the properties of a Gaussian measure into the Heisenberg commutation relations equipped with the Fock property can be generalized to arbitrary measures. For example in [3], Accardi, Kuo and Stan have found characterization theorems for the symmetry and factorizability of a probability measure on  $\mathbb{R}^d$  in terms of the corresponding creation, annihilation and number operators.

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# Vita

Suat Namli was born on June 13, 1978, in the Turkish Republic. He began his undergraduate studies in 1996 at Bilkent University in Ankara, Turkey, where in 2001 he earned a Bachelor of Science degree in mathematics. In August 2001 he came to Louisiana State University to pursue graduate studies in mathematics. He earned a Master of Science degree in mathematics from Louisiana State University in 2003. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will hopefully be awarded in December 2007.