QUASICONTINUOUS DERIVATIVES AND VISCOSITY FUNCTIONS

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Rodica Cazacu
B.S., University of Bucharest, 1993
M.S., Louisiana State University, 2001
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Abstract

In this work we demonstrate how the continuous domain theory can be applied to the theory of nonlinear optimization, particularly to the theory of viscosity solutions. We consider finding the viscosity solution for the Hamilton-Jacobi equation $H(x, \nabla y) = g(x)$, with continuous hamiltonian, but with possibly discontinuous right-hand side. We begin by finding a new function space $Q(X, L)$, the space of equivalence classes of quasicontinuous functions from a locally compact set X to a bicontinuous lattice L and we will define on $Q(X, L)$ the *qo-topology*, which is a variant of classical order topology defined on complete lattices. On this new function space we will show that there exist closed extensions of some differential operators, like the usual gradient and the operator defined by the continuous hamiltonian H. The domain of the closure of the corresponding operator will coincide with the set of viscosity solutions for the Hamilton-Jacobi equation when the hamiltonian is convex in the second argument.

Introduction

In this dissertation we demonstrate how the continuous domain theory pioneered by Dana Scott can be applied to the theory of nonlinear optimization, particularly to the theory of viscosity solutions.

In his paper [Sa], S. Samborski considers certain partial differential equations such as Hamilton-Jacobi equation of the form

$$H(x, \nabla y) = g(x)$$

for which the desired solutions are not differentiable, or not even continuous, but rather are so-called the viscosity solutions, defined by M. G. Crandal in [Cr]. In this work we consider the same problem, but from the point of view of domain theory, and find this to be more natural. We will find a new function space on which we will extend the partial differential operator $\mathcal{D}_H$ defined by the hamiltonian H to an operator $\mathcal{D}$, such that the solutions of the equation $\mathcal{D}y = g$ coincide with the viscosity solutions of the equation $\mathcal{D}_H y = g$. On our new function space we will define a new topology, called the *qo-topology*, which is a variant of classical order topology defined on complete lattices.

In the first chapter of this dissertation we define the new function space $Q(X, L)$ as the set of equivalence classes of quasicontinuous functions from a locally compact set X to a bicontinuous lattice L. We show that there is a one-to-one correspondence between the classes of quasicontinuous functions, $Q(X, L)$, and the set of maximal elements of the domain $[X \longrightarrow \mathbb{L}]$ of approximate maps from X to $\mathbb{L}$, and we will define on our set a new topology, the *qo-topology.* We will define the convergence in the space using the Scott and Lawson convergences.

In the second chapter we will look at the case when $X \subseteq \mathbb{R}^m$ is locally compact locally convex with dense interior, and $L = \overline{\mathbb{R}}$. We will show that the usual gradient operator $\nabla : C^1(X, \mathbb{R}) \subseteq Q(X, \mathbb{R}) \to Q(X, \mathbb{R}^m)$ can be extended by closure in $Q(X, \mathbb{R})$. Any class $f \in Q(X, \mathbb{R})$ from the domain of this closed extension consists of a unique representative which is locally Lipschitz, and the "strong derivative" of this representative is well defined on a dense set, and it defines the class of $\overline{\nabla} f$.

As for the last chapter, we will consider the Hamilton-Jacobi equation

$$H(x, \nabla y) = g(x)$$

with continuous hamiltonian, but with possibly discontinuous right-hand side, and we will show that the viscosity solutions in the sense of [Cr] from the space Q can always be obtained when we close up the operator defined by the hamiltonian H on Q. We will show also that the extension by closure (from $C^1_{\min} \subseteq Q$, the set of functions that can be represented as a minimum of finitely many differentiable functions) of the corresponding operator given by a continuous hamiltonian $H(x, p)$ convex in the second argument coincides with the viscosity solutions.

Since domains are ordered structures, the topology that has arisen has been a distinctive topology that has been combined with the study of partial orders. Here are few definitions and facts from [LG] and [KL] we will use in this paper.

Definition 0.1. A partially ordered set P is said to be *directed-complete* and is called a *dcpo (directed complete partially ordered set)* if every directed set D ($a, b \in D$ implies there exists $c \in D$ with $a \leq c$, $b \leq c$) has a supremum, denoted $\bigvee^{\uparrow} D$ (where the upward arrow denotes that the supremum is taken over a directed set). We assume always that directed sets are non-empty. If the empty set is also required to have a supremum, then D must have a least or bottom element, denoted $\perp$. A *pointed* dcpo is one with a bottom element.

Intuitively we say that state A approximates state B if any computation of B yields the information of state A at some finite stage. One of the important insights of the theory of "continuous partial orders" that has emerged in the last thirty years is the mathematical formalization and detailed investigation of a suitable notion of approximation.

Definition 0.2. Let P be a partially ordered set. For $x \leq y \in P$, we say that x *approximates* y, written $x \ll y$, if

$$\text{for any } D \text{ directed, } w = \bigvee^{\uparrow} D,\ y \leq w \Rightarrow x \leq d,\ \text{ for some } d \in D.$$

A *continuous poset* is a partially ordered set P in which each element is the directed supremum of all elements which approximate it, i.e.,

$$\forall x \in P,\ x = \bigvee^{\uparrow}\{y \in P \colon y \ll x\}.$$

A continuous poset which is also a dcpo is called a *continuous domain* or *continuous dcpo*. The study of these ordered structures is called *"domain theory"*.

An alternate characterisation of continuous posets is given in [KL] Proposition 2.2 as follows:

Proposition 0.3. *Let D be a directed subset of a partially ordered set P with supremum x. If $d \ll x$ for each $d \in D$, then $\{y \in P \colon y \ll x\}$ is directed with supremum x, and D is a cofinal subset (i.e., given $y \ll x$, there exists $d \in D$ such that $y \leq d$). Hence P is continuous if for each $x \in P$, there exists a directed set D_x with supremum x such that $y \ll x$ for each $y \in D_x$.*

Just as topological spaces can alternately be defined in terms of a basis of open sets, a continuous domain can be defined in terms of an appropriate notion of basis. Countable bases play a fundamental role with respect to the development of recursive and computable notions in the context of continuous domains.

Definition 0.4. Let P be a dcpo. A subset B of P is a *basis* for P if for each $x \in P$, there exists a directed set $B_x \subseteq B$ such that each element of B_x approximates x and $\bigvee^{\uparrow} B_x = x$. An ω-*continuous* domain is a dcpo which possesses a countable basis.

Besides ω-continuous domains there are other important subclasses of continuous domains.

Definition 0.5. Let P be a partially ordered set. Then P is *bounded-complete* if each pair $x, y \in P$ which is bounded above has a least upper bound and is *meet-complete* if every non-empty subset has a greatest lower bound. A meet-complete dcpo is also called a *complete semilattice*. A partially ordered set L is a *complete lattice* if every subset has a supremum and infimum in L.

The first class of continuous domains to be studied were the continuous lattices, those continuous domains which are also complete lattices.

Definition 0.6. A continuous domain which is a complete semilattice and contains a largest element $\top$ is actually a complete lattice and is called a *continuous lattice*.

We think of the elements of an ascending sequence as providing increasingly better approximations to the supremum of the sequence. But, from the viewpoint of the information ordering, they also provide increasingly better information about states below the supremum. These considerations yield a notion of convergence that can be precisely captured topologically by the *Scott* (σ) *topology*, named in honor of Dana Scott, who carried out ground-breaking work concerning domain theory and its applications in the 1970's. The Scott topology is admirably suited for capturing many aspects of domain theory.

Definition 0.7. Let P be a dcpo. A subset U is *Scott open* if

- $U = \uparrow U := \{z \in P \colon \exists x \in U,\ x \leq z\}$, and

- for any directed D, if $\bigvee^{\uparrow} D \in U$, then D is eventually in U, i.e., there exists $b \in D$ such that $d \in U$ for $b \leq d$.

The Scott open sets form a topology called the *Scott topology.* Dually a subset A is *Scott closed* if

- $A = \downarrow A := \{y \in P \colon \exists x \in A,\ y \leq x\}$,

- D directed, $D \subseteq A \Rightarrow \bigvee^{\uparrow} D \in A$.

We will denote by L^{op} the set L on which we have the opposite order.

Definition 0.8. Let L be a complete lattice. The topology generated by the Scott open sets and their duals (the Scott open subsets for L^{op}) is called the *biScott topology.*

Given a topology on a dcpo P, a directed set D is said to *converge to* $x \in P$ if given any open set U containing x, there exists $b \in D$ such that $d \in U$ if $b \leq d$. In the Scott topology a directed set converges to the elements it "computes."

The Scott topology is very natural and useful in the study of continuous domains. Via the Scott topology fundamental concepts of domain theory have alternate topological descriptions. However, it departs radically from classical topology since it is a non-Hausdorff topology. But it is precisely such topologies that lend themselves to the study of partially ordered sets. These developments have become a driving impetus for a development of a new kind of topology that we might call "order-theoretic topology," the study of topological spaces, in general non-Hausdorff spaces, with close links to a partial order on the space.

Definition 0.9. Let X be a topological space. The *order of specialization* of X is defined by

$$x \leq y \Leftrightarrow x \in \overline{\{y\}}.$$

Note that in general the order of specialization is only a quasiorder (reflexive and transitive), that it is a partial order precisely when X is a T_0 space, and that it is the diagonal relation precisely when X is T_1. Thus the order of specialization becomes mathematically interesting precisely in the context of T_0-spaces.

Definition 0.10. In a partially ordered set P, sets A such that $A = \uparrow A$ are called *upper* sets and sets B such that $B = \downarrow B$ are called *lower* sets. We denote $\downarrow\{x\}$ by $\downarrow x$ and $\uparrow\{x\}$ by $\uparrow x$. We note that in the order of specialization on a T_0-space, closed sets are always lower sets and open sets are always upper sets. Since $\overline{\{x\}} = \downarrow x$, it is easy to see that a lower set is the union of all the point closures it contains, and hence that a set is an upper set if and only if it is the intersection of open sets. The latter are also called *saturated sets*.

Definition 0.11. If P is a partially ordered set, then a topology on P is called *compatible* if its order of specialization agrees with the original partial order.

Given a partially ordered set P, there are in general a host of topologies for which the order of specialization agrees with the given order. The finest of these is the *Alexandroff discrete* topology consisting of all upper sets, and the coarsest of these is the *lower interval* topology, for which the sets $\{\downarrow x \colon x \in P\}$ form a subbasis for the closed sets.

The lower interval topology has also been called the weak topology since it is the weakest compatible topology. In [LG] it is called the upper topology.

Definition 0.12. Let (X, τ) be a topological space. A topology τ^* on X is called a *dual topology* if the order of specialisation is $\geq$, the reverse or opposite of the ordeer of specialisation for τ.

In a continuous domain, there are close connections between the Scott open sets and the approximation relation. For example, as in [KL] Proposition 3.3, if P is a continuous domain equipped with the Scott topology the sets $\twoheaduparrow x$, $x \in P$ form an open basis for the topology, where

$$\twoheaduparrow x := \{y \in P \colon x \ll y\}.$$

The directed complete partially ordered sets form the objects of a category **DCPO**. The appropriate morphisms are the *continuous* functions, the order preserving functions which also preserve suprema of directed sets. Such functions may be viewed as the "computation-preserving" functions. They have a natural topological characterization, one which provides another motivation for the Scott topology. The next proposition gives a characterisation of Scott continuous function. See [LG] Proposition II-2.1.

Proposition 0.13. *Let P, Q be directed complete partially ordered sets equipped with the Scott topology and let $f \colon P \to Q$ be a function. The following are equivalent:*

1. *The function f is order preserving and preserves directed suprema.*

2. *The function f is (Scott) continuous.*

Definition 0.14. Let $f : X \to L$ be a function from a topological space X to a complete lattice L. We will say that f is lower semicontinuous if it is Scott continuous, and we denote the set of such functions by $LSC(X, L)$. We consider the point-wise order on this set, i.e. $f \leq g$ if and only if $f(x) \leq g(x)$ for any $x \in X$. Dually we define the set of *upper semicontinuous functions* as the Scott continuous functions $f : X \to L^{op}$, and denote that by $USC(X, L)$.

Proposition 0.15. *If X is locally compact space and L is bounded complete domain then $LSC(X, L)$ is a bounded complete domain. In this case the Scott topology and the compact-open topology agree on $LSC(X, L)$, and hence the evaluation map $E : LSC(X, L) \times X \to L$ is continuous.*

This result is a consequence of [LG] Proposition II-4.6. Continuous domains admit a natural topology which refines the Scott topology and provides the structure of an ordered topological space.

Definition 0.16. Let P be a partially ordered set. The *upper interval* topology on P is defined by taking all sets $P \setminus \uparrow x$ (complements of principal filters), $x \in P$, as a subbasis of open sets. The *Lawson* (λ)-topology on P is defined as the join of the Scott and the upper interval topologies.

The next result is Proposition 7.1 in [KL]. We use it in the first chapter of this dissertation.

Proposition 0.17. *If P is a continuous domain, then P equipped with the λ- topology is an ordered topological space with a regular topology. If P is ω-continuous, then the λ-topology is separable metrizable.*

Definition 0.18. Let L be a complete lattice. The *interval topology* on L is the join of the lower topology and its dual, the upper topology. Hence, the set of principal filters and principal ideals forms a subbasis for the closed sets for the interval topology.

The theory of T_0-spaces provides a convenient mathematical framework for relating topological and order theoretic notions. But there is an earlier approach to relating ordered and topological structures that dates back to the work of L. Nachbin [Na] in the middle of the last century.

Definition 0.19. An *ordered topological space*, or more briefly a *pospace*, is a topological space X equipped with a partial order with closed graph, i.e., the set $\{(x, y)\colon x \leq y\}$ is a closed subset of $X \times X$ (equipped with the product topology).

Since in an ordered topological space the diagonal of $X \times X$ is given by $\leq \cap \geq$, and is thus closed, it follows that an ordered topological space is always Hausdorff.

Finally a few things about the set of maximal elements of a continuous domain, which we will use in our work. We consider ω-continuous domains P satisfying the condition

$$p \in P \Rightarrow \exists A \text{ Scott closed in } P,\ \uparrow\! p \cap \mathrm{Max}(P) = A \cap \mathrm{Max}(P), \tag{1}$$

where $\mathrm{Max}(P)$ is the set of elements in P which are maximal in the partial order. Alternately

$$\sigma\text{−topology}|\mathrm{Max}(P) = \lambda\text{-topology}|\mathrm{Max}(P), \tag{2}$$

i.e., the Scott and Lawson topologies restricted to the set of maximal elements agree. Indeed the subbasic closed sets in the λ-topology on P are either Scott closed or of the form $\uparrow\! p$ for some $p \in P$, and from this it follows easily that (1) and (2) are equivalent. In this case, $\mathrm{Max}(P)$ is a separable metric space, since the λ-topology is separable metric for ω-continuous domains (See [LG] Corollary III-4.6.).

Definition 0.20. A separable metric space X is called a *maximal point space* if there exists an ω-continuous domain P satifying condition (1) (or equivalently (2)) such that X is homeomorphic to $\mathrm{Max}(P)$ equipped with the relative Scott topology. In this case the embedding $X \leftrightarrow \mathrm{Max}(P) \hookrightarrow P$ is called a *domain environment* or *computational environment* for X.

Maximal point spaces were studied by Kamimura and Tang [KT] for the case that the domain environments were Scott-continuous retracts of Scott domains. They called such spaces "total spaces."

1. Approximate and Quasicontinuous Maps

1.1 Bicontinuous Lattices

Definition 1.1. A complete lattice L is *linked bicontinuous*, or simply *bicontinuous* for short, if it satisfies:

(1) L and L^{op} are continuous lattices;

(2) The interval, biScott, Lawson, and dual Lawson topologies all agree on L.

A variety of equivalent conditions appear in [LG] Proposition VII-2.9, for example the following :

(3) $(L, \vee, \wedge)$ is a compact topological lattice with a basis of open sets that are sublattices. In this case the topology must be the biScott.

As in [LG] Proposition VII-2.10, for distributive lattices, the bicontinuous lattices are precisely the completely distributive ones.

We restrict our attention to *ω-bicontinuous lattices*, those bicontinuous lattices L that are ω-continuous, that is, have a countable base (in the sense of continuous lattices). This is equivalent to assuming that the biScott topology is metrizable, and hence equivalent to L^{op} being ω-continuous (see [KL] Proposition 7.1).

Primary Example. Let $\overline{\mathbb{R}} = [-\infty, \infty]$, the extended reals, and $\overline{\mathbb{R}}^n$ extended n-dimensional euclidian space. Observe that $\overline{\mathbb{R}}^n$ is a completely distributive lattice with respect to the coordinatewise order:

$$(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n) \Leftrightarrow \forall i, x_i \leq y_i.$$

We observe that the biScott topology is the product topology, which is metrizable.

Note. The preceding observations remain valid for $\overline{\mathbb{R}}^{\mathbb{N}}$.

1.2 Domain Environments

Definition 1.2. A *domain environment* for a topological space X is a homeomorphic embedding $X \hookrightarrow Max(D)$ onto the set of maximal points of a continuous domain D equipped with the relative Scott topology.

Remark 1.3. A natural domain environment $\mathbb{L}$ for a bicontinuous lattice L (always endowed with the biScott=Lawson topology) consists of all nonempty order intervals

$$[u, v] := \{x \in L \mid u \leq x \leq v\},$$

where the order intervals are ordered by reverse inclusion, the "information order".

Lemma 1.4. *Let L be a bicontinuous lattice, and $\mathbb{L}$ the set of all order intervals. Let $a_1, a_2, b_1, b_2 \in L$. The following are equivalent:*

(i) $[a_1, b_1] \ll [a_2, b_2]$*;*

(ii) $[a_2, b_2] \subseteq int[a_1, b_1]$*;*

(iii) $a_1 \ll a_2$ *in* L*, and* $b_1 \ll b_2$ *in* L^{op}*.*

Proof. (i) $\Rightarrow$ (ii): Recall that $[a_1, b_1] \ll [a_2, b_2]$ in $\mathbb{L}$ if and only if, for any nonempty directed subset $D \subseteq \mathbb{L}$ for which $\sup D$ exists, and $[a_2, b_2] \leq \sup D$, there exists $[d_1, d_2] \in D$ such that $[a_1, b_1] \leq [d_1, d_2]$.

Since L is continuous, $D = \{x \in L : x \ll a_2\}$ is a directed set with $\sup D = a_2$. Dually, $D^* = \{y \in L : y \gg b_2\}$ has a directed $\inf D^* = b_2$. Thus the set $A = \{[x, y] \in \mathbb{L} : x \in D,\ y \in D^*\}$ is directed in $\mathbb{L}$ and

$$\sup A = \bigcap_{(x,y) \in D \times D^*} [x, y] = [a_2, b_2].$$

Since $[a_1, b_1] \ll [a_2, b_2]$, there exists $[x, y] \in A$ such that $[a_1, b_1] \leq [x, y] \leq [a_2, b_2]$, or, as sets in L, we have that $[a_2, b_2] \subseteq [x, y] \subseteq [a_1, b_1]$. Therefore for any $z \in [a_2, b_2]$ we have $a_1 \leq x \ll a_2 \leq z$, thus $a_1 \ll z$, which by [LG], Proposition II-1.10(i), implies that $z \in int(\uparrow a_1)$. Dually $z \in int(\downarrow b_1)$, so $z \in int(\uparrow a_1 \cap \downarrow b_1) = int[a_1, b_1]$. In conclusion, since z is an arbitrary element of $[a_2, b_2]$, we have $[a_2, b_2] \subseteq int[a_1, b_1]$.

(ii) $\Rightarrow$ (iii): Let $D = \{d \in L : d \ll a_2\} = \Downarrow a_2$. Since L is bicontinuous D is directed and $\sup D = a_2$, so we can say that D converges to a in the Scott topology, and trivially in the dual Scott topology. Thus it converges in the biScott topology, so, since $int[a_1, b_1]$ is open, there exists $d \in D$ such that $a_1 \leq d \ll a_2$, which implies $a_1 \ll a_2$. A similar proof will show that $b_1 \ll b_2$ in L^{op}.

(iii) $\Rightarrow$ (i): Let D be a directed subset of $\mathbb{L}$ such that $\sup D = [s_1, s_2]$ and $[a_2, b_2] \leq [s_1, s_2]$. Let $D_1 = \{d \in L : \exists b \in L, [d, b] \in D\}$. For any $d \in D_1$, since $\sup D = [s_1, s_2]$ and there exists a $b \in L$ such that $[d, b] \in D$, we have that $d \leq s_1$. It is easy to see that D_1 is a directed set and $\sup D_1 = s_1$. From $[a_2, b_2] \leq [s_1, s_2]$ we conclude $a_2 \leq s_1$, and knowing that $a_1 \ll a_2$ we can find a $d_1 \in D$ such that $a_1 \leq d_1$. We also have that there exists a $b \in L$ such that $[d_1, b] \in D$. With a similar proof we can find a $d_2 \in L$ such that $b_1 \leq d_2$ in L^{op}, and for which there exists a $c \in L$ with $[c, d_2] \in D$. We know that D is directed, and $[d_1, b], [c, d_2] \in D$, which means that there exists a $[a, b] \in D$ such that $[d_1, b] \leq [a, b]$ and $[c, d_2] \leq [a, b]$, or in other words, there exists $[a, b] \in D$ such that $[a, b] \subseteq [d_1, b] \cap [c, d_2]$. It is clear that $[a_1, b_1] \leq [a, b]$, and so (i) is true. □

Theorem 1.5. *The set* $\mathbb{L}$ *is a bounded complete continuous domain.*

Proof. By definition, $\mathbb{L}$ is a bounded complete continuous domain if it is a complete semilattice which is a domain as a partially ordered set (poset). We will show the following:

(1) Any directed subset in $\mathbb{L}$ has a supremum;

(2) Any subset that is bounded above has a least upper bound;

(3) $x = \bigvee^{\uparrow} \Downarrow x$.

Let D be a directed subset of $\mathbb{L}$. That means for any $[a_1, a_2]$, $[b_1, b_2] \in D$ there exists $[d_1, d_2] \in D$ such that $[a_1, a_2] \leq [d_1, d_2]$ and $[b_1, b_2] \leq [d_1, d_2]$, and this implies $[d_1, d_2] \subset [a_1, a_2] \cap [b_1, b_2]$, so

$$[s_1, s_2] = \bigcap_{[a,b]\in D} [a, b] \neq \emptyset,$$

where s_1 is the directed sup of $D_1 = \{a \in L : \exists b \in L,\ [a, b] \in D\}$, and s_2 is the directed inf of $D_2 = \{b \in L : \exists a \in L,\ [a, b] \in D\}$. Since D is directed we have $s_1 \leq s_2$. It is evident that $[s_1, s_2] = \sup D$, and (1) is satisfied.

Let $A \subset \mathbb{L}$, and $[s_1, s_2] \in \mathbb{L}$ such that $[s_1, s_2]$ is an upper bound of A. Let $A_1 = \{a \in L : \text{there exists } b \in L, [a, b] \in A\}$ and $A_2 = \{b \in L : \text{there exists } a \in L, [a, b] \in A\}$. Since L is a bicontinuous lattice, then there exist a_1 and $b_1 \in L$ such that $a_1 = \sup A_1$ and $b_1 = \inf A_2$. Since $[a, b] \leq [s_1, s_2]$ for any $[a, b] \in A$, we have $a \leq a_1 \leq s_1 \leq s_2 \leq\ b_1 \leq b$, and so $[a_1, b_1]$ is the least upper bound of A.

We show now that the set $\twoheaddownarrow[a, b] = \{[u, v] : [u, v] \ll [a, b]\}$ is a directed set. Let $[u, v]$, $[u', v'] \in \twoheaddownarrow[a, b]$. By Lemma 1.4 u and $u' \ll a$, and thus $u \vee u' \ll a$ by [LG] Proposition I-1.2. Dually $v \wedge v' \ll b$ in L^{op}. Again by Lemma 1.4, $[u \vee u', v \wedge v'] \ll [a, b]$, which shows that $\twoheaddownarrow[a, b]$ is directed.

By Lemma 1.4 $[u, v] \ll [a, b]$ if and only if $u \ll a$ in L and $v \ll b$ in L^{op}, or, we can rewrite this as $[u, v] \in \twoheaddownarrow[a, b]$ if and only if $u \in \twoheaddownarrow a$ and $v \in \twoheaduparrow b$. Since L is bicontinuous, the set $\twoheaddownarrow a$ is directed in L and $a = \sup \twoheaddownarrow a$. Dually $\twoheaduparrow b$ has directed inf $= b$. Thus

$$\sup \twoheaddownarrow[a, b] = \bigcap_{(u,v)\in \twoheaddownarrow a \times \twoheaduparrow b} [u, v] = [a, b].$$

$\square$

Theorem 1.6. *The map*

$$u \mapsto [u, u] : L \longrightarrow \mathbb{L}$$

is a homeomorphic embedding, hence a domain environment for $(L, biScott)$*, representing* L *as the degenerate intervals* $[u, u]$.

Proof. We want to show that the map is one-to-one, continuous, open and its image is the set of maximal elements of $\mathbb{L}$.

If $x, y \in L$, $x \neq y$, then $[x, x] = \{x\} \neq \{y\} = [y, y]$, and so the map is one-to-one.

Let U open in $\mathbb{L}$, $[x, x] \in U$. Since U is Scott open there exists an order interval $[a, b] \in U$ such that $[a, b] \ll [x, x]$, and so, by Lemma 1.4, $[x, x] \subset int[a, b]$. The set $\Uparrow a \bigcap \Uparrow^{op} b$ is an open set in L. Let $c \in \Uparrow a \bigcap \Uparrow^{op} b$. We will like to show that $[c, c] \in U$, and so the map is continuous. We have that $c \in \Uparrow a$, which implies that $a \ll c$, and that $c \in \Uparrow^{op} b$, which means $b \ll^{op} c$. Using Lemma 1.4 we can conclude that $[a, b] \ll [c, c]$, and so $[c, c] \in U$.

We will like to show now that the image is the set of maximal elements, and for that we will show that $[a, b]$ is maximal in $\mathbb{L}$ if and only if $a = b$. Let $[a, b] \in \mathbb{L}$ be a maximal element. If $a \neq b$ then $[a, b] < [a, a]$ in $\mathbb{L}$, which contradicts the maximality of $[a, b]$, so $a = b$, and any maximal element of $\mathbb{L}$ must be of the form $[a, a]$ with $a \in L$. It is clear that each $[a, a]$ is maximal.

Next we will show that the map

$$u \mapsto [u, u] : L \longrightarrow (\mathbb{L} \cap \text{Set of maximal elements})$$

is open, and since we saw that is also a bijective map, that will make it an homeomorphism. Since L is a continuous lattice, there exist $a = \inf L$ and $b = \sup L$. The sets $\Downarrow x$ and $\Uparrow x \subseteq L$ are basic open sets in the biScott topology, where $x \in L$. It is enough to show that the images of these sets are intersections of open sets in $\mathbb{L}$ and the set of maximal elements of $\mathbb{L}$.

We will show that the image of $\downdownarrows x$ by this map is the set of maximal elements in the open set $\Uparrow[a,x] \subseteq \mathbb{L}$ and the image of $\upuparrows x$ is the set of maximal elements in the open set $\Uparrow[x,b] \subseteq \mathbb{L}$.

We already showed that a maximal element of $\mathbb{L}$ must be of the form $[y,y]$, where $y \in L$. Let $y \in \upuparrows x$, then, since L is bicontinuous, $y \in int[x,b]$, and by Lemma 1.4 this is equivalent to $[x,b] \ll [y,y]$, or $[y,y] \in \Uparrow[x,b]$. Therefore the imagine of $\upuparrows x$ by this map is a subset of $\Uparrow[x,b]$, which is an open set of $\mathbb{L}$. It is left to show now that for any maximal element $[z,z] \in \Uparrow[x,b]$ we have $z \in \upuparrows x$. Let $[z,z] \in \Uparrow[x,b]$. Then $[x,b] \ll [z,z]$, and, by Lemma 1.4, $x \ll z$, which means $z \in \upuparrows x$.

A similar proof, using the duality, will show that the image of $\downdownarrows x$ is the set of maximal elements in the open set $\Uparrow[a,x] \subseteq \mathbb{L}$. □

Remark 1.7. Note by Lemma 1.4 that a (finitary) approximation $[v,w] \ll [u,u]$ of $u = [u,u]$ is an order interval $[v,w]$ containing u in its interior.

1.3 Approximate Functions

Intuitively an "approximate" or "fuzzy" function is one for which we have incomplete information. One way of modelling such functions is to assume that we only know $f(x)$ up to an interval of values.

Definition 1.8. An *approximate function* f from a topological space X into a bicontinuous lattice L is a function $f : X \longrightarrow \mathbb{L}$. The approximate function f is continuous if it is continuous into the Scott topology of $\mathbb{L}$.

Since each $f(x)$ is an order interval, we can write $f(x) = [\alpha(x), \beta(x)]$, where $\alpha, \beta : X \longrightarrow L$. In this case we write the interval function $f = [\alpha, \beta]$.

Theorem 1.9. *The approximate function $f = [\alpha, \beta]$ is continuous if and only if α is Scott-continuous (or lower semicontinuous) and β is dually Scott-continuous (or upper semicontinuous).*

Proof. Suppose that f is continuous. We show that $\alpha : X \longrightarrow L$ is Scott-continuous and $\beta : X \longrightarrow L$ is dually Scott-continuous. Let V be Scott open in L such that $\alpha(x) \in V$ and let W be Scott open in L such that $\beta(x) \in W$. By continuity of L there exists $a \in V$ such that $\alpha(x) \in \twoheaduparrow a$ and $b \in W$ such that $\beta(x) \in \twoheaduparrow^{op} b$.

Let $\twoheaduparrow[a, b] = A$. We have that A is open in $\mathbb{L}$, and since f is continuous in the Scott topology of $\mathbb{L}$, $f^{-1}(A)$ is open in X. Let $z \in f^{-1}(A)$. We have that $f(z) = [\alpha(z), \beta(z)] \in \twoheaduparrow[a, b]$, which is equivalent to $[a, b] \ll [\alpha(z), \beta(z)]$, and by Lemma 1.4 this is equivalent to $a \ll \alpha(z)$ and $b \ll^{op} \beta(z)$. So we have that $\alpha(z) \in \twoheaduparrow a$ and $\beta(z) \in \twoheaduparrow^{op} b$, which establishes the continuity of α and β.

Now we show that f is continuous if α is Scott-continuous and β is dually Scott-continuous. Let U be open in $\mathbb{L}$ such that $f(x) = [\alpha(x), \beta(x)] \in U$. Since U is open in the Scott topology of $\mathbb{L}$, there exists an order interval $[a, b] \in U$ such that $[\alpha(x), \beta(x)] \in \twoheaduparrow[a, b]$.

Let $O = \alpha^{-1}(\twoheaduparrow a) \cap \beta^{-1}(\twoheaduparrow^{op} b)$. By construction O is open in X, and $O \neq \emptyset$ because $x \in O$. Let $z \in O$. Since $\alpha(z) \in \twoheaduparrow a$ and $\beta(z) \in \twoheaduparrow^{op} b$ we have that $a \ll \alpha(z)$ and $b \ll^{op} \beta(z)$, and by Lemma 1.4 this is equivalent to saying that $[a, b] \ll [\alpha(z), \beta(z)]$. This means that $f(z) = [\alpha(z), \beta(z)] \in \twoheaduparrow[a, b]$, and so we have $f(O) \subseteq \twoheaduparrow[a, b]$. □

1.4 The Domain of Approximate Functions

Proposition 1.10. *The set of all continuous approximate functions from a locally compact X to a bicontinuous lattice L (which is the set of continuous functions from X into $\mathbb{L}$) ordered by the pointwise order is a bounded complete domain $[X \longrightarrow \mathbb{L}]$, called the domain of approximate functions.*

Proof. From Theorem 1.5 we know that $\mathbb{L}$ is a bounded continuous domain, and this makes the set of approximate functions to be one. See [LG] Proposition II-4.6. □

Let X be a locally compact separable metric space, and L be a bicontinuous lattice. We have also the spaces of lower semicontinuous functions $(LSC(X,L), \leq)$ and the space of upper semicontinuous functions $(USC(X,L), \geq)$, where the order for both of them is the pointwise order. We define

$$\widehat{\mathbb{L}} = \{(f,g) \in LSC(X,L) \times USC(X,L) : f \leq g\}.$$

For $LSC(X,L) \times USC(X,L)$ we will consider the order given by

$(f_1, g_1) \leq (f_2, g_2)$ if and only if $f_1 \leq f_2$ in $LSC(X,L)$ and $g_1 \leq g_2$ in $USC(X,L)$.

Proposition 1.11. *The set* $\widehat{\mathbb{L}}$ *is a Scott closed bounded complete subdomain, and it is homeomorphic to the domain of the continuous approximate functions,* $[X \to \mathbb{L}]$.

Proof. The set $\widehat{\mathbb{L}} \subseteq LSC(X,L) \times USC(X,L)$ is closed under directed sups and arbitrary infs, so, by [LG] Theorem I-2.6, it is a Scott closed bounded complete subdomain of the domain $LSC(X,L) \times USC(X,L)$.

For the second part of the Proposition let $O : [X \to \mathbb{L}] \to \widehat{\mathbb{L}}$ be defined by $O([f,g]) = (f,g)$ for any $[f,g] \in [X \to \mathbb{L}]$. Since $[f,g] \in [X \to \mathbb{L}]$ then $f \in LSC(X,L)$, $g \in USC(X,L)$ and $f \leq g$, which makes our application well defined. If $[f_1,g_1], [f_2,g_2] \in [X \to \mathbb{L}]$ such that $[f_1,g_1] \neq [f_2,g_2]$ then at least one of these is true: $f_1 \neq f_2$ or $g_1 \neq g_2$, and this implies $(f_1,g_1) \neq (f_2,g_2)$, which makes the application O one-to-one. If $(f,g) \in \widehat{\mathbb{L}}$ then it is clear that $[f,g] \in [X \to \mathbb{L}]$, so O is surjective.

One sees directly that this one-to-one correspondence is an order isomorphism, hence a homeomorphism for the Scott and biScott topologies.

Let $[f,g] \in [X \to \mathbb{L}]$. Then $\Uparrow[f,g]$ is a basic open set in the domain. We show that the image of this set is an open set in the topology of $\widehat{\mathbb{L}}$. Using Lemma 1.4 we get

$$\begin{aligned} O(\Uparrow[f,g]) &= \{(\alpha,\beta) : [f,g] \ll [\alpha,\beta]\} \\ &= \{(\alpha,\beta) : f \ll \alpha,\ g \ll^{op} \beta,\ \alpha \leq \beta\} \\ &= \Uparrow f \times \Downarrow g \cap \widehat{\mathbb{L}} \end{aligned}$$

and the last set is open in the topology of $\widehat{\mathbb{L}}$.

Therefore the application O is a homeomorphism. □

Definition 1.12. For any function $f : X \longrightarrow L$, we define

$$f_*(x) := \sup\{\inf f(U) : x \in U,\ U \text{ is open}\}$$

and

$$f^*(x) := \inf\{\sup f(U) : x \in U,\ U \text{ is open}\}.$$

Proposition 1.13. *Let $f : X \longrightarrow L$ be a function, and f_*, f^* be defined as in Definition 1.12. The following are true:*

(i) $f_* \leq f \leq f^*$;

(ii) *f_* is lower semicontinuous and f^* is upper semicontinuous;*

(iii) *f is lower semicontinuous if and only if $f = f_*$;*

(iv) *f upper semicontinuous if and only if $f = f^*$;*

(v) *f is continuous if and only if $f = f^* = f_*$;*

(vi) *f_* is the largest lower semicontinuous function such that $f_* \leq f$;*

(vii) *f^* is the smallest upper semicontinuous function such that $f \leq f^*$.*

Proof. (i): For any U open containing x the next inequalities hold:

$$\inf f(U) \leq f(x) \leq \sup f(U).$$

If we take the sup for the first inequality and the inf for the second, over all open sets U containing x, we get (i).

(ii): We will prove that f_* is continuous in the Scott topology, which is equivalent to saying that f_* is lower semicontinuous. Let $V \subset L$ be open such that $f_*(x) \in V$. There exists $a \in V$ such that $f_*(x) \in \twoheaduparrow a$, hence $a \ll f_*(x)$. Since L is continuous, by Theorem I-1.9 [LG], there exists $b \in L$ such that $a \ll b \ll f_*(x)$. It is easy to see that the set $D = \{\inf f(U) : U \text{ open}, x \in U\}$ is a directed set in L and $f_*(x) = \sup D$. This implies that $b \leq \inf f(U)$ for some U open, $x \in U$, and for any $z \in U$ we get $b \leq f_*(z)$, which means $f_*(U) \subseteq \uparrow b \subseteq \twoheaduparrow a \subseteq V$.

Dually, we can prove that f^* is continuous in the dual Scott topology, which means upper semicontinuous.

(iii): Since f_* is lower semicontinuous, if $f = f_*$ then f is lower semicontinuous.

Conversely, suppose f is lower semicontinuous, or equivalently Scott continuous. Suppose $f_*(x) < f(x)$ for some x. Then there exists $a \in L$, $a \ll f(x)$ and $a \nleq f_*(x)$. So $f(x) \in \twoheaduparrow a$, where $\twoheaduparrow a$ is an open set in the Scott topology, hence in the biScott topology. Therefore there exists U open in X, $x \in U$ such that $f(U) \subseteq \twoheaduparrow a$, so $a \ll f(z)$ for any $z \in U$ and $a \leq \inf f(U) \leq f_*(x)$, which contradicts the fact that a was chosen such that $a \nleq f_*(x)$. That means $f_* = f$ if f is lower semicontinuous.

(iv): Dual to (iii).

(v): If f is continuous then it is both upper and lower semicontinuous, so, using (iii) and (iv), $f = f^*$ and $f = f_*$, hence $f_* = f = f^*$.

Conversely, if the double equality holds then f is both lower and upper semicontinuous by (iii) and (iv), hence continuous.

(vi): It is easy to see that if f and $g : X \longrightarrow L$ are such that $g \leq f$ then $g_* \leq f_*$. If g is a lower semicontinuous function with this property, then, by (iii), $g_* = g$, and we get $g \leq f_*$, which means f_* is the largest lower semicontinuous function such that $f_* \leq f$.

(vii): The proof is similar to the proof of (vi), in the dual topology of L. $\square$

Proposition 1.14. *The maximal elements in the domain* $[X \longrightarrow \mathbb{L}]$ *of approximate functions have the form* $f(x) = [\alpha(x), \beta(x)]$*, where* $\alpha^* = \beta$ *and* $\beta_* = \alpha$*. These include the continuous functions.*

Proof. We know that the elements of the domain $[X \longrightarrow \mathbb{L}]$ are continuous approximate functions, which, by Theorem 1.9, means that α is lower semicontinuous, and β is upper semicontinuous. That is, $\alpha_* = \alpha \leq \alpha^*$ and $\beta_* \leq \beta = \beta^*$.

Let f be a maximal element of the domain $[X \longrightarrow \mathbb{L}]$. Since $\alpha \leq \beta$ we have that $\alpha^* \leq \beta^* = \beta$. Thus $[\alpha(x), \alpha^*(x)] \subseteq [\alpha(x), \beta(x)]$, which means that $[\alpha(x), \beta(x)] \leq [\alpha(x), \alpha^*(x)]$. If f is maximal then we must have $[\alpha(x), \beta(x)] = [\alpha(x), \alpha^*(x)]$, and that gives us $\alpha^* = \beta$. A similar proof yields that $\alpha = \beta_*$ if f is maximal in the domain.

Now suppose that $f = [\alpha, \beta]$ is such that $\alpha^* = \beta$ and $\beta_* = \alpha$. Suppose that $f \leq g = [\alpha_1, \beta_1]$. Then $\alpha \leq \alpha_1 \leq \beta_1 \leq \beta$ implies $\beta = \alpha^* \leq \alpha_1^* \leq \beta_1^* = \beta_1 \leq \beta$, so $\alpha_1^* = \beta_1$ and $\beta = \beta_1$. Similarly $\alpha_1 = (\beta_1)_*$ and $\alpha = \alpha_1$. That means $f = g$, so f is maximal in the domain. $\square$

1.5 Quasicontinuous Functions

Definition 1.15. A function $f : X \longrightarrow Y$ is called *quasicontinuous* at x if for any open set V containing $f(x)$ and any U open containing x, there exists a nonempty open set $W \subseteq U$ such that $f(W) \subseteq V$. A function is called *quasicontinuous* if it is quasicontinuous at every x in the domain.

In the following X is assumed to be a locally compact Hausdorff space and L is an ω-bicontinuous lattice equipped with the biScott topology.

Lemma 1.16. *If $f : X \longrightarrow L$ is quasicontinuous, then f is continuous at a dense G_δ-set of points.*

Proof. We will show first that any function $f : X \longrightarrow Y$, where Y is a metric space, is continuous on a G_δ-set.

Let $Osc(f, x) = \inf\{diam f(U) : x \in U \text{ open}\}$. We have that f is continuous at x if and only if $Osc(f, x) = 0$. The set $A_\varepsilon = \{x : Osc(f, x) \geq \varepsilon\}$ is closed in X and so its complement $B_\varepsilon = \{x : Osc(f, x) < \varepsilon\}$ is open in X. It is clear that we have

$$\bigcap_{n \geq 1} B_{1/n} = \{x \in X : f \text{ is continuous at } x\}$$

and thus this set is a G_δ-set.

In our case we have L an ω-bicontinuous lattice equipped with biScott topology, which means that the biScott topology is metrizable. Also X is a locally compact Hausdorff space, thus a Baire space. If we show that for a quasicontinuous function f, B_ε is dense in X for any ε, then by Baire's theorem we will have that our countable intersection of open dense subsets is a dense subset of X.

Let $\varepsilon > 0$. Let $U \neq \emptyset$ be open in X. We show that $U \cap B_\varepsilon \neq \emptyset$. Let $x \in U$ and let V open in L such that $f(x) \in V$ and $\text{diam} V < \varepsilon$. Since f is quasicontinuous at x there exists a nonempty $W \subseteq U$ such that $f(W) \subseteq V$, which means that $diam f(W) < \varepsilon$. Let $z \in W \subseteq U$. By definition, $Osc(f, z) \leq diam f(W) < \varepsilon$, and that says $z \in B_\varepsilon \cap U$. Since U was chosen to be an arbitrary nonempty open set in X we can say that the intersection of B_ε with any open in X is nonempty, and that means B_ε is dense in X for any $\varepsilon > 0$.

In conclusion the set of points of continuity for the quasicontinuous $f : X \longrightarrow Y$ is a dense G_δ-set of points. $\square$

Remark 1.17. The reverse of Lemma 1.16 is not true. For example

$$f(x) = \begin{cases} -1 & , \text{if } x < 0 \; ; \\ 0 & , \text{if } x = 0; \\ 1 & , \text{if } x > 0 \; . \end{cases} \tag{1.1}$$

is continuous on a dense G_δ-set but is not quasicontinuous.

Lemma 1.18. *If f and g are two quasicontinuous functions that agree on a dense set, then the two functions agree on a dense G_δ-set.*

Proof. By the preceding lemma, f and g are continuous on dense G_δ-sets. If the set on which they agree is dense, then the intersection of the sets of continuity will be a dense G_δ-set, so they agree on a dense G_δ-set, which is the set of the common points of continuity. □

Theorem 1.19. *Let $f, g : X \longrightarrow L$ be quasicontinuous maps. The following are equivalent:*

(1) *f, g agree on a dense (G_δ) set (the set of common points of continuity);*

(2) *$f^* = g^*$, $f_* = g_*$;*

(3) *The graph closure of f and g agree in $X \times L$.*

Before proving the theorem we will prove some more results that will be helpfull in the proof of the theorem.

Lemma 1.20. *Let $f : X \longrightarrow L$ be quasicontinuous and let D be dense in X. We have for a nonempty open set U that*

$$\sup\{f(x) : x \in U\} = \sup\{f(x) : x \in U \cap D\}$$

and

$$\inf\{f(x) : x \in U\} = \inf\{f(x) : x \in U \cap D\}.$$

Proof. It is clear that

$$\sup\{f(x) : x \in U\} \geq \sup\{f(x) : x \in U \cap D\}.$$

If equality fails then $\sup\{f(x) : x \in U\} > \sup\{f(x) : x \in U \cap D\}$. Let $d = \sup\{f(x) : x \in U \cap D\}$. Then there exists an $x \in U \setminus D$ such that $f(x) \notin \downarrow d$. We have that $\downarrow d$ is a closed set in the biScott topology of L. That means $L \setminus \downarrow d$ is open in L, and $f(x) \in L \setminus \downarrow d$. Since f is quasicontinuous at x and $x \in U$ open, there exists a nonempty $W \subseteq U$ such that $f(W) \subseteq L \setminus \downarrow d$. Since D is dense in X we have that $D \cap W \neq \emptyset$, and so there exists $a \in D \cap W \subseteq D \cap U$ such that $f(a) \in L \setminus \downarrow d$, which means $f(a) \nleq d$. This contradicts the fact that $d = \sup\{f(x) : x \in U \cap D\}$. Hence the two suprema are equal.

For the other equality there is a similar proof. □

Lemma 1.21. *If f is a function continuous at x, then $\overline{Gr(f)} \cap (\{x\} \times L) = \{(x, f(x))\}$.*

Proof. It is clear that $(x, f(x)) \in \overline{Gr(f)} \cap (\{x\} \times L)$. Suppose that there exists a $y \neq f(x)$ such that $(x, y) \in \overline{Gr(f)} \cap (\{x\} \times L)$. Since L is Hausdorff there are two open sets W, $V \subset L$ such that $y \in W$, $f(x) \in V$ and $W \cap V = \emptyset$. We have f continuous at x, so let $U \subset X$ be open containing x such that $f(U) \subseteq V$. Since $(x, y) \in \overline{Gr(f)}$ and the open $U \times W \subset X \times L$ contains (x, y) we have $U \times W \cap Gr(f) \neq \emptyset$, which means that there exists $(z, f(z)) \in U \times W$. But $f(z) \in f(U) \subseteq V$, and from this, $f(z) \in W \cap V = \emptyset$, a contradiction. That means there is no such $(x, y) \in \overline{Gr(f)} \cap (\{x\} \times L)$. □

Now we can prove Theorem 1.19.

Proof. (1) → (2): Let $D = \{x : f(x) = g(x)\}$. The set D is a dense set, by (1). Since $f^*(x) = \inf\{\sup f(U) : x \in U, U \text{ open}\}$, we can use Lemma 1.20 and obtain

$$\begin{aligned}
\sup\{f(x) : x \in U \text{ open}\} &= \sup\{f(x) : x \in U \cap D, U \text{ open}\} \\
&= \sup\{g(x) : x \in U \cap D, U \text{ open}\} \\
&= \sup\{g(x) : x \in U \text{ open}\}
\end{aligned}$$

for any open U containing x. Taking the infimum over all such U we get that $f^* = g^*$.

In the same way we can show that $f_* = g_*$.

(2) → (1): If f and g are quasicontinuous then, by Lemma 1.16, they are continuous on dense G_δ-sets. If f is continuous at x then,by Lemma 1.13(v), $f(x) = f_*(x) = f^*(x)$. The same holds for g if it continuous at x. By (2) $f^*(x) = g^*(x)$, and we have that $f(x) = g(x)$ if x is a continuity point for both f and g , hence for x is in a dense G_δ-set.

(1) → (3): Let $(x, y) \in \overline{Gr(f)}$, and $U \times V$ be a basic open subset of $X \times L$ such that $(x, y) \in U \times V$. Then there exists $z \in U$ such that $(z, f(z)) \in (U \times V) \cap Gr(f)$. Since f is quasicontinuous there exists a nonempty open $W \subseteq U$ such that $f(W) \subseteq V$. We assume $f = g$ on a dense set; that means there exists $a \in W$ so that $f(a) = g(a)$, and we get $(a, g(a)) = (a, f(a)) \in U \times V$, so $U \times V \cap Gr(g) \neq \emptyset$, and since $U \times V$ was an arbitrary basic open set, we can conclude that $(x, y) \in \overline{Gr(g)}$. Thus $\overline{Gr(f)} \subseteq \overline{Gr(g)}$.

A similar proof will show the other inclusion.

(3) → (1): Let $x \in X$ such that $f(x) \neq g(x)$ and $D = \{x : f(x) = g(x)\}$. We want to show that $x \in \overline{D}$.

Let $U \subset X$ be an open set containing x. Since f is quasicontinuous, by Lemma 1.16 it is continuous on a G_δ-set. That means there exists $z \in U$ such that f is continuous at z and, using Lemma 1.21 we have that

$$\{(z, f(z))\} = \overline{Gr(f)} \cap (\{z\} \times L) = \overline{Gr(g)} \cap (\{z\} \times L).$$

We know that $(z, g(z)) \in Gr(g) \cap (\{z\} \times L) \subseteq \overline{Gr(g)} \cap (\{z\} \times L) = \{(z, f(z))\}$, and so $(z, g(z)) = (z, f(z))$, which gives us $f(z) = g(z)$ and $z \in D \cap U \neq \emptyset$. □

1.6 Quasicontinuous Extensions of Lower Semicontinuous Functions

Lemma 1.22. *Let Y be a topological space with a countable basis $\mathcal{B}$. Define $Y[\mathcal{B}]$ to be Y endowed with the topology generated by the subbase consisting of all members of $\mathcal{B}$ and all complements of members of $\mathcal{B}$. If X is a Baire space (every nonempty open set is of second category) and $f : X \to Y$ is continuous, then $f : X \to Y[\mathcal{B}]$ is continuous at a dense G_δ-set of points.*

Proof. For each $B \in \mathcal{B}$, the set $f^{-1}(B) \cup X \setminus \overline{f^{-1}(B)}$ is an open dense subset of X, so the set $D = \bigcap\{f^{-1}(B) \cup X \setminus \overline{f^{-1}(B)} \colon B \in \mathcal{B}\}$ is a dense G_δ-subset of X. One verifies readily that $f : X \to Y[\mathcal{B}]$ is continuous at each point of D. □

Lemma 1.23. *Let X be a Baire space, L an ω-continuous domain, and $f : X \to L$ a Scott-continuous map. Then f from X into L equipped with the Lawson topology is continuous at a dense G_δ-set of points.*

Proof. Let A be a countable basis for L, that is, a countable subset of L such that every member of L is the supremum of an increasing sequence in A. Then $\mathcal{B} := \{\mathord{\uparrow\!\!\!\uparrow} x \colon x \in A\}$ is a countable basis for the Scott topology. Thus $f : X \to L[\mathcal{B}]$ is continuous at a dense G_δ-set X_0 of points of X by Lemma 1.22. If $y \in L$, then y is the supremum of a directed sequence $\{x_n\} \subseteq A$ such that $x_n \ll y$ for each n. Then $L \setminus \uparrow y = \bigcup_n L \setminus \mathord{\uparrow\!\!\!\uparrow} x_n$ is open in $L[\mathcal{B}]$. Since the Scott-open sets together with the sets $L \setminus \uparrow y$, $y \in L$, form a subbase for the Lawson topology, it follows that the Lawson topology $\lambda(L)$ is contained in the topology of $L[\mathcal{B}]$. Hence $f : X \to (L, \lambda(L))$ is also continuous on X_0. $\square$

Proposition 1.24. *Let D be a dense subset of X and let $f : D \to Y$ be continuous, where Y is a compact space. Let $\overline{f}$ be the closure in $X \times Y$ of $Gr(f) := \{(x, y) : x \in D,\ y = f(x)\}$. Then*

(i) *$\overline{f}$ restricted to*

$$E := \{x \in X : |\{x\} \times Y \cap \overline{f}| = 1\}$$

is a continuous extension of f;

(ii) *for each $x \in X$, there exists $y_x \in Y$ such that $(x, y_x) \in \overline{f}$ and any such choice defines a quasicontinuous function g which extends $\overline{f}$ restricted to E and is continuous at points of E.*

(iii) *$\overline{f}$ restricted to E is the (unique) largest extension of f to a continuous function.*

(iv) *Every quasicontinuous function extending f arises via the construction in (ii) and has E as its set of points of continuity.*

Proof. (i) Since $\mathrm{Gr}(f)$ is the inverse image of the diagonal Δ under the map $f \times 1_Y : D \times Y \to Y \times Y$, it is closed in $D \times Y$. Hence the restriction of its closure $\overline{f}$ to $D \times Y$ is equal to f. Thus $D \subseteq E$. Clearly $\overline{f}$ restricted to E defines a function from E into Y that has closed graph. Since Y is compact, the closed graph property implies that the function is continuous.

(ii) The projection $\pi_X : X \times Y \to X$ is a closed map since the factor Y is compact. Hence the image of $\overline{f}$ is a closed subset of X containing D, and thus is all of X by the denseness of D. Hence the first assertion of (ii) follows.

Clearly any function g defined as in (ii) must extend $\overline{f}$ restricted to E, since only one choice of y_x is possible for each $x \in E$. Let $x \in E$ and let $g(x) \in V$. We claim that $\{u \in X \colon g(u) \in V\}$ contains a neighborhood of x. Otherwise there exists a next $x_\alpha \to x$ such that $g(x_\alpha) \notin V$ for each α. By compactness of Y some subnet of $g(x_\alpha)$ must converge, and hence some subnet of $(x_\alpha, g(x_\alpha)$ must converge to some (x, y) in $\overline{f}$, since the latter is closed. It then follows from the definition of E that $y = f(x)$, but this is impossible since $g(x_\alpha) \notin V$ for any α. Thus g is continuous at each point of E.

Now let $x \in U$, U open in X, and let $g(x) = y_x \in V$, V open in Y. Since $(x, g(x))$ is in the closure of $\mathrm{Gr}(f)$, it follows that there exists $(x', f(x')) \in U \times V$. By the preceding paragraph g is continuous at x, and thus there exists U' open containing x' and such that $g(U') \subseteq V$. Then $U \cap U'$ is an open nonempty subset of U that is carried by g into V. The quasicontinuity of g now follows.

(iii) Let g be any continuous extension of f to some superset C of D, and let $\overline{g}$ be its closure in $X \times Y$. By part (i) applied to g, the restriction of $\overline{g}$ to C must yield the function g. Since $\overline{f} \subseteq \overline{g}$, and the "domain" of $\overline{f}$ is all of X by part (ii), it follows that the restriction of $\overline{f}$ to C defines a function, and hence by the definition of E in part (i), we have $C \subseteq E$. Again since $\overline{f} \subseteq \overline{g}$, the two define the same function on C.

(iv) Let $q : X \to Y$ be a quasicontinuous function that extends $f : D \to Y$. Let $(x, q(x)) \in U \times V$, where U is open in X and V is open in Y. By quasicontinuity there exists a nonempty open set $U' \subseteq U$ such that $q(U') \subseteq V$. Pick $d \in U' \cap D$. Then $(d, f(d)) = (d, q(d)) \in U \times V$. It follows that $(x, q(x)) \in \overline{f}$, and hence that q must arise via the construction in part (ii). Then from (ii) q is continuous at points of E. Since q restricted to its set of points of continuity is an extension of f, it follows that the points of continuity are contained in E. $\square$

Note. Dually we can get similar results for the upper semicontinuous functions in the dual topologies.

1.7 The Quasicontinuous Function Space

Let X be locally compact T_2 and L an ω-bicontinuous lattice.

Definition 1.25. Let f, g be two quasicontinuous functions. We write $f \sim g$ if f and g satisfy the equivalent conditions of Theorem 1.19. We can observe easily that this relation is an equivalence relation. Let $[f] = \{g : g : X \longrightarrow L, \text{quasicontinuous}, f \sim g\}$ be the class of f. We will denote by $Q(X, L)$ the space of such equivalence classes.

Note. A continuous function has a singleton equivalence class.

Theorem 1.26. *The association $[f] \longleftrightarrow [f_*, f^*]$ is a one-to-one correspondence between the classes of quasicontinuous functions, $Q(X, L)$, and the maximal elements of the domain $[X \longrightarrow \mathbb{L}]$ of approximate maps from X to L.*

Lemma 1.27. *If $f : X \longrightarrow L$ is a quasicontinuous function then the following equalities hold:*

$$(f_*)^* = f^* \tag{1.2}$$

$$(f^*)_* = f_* \tag{1.3}$$

Proof. By Proposition 1.13 we have that $f_* \leq f$, which implies

$$(f_*)^* \leq f^*.$$

We know that f is quasicontinuous, which, by Lemma 1.16, means that f is continuous on a G_δ-set D, so $f_* = f = f^*$ on D.

Using Lemma 1.20 we get

$$\begin{aligned} f^*(x) &= \inf\{\sup f(U) : x \in U,\ U \text{ is open}\} \\ &= \inf\{\sup f(U \cap D) : x \in U,\ U \text{ is open}\} \\ &= \inf\{\sup f_*(U \cap D) : x \in U,\ U \text{ is open}\} \\ &\leq \inf\{\sup f_*(U) : x \in U,\ U \text{ is open}\} \\ &= (f_*)^*(x). \end{aligned}$$

Therefore we have (1.2). Similarly we get (1.3). □

Proof. (of Theorem 1.26) First we show that we have a well defined function from $Q(X, L)$ into the maximal elements of the domain $[X \longrightarrow \mathbb{L}]$. Let $f \sim g$. That means $[f] = [g]$ and we want to show that $[f_*, f^*] = [g_*, g^*]$. But $f \sim g$ if and only if $f_* = g_*$ and $f^* = g^*$ which means the two intervals are equal.

Secondly we show that for any quasicontinuous f, $[f_*, f^*] \in [X \longrightarrow \mathbb{L}]$ and is also an maximal element of the domain. By Theorem 1.9, $[f_*, f^*] \in [X \longrightarrow \mathbb{L}]$ because f_* is lower semicontinuous and f^* is upper semicontinuous. Using Lemma 1.27 and Theorem 1.14 we can conclude that the image of $[f]$ is a maximal element of the domain.

We show now that the application is a bijection between the equivalence classes and the maximal elements of the domain.

Let $[\alpha, \beta] \in [X \longrightarrow \mathbb{L}]$ be a maximal element. Since $[\alpha, \beta] \leq [\alpha, \alpha^*]$, we have $\beta = \alpha^*$. Similarly $\alpha = \beta_*$. By Lemma 1.16 α is continuous on a dense $G_\delta-$set D and by Proposition 1.24 there exists a quasicontinuous function f that extends $\alpha\ |_D$. Since α is lower semicontinuous, using Lemma 1.20 we have $f_* = \alpha$. Also, since $\alpha = \beta_*$, we can use again Lemma 1.20 to get $f^* = \beta$. By our application $[f] \longmapsto [f_*, f^*] = [\alpha, \beta]$, so the application is surjective.

It is also clear that if we have f and g such that they are not in the same class of equivalence that means at least one of the following is true: $f_* \neq g_*$ or $f^* \neq g^*$. This means that $[f_*, f^*] \neq [g_*, g^*]$ and so the two images are not equal, so the application is injective. □

Remark 1.28. By writing $f \in Q(X, L)$ we will understand the equivalence class of the quasicontinuous function f, or the maximal element $[f_*, f^*]$ of the domain $[X \to \mathbb{L}]$. Therefore for $x \in X$ we have

$$f(x) = [f](x) = [f_*(x), f^*(x)] \in \mathbb{L}.$$

Proposition 1.29. *Let L be a bicontinuous lattice, and let $[f], [g] \in Q(X, L)$ such that $[f] \neq [g]$. Then there exist $U \subseteq X$, a, $b \in L$, $b \not\leq a$ such that for any $x \in U$ $[f](x) \in \Uparrow[\bot, a]$ and $[g](x) \in \Uparrow[b, \top]$ (or vice-versa), where $\bot = \inf L$ and $\top = \sup L$.*

Proof. Let $[f], [g] \in Q(X, L)$ be such that $[f] \neq [g]$. Then by Theorem 1.19 there exists $x \in X$ such that f is continuous at x, g is continuous at x and $f(x) \neq g(x)$, say $g(x) \not\leq f(x)$. Since L is bicontinuous, we can find $a, b \in L$ such that $b \not\leq a$ and $f(x) \in \Downarrow a$, $g(x) \in \Uparrow b$. Since f and g are continuous at x we have that $[f](x) = [f(x), f(x)]$ and $[g](x) = [g(x), g(x)]$, and also we can see that $[f](x) \in \Uparrow[\bot, a]$, $[g](x) \in \Uparrow[b, \top]$, which are open sets in $\mathbb{L}$. Since $[f], [g]$ are continuous there exist $U_1, U_2 \subseteq X$, open sets, $x \in U_1$, $x \in U_2$ such that $[f](U_1) \subseteq \Uparrow[\bot, a]$ and $[g](U_2) \subseteq \Uparrow[b, \top]$. We can take $U = U_1 \cap U_2$. □

Example 1.30. All the step functions which are continuous from one side at the points of discontinuity are quasicontinuous functions.

Proposition 1.31. *A uniform limit of quasicontinuous functions is quasicontinuous.*

Proof. Let $f_n : X \to L$ be quasicontinuous for every n such that $f_n \to f$ uniformly. Let $x \in X$, $U \subseteq X$ be open, $x \in U$ and $V \subseteq L$ be open, $f(x) \in V$. Let $\varepsilon > 0$ such that the open ball $B_\varepsilon(f(x)) \subseteq V$. Because of the uniform convergence, for $\frac{\varepsilon}{2}$ there exists $N_{\frac{\varepsilon}{2}}$ such that $\|f_n(y) - f(y)\| < \frac{\varepsilon}{2}$ for any y and any $n \geq N_{\frac{\varepsilon}{2}}$. Let $N \geq N_{\frac{\varepsilon}{2}}$. Then $f_N(x) \in B_{\frac{\varepsilon}{2}}(f(x))$, and, since f_N is quasicontinuous, there exists a nonempty open $U_0 \subseteq U$ such that $f_N(U_0) \subseteq B_{\frac{\varepsilon}{2}}(f(x))$. We prove that $f(U_0) \subseteq B_\varepsilon(f(x)) \subseteq V$.

Let $z \in U_0$. Then

$$\|f(z) - f(x)\| \leq \|f(z) - f_N(z)\| + \|f_N(z) - f(x)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

wich means that $f(z) \in B_\varepsilon(f(x))$. Since z is an arbitrary element of U_0, $f(U_0) \subseteq B_\varepsilon(f(x)) \subseteq V$; therefore f is quasicontinuous. □

Definition 1.32. A *regulated function* is a uniform limit of step functions.

Corollary 1.33. *A regulated function obtained using step functions that are continuous from one side at the points of discontinuity is quasicontinuous.*

Proof. This is a direct consequence of Proposition 1.31 and Example 1.30. □

1.8 The Topology of $Q(X, L)$

Considering the identification between the classes of quasicontinuous maps from X into L and the maximal elements of the domain $[X \longrightarrow \mathbb{L}]$, we have now on the space of quasicontinuous classes, $Q(X, L)$, a natural function space topology, namely the relative Scott-topology from the domain model of approximate maps. When this is viewed from the right-hand side of the one-to-one correspondence from Theorem 1.26, then this topology is rather well understood from the theory of domains. When restricted to the continuous functions, it is the compact-open topology. Let $(f_i)_i \subset Q(X, L)$ and $f \in Q(X, L)$. We will say that

$f_i(x) \to f(x)$ if for any $[a,b] \in \mathbb{L}$, $[f_*(x), f^*(x)] \in \Uparrow[a,b]$, there exists $N > 0$ such that $[(f_i)_*(x), (f_i)^*(x)] \in \Uparrow[a,b]$ for all $i \geq N$.

Proposition 1.34. *Let $(f_n)_n \subseteq Q(X,L)$ and $f \in Q(X,L)$. The following are equivalent:*

(1) *$(f_n)_* \longrightarrow f_*$ in the Scott topology, and $(f_n)^* \longrightarrow f^*$ in the dual-Scott topology;*

(2) *$f_* = \sup_n \left(\inf_{n \leq m} (f_m)_* \right)_* = \underline{\lim}(f_n)_*$ and $f^* = \inf_n \left(\sup_{n \leq m} (f_m)^* \right)^* = \overline{\lim}(f_n)^*$;*

(3) *$[(f_n)_*, (f_n)^*] \longrightarrow [f_*, f^*]$ in the relative Scott topology of the set of maximal elements of the domain $[X \to \mathbb{L}]$;*

(4) *There exist an increasing sequence $(g_n)_n \subseteq LSC(X,L)$ and a decreasing sequence $(h_n)_n \subseteq USC(X,L)$ such that $f_* = \sup_n g_n$, $f^* = \inf_n h_n$ and $g_n \leq (f_n)_* \leq (f_n)^* \leq h_n$, for each n.*

Proof. (1) $\Leftrightarrow$ (2). From the definition of Scott convergence we have that $(f_n)_* \longrightarrow f_*$ if and only if $f_* \leq \underline{\lim}(f_n)_*$, and similarly for the dual Scott convergence. The only thing that must be proved is that

$$f_* = \sup_n \left(\inf_{n \leq m} (f_m)_* \right)_*$$

and

$$f^* = \inf_n \left(\sup_{n \leq m} (f_m)^* \right)^*,$$

where the inequalities follow from [LG] Proposition III-3.12. We have that

$$\left[\sup_n \left(\inf_{m \leq n} (f_n)_* \right)_*, \inf_n \left(\sup_{m \leq n} (f_n)^* \right)^* \right] \in [X \to \mathbb{L}],$$

and

$$[f_*, f^*] \leq \left[\sup_n \left(\inf_{m \leq n} (f_n)_* \right)_*, \inf_n \left(\sup_{m \leq n} (f_n)^* \right)^* \right] \text{ in } [X \to \mathbb{L}].$$

Since $[f_*, f^*]$ is a maximal element of the domain $[X \to \mathbb{L}]$, we must have the equality of the two intervals, therefore the equalities we want.

(2) $\Leftrightarrow$ (3). We will show first that in the domain $[X \to \mathbb{L}]$, for any $n > 0$ we have the equality

$$\inf_{n \le m} [(f_m)_*, (f_m)^*] = \left[\left(\inf_{n \le m} (f_m)_* \right)_*, \left(\sup_{n \le m} (f_m)^* \right)^* \right]. \tag{1.4}$$

Let $n > 0$. Then we have

$$\left[\inf_{n \le m} (f_m)_*, \sup_{n \le m} (f_m)^* \right] \le [(f_m)_*, (f_m)^*], \text{ for each } m \ge n.$$

Using Proposition 1.13

$$\left[\left(\inf_{n \le m} (f_m)_* \right)_*, \left(\sup_{n \le m} (f_m)^* \right)^* \right] \le [(f_m)_*, (f_m)^*], \text{ for each } m \ge n. \tag{1.5}$$

Let $[\alpha, \beta]$ an element of the domain $[X \to \mathbb{L}]$ such that

$$[\alpha, \beta] \le [(f_m)_*, (f_m)^*], \text{ for each } m \ge n,$$

which implies

$$\alpha \le (f_m)_* \text{ and } (f_m)^* \le \beta \text{ for each } m \ge n;$$

thus

$$\alpha \le \inf_{n \le m} (f_m)_* \text{ and } \sup_{n \le m} (f_m)^* \le \beta.$$

Since α is lower semicontinuous and β is upper semicontinuous, we can get

$$\alpha = \alpha_* \le \left(\inf_{n \le m} (f_m)_* \right)_* \text{ and } \left(\sup_{n \le m} (f_m)^* \right)^* \le \beta^* = \beta,$$

so we have

$$[\alpha, \beta] \le \left[\left(\inf_{n \le m} (f_m)_* \right)_*, \left(\sup_{n \le m} (f_m)^* \right)^* \right],$$

which means that

$$\left[\left(\inf_{n \le m} (f_m)_* \right)_*, \left(\sup_{n \le m} (f_m)^* \right)^* \right]$$

is the largest element of the domain $[X \to \mathbb{L}]$ satisfying 1.5. Therefore we have 1.4.

The set

$$D = \left\{ \left[\left(\inf_{n \le m} (f_m)_* \right)_*, \left(\sup_{n \le m} (f_m)^* \right)^* \right] : n > 0 \right\}$$

is directed in the domain $[X \to \mathbb{L}]$ and its supremum is given by

$$\left[\sup_n \left(\inf_{n \le m} (f_m)_* \right)_*, \inf_n \left(\sup_{n \le m} (f_m)^* \right)^* \right].$$

Thus if we suppose that (2) is satisfied then we have

$$[f_*, f^*] = \left[\sup_n \left(\inf_{n \le m} (f_m)_* \right)_*, \inf_n \left(\sup_{n \le m} (f_m)^* \right)^* \right] = \underline{\lim}[(f_n)_*, (f_n)_*],$$

which is equivalent to (3), and so (2) $\Rightarrow$ (3).

If we assume (3) is satisfied, then that is equivalent to

$$[f_*, f^*] = \underline{\lim}[(f_n)_*, (f_n)_*] = \left[\sup_n \left(\inf_{n \le m} (f_m)_* \right)_*, \inf_n \left(\sup_{n \le m} (f_m)^* \right)^* \right],$$

which implies

$$f_* = \sup_n \left(\inf_{n \le m} (f_n)_* \right)_*$$

and

$$f^* = \inf_n \left(\sup_{n \le m} (f_m)^* \right)^*.$$

Thus $f_* = \underline{\lim}(f_n)_*$ and $f^* = \overline{\lim}(f_n)^*$, so (2) is satisfied too, and (3) $\Rightarrow$ (2).

(2) $\Leftrightarrow$ (4). Suppose that (2) is true. For each $n \ge 1$ let

$$g_n = \left(\inf_{n \le m} (f_m)_* \right)_* \text{ and } h_n = \left(\sup_{n \le m} (f_m)^* \right)^*.$$

It is clear that each g_n is lower semicontinuous and each h_n is upper semicontinuous. Since $n_1 \le n_2$ implies

$$\inf_{n_1 \le m} (f_m)_* \le \inf_{n_2 \le m} (f_m)_*$$

and

$$\sup_{n_1 \le m} (f_m)^* \ge \sup_{n_2 \le m} (f_m)^*,$$

we have $g_{n_1} \leq g_{n_2}$ and $h_{n_1} \geq h_{n_2}$, which means $(g_n)_n$ is increasing and $(h_n)_n$ is decreasing. It is also clear that we have $g_n \leq (f_n)_* \leq (f_n)^(*) \leq h_n$ for each $n > 0$.

For the other implication, let $(g_n)_n$ and $(h_n)_n$ like in (3). Since $(g_n)_n$ is increasing, we have $g_n \leq g_m \leq f_m$ for every $m \geq n$, which implies

$$g_n \leq \inf_{n \leq m} (f_m)_*,$$

and, since $g_n \in LSC(X, L)$,

$$g_n \leq \left(\inf_{n \leq m} (f_m)_* \right)_* .$$

Therefore

$$f_* \leq \sup_n \left(\inf_{n \leq m} (f_m)_* \right)_* .$$

Similarly we get

$$f^* = \inf_n \left(\sup_{n \leq m} (f_m)^* \right)^* = \overline{\lim}(f_n)^*,$$

and because $f \in Q(X, L)$, $[f_*, f^*]$ is a maximal element of the domain $X \to \mathbb{L}]$, hence we have (2). □

Definition 1.35. The topology defined on $Q(X, L)$ from the convergence given in Proposition 1.34 will be called the *quasiorder topology*, or *qo-topology* for short, since by condition (4) it is a variant of the classical order topology defined on complete lattices.

Corollary 1.36. *The qo-topology on $Q(X, L)$ is Hausdorff.*

Proof. Proposition 1.34(2) implies uniquesness of limits, hence Hausdorffness. □

Proposition 1.37. *Let $f_n \to f$ in $Q(X, L)$ and $x_n \to x$ in X. Then the following are true:*

$$(f_n)_*(x_n) \to f_*(x) \text{ in the Scott topology,}$$

$$(f_n)^*(x_n) \to f^*(x) \text{ in the dual Scott topology.}$$

In particular, if $f^*(x) = f_*(x) = f(x)$, *then* $(f_n)^*(x_n), (f_n)_*(x_n) \to f(x)$ *in* L.

Proof. This is a consequence of Proposition 1.34 and [LG] Theorem II-4.10. □

Remark 1.38. Considering the set $\widehat{\mathbb{L}}$ defined in Proposition 1.11, the result of this proposition and the fact that on the set of maximal elements the Scott (σ) topology and the Lawson (λ) topology agree, by [JL] Corollary 3.4, we can use also the product Lawson topology on $Q(X, L)$, if we consider each class $[f] \in Q(X, L)$ as the pair $(f_*, f^*) \in \widehat{\mathbb{L}}$.

Proposition 1.39. *Let* $(u_n)_n \subseteq LSC(X, \overline{\mathbb{R}})$ *and* $U \in LSC(X, \overline{\mathbb{R}})$. *Then* $u_n \to U$ *in* $LSC_\lambda(X, \bar{\mathbb{R}})$ *if and only if the following are true:*

(1) *If* $x_n \to x \in X$, *then* $U(x) \leq \liminf_n u_n(x_n)$;

(2) *For* $x \in X$ *there exists* $z_n \to x$ *such that* $u_n(z_n) \to U(x)$.

Proof. ($\Rightarrow$). Suppose $u_n \to U$ in $LSC_\lambda(X, \bar{\mathbb{R}})$. Then (1) is a consequence of the Scott convergence and the continuity of the evaluation function $E : LSC_\sigma(X, \bar{\mathbb{R}}) \times X \to \bar{\mathbb{R}}_\sigma$.

(2) For each n, set

$$\beta_n = \inf\{d(y, x) + d(u_n(y), u(x)) : y \in X\}.$$

We will prove that $\beta_n \to 0$.

Let $\varepsilon > 0$. Pick B open in $LSC_\sigma(X, \bar{\mathbb{R}})$ containing U and $0 < \delta < \frac{\varepsilon}{2}$ such that $E(B \times B_\delta(x)) \subseteq (U(x) - \frac{\varepsilon}{2}, \infty]$. Define $Q : X \to \bar{\mathbb{R}}$ by $Q(B_\delta(x)) = U(x) + \frac{\varepsilon}{2}$, $Q(y) = -\infty$ otherwise. Then $U \notin \uparrow Q$ since $U(x) < Q(x)$. Thus there exists N such that $u_n \notin \uparrow Q$ and $u_n \in B$ for $n \geq N$. Then for $n \geq N$,

$$U(x) - \frac{\varepsilon}{2} < u_n(z) < U(x) + \frac{\varepsilon}{2}$$

for some $z \in B_\delta(x)$. Thus

$$d(z,x) + d(u_n(z), U(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and hence $\beta_n < \varepsilon$ for $n \geq N$, thus $\beta_n \to 0$.

Now choose for each n, a point z_n such that

$$d(z_n, x) + d(u_n(z_n), U(x)) < \beta_n + \frac{1}{n}.$$

It follows that $z_n \to x$ and $u_n(z_n) \to U(x)$.

($\Leftarrow$). Suppose $(u_n)_n \subseteq LSC(X, \mathbb{R})$ and $U \in LSC(X, \mathbb{R})$ such that we have (1) and (2). It is clear that (1) implies $u_n \to U$ in $LSC_\sigma(X, \mathbb{R})$. Let $F \in LSC(X, \mathbb{R})$ such that $U \in LSC(X, \mathbb{R}) \setminus \uparrow F$, a basic open set in the $\lambda-$topology. Therefore $F \nleq U$, or equivalently, there exists $x \in X$ such that $F(x) \nleq U(x)$ in $\mathbb{R}$, which means $U(x) < F(x)$. Then there exists $a \in \mathbb{R}$ such that $U(x) < a < F(x)$. By (2) there exists $(z_n)_n \subseteq \mathbb{R}$ such that $z_n \to x$ and $u_n(z_n) \to U(x)$. Since $U(x) \in [-\infty, a) \subseteq \overline{\mathbb{R}}$ is open, there exists $N_1 > 0$ such that for every $n \geq N_1$ $u_n(z_n) \in [-\infty, a)$.

Since F is lower semicontinuous and $F(x) \in (a, \infty]$, there exists an open $W \subseteq X$, $x \in W$ such that $F(W) \subseteq (a, \infty]$, and since $z_n \to x$ there exists $N_2 > 0$ such that $z_n \in W$ for any $n \geq N_2$. Then for every $n \geq N = \max(N_1, N_2)$ we have $F(z_n) \nleq u_n(z_n)$, which implies that for any $n \geq N$ $F \nleq u_n$, or, equivqlently, $u_n \in LSC(X, \mathbb{R}) \setminus \uparrow F$. Therefore we have $u_n \to U$ in $LSC_\lambda(X, \bar{\mathbb{R}})$. $\square$

2. Generalized Derivatives

In this chapter we will use the bicontinuous lattice $\overline{\mathbb{R}}$, but since we will work mostly with finite-valued functions, we will use $\mathbb{R}$ for denoting that, i.e., $Q(X,\mathbb{R})$ denotes the members of $Q(X,\overline{\mathbb{R}})$ with $f_*(X) \cup f^*(X) \subseteq \mathbb{R}$.

2.1 Generalized Gradient

Let X be a locally compact locally convex subset of $\mathbb{R}^m$ with dense interior. We consider the partial derivative operator

$$\frac{\partial}{\partial x_k} : C^1(X,\mathbb{R}) \longrightarrow C^0(X,\mathbb{R}) \subseteq Q(X,\mathbb{R}) \tag{2.1}$$

and the gradient operator

$$\nabla : C^1(X,\mathbb{R}) \longrightarrow C^0(X,\mathbb{R}^m) \subseteq Q(X,\mathbb{R}^m),\ \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_m}\right). \tag{2.2}$$

Lemma 2.1. *Suppose $f_n \to f$ in $Q(X,\mathbb{R})$, $f_n \in C^1(X,\mathbb{R}) \subseteq Q(X,\mathbb{R})$, and $\nabla f_n \to F$ in $Q(X,\mathbb{R}^m)$. Then $f \in Q(X,\mathbb{R})$ has a unique representative, which is a locally Lipschitz function.*

Proof. Let $x \in X$. Since $F_*(X) \cup F^*(X) \subseteq \mathbb{R}^m$, we may pick $a, b \in \mathbb{R}^m$ such that $b \gg F^*(x)$ and $a \ll F_*(x)$. By continuity of the evaluation map $E : LSC_\sigma(X,\overline{\mathbb{R}}^m)\times X \to \overline{\mathbb{R}}_\sigma$ there exist $W \subseteq LSC(X,\overline{\mathbb{R}}^m)$ open, $F_* \in W$, and $U_1 \subseteq X$, $x \in U_1$, such that $E(W \times U_1) \subseteq \mathop{\Uparrow} a$. Since $\nabla f_n \to F$ in $Q(X,\overline{\mathbb{R}}^m)$ and $f_n \in C^1(X,\mathbb{R})$ for all n, there exists $N_1 > 0$ such that $\nabla f_n \in W$ for all $n \geq N_1$, so we get

$$\nabla f_n(u) \in \mathop{\Uparrow} a \text{ for each } u \in U_1 \text{ and each } n \geq N_1. \tag{2.3}$$

Similarly, using the continuity of the evaluation map $E : USC_\sigma(X,\overline{\mathbb{R}}^n)\times X \to \overline{\mathbb{R}}_{\sigma^*}$, where σ^* is the dual Scott topology, we can find an open set $U_2 \subseteq X$, $x \in U_2$ and

$N_2 > 0$ such that

$$\nabla f_n(u) \in \mathop{\downarrow\!\!\!\downarrow} b \text{ for each } u \in U_2 \text{ and each } n \geq N_2. \qquad (2.4)$$

Let $N = \max\{N_1, N_2\}$ and $U = U_1 \cap U_2$. From (2.3) and (2.4) we have

$$\nabla f_n(u) \in \mathop{\downarrow\!\!\!\downarrow} b \cap \mathop{\uparrow\!\!\!\uparrow} a \text{ for each } u \in U \text{ and each } n \geq N. \qquad (2.5)$$

Let $M > 0$ such that $[a, b] \subseteq B_M(0)$, the open ball in $\mathbb{R}^m$ around 0 of radius M. Then

$$\|\nabla f_n(u)\| \leq M \text{ for each } u \in U \text{ and each } n \geq N. \qquad (2.6)$$

We can choose U such that U is also convex, so that we can apply the Mean Value Theorem for differentiable functions on $\mathbb{R}^m$. Therefore for each $n > N$ and each $u, v \in U$ there exists $0 < t_n < 1$ such that

$$f_n(u) - f_n(v) = \langle \nabla f_n(\xi_n), u - v \rangle ,$$

where $\xi_n = t_n u + (1 - t_n)v \in U$, and by (2.6) we get

$$\|f_n(u) - f_n(v)\| \leq \|\nabla f_n(\xi_n)\| \|u - v\| \leq M\|u - v\|,$$

for each $n \geq N$, which means that $\mathcal{F} = \{f_n|_U : n \geq N\}$ is an equicontinuous family of functions.

Using the same arguments we use for ∇f_n to find (2.6), we can find U_0 open containing x, $N_0 > 0$, $M_0 > 0$ such that $\{f_n(y) : n \geq N_0\} \subseteq (-M_0, M_0) \subseteq \mathbb{R}$ for each $y \in V = U_0 \cap U$, which makes the closure of $\{f_n(y) : n \geq N_0\}$ compact in $\mathbb{R}$. Thus we are in the setting of [Ro] Ascoli's Theorem, so we obtain a subsequence of $\{f_n|_V : n \geq N\}$, (f_{n_k}), which converges pointwise to a continuous function g, the convergence being uniform on each compact subset of V. Equivalently, we can say that $(f_{n_k}) \to g$ in the compact-open topology, so in the Scott topology. Since all f_n are M-Lipschitz on U, then g is M-Lipschitz on U also.

The convergence $f_n \to f$ in $Q(X, \overline{\mathbb{R}})$ makes $f_{n_k} \to f$ in $Q(X, \overline{\mathbb{R}})$, and since $Q(X, \overline{\mathbb{R}})$ is Hausdorff, $f|_V = g|_V$ in $Q(X, \overline{\mathbb{R}})$, so the class $[f]$ has a unique representative, which is locally Lipschitz. □

Theorem 2.2. *Closing up the operator* ∇ *of* (2.2) *in* $Q(X, \mathbb{R}) \times Q(X, \mathbb{R}^m)$ *gives another operator with domain* $Q^1(X, \mathbb{R})$*. Each member* $[f]$ *of the domain of the extended operator has a unique representative* f *that is a locally Lipschitz map from* X *to* $\mathbb{R}$*. Its image* $\overline{\nabla} f$ *in* $Q(X, \mathbb{R}^m)$ *is called a* generalized gradient*.*

Proof. For the first part of the theorem it is sufficient to prove that the operator given in (2.1) is preclosed. Assume that there exist two sequences $f_i \to f$ and $g_i \to f$ in $Q(X, \mathbb{R})$, $f_i \in C^1(X, \mathbb{R})$, $g_i \in C^1(X, \mathbb{R})$ such that

$$\frac{\partial}{\partial x_k} f_i \to F, \ \frac{\partial}{\partial x_k} g_i \to G \ F, G \in Q(X, \mathbb{R}) \text{ and } F \neq G. \tag{2.7}$$

Since $F, G \in Q(X, \overline{\mathbb{R}})$, we can consider them as maximal elements of the domain $[X \longrightarrow [\overline{\mathbb{R}}]]$, where $[\overline{\mathbb{R}}] = \big\{[a, b] : a, b \in \overline{\mathbb{R}}, a \leq b\big\}$. From Proposition 1.29 there exists a nonempty open $U \subseteq X$, $a, b \in \mathbb{R}$, $a < b$ such that $F(x) \subseteq [-\infty, a)$ and $G(x) \subseteq (b, \infty]$ (or $F(x) \subseteq (b, \infty]$ and $G(x) \subseteq [-\infty, a)$) for every $x \in U$. Since f_i and $g_i \in C^1(X, \overline{\mathbb{R}})$, then $\dfrac{\partial}{\partial x_k} f_i$ and $\dfrac{\partial}{\partial x_k} g_i$ are continuous functions, and that means $\left[\dfrac{\partial f_i}{\partial x_k}\right](x) = \left[\dfrac{\partial f_i}{\partial x_k}(x), \dfrac{\partial f_i}{\partial x_k}(x)\right]$, respectively $\left[\dfrac{\partial g_i}{\partial x_k}\right](x) = \left[\dfrac{\partial g_i}{\partial x_k}(x), \dfrac{\partial g_i}{\partial x_k}(x)\right]$.

By continuity of the evaluation map $E : LSC_\sigma(X, \overline{\mathbb{R}}) \times X \to \overline{\mathbb{R}}_\sigma$ there exist $W \subseteq LSC(X, \overline{\mathbb{R}})$ open, $G_* \in W$, and $U_1 \subseteq U$, $x \in U_1$ open such that $E(W \times U_1) \subseteq (b, \infty]$. Since we have (2.7), there exists $N > 0$ such that

$$\frac{\partial g_i}{\partial x_k}(U_1) \subseteq (b, \infty], \text{ for every } i \geq N_1.$$

Similarly we can use the continuity of the evaluation map $E : USC_\sigma(X, \overline{\mathbb{R}}) \times X \to \overline{\mathbb{R}}_{\sigma^*}$ and (2.7) for partials of f_i to find $U_2 \subseteq U$, $x \in U_2$ and $N_2 > 0$ such that

$$\frac{\partial f_i}{\partial x_k}(U_2) \subseteq [-\infty, a), \text{ for every } i \geq N_2.$$

Let $N = \max\{N_1, N_2\}$ and $V = U_1 \cap U_2 \subseteq U$. Then for each $y \in V$ and each $n \geq N$

$$\frac{\partial f_i}{\partial x_k}(y) < a \text{ and } \frac{\partial g_i}{\partial x_k}(y) > b. \tag{2.8}$$

Since

$$\frac{\partial f_i}{\partial x_k}(y) = \lim_{h \to 0} \frac{f_i(x + he_k) - f_i(x)}{h},$$

there exists $h_1 > 0$ such that $\frac{f_i(x + he_k) - f_i(x)}{h} < a$ for every $0 < h \leq h_1$. Similarly we can find $h_2 > 0$ such that $\frac{g_i(x + he_k) - g_i(x)}{h} > b$ for every $0 < h \leq h_2$. Let h such that $0 < h \leq \min(h_1, h_2)$. We have

$$g_i(x + he_k) - g_i(x) - (f_i(x + he_k) - f_i(x)) > h(b - a) > 0. \tag{2.9}$$

By Lemma 2.1 we know that f must be locally Lipschitz, and since $f_i, g_i \to f$, for any $\varepsilon > 0$ and for any y we can choose $N > 0$ so that $|f_i(y) - g_i(y)| < \frac{\varepsilon}{2}$ for all $i \geq N$. By choosing the proper $N_3, N_4 > 0$ for $\varepsilon = \frac{h(b-a)}{2}$ and respectively $x + he_k$, x, we get that for any $i \geq N' = \max(N_3, N_4)$ $|g_i(x+he_k) - g_i(x) - (f_i(x+he_k) - f_i(x))| \leq |g_i(x + he_k) + f_i(x + he_k)| + |g_i(x) + f_i(x)| < \frac{h(b-a)}{2} + \frac{h(b-a)}{2} = h(b - a)$. For any $i \geq \max(N, N')$ this contradicts (2.9).

Let f in the domain of the extended operator on $Q(X, \mathbb{R})$. Then, there exists $(f_i)_i \subseteq C^1(X, \mathbb{R})$ such that $f_i \to f$ in $Q(X, \overline{\mathbb{R}})$ and $\nabla f_i \to F$ in $Q(X, \overline{\mathbb{R}}^m)$. We can apply Lemma 2.1, so $f \in Q^1(X, \mathbb{R})$ has a unique representative, which is Lipschitz. □

Example 2.3. Consider the absolute value function on the interval $X = [-1, 1]$. It admits an extended derivative $[g]$ that is the sign function, with either the value 1 or -1 at 0, i.e., $[g](0) = \{-1, 1\}$.

2.2 The Strong Derivative

Recall that the *strong derivative* of a function $f : U \subseteq \mathbb{R} \to \mathbb{R}$ is given by

$$\lim_{\substack{u,v\to x\\ u\neq v}} \frac{f(u)-f(v)}{u-v},$$

if the limit exists.

Let $U \subseteq \mathbb{R}^m$ be locally compact with dense interior, and let $f : U \to \overline{\mathbb{R}}$. and $x \in U$. We will say that

$$\lim_{\substack{u,v\to x\\ u\neq v}} \frac{f(u)-f(v)}{\|u-v\|}$$

is the strong derivative of f if the limit exists.

Theorem 2.4. *Let $U \subset \mathbb{R}^m$ be locally compact with dense interior, and $f_n \in C^1(U,\overline{\mathbb{R}}) \subseteq Q(U,\mathbb{R})$ such that $f_n \to f$ in $Q(X,\mathbb{R})$ and $\nabla f_n \to G$ in $Q(U,\mathbb{R}^m)$. Then the strong derivative of f exists, and it is equal to G, on a dense G_δ-set $D \subseteq X$. In particular $\nabla f = G$ on D so we can say that the gradient of f is given by*

$$\overline{\nabla} f = [(\nabla f)_*, (\nabla f)^*] = [G_*, G^*],$$

where

$$(\nabla f)_*(x) = \sup_{x\in U\ open} \inf \{\nabla f(y) : y \in U \cap D\}, \tag{2.10}$$

and

$$(\nabla f)^*(x) = \inf_{x\in U\ open} \sup \{\nabla f(y) : y \in U \cap D\}, \tag{2.11}$$

further more $[\overline{\nabla} f] = [G]$.

Proof. We can talk about the strong derivative of f, since f is a locally Lipschitz function, so its class has an unique representative.

Let $x \in \text{int}U$ be such that g is continuous at x, $u, v \in U$. Also, the fact that $f_n \to f$ in $Q(U,\mathbb{R})$ is equivalent to $f_n(x) \to f(x)$ for any $x \in U$ implies that for any $\varepsilon > 0$ there exists $N > 0$ such that

$$\frac{|f(u) - f(v) - (f_i(u) - f_i(v))}{\|u - v\|} < \frac{\varepsilon}{2} \text{ for every } i \geq N,$$

for $u \neq v$ close enough to x. Also, since each f_i is differentiable, we can apply the Mean Value theorem on $\mathbb{R}^m$ for each of them for u and v close enough to x, but not equal. Therefore, there exists $\xi_i = (1-t)u + tv$ for some $0 < t < 1$ such that

$$f_i(u) - f_i(v) = \langle \nabla f_i(\xi_i), u - v \rangle .$$

Then we have

$$\begin{aligned} \frac{|f(u) - f(v) - \langle G(x), u - v \rangle}{\|u - v\|} &\leq \frac{|f(u) - f(v) - (f_i(u) - f_i(v))}{\|u - v\|} \\ &+ \frac{|f_i(u) - f_i(v) - \langle \nabla f_i(\xi_i), u - v \rangle}{\|u - v\|} \\ &+ \frac{\| \langle \nabla f_i(\xi_i) - G(x), u - v \rangle \|}{\|u - v\|}. \end{aligned}$$

The midle term of the right-hand side of the inequality is zero. Also, since, when $u, v \to x$ $\xi_i \to x$ too, and because $\nabla f_n \to G$ in $Q(U, \mathbb{R}^m)$, by Proposition 1.37 for any $\varepsilon > 0$ there exists $N_2 > 0$ such that $\|\nabla f_i(\xi_i) - G(x)\| < \frac{\varepsilon}{2}$. Also we have

$$\frac{\| \langle \nabla f_i(\xi_i) - G(x), u - v \rangle \|}{\|u - v\|} \leq \frac{\|\nabla f_i(\xi_i) - G(x)\| \|u - v\|}{\|u - v\|} = \|\nabla f_i(\xi_i) - G(x)\| < \frac{\varepsilon}{2},$$

hence

$$\frac{|f(u) - f(v) - \langle G(x), u - v \rangle}{\|u - v\|} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore

$$\lim_{\substack{u,v \to x \\ u \neq v}} \frac{f(u) - f(v)}{\|u - v\|} = \frac{\langle G(x), u - v \rangle}{\|u - v\|},$$

so the strong derivative of f exists for x a continuity point for G, and, since $G \in Q(X, \mathbb{R}^m)$, G is continuous on a $G_\delta-$set D.

For $x \in D$ we have also

$$\nabla f(x) = G(x),$$

and since D is dense we can define

$$(\nabla f)_* = (x \to \nabla f(x) | x \in D \subseteq X)_*$$

and

$$(\nabla f)^* = (x \to \nabla f(x) | x \in D \subseteq X)^*.$$

Hence, by Lemma 1.20, we have

$$(\nabla f)_* = G_* \text{ and } (\nabla f)^* = G^*.$$

Finally, it is clear that

$$\overline{\nabla} f = [(\nabla f)_*, (\nabla f)^*] = G \text{ in } Q(X, \overline{\mathbb{R}}),$$

and the theorem is proved.

□

3. Viscosity Functions

Let X be a locally compact subset of $\mathbb{R}^n$ that has dense interior, and let $H : X \times \mathbb{R}^n \to \mathbb{R}$ be a function convex in the second argument. In this chapter we will try to solve the equation

$$H(x, \nabla y(x)) = h(x) \tag{3.1}$$

so that if y_1 and y_2 are two solutions of it, then $\min(y_1, y_2)$ is also a solution.

We will consider in the space $Q(X, \overline{\mathbb{R}})$ a differential operator defined by the left-hand side of (3.1), which is defined on the set $C^1(X, \mathbb{R}) \subseteq Q(X, \mathbb{R})$. But, since we seek the desired condition on the set of solutions of the equation (3.1), we need a new condition on the domain of the new operator, namely the domain of a desired extension must be stable with respect to the operation min. Thus we will first consider the set denoted by $C^1_{\min}(X, \mathbb{R})$, which is the set of functions f for which there exists $k > 0$ and $f_i \in C^1(X, \mathbb{R})$ for each $0 < i \leq k$ such that

$$f = \min(f_1, f_2, \ldots, f_k).$$

3.1 Continuous Hamiltonians

Recall that if $\alpha \in LSC(X, \mathbb{R})$ and $\partial_- \alpha(x) = \{\zeta \in \mathbb{R}^m : \alpha(y) \geq \alpha(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2,$ for some $\sigma > 0$ and y close enough to $x\}$ is the subgradient of α at x, then the subset $\partial_- \alpha(x) \neq \emptyset$ for x in a dense subset of X. The same is true for $\beta \in USC(X, \mathbb{R})$ and its supergradient $\partial_+ \beta$.

Proposition 3.1. *Let $H : X \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuous. For $f \in Q(X, \mathbb{R})$ let $D_1 = \{x : \partial_- f_*(x) \neq \emptyset\}$ and $D_2 = \{x : \partial_+ f^*(x) \neq \emptyset\}$, dense subsets of X. We define*

$$\mathcal{D}_- : Q(X, \mathbb{R}) \to LSC(X, \overline{\mathbb{R}})$$

by

$$\mathcal{D}_- f = \left(x \to \inf_{\substack{a \in \partial_- f_*(x) \\ \partial_- f_*(x) \neq \emptyset}} H(x, f_*(x), a) \mid x \in D_1 \subset X \right)_* , \tag{3.2}$$

and

$$\mathcal{D}_+ : Q(X, \mathbb{R}) \to USC(X, \overline{\mathbb{R}})$$

by

$$\mathcal{D}_+ f = \left(x \to \sup_{\substack{b \in \partial_+ f^*(x) \\ \partial_+ f^*(x) \neq \emptyset}} H(x, f^*(x), b) \mid x \in D_2 \subset X \right)^* . \tag{3.3}$$

Let $\Delta \subseteq Q(X, \mathbb{R})$, $\Delta = \{f \in Q(X, \mathbb{R}) : \mathcal{D} f = [\mathcal{D}_- f, \mathcal{D}_+ f] \in Q(X, \mathbb{R})\}$. *Then* $\mathcal{D}$ *is a closed operator in* $Q(X, \mathbb{R})$ *with domain* Δ.

For proving this proposition we will need the next result.

Lemma 3.2. *Let* $U \in \mathbb{R}^n$ *be locally compact,* $f : U \to \mathbb{R}$ *be lower semicontinuous,* $x \in U$ *such that* $\partial_- f(x) \neq \emptyset$ *and* $a \in \partial_- f(x)$. *Suppose also that* $(f_i) \in LSC(X, \mathbb{R})$ *is a sequence such that* $f_i \to f$ *in* $LSC_\lambda(X, \mathbb{R})$. *Then there exists* $x'_i \in U$, $\partial_- f_i(x'_i) \neq \emptyset$ *and* $a'_i \in \partial_- f_i(x'_i)$ *such that*

$$x'_i \to x, \ f_i(x'_i) \to f(x) \text{ and } a'_i \to a. \tag{3.4}$$

Proof. This is a particular case of Proposition 8.1 from [Cr], applied to lower semicontinuous functions. By Proposition 1.39 the Lawson convergence in $LSC(X, \mathbb{R})$ is equivalent with the two conditions that Proposition 8.1 from [Cr] has in its hypothesis for the lower semicontinuous case. □

Proof. (Of Proposition 3.1) Let $(f_i) \subseteq \Delta$ such that $f_i \to f$, $f \in \Delta$, $\mathcal{D} f_i \to F$ in $Q(X, \mathbb{R})$. We will show that $F = \mathcal{D} f$ in $Q(X, \mathbb{R})$. Suppose $F \neq \mathcal{D} f$. By Proposition 1.29 there exist a nonempty open $U \subseteq X$, b_1, $b_2 \in \mathbb{R}$, $b_1 < b_2$ such that $\mathcal{D} f(x) \subseteq [-\infty, b_1)$ and $F(x) \subseteq (b_2, \infty]$ for any $x \in U$ or vice versa. Therefore in U we have $F_* > b_2$ and $\mathcal{D}_+ f < b_1$. Let $x \in U$. Then $F_*(x) > b_2$ and $\mathcal{D}_+ f(x) < b_1$. Using the continuity of the evaluation map $E : LSC_\sigma(X, \mathbb{R}) \times X \to \mathbb{R}_\sigma$ we find $O \subseteq LSC(X, \mathbb{R})$ open, $F_* \in O$ and $U_1 \subseteq X$ open, $x \in U_1$ such that $E(O \times U_1) \in (b_2, \infty]$. Since $\mathcal{D} f_i \to F$ in $Q(X, \mathbb{R})$ then $\mathcal{D}_- f_i \to F_*$ in $LSC_\sigma(X, \mathbb{R})$. Therefore there exists $N_1 > 0$ such that

$$\mathcal{D}_- f_i(y) > b_2, \text{ for each } i \geq N_1, y \in U_1.$$

For every $y \in U$ we have

$$\mathcal{D}_+ f(y) < b_1.$$

Let $W = U \cap U_1$. Thus

$$\mathcal{D}_- f_i(y) > b_2 \text{ and } \mathcal{D}_+ f(y) < b_1, \text{ each } i \geq N_1, y \in U,$$

which implies that for every $y, y' \in W$ and $n \geq N_1$ we have

$$\mathcal{D}_- f_i(y') - \mathcal{D}_+ f(y) > b_2 - b_1 = c.$$

Because $\mathcal{D} f \in Q(X, \mathbb{R})$ we have $\mathcal{D}_+ f(y) \geq \mathcal{D}_- f(y)$ for any $y \in W$, so we get

$$\mathcal{D}_- f_i(y') - \mathcal{D}_- f(y) > c, \text{ for each } y, y' \in W \text{ and } i \geq N_1.$$

For a nonempty $W' \subseteq W$ let $\mu = \inf \mathcal{D}_- f(W')$. It follows from the fact that $\inf(F_*(W)) = \inf(F(W))$ for any open set W and any function F and the definition of $\mathcal{D}_- f$ that $\inf\{H(x, f_*(x), a) : x \in W',\ a \in \partial_- f_*(x)\} = \mu$.

Then there exist $x \in W'$, $a \in \partial_- f_*(x) \neq \emptyset$ such that

$$\mathcal{D}_- f_i(y') - H(x, f_*(x), a) > c/2, \text{ for every } y' \in W \text{ and } i \geq N_1.$$

From the definition of $\mathcal{D}_-(f_i)_*$, for any $x' \in W'$ for which $\partial_-(f_i)_*(x') \neq \emptyset$, we have

$$H(x', (f_i)_*(x'), a') \geq \mathcal{D}_- f_i(x'),$$

for every $a' \in \partial_-(f_i)_*(x')$. Therefore, from the last two inequalities we get that:

Statement 3.3. For all nonempty open subsets W' of W and for any $i \geq N_1$, there exists $(x, a) \in W' \times \mathbb{R}^n$, $a \in \partial_- f_*(x) \neq \emptyset$ such that for every $(x', a') \in W' \times \mathbb{R}^n$, $a' \in \partial_-(f_i)_*(x') \neq \emptyset$ we have

$$H(x', (f_i)_*(x'), a') - H(x, f_*(x), a) > c/2. \tag{3.5}$$

We will apply now Lemma 3.2, knowing from Remark 1.38 that the Lawson topology and the Scott topology agree on the set $\widehat{\mathbb{L}}$.

Statement 3.4. For any $\varepsilon > 0$, for every $(x, a) \in W' \times \mathbb{R}^n$, $a \in \partial_- f_*(x') \neq \emptyset$ there exists $N_2 > 0$ such that for every $i \geq N_2$, there exists $(x', a') \in W' \times \mathbb{R}^n$ where $a \in \partial_-(f_i)_*(x')$ with the property

$$\|x - x'\| < \varepsilon,\ |f_*(x) - (f_i)_*(x')| < \varepsilon,\ \|a - a'\| < \varepsilon. \tag{3.6}$$

We will use now the continuity of H, which implies that for any $\eta > 0$ there exists $\varepsilon > 0$ such that for any $i > 0$

$$\max\{\|x - x'\|, |f_*(x) - (f_i)_*(x')|, \|a - a'\|\} < \varepsilon$$

implies

$$|H(x, f_*(x), a) - H(x', (f_i)_*(x'), a')| < \eta.$$

Choosing $\eta < c/2$ we obtain an $\varepsilon = \varepsilon(\eta)$, and for this ε, using Statement 3.4 we can find an $N > 0$, for which there exists $W' \subseteq W$ nonempty open such that for every $i \geq N$, every $(x, a) \in W' \times \mathbb{R}^n$, $a \in \partial_- f_*(x)$, there exists $(x', a') \in W' \times \mathbb{R}^n$, where $a' \in \partial_-(f_i)_*(x')$ such that we have (3.6), which implies

$$|H(x, f_*(x), a) - H(x', (f_i)_*(x'), a')| < c/2. \tag{3.7}$$

We can observe that (3.7) is in contradiction with (3.5), and that means the operator $\mathcal{D}$ is closed. $\square$

Remark 3.5. In the begining of the proof of Proposition 3.1 we considered only one case of Proposition 1.29. For the other case the proof is similar to this one, only we work in $USC(X, \mathbb{R})$, and we use the exact form of Proposition 8.1 from [Cr].

3.2 Viscosity Functions

Definition 3.6. A function $\varphi : X \to \mathbb{R}$ is a *(discontinuous) viscosity solution* of $H(x, f, \nabla f) = g(x)$ if for any $x \in X$ such that $\partial_- \varphi_*(x) \neq \emptyset$, for any $a \in \partial_- \varphi_*(x)$ the inequality

$$H_*(x, \varphi_*(x), a) \geq g_*(x) \tag{3.8}$$

is true, and for any $x \in X$ such that $\partial_+ \varphi^*(x) \neq \emptyset$, for any $b \in \partial_+ \varphi^*(x)$ the inequality

$$H^*(x, \varphi^*(x), b) \leq g^*(x) \tag{3.9}$$

is true. We will call such a function a *viscosity function*.

Remark 3.7. If f and g are elements of $Q(X, \mathbb{R})$, then either none or all representatives of the class of f are viscosity solutions of the equation (3.1).

Proposition 3.8. *Let $H : X \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $\mathcal{D}$ be the operator in $Q(X, \mathbb{R})$ with the domain Δ defined in Proposition 3.1, and $g \in Q(X, \mathbb{R})$. Then every solution $f \in \Delta$ of the equation $\mathcal{D}y = g$ is a viscosity solution of the equation*

$$H(x, y(x), \nabla y(x)) = g(x). \tag{3.10}$$

Proof. Let $f \in \Delta$ be such that $\mathcal{D}f = g$. Then we have

$$\mathcal{D}_- f = g_* \text{ and } \mathcal{D}_+ f = g^*.$$

By definition of $\mathcal{D}_-$ in Proposition 3.1, for any $x \in X$ such that $\delta_- f_*(x) \neq \emptyset$ we have that

$$g_*(x) = \mathcal{D}_- f(x) \leq \inf_{a \in \partial_- f_*(x)} H(x, f_*(x), a),$$

which implies that for any $a \in \partial_- f_*(x)$

$$H(x, f_*(x), a) \geq g_*(x).$$

Since H is continuous, $H = H_*$, which implies that inequality (3.8) from Definition 3.6 is true.

Similarly we can obtain inequality (3.9), therefore f is a viscosity solution. $\square$

Proposition 3.9. *Let $H : X \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $\mathcal{D}$ be the operator in $Q(X, \mathbb{R})$ with the domain Δ defined in Proposition 3.1, $f, g \in Q(X, \mathbb{R})$. If f is a solution of equation (3.10) and on any open set of X the restriction of f is not a viscosity solution of any equation*

$$H(x, y(x), \nabla y(x)) = g(x) + \alpha \tag{3.11}$$

with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, then $f \in \Delta$ and $\mathcal{D}f = g$.

Proof. Let $f \in Q(X, \mathbb{R})$ be a viscosity solution of equation (3.10) such that on any open set of X the restriction of f is not a viscosity solution of any equation (3.11). Then we can define $\mathcal{D}_- f$ and $\mathcal{D}_+ f$ as in Proposition 3.1. To show $f \in \Delta$ we must show that $\mathcal{D}f = [\mathcal{D}_- f, \mathcal{D}_+ f] \in Q(X, \mathbb{R})$, which means, by Lemma 1.27, we have to prove that

$$(\mathcal{D}_- f)^* = \mathcal{D}_+ f \text{ and } (\mathcal{D}_+ f)_* = \mathcal{D}_- f.$$

This is true if we show $\mathcal{D}_- f = \mathcal{D}_+ f$ on a dense set, and then we use Lemma 1.20.

Suppose that $\mathcal{D}_- f$ and $\mathcal{D}_+ f$ are not equal on a dense set. Then there exists a nonempty open subset U, where $\mathcal{D}_- f(x) \neq \mathcal{D}_+ f(x)$. Since f is a viscosity solution we have inequality (3.8), which implies that for any x for which $\delta_- f_* \neq \emptyset$

$$\inf_{a \in \partial_- f_*(x)} H(x, f_*(x), a) \geq g_*(x)$$

is true on a dense set. From the definition of $\mathcal{D}_- f$, and this inequality we get

$$\mathcal{D}_- f(x) \geq g_*(x) \text{ for any } x \in X.$$

Similarly we can get

$$\mathcal{D}_+ f(x) \leq g^*(x) \text{ for any } x \in X.$$

Therefore, if we suppose that

$$\mathcal{D}_- f \leq \mathcal{D}_+ f$$

on X then

$$g_* \leq \mathcal{D}_- f \leq \mathcal{D}_+ f \leq g^* \text{ on } X. \tag{3.12}$$

Since $g \in Q(X, \mathbb{R})$, it is continuous on a dense set, thus $g_* = g^*$ on that dense set, which implies $\mathcal{D}_- f = \mathcal{D}_+ f$ on the same dense set. Since we suppose $\mathcal{D}_- f$ and $\mathcal{D}_+ f$ do not coincide on a dense set, there exist a nonempty open $U \subseteq X$, $\alpha > 0$, such that

$$\mathcal{D}_- f(x) \geq g^*(x) + \alpha \text{ or } \mathcal{D}_+ f(x) \leq g_*(x) - \alpha$$

for any $x \in U$. Suppose $\mathcal{D}_- f(x) \geq g^*(x) + \alpha$, the other case is similar to this one.

Then $g^* \geq g_*$ on X implies

$$\mathcal{D}_- f(x) \geq g_*(x) + \alpha$$

on U, which means that for any $x \in U$ such that $\partial_- f_*(x) \neq \emptyset$, for any $a \in \partial_- f_*(x)$,

$$H(x, f_*(x), a) \geq \inf_{a \in \partial_- f_*(x)} H(x, f_*(x), a) \geq \mathcal{D}_- f(x) \geq g_*(x) + \alpha. \tag{3.13}$$

Since $\alpha > 0$ we have

$$\mathcal{D}_+ f(x) \leq g^*(x) \leq g^*(x) + \alpha \text{ for any } x \in X.$$

Therefore, for $x \in U \subseteq X$ for which $\partial_+ f^*(x) \neq \emptyset$, and any $b \in \partial_+ f^*(x)$

$$H(x, f^*(x), b) \leq \sup_{b \in \partial_+ f^*(x)} H(x, f^*(x), b) \leq \mathcal{D}_+ f(x) \leq g_*(x) + \alpha. \tag{3.14}$$

Inequalities (3.13) and (3.14) make $f|_U$ a viscosity solution of equation (3.11), which contradicts the hypothesis of this proposition. Therefore $\mathcal{D}_- f = \mathcal{D}_+ f$ on a dense set, and also we must have $\mathcal{D}_- f \leq \mathcal{D}_+ f$ on that dense set, which make $\mathcal{D} f$ an element of $Q(X, L)$.

The last equality is a consequence of (3.12) and the fact that both g and $\mathcal{D} f$ are elements of $Q(X, \mathbb{R})$, so, by Theorem 1.26, maximal elements of the domain $[X \longrightarrow [\mathbb{R}]]$, where $[\mathbb{R}]$ is the domain of closed intervals on $\mathbb{R}$. □

We will call the operator $\mathcal{D}$ defined in Proposition 3.1 on the space $Q(X, \mathbb{R})$ the *viscous extension* of the operator defined by the hamiltonian H.

3.3 Convex Hamiltonians and Viscosity Functions

Theorem 3.10. *Let $H : X \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function convex in the second argument. For every $f \in C^1_{\min}(X, \mathbb{R})$ the map $x \to H(x, \overline{\nabla} f(x))$ is well defined on a dense subset of X and it uniquely determines an element $\mathcal{D} f \in Q(X, \mathbb{R})$. The operator $\mathcal{D} : C^1_{\min}(X, \mathbb{R}) \to Q(X, \mathbb{R})$ is preclosed, so it extends to another operator.*

Proof. Let $f \in C^1_{\min}(X, \mathbb{R})$. Then f is differentiable on a dense open set D, therefore, on that set, ∇f is continuous, so, in $Q(D, \mathbb{R})$ $[\nabla f]$ has a unique representative. Then the map $x \to H(x, \nabla f(x))$ is well defined on the dense set D.

We will prove that the set $C^1_{\min}(X, \mathbb{R})$ is a subset of the domain Δ from Proposition 3.1. Let $f = \min(f_1, f_2, \ldots, f_k)$, where $f_i \in C^1(X, \mathbb{R})$ for $1 \leq i \leq k$. We may assume that this family is one of minimal cardinality needed to represent f. Let $D_i = \{x : f(x) = f_i(x)\}$, a closed set, and let D_i° denote its interior for $1 \leq i \leq k$.

Since $X = \bigcup_{i=1}^k D_i^\circ$ is a dense open set on which f is C^1, and hence $X = \bigcup_{i=1}^k \overline{D_i^\circ}$. It follows that each $D_i^\circ \neq \emptyset$ (otherwise we can represent f without f_i).

If $x \notin D$, write $f(x) = f_{i_1}(x) = \cdots = f_{i_l}(x)$, where $x \in \overline{D_{i_j}^\circ}$ for each $1 \leq j \leq l$. Then $\partial_- f(x) = \emptyset$, hence in the definition of $\mathcal{D}_-$ in Proposition 3.1 we obtain

$$\mathcal{D}_- f(x) = (y \to \mathcal{D}f(y))_*|_{y=x}.$$

On the other hand, $\partial_+ f(x)$ is the convex hull of the vectors $\nabla f_{i_1}(x), \ldots, \nabla f_{i_l}(x)$. By the convexity of $H(x, \cdot)$, $\sup\{H(x,b) : b \in \partial_+ f(x)\}$ occurs at some $H(x, \nabla f_{i_j}(x))$ for some i_j, $1 \leq j \leq l$. Since $x \in \overline{D_{i_j}^\circ}$, it follows that

$$H(x, \nabla f_{i_j}(x)) = \lim_{n \to \infty} H(x_n, \nabla f_{i_j}(x_n))$$

for some sequence $(x_n)_n \subseteq D_{i_j}^\circ$, $x_n \to x$. It follows that

$$\mathcal{D}_+ f(x) = (x \to \mathcal{D}f(x)|x \in D)^*.$$

Since f is continuously differentiable on the dense set D then $\mathcal{D}f$ is continuous on the dense set D. Therefore $\mathcal{D}f = [\mathcal{D}_- f, \mathcal{D}_+ f] \in Q(X, \mathbb{R})$, which implies $f \in \Delta$. That means we can apply Proposition 3.1 to the particular case of our theorem, for

$$\mathcal{D} : C^1_{\min}(X, \mathbb{R}) \to Q(X, \mathbb{R});$$

therefore this operator is preclosed on its domain. □

Theorem 3.11. *Let $H : X \times \mathbb{R}^n \to \mathbb{R}$ be a uniformly continuous function convex in the second argument. Then the extension $\mathcal{D}$ by closure from $C^1_{\min}(X, \mathbb{R})$ of the operator $f \to H(\cdot, \nabla f(\cdot))$ coincides with the viscous extension. Moreover, $f \in Q(X, \mathbb{R})$ is a solution of the equation $\mathcal{D}y = g$ if and only if it is a viscosity solution of the equation*

$$H(x, \nabla y(x)) = g(x). \tag{3.15}$$

Proof. For the first part of the theorem one shall prove that the domain Δ of the viscosity extension generated by the continuous hamiltonian $H(x,p)$ convex in the second argument is the domain of the closed extension of the corresponding operator defined in Theorem 3.10, from $C^1_{\min}(X,\mathbb{R})$. For this, one can prove that for each $f \in \Delta$ there exists a sequence $(f_i)_i \subseteq C^1_{\min}(X,\mathbb{R})$, such that $f_i \to f$ and $\mathcal{D}f_i \to \mathcal{D}f$ in $Q(X,\mathbb{R})$. The construction of this sequence is rather technical and we will omit it here, see [Sa] Theorem 6.2.

The last part of the theorem is a direct consequence of the previous results. The equation (3.15) is a particular case of equation (3.10), therefore we can apply Propositions 3.8 and 3.9. The conditions in Proposition 3.9 are satisfied for continuous hamiltonians of the type $H(x,p)$ convex in the second argument. $\square$

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Vita

Rodica Cazacu was born on November 24 1969, in Ianca, Olt, Romania. She finished her undergraduate studies at University of Bucharest, in June 1993. In January 1999 she came to Louisiana State University to pursue graduate studies in mathematics. She earned a Master of Science degree in mathematics from Louisiana State University in May 2001. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in December 2005.