STOCHASTIC NAVIER-STOKES EQUATIONS WITH FRACTIONAL BROWNIAN MOTIONS

A Dissertation

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Abstract

The aim of this dissertation is to study stochastic Navier-Stokes equations with a fractional Brownian motion noise. The second chapter will introduce the background results on fractional Brownian motions and some of their properties. The third chapter will focus on the Stokes operator and the semigroup generated by this operator. The Navier-Stokes equations and the evolution equation setup will be described in the next chapter. The main goal is to prove the existence and uniqueness of solutions for the stochastic Navier-Stokes equations with a fractional Brownian motion noise under suitable conditions. The proof is given with full details for two separate cases based on the value of the Hurst parameter H: $1/2 < H < 1$ and $1/8 < H < 1/2$.

Chapter 1
Introduction

The Navier-Stokes equations perturbed by a random noise term, such as white noise, can be used as a model to explain the random fluctuations observed in the velocity profile of viscous incompressible fluid flows. Such a perturbed system is a nonlinear stochastic partial differential equation known as stochastic Navier-Stokes equations (SNSE). Probabilistic analysis of such equations yields answers to certain hydrodynamical problems and lends insight into turbulence theory.

We consider the following stochastic Navier-Stokes equation driven by a fractional Brownian motion W^H:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = \Phi \, \frac{dW^H(t)}{dt} \tag{1.1}$$

and

$$\nabla \cdot u = 0 \tag{1.2}$$

with

$$u(t, x) = 0 \ \forall \ x \in \partial G,$$

$$u(0, x) = u_0(x) \ \forall \ x \in G.$$

and W^H, a space-time fractional Brownian motion in a suitable Hilbert space with Hurst parameter $H \in (0, 1)$ as in Tindel, Tudor and Viens [21]. Recall that a centered real-valued Gaussian process $\{\beta^H\}$ is called a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ if it has covariance function given by

$$R_H(t, s) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).$$

Hereafter, we will suppress the superscript H from the process. For our problem, we define $W(t) = \sum_{n=0}^{\infty} e_n\beta_n(t)$ where β_n are real independent, identically distributed fBms and $\{e_n\}$ is the sequence of eigenfunctions of the Stokes operator on G. Next, we discuss stochastic integration with respect to fBms.

The equation (1.5) is a nonlinear fBm driven equation and hence a complex object. It is especially so when one recalls that fBms are not semimartingales and have long range dependence. Therefore, the usual methods of solvability of SNSEs do not apply to the present system. Stochastic integration with respect to fBms has been developed by several authors (see Nualart [13] and the reference therein). If $H > 1/2$, the stochastic integral with respect to fBm can be defined as a pathwise Riemann-Stieltjes integral which exists if the integrand has Hölder-continuous trajectories of order larger than $1 - H$. If $H < 1/2$, one defines a certain symmetric stochastic integral under suitable integrability conditions on the integrand. In fact, in this case

$$\int_0^T u(s)d\beta_s := \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^T u(s)(\beta_{s+\epsilon} - \beta_{s-\epsilon})ds$$

where the limit is taken in the sense of limit in probability.

Since the stochastic system (1.1) and (1.2) is nonlinear with non-Lipschitz unbounded coefficients, the simplest of perturbations by fBms is considered. In the integral form, such noise terms are assumed to be Gaussian integrals which are briefly described below.

Let $\mathcal{S}$ denote the linear space of step functions on $[0, T]$ of the form

$$\phi(t) = \sum_{j=1}^{n} a_j 1_{(t_j, t_{j+1}]}(t)$$

where $t_1 = 0 < t_1 < \cdots < t_n = T$. Let $\mathcal{E}$ denote the closure of $\mathcal{S}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{E}} = R(t, s).$$

Define for $\phi \in \mathcal{S}$, the Wiener integral

$$\int_0^T \phi_s d\beta_s = \sum_{j=1}^{n} a_j(\beta_{t_{j+1}} - \beta_{t_j}). \tag{1.3}$$

The mapping $\phi \to \int_0^T \phi_s d\beta_s$ is an isometry between $\mathcal{S}$ and $L^2(\Omega)$. Therefore, it can be extended to an isometry between $\mathcal{E}$ and the first Wiener-Itô chaos of $\{\beta_t\}$. Thus the Wiener inegral of ϕ with respect to β can be defined.

It can be shown that if $\phi_1, \phi_2 \in \mathcal{E}$ with

$$\int_0^T \int_0^T |\phi_1(s)|\,|\phi_2(t)|\,|t-s|^{2H-2} ds dt < \infty$$

then

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{E}} = H(2H-1) \int_0^T \int_0^T |\phi_1(s)|\,|\phi_2(t)|\,|t-s|^{2H-2} ds dt. \tag{1.4}$$

Let $\Phi : [0,T] \to L_2(H;H)$, where L_2 denotes the space of Hilbert-Schmidt operators. Define

$$\int_0^t \Phi_s dW_s = \sum_{n=1}^{\infty} \int_0^t \Phi_n e_n d\beta_n(s).$$

The above sum is finite when $\sum_n |\,||\Phi e_n||_{\mathcal{E}}\,|_H^2 < \infty$. Thus $\int_0^t \Phi_s dW_s$ is an H-valued Gaussian random variable.

Using such an additive noise term in (1.1), we investigate the existence and uniqueness of mild solutions. In fact, by considering mild solutions of the stochastic system, we bypass the need for the noise term to be a semimartingale. Instead, suitable integrability conditions would suffice for solvability of the system by using the techniques developed by Da Prato and Zabczyk [4].

The conditions on the noise coefficient will be ascertained by our analysis. As explained in Section 1, the system (1.1) and (1.2) can be cast as an abstract evolution equation:

$$du + [\nu A\, u + B(u)]\, dt = \Phi\, dW^H(t). \tag{1.5}$$

The case when $H > 1/2$ is technically simpler though the other case shows intermittency and hence more important in applications. When $H < 1/2$, we plan to take advantage of the representation of the Wiener integral with respect to fBm as a usual stochastic integral with respect to a Wiener process. In fact, let

$$K(t,s) = c_H\left(\frac{t}{s}\right)^{H-1/2}(t-s)^{H-1/2} + s^{1/2-H}F\left(\frac{t}{s}\right)$$

where

$$F(z) = c_H(1/2-H)\int_0^{z-1} r^{H-3/2}(1-(1+r)^{H-1/2})dr$$

and c_H is a constant. Consider the operator K^* on $L^2([0,T])$ for every $s < t$:

$$(K_t^*\phi)(s) = K(t,s)\phi(s) + \int_s^t (\phi(r)-\phi(s))\frac{\partial K}{\partial r}(r,s)dr$$

One can write

$$\int_0^T \phi_s d\beta_s^H = \int_0^T (K_t^*\phi)_s dB_s \tag{1.6}$$

where $\{B_s\}$ is a standard Wiener process. Such a representation in the space-time set up is easy to use since we deal only with Wiener integrals.

It is worthwhile to note that stochastic partial differential equations perturbed by an fBm noise has received considerable interest in recent years and has been studied by several authors (see Tindel, Tudor and Viens [21], Maslowski and Schmalfuss, and Nualart [13]). The theory is well-developed in finite dimensions and the stochastic calculus of variations with respect to fBm has also been developed in recent papers.

Chapter 2

Fractional Brownian Motion (fBm) and Stochastic Integration with Respect to fBm

2.1 Brownian Motion

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let S be a complete separable metric space with $\mathcal{B}$ as the σ-field of Borel sets in S.

Definition 2.1. A *stochastic process* is a collection $X = \{X(t,\omega); t \in T, \omega \in \Omega\}$ of S-valued random variables defined on the probability space $(\Omega,\mathcal{F},P)$ with index set T.

Remark 2.2. The stochastic process that we consider will be jointly measurable on the product space $([0,\infty) \times \Omega, \mathcal{B}_{[0,\infty)} \times \mathcal{F})$. In other words,

1. for each fixed t, $X(t,\cdot)$ is a random variable;
2. for each fixed ω, $X(\cdot,\omega)$ is a measurable function of t, called a *sample path*.

If there is no confusion, we denote $X(t,\omega)$ by $X(t)$ or X_t.

Definition 2.3. A real-valued stochastic process $B(t,\omega)$ is called a *Brownian motion* if it satisfies:

1. $P\{\omega; B(0,\omega) = 0\} = 1$.
2. For any $0 \le s < t$, the random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, i.e., for any $a < b$,

$$P\{a \le B(t) - B(s) \le b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-x^2/2(t-s)} dx.$$

3. $B(t, \omega)$ has independent increments, i.e., for any $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables

$$B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1}),$$

are independent.

4. Almost all sample paths of $B(t, \omega)$ are continuous functions, i.e.,

$$P\{\omega; B(\cdot, \omega) \text{ is continuous}\} = 1.$$

Definition 2.4. A *filtration on* T is an increasing family $\{\mathcal{F}_t | t \in T\}$ of sub σ-fields of $\mathcal{F}$. A stochastic process $X_t, t \in T$, is said to be *adapted* to $\{\mathcal{F}_t | t \in T\}$ if for each t the random variable X_t is $\mathcal{F}_t$-measurable.

Remark 2.5. A σ-field $\mathcal{F}$ is called *complete* if $A \in \mathcal{F}$ and $P(A) = 0$ imply that $B \in \mathcal{F}$ for any subset B of A. We will always assume that all σ-fields $\mathcal{F}_t$ are complete.

Definition 2.6. Let X_t be a stochastic process adapted to a filtration $\{\mathcal{F}_t\}$ and $E|X_t| < \infty$ for all $t \in T$. Then X_t is called a *martingale* with respect to $\{\mathcal{F}_t\}$ if for any $s \leq t$ in T,

$$E\{X_t | \mathcal{F}_t\} = X_s, \qquad \text{a.s. (almost surely).} \tag{2.1}$$

The concept of the martingale is a generalization of the sequence of partial sums arising from a sequence $\{X_n\}$ of independent and identically distributed random variables with mean 0. Let $S_n = X_1 + \ldots + X_n$. Then the sequence $\{S_n\}$ is a martingale.

We may also define *submartingale* and *supermartingale* by replacing the equality in (2.1) with $\geq$ and $\leq$, respectively; namely, for any $s \leq t$ in T,

$$E\{X_t|\mathcal{F}_s\} \geq X_s, \quad \text{a.s. (submartingale)},$$
$$E\{X_t|\mathcal{F}_s\} \leq X_s, \quad \text{a.s. (supermartingale)}.$$

Remark 2.7. If we are given $B(t); 0 \leq t < \infty$ but no filtration, and if we know that B has stationary, independent increments and that $B(t) = B(t) - B(0)$ is normal with mean zero and variance t, then $\{B_t, \mathcal{F}_t^B; 0 \geq t < \infty\}$ is a Brownian motion as well as a martingale. Moreover, if $\mathcal{F}_t$ is a "larger" filtration in the sense that $\mathcal{F}_t^B \subset \mathcal{F}_t$ for $t \geq 0$, and if $B_t - B_0$ is independent of $\mathcal{F}_s$ whenever $0 \leq s < t$, then $(B_t, \mathcal{F}_t)$, $0 \leq t < \infty$ is a Brownian motion as well as a martingale.

Proposition 2.8. *Every Brownian motion B is a square integrable martingale and $\langle B \rangle_t = t$, $t \geq 0$. For any $t \geq 0$,*

$$E(B_t^2) = t.$$

Moreover, for any $0 \leq s < t$,

$$E(B_t B_s) = s.$$

2.2 The Weak Convergence

Definition 2.9. Let (S, μ) be a metric space with Borel σ-field $\mathcal{B}(S)$. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of probability measures on $(S, \mathcal{B}(S))$, and let P be another measure on this space. We say that $\{P_n\}_{n=1}^{\infty}$ *converges weakly to* P and write as $P_n \xrightarrow{w} P$, if and only if

$$\lim_{n \to \infty} \int_S f(s) dP_n(s) = \int_S f(s) dP(s)$$

for every bounded, continuous real-valued function f on S.

It follows, in particular, that the *weak limit* P is a probability measure, and that it is unique.

Definition 2.10. Let $\{(\Omega_n, \mathcal{F}_n, P_n)\}_{n=1}^{\infty}$ be a sequence of probability spaces, and on each of them consider a random variable X_n with values in a complete, separable metric space (S, d). Let $(\Omega, \mathcal{F}, P)$ be a probability space, on which a random variable X with values in (S, d) is given. We say that $\{X_n\}_{n=1}^{\infty}$ *converges to* X *in distribution*, and write as $X_n \xrightarrow{\mathcal{D}}$ X, if the sequence of measures $\{P_n X_n^{-1}\}_{n=1}^{\infty}$ converges weakly to the measure PX^{-1}.

Equivalently, $X_n \xrightarrow{\mathcal{D}} X$ if and only if

$$\lim_{n \to \infty} E_n f(X_n) = Ef(X)$$

for every bounded, continuous real-valued function f on S, where E_n and E denote expectations with respect to P_n and P, respectively.

2.3 The Fractional Brownian Motion

Definition 2.11. A centered Gaussian process $\{B^H(t)\}$ is a *fractional Brownian motion* (fBm for short) with Hurst parameter $H \in (0, 1)$ provided

$$E(B_t^H B_s^H) = R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

In particular, if $H = \frac{1}{2}$, it is a standard Brownian motion.

From on now, let us denote by B_t as the fBm with Hurst parameter H for simplicity.

Proposition 2.12. *Fractional Brownian motions have the self-similarity property: For any constant $a > 0$, the process $\{a^{-H} B_{at}, t \geq 0\}$ and $\{B_t, t \geq 0\}$ have the same distribution.*

Proof. This can be obtained immediately from

$$\begin{aligned} E(B_{at}B_{as}) &= R(at, as) \\ &= a^{2H} R(t, s). \end{aligned}$$

□

Proposition 2.13. *Fractional Brownian motions have stationary increments.*

Proof.

$$\begin{aligned} E[(B_t - B_s)^2] =& E(B_t^2) + E(B_s^2) - 2E(B_t B_s) \\ =& t^{2H} + s^{2H} - (t^{2H} + s^{2H} - |t-s|^{2H}) \\ =& |t-s|^{2H}. \end{aligned}$$

□

Theorem 2.14. *Kolmogorov's continuity theorem.* *Let X_t, $0 \leq t \leq 1$, be a measurable stochastic process. Assume that there exist constants α, β, $K > 0$ satisfying the inequality*

$$E|X_t - X_s|^{\alpha} \leq K|t-s|^{1+\beta}, \qquad \forall 0 \leq t \leq 1.$$

Then X_t has a continuous realization; namely, there exists Ω_0 such that $P(\Omega_0) = 1$ and for each $\omega \in \Omega_0$, $X(t, \omega)$ is a continuous function of t.

Remark 2.15. For all $p \geq 1$,

$$E(B_t - B_s)^{2p} \leq c_p |t-s|^{2Hp}.$$

If $2Hp > 1$, using the stationary increments property and the Kolmogorov's continuity, we can show that fBm $\{B_t\}$ has a version with continuous trajectories.

Moreover, the parameter H controls the regularity of the trajectories, which are Holder continuous of order $H - \epsilon$ for any $\epsilon > 0$; namely, for all $\epsilon > 0$ and $T > 0$, there exists a nonnegative random variable $K_{\epsilon,T}$ such that

$$E(|K_{\epsilon,T}|^p) < \infty$$

for all $p \geq 1$, and

$$|B_t - B_s| \leq K_{\epsilon,T}|t - s|^{H-\epsilon}$$

for all $s, t \in [0, T]$. This result can be obtained by using a standard result known as the Garsia-Rodemich-Rumsey theorem.

Remark 2.16. As for the covariance, if $H = \frac{1}{2}$, then $R_{\frac{1}{2}}(t - s) = t \wedge s$ and B is a standard Brownian motion in which case the increments of the process in disjoint intervals are independent.

However, if $H \neq \frac{1}{2}$, the increments are not independent. To see this, let $s < s + h < t < t + h$ and $t - s = nh$, the covariance between two increments B_{s+h-B_s} and $B_{t+h} - B_t$ is

$$\begin{aligned}
\rho(n) &= E(B_{s+h} - B_s)(B_{t+h} - B_t) \\
&= \frac{1}{2}[(s+h)^{2H} + (t+h)^{2H} - (nh)^{2H} - (s+h)^{2H} - t^{2H} + ((n-1)h)^{2H} \\
&\qquad - s^{2H} - (t+h)^{2H} + ((n-1)h)^{2H} + s^{2H} + t^{2H} - (nh)^{2H}] \\
&= \frac{h^{2H}}{2}[(n+1)^{2H} - 2n^{2H} + (n+1)^{2H}] \\
&\approx \frac{h^{2H}}{2} \cdot 2H(2H-1)n^{2H-2}, \qquad 0 < H < 1 \\
&\to 0,
\end{aligned}$$

as $n \to \infty$.

If $H > \frac{1}{2}$, then $\rho(n) > 0$, we say the increments $B_{s+h} - B_s$ and $B_{t+h} - B_h$ are positively correlated; moreover,

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

Such a property of the process is known as aggregation behavior.

If $H < \frac{1}{2}$, then $\rho(n) < 0$, we say these two increments are negatively correlated; moreover,

$$\sum_{n=1}^{\infty} \rho(n) < \infty,$$

which is known as intermittency.

Proposition 2.17. *If $H \neq \frac{1}{2}$, B_t is not a semimartingale.*

Proof. To prove this, let $p > 0$. Denote

$$\begin{aligned} Y_{n,p} &= n^{pH-1} \sum_{j=1}^{n} |B_{\frac{j}{n}} - B_{\frac{j-1}{n}}|^p \\ &= (\frac{1}{n})^{-H} \frac{1}{n} \sum_{j=1}^{n} |B_{\frac{j}{n}} - B_{\frac{j-1}{n}}|^p n^p \end{aligned}$$

and

$$\tilde{Y}_{n,p} = \frac{1}{n} \sum_{j=1}^{n} |B_j - B_{j-1}|^p.$$

By the self-similar property by taking $a = \frac{1}{n}$, the sequences $\{Y_{n,p}, n \leq 1\}$ and $\{\tilde{Y}_{n,p}, n \leq 1\}$ have the same distribution, i.e.

$$Y_{n,p} = \tilde{Y}_{n,p} \quad \text{in distribution.}$$

Since the stationary sequence $\{|B_j - B_{j-1}|^p : 1 \le j \le n\}$ is identically distributed and asymptotically independent, by the ergodic theorem,

$$\tilde{Y}_{n,p} = \frac{1}{n}\sum_{j=1}^{n}|B_j - B_{j-1}|^p$$
$$\to E(|B_1|^p) \quad \text{a.s. in } L^1$$

as $n \to \infty$. Therefore,

$$Y_{n,p} = n^{pH-1}\sum_{j=1}^{n}|B_{\frac{j}{n}} - B_{\frac{j-1}{n}}|^p$$
$$\to E(|B_1|^p) \quad \text{in probability.}$$

Let

$$V_{n,p} = \sum_{j=1}^{n}|B_{\frac{j}{n}} - B_{\frac{j-1}{n}}|^p.$$

If $H > \frac{1}{2}$, for $p = 2$ we get $pH > 1$ and thus

$$V_{n,2} \to 0 \quad \text{in probability}$$

as $n \to \infty$, which means that the quadratic variation is zero; However, for $p = 1$ we get $pH < 1$ and thus

$$V_{n,1} \to \infty \quad \text{in probability}$$

as $n \to \infty$, in which case the total variation is infinity. If $H < \frac{1}{2}$, then for both cases $p = 2$ and $p = 1$,

$$V_{n,p} \to \infty \quad \text{in probability}$$

and thus the total variation is infinity.

Therefore, we see that for $H \neq \frac{1}{2}$, the fBm B_t is not a semimartingale.

□

Now we consider the representation of fBm on an interval. Let t be in a time interval $[0,T]$, and let $\{B_t, t \in [0,T]\}$ be a fBm with Hurst parameter $H \in (0,1)$. Denote by $\mathcal{E}_{\mathcal{H}}$ the linear space of step functions on $[0,T]$ of the form

$$\varphi(t) = \sum_{j=1}^{n} a_{j-1} 1_{(t_{j-1},t_j]}(t)$$

where $t_1, t_2, \ldots, t_n \in [0,T]$, $n \in \mathbb{N}$, $a_j \in \mathbb{R}$, and by $\mathcal{H}$ the closure of $\mathcal{E}_H$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s).$$

Also define the Wiener integral of $\varphi \in \mathcal{E}_H$ with respect to the fBm as

$$\int_0^T \varphi_s dB^H(s) = \sum_{j=1}^{n} a_{j-1}(B_{t_j} - B_{t_{j-1}}).$$

Then the mapping:

$$\varphi = \sum_{j=1}^{n} a_{j-1} 1_{(t_{j-1},t_j]}(t) \to \int_0^T \varphi_s dB^H(s)$$

is an isometry between $\mathcal{E}_H$ and the linear space $L = span\{B_t, t \in [0,T]\}$ viewed as a subspace of $L^2(T)$ where $L^2(T)$ denotes $L^2([0,T])$:

$$I: \quad \mathcal{E}_H \to L.$$

This mapping can be extended to the closure $\mathcal{H}$ with respect to the above inner product:

$$I: \quad \mathcal{H} \to \bar{L},$$

where $\bar{L}$ denotes $L^2(\Omega)$, the closure of L.

We already know that $B^{\frac{1}{2}}$ is a standard Brownian motion. Moreover, B^H has the Wiener integral representation:

$$B_t^H = \int_0^t K(t,s) dW_s$$

where W_s is a standard Wiener process, and the kernel $K(t,s)$ can be determined.

For $s < t$, consider the operator K^* in $L^2(T)$ given as:

$$(K_t^*\varphi)(s) = K(t,s)\varphi(s) + \int_s^t (\varphi(r) - \varphi(s))\frac{\partial K}{\partial r}(r,s)dr.$$

When $H > \frac{1}{2}$, this operator K_t^* has a simpler expression:

$$(K_t^*\varphi)(s) = \int_s^t \varphi(r)\frac{\partial K}{\partial r}(r,s)dr.$$

Then K_t^* is a isometry between $\mathcal{H}$ and $L^2(T)$. Therefore, we have the following equivalence between the Wiener integral with respect to the fBm and the Wiener integral with respect to the Wiener process W:

$$\int_T \varphi(s)dB^H(s) = \int_T (K_t^*)(s)dW(s),$$

which holds for every $\varphi \in \mathcal{H}$ if and only if $K_t^*\varphi \in L^2(T)$. Note that

$$K_\tau^*[\varphi 1_{[0,t]}](s) = K_t^*[\varphi](s)1_{[0,t]}(s)$$

for any $s, t \in [0,T]$. Then if the definite stochastic integral $\int_0^t \varphi(s)dB^H(s)$ is definded, then we have

$$\int_0^t \varphi(s)dB^H(s) = \int_0^t (K_t^*\varphi)(s)dW(s) \tag{2.2}$$

for every $t \in [0,T]$ and $\varphi 1_{[0,t]} \in \mathcal{H}$ if and only if $K_t^*\varphi \in L^2(T)$. Also, for the case $H > \frac{1}{2}$, if $\phi, \chi \in \mathcal{H}$ satisfy

$$\int_T \int_T |\phi(s)||\chi(t)||t-s|^{2H-2}dsdt < \infty,$$

then their scalar product in $\mathcal{H}$ is given by:

$$\langle \phi, \chi \rangle_{\mathcal{H}} = H(2H-1)\int_0^T \int_0^T \phi(s)\chi(t)|t-s|^{2H-2}dsdt.$$

Here, we only work with Wiener integrals over Hilbert spaces. In this case, we note that if $u \in L^2(T,V)$ is a deterministic function, then the relation given by 2.2 holds, and the Wiener integral on the right-hand side is well defined in $L^2(V)$ if $K^*u \in L^2(T \times V)$.

2.4 Infinite Dimensional fBm and Stochastic Integration

Let U be a real and separable Hilbert space and let Q be a self-adjoint and positive operator on U. Note that $Q = Q^* > 0$. It is typical and usually convenient to assume that Q is nuclear; namely, Q is a compact operator, $Q = \sum_{n=1}^{N} \lambda_n \langle f_n, \cdot g_n \rangle$ where f_n's and g_n's are (not necessarily complete) orthonormal sets, λ_n's are a set of real numbers satisfying $\lambda_n \to 0$ as $n \to 0$, and $\sum_{n=1}^{N} \lambda_n < \infty$. Here, $\langle \cdot, \cdot \rangle$ denotes the scalar product on the Hilbert space, and the sequence $\{\lambda_n\}_{n \geq 0}$ is well known as the eigenvalues.

Moreover, let e_n denote the corresponding eigenvectors. Then $\{e_n\}$ form an orthonormal basis in U. We define the infinite dimensional fBm on U with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t) \tag{2.3}$$

where β_n^H are real, independent fBm's. This process is a U-valued Gaussian process, it starts from 0, has zero mean and covariance

$$E(B_Q^H(t))(B_Q^H(s)) = R(s,t)Q, \quad \text{for every } s,t \in T.$$

We realize that the assumption Q being nuclear is not convenient. We may wish to consider the case of a genuine *cylindrical* fBm on U by setting $\lambda_n \equiv 1$, that is,

$$B^H(t) = \sum_{n=0}^{\infty} e_n \beta_n^H(t).$$

Remark 2.18. Following the standard approach as for $H = \frac{1}{2}$, it is possible to define a generalized fBm on U (for example, in the sense of generalized functions if U is a space of functions) by the right-hand side of (2.3) for any fixed complete orthonormal basis $(e_n)_n$ in U, and any fixed sequence of positive numbers $(\lambda_n)_n$, even if $\sum_{n=0}^{\infty} \lambda_n = \infty$. Although for any fixed t the series (2.3) is not convergent in $L^2(\Omega \times U)$, we can always consider a Hilbert space U_1 such that $U \subset U_1$ and such that this inclusion is a Hilbert-Schmidt operator. In this way, $B^H(t)$ given by (2.3) is a well-defined U_1-valued Gaussian stochastic process.

Now let V be another real separable Hilbert space, B^H the process defined above, defined as a U_1-valued process if necessary, and $(\Phi_s)_{s \in T}$ a deterministic function with values in $\mathcal{L}_2(U, V)$, the *Hilbert-Schmidt operators from U to V*. The stochastic integral of Φ with respect to B^H is defined by

$$\int_0^t \Phi_s dB^H(s) = \sum_{n=0}^{\infty} \int_0^t \Phi_s e_n d\beta_n^H(s) = \sum_{n=0}^{\infty} \int_0^t (K^*(\Phi e_n))_s d\beta_n(s) \tag{2.4}$$

where β_n is the standard Brownian motion; and moreover, the above sum is finite when

$$\sum_n \|K^*(\Phi e_n)\|_{L^2(T;V)^2} = \sum_n |\|\Phi e_n\|_{\mathcal{H}}|_V^2 < \infty.$$

In this case the integral (2.4) is well defined as a V-valued Gaussian random variable. However, the linear additive equation in its evolution form can have a solution even if $\int_0^t \Phi_s dB^H(s)$ is not properly defined as a V-valued Gaussian random variable.

In order to define this stochastic integral in a larger Hilbert space than V, a remark similar to Remark 2.18 also applies. In particular, there is no reason to assume that $\Phi \in L_2(U, V)$.

Chapter 3
The Stokes Operator

3.1 Preliminaries

Let $C_c^\infty(U)$ denote the space of infinitely differentiable functions $\phi : U \to \mathbb{R}$, with compact support in U. We sometimes call a function ϕ belonging to $C_c^\infty(U)$ a *test function.*

Assume we are given a function $u \in C^1(U)$. Then if $\phi \in C_c^\infty(U)$, we see from the integration by parts formula that

$$\int_U u\phi_{x_i} dx = -\int_U u_{x_i}\phi dx, \qquad i = 1, \ldots, n. \tag{3.1}$$

Since ϕ has compact support in U, there are no boundary terms and thus ϕ vanishes near ∂U. More generally, if k is a positive integer, $u \in C^k(U)$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of order $|\alpha| = \alpha_1 + \ldots + \alpha_n = k$, then

$$\int_U uD^\alpha\phi dx = (-1)^{|\alpha|}\int_U D^\alpha u\phi dx.$$

This equality holds because

$$D^\alpha\phi = \frac{\partial\alpha_1}{\partial x_1^{\alpha_1}} \cdots \frac{\partial\alpha_n}{\partial x_n^{\alpha_n}}\phi$$

and we can apply formula (3.1) for $|\alpha|$ times.

Definition 3.1. Suppose $u, v \in L^1_{loc}(U)$, and α is a multi-index. We say that v is the α^{th}*-weak partial derivative of* u*, written as*

$$D^\alpha u = v,$$

provided

$$\int_U uD^\alpha\phi dx = (-1)^{|\alpha|}\int_U v\phi dx \tag{3.2}$$

for all test functions $\phi \in C_c^\infty(U)$.

In other words, if we are given u and if there exists a function v which verifies (3.2) for all ϕ, then we say that $D^\alpha u = v$ in the weak sense. If there does not exist such a function v, then u does not possess a weak α^{th}-partial derivative. See Evans [7].

Proposition 3.2. ***Uniqueness of weak derivative.*** *A weak* α^{th}*-partial derivative of* u*, if it exists, is uniquely defined up to a set of measure zero.*

Proof. Assume that $v, \tilde{v} \in L^1_{loc}(U)$ satisfy

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi dx = (-1)^{|\alpha|} \int_U \tilde{v}\phi dx$$

for all $\phi \in C_c^\infty(U)$. Then

$$\int_U (v - \tilde{v})\phi = 0$$

for all $\phi \in C_c^\infty(U)$. Hence $v - \tilde{v} = 0$ a.e.

□

Example 3.3. Let $n = 1$, $U = (0, 2)$, and

$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 \leq x < 2. \end{cases}$$

Define

$$v(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } 1 < x < 2. \end{cases}$$

Let us show $u' = v$ in the weak sense. To see this, choose any $\phi \in C_c^\infty(U)$. We must demonstrate

$$\int_0^2 u\phi' dx = -\int_0^2 v\phi dx.$$

Meanwhile we calculate that

$$\int_0^2 u\phi' dx = \int_0^1 x\phi' dx + \int_1^2 \phi' dx$$
$$= -\int_0^1 \phi dx + \phi(1) - \phi(1)$$
$$= -\int_0^2 v\phi dx$$

as required.

Definition 3.4. Let G be an open set. We say that G has the *segment property* if the boundary of G, ∂G, has a locally finite open cover $(U_j)_{j\in I}$, and for each j there exists a direction $w_j \in S^{n-1}$ and $\epsilon_j > 0$ such that for $x \in U_j \cap \bar{G}$, $x_t = x + t_{w_j} \in G$ for $0 < t < \epsilon_j$.

We denote $L^p(G) = \{f : G \to \mathbb{R}, \text{measurable}, \int |f(x)|^p dx < \infty\}$, with the scalar product $(\cdot, \cdot)$ in $L^2(G)$.

If G has the segment property, then the notions of a weak derivative in the sense of distributions and in the L^p sense coincide. We still denote $D_j = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$ and

$$D^\alpha = \frac{\partial \alpha_1}{\partial x_1^{\alpha_1}} \cdots \frac{\partial \alpha_n}{\partial x_n^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Definition 3.5. The Sobolev space

$$W^{m,p}(G) = \{f | D^\alpha f \in L^p, |\alpha| < m\}.$$

consists of all locally summable functions $f : G \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \le m$, $D^\alpha f$ exists in the weak sense and belongs to $L^p(G)$.

Remark 3.6. For $p = 2$, we usually denote $H^m(G) = W^{m,2}(G)$. Note that $H^m(G)$ is a Hilbert space and in particular, $H^0(G) = L^2(G)$. The equipped scalar product for $u, v \in W^{m,p}(G)$ is:

$$(u, v)_{m,p} = \sum_{|\alpha| \leq m} \int_G (D^\alpha u)(D^\alpha v)(x) dx.$$

Then we can define its *norm* in $W^{m,p}(G)$ to be:

$$\|u\|_{W^{m,p}(G)} := \begin{cases} (\sum_{|\alpha| \leq m} \int_G |D^\alpha u|^p dx)^{1/p}, & \text{if } 1 \leq p < \infty \\ \sum_{|\alpha| \leq m} \text{ess sup}_G |D^\alpha u|, & \text{if } p = \infty. \end{cases}$$

Remark 3.7. The spaces $W^{m,p}(G)$ are Banach spaces. We will use the same notation $L^p(G)$, $H^m(G)$, $W^{m,p}(G)$ for vectorial counterparts. For instance, the scalar product in $H^m(G)^n$ will be denoted by:

$$(\ ,\ )_{m,G} : \quad (u, v)_{m,G} = \sum_{|\alpha| \leq m} \int_G D^\alpha u \cdot D^\alpha v dx,$$

where $\cdot$ signifies scalar prodect in $\mathbb{R}^n$.

Definition 3.8. We denote by

$$W_0^{m,p}(G)$$

the *closure* of $C_c^\infty(G)$ in $W^{m,p}(G)$.

Thus $u \in W_0^{m,p}(G)$ if and only if there exist functions $u_j \in C_c^\infty(G)$ such that $u_j \to u$ in $W^{m,p}(G)$. We interpret $W_0^{m,p}(G)$ as comprising those functions $u \in W^{m,p}(G)$ such that

$$D^\alpha u = 0 \text{ on } \partial U \quad \text{for all } |\alpha| \leq m - 1.$$

The following properties are useful, which can be found in Chapter 1 of Constantin and Foias [3].

Proposition 3.9. *Let G satisfy the segment property. Then $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(G)$, for $1 \leq p < \infty$.*

Proof. Let $u \in W^{m,p}(G)$. We first approximate u in $W^{m,p}(G)$ by a sequence of elements in $W^{m,p}(G)$ with compact support by considering $u_m(x) = \phi(\frac{x}{m})u(x)$, where $\phi(x) = 1$ if $|x| \leq 1$, $\phi \in C_c^\infty(\mathbb{R}^n)$, supp $\phi \subset \{x : |x| \leq 2\}$. Then using a partion of unity we may assume that the support of u is compact, and either is included in G or is one of the locally finite open covers U_j from the definition of the segment property. If the support of u is contained in G then a standard convolution with a mollifier will provide the approximation.

We may assume that the support of u is compact and included in some open set $V_j \subset\subset U_j$. Let $\tilde{u}$ be the extension of u defined by setting $\tilde{u}$ to be zero outside $\bar{G}$. Then $\tilde{u} \in W^{m,p()\mathbb{R}^n \ (\partial G \cap V_j)}$. We approximate $\tilde{u}$ by $u_t = \tilde{u}(\cdot + t_{w_j})$ for small t. By doing this we push the singular set $\partial G \cap V_j$ to $\partial G \cap V_j - t_{w_j}$: $u_t \in W^{m,p}(\mathbb{R}\ (\partial G \cap V_j - t_{w_j}))$. From the segment property, this set does not touch $\bar{G}$. Thus, $u_t \in W^{m,p}(U)$ for some open neighborhood U of $\bar{G} \cap V_j$. A convolution of u_t with some mollifier will produce a $C_c^\infty(\bar{G})$ function near u.

□

Proposition 3.10. ***(The Poincaré inequality.)*** *If G is bounded in some direction; that is, if there exists a straight line in $\mathbb{R}^n$ such that the projection of G on it is bounded, then*

$$\|u\|_{L^2(G)} \leq C(G)\|\nabla u\|_{L^2(G)}, \quad \textit{for all } u \in H_0^1(G).$$

For the convenience of notations, let us denote by $|u|$ and $\|u\|$ for short, respectively, for $\|u\|_{L^2(G)}$ and the Dirichlet norm $\|\nabla u\|_{L^2(G)} = (\int_G \sum_{j=1}^n |D_j u|^2 dx)^{1/2}$.

Now let G have the segment property. Let $E(G)$ denote the space

$$E(G) = \{u : u \in L^2(G)^n, \nabla \cdot u \in L^2(G)\},$$

where $\nabla \cdot u = \text{div } u = \sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j}$ is the sum of partial derivatives taken in the sense of distributions in G. Then $E(G)$ is a Hilbert space with the scalar product

$$[u, v] = (u, v) + \int_G (\nabla \cdot u)(\nabla \cdot v) dx.$$

Proposition 3.11. *The set* $(C_c^\infty(G))^n$ *is dense in* $E(G)$.

Proof. Same method as for Proposition 3.9 yields the proof of this proposition.

□

We now impose a much more restrictive assumption on G. We say that G is *of class* C^r if there exists a locally finite open cover $(U_j)_{j \in I}$ of ∂G and C^r diffeomorphisms $\psi_j : U_j \to D$, where D is the unit open disk in $\mathbb{R}^n$, $D = \{x : |x| < 1\}$ such that

$$\psi_j(U_j \cap G) = D_+ = \{x : x \in D, x_n > 0\},$$

and

$$\psi_j(U_j \cap \partial G) = D_0 = \{x : x \in D, x_n = 0\}.$$

Here, a C^r-diffeomorphism is a bijective map f between two manifolds such that both f and its inverse f^{-1} are r times continuously differentiable.

Suppose now that G is bounded and of class C^2. The trace operator $\gamma_0 : H^1(G) \to L^2(\partial G)$ is a bounded linear operator agreeing with the restriction operation $u \mapsto u|_{\partial G}$ for continuously differentiable functions on $\bar{G}$. The kernel of γ_0 is $H_0^1(G)$. The image is denoted by $H^{1/2}(\partial G)$, and is a Hilbert space.

Here, we consider the Lebesgue measure on ∂G. Then there exists a lifting operator

$$l_G : H^{1/2}(\partial G) \to H^1(G)$$

satisfying $\gamma_0 l_G = \text{Identity in } H^{1/2}(\partial G)$. See Constantin and Foias [3].

We define $H^{-1/2}(\partial G)$ as the *dual space* of $H^{1/2}$. We want to define the normal component $u \cdot n_G$ of elements of $E(G)$. The notation n_G stands for the outer normal to ∂G.

Proposition 3.12. *Let G be an open bounded set of class C^2. There exists a continuous linear operator $\gamma : E(G) \to H^{-1/2}(G)$ such that $\gamma(u) = u \cdot n_G$ for every $u \in C^\infty(G)^n$. The Stokes formula*

$$(u, \nabla w) + (\nabla \cdot u, w) = \langle \gamma(u), \gamma_0(w) \rangle \tag{3.3}$$

holds for every $u \in E(G)$ and $w \in H^1(G)$.

Proof. The idea of the proof is to use the lifting operator $l_G : H^{1/2}(\partial G) \to H^1(G)$ to define the element $\gamma(u)$ of the dual $H^{-1/2}(\partial G)$ of $H^{1/2}(\partial G)$ by (3.3):

$$\langle \gamma(u), \Phi \rangle := (u, \nabla l_G \Phi) + (\nabla \cdot u, l_G \Phi)$$

for all $\Phi \in H^{1/2}(\partial G)$, and fixed $u \in E(G)$. Since

$$l_G \Phi \in H^{1/2}(G)$$

and

$$\|l_G(\Phi)\|_{H^1(G)} \leq c \|\Phi\|_{H^{1/2}(\partial G)},$$

we see that

$$|\langle \gamma(u), \Phi \rangle| \leq c \|\Phi\|_{H^{1/2}(\partial G)} \|u\|_{E(G)}.$$

Thus, γ maps from $E(G)$ to $H^{-1/2}(\partial G)$, and it is a bounded and linear map. If u is a $C^\infty(\bar{G})^n$ function and Φ is the restriction to ∂G of a $C^\infty(\bar{G})$ function, w, then the divergence theorem (Stokes formula) implies that

$$\int_{\partial G} (u \cdot n_G) \Phi dx = (u, \nabla w) + (\nabla \cdot u, w).$$

Since $w - l_G(\Phi)$ is in the kernel of γ_0, namely,

$$w_0 = w - l_G(\Phi) \in H_0^1(G),$$

and since $(u, \nabla w_0) + (\nabla \cdot u, w_0) = 0$, for any $w_0 \in H_0^1(G)$, it follows that

$$\int_{\partial G} (u \cdot n_G)\Phi dx = \langle \gamma(u), \Phi \rangle.$$

Now the functions Φ which are restrictions of $C^\infty(\bar{G})$ functions are dense in $H^{1/2}(\partial G)$. Therefore, for any smooth u, $u \cdot n_G = \gamma(u)$.

□

Following [3], let us now denote by $\mathcal{V}$ the set

$$\mathcal{V} = \{u : u \in C_c^\infty(G)^n \text{and } \nabla \cdot u = 0\}.$$

Let us denote by H and V the closure of $\mathcal{V}$ in $L^2(G)^n$ and $H_0^1(G)^n$, respectively; namely,

$$H = \text{Closure of } \mathcal{V} \text{ in } L^2(G)^n,$$

and

$$V = \text{Closure of } \mathcal{V} \text{ in } H_0^1(G)^n.$$

Proposition 3.13. *Let $G \in \mathbb{R}^n$ be a locally Lipschitz bounded open set. Then*

$$H = \{u : u \in L^2(G)^n, \nabla \cdot u = 0, \gamma(u) = 0\}, \tag{3.4}$$

$$\textit{and} \quad H^\perp = \{u : u \in L^2(G)^n, u = \nabla p, p \in H^1(G)\}. \tag{3.5}$$

Proof. For $H^\perp$, if $u = \nabla p$ with $p \in H^1(G)$, then $(u, v) = 0$ for all $v \in \mathcal{V}$ and $u \in H^\perp$. On the other hand, if $(v, v) = 0$ for all $v \in \mathcal{V}$, then $u = \nabla p$, $p \in H^1(G)$.

For H, let $H^\sim$ denote the right hand side of (3.4). If u belongs to H, then u is the limit in $L^2(G)^n$ of a sequence of functions in $\mathcal{V}$. Thus, $\nabla \cdot u = 0$. Therefore,

$u \in E(G)$ and the convergence of the functions of $\mathcal{V}$ to u takes place in $E(G)$. Since $\gamma : E(G) \to H^{-1/2}$ is continuous, we have that $\gamma(u) = 0$. Note that $H \subset H^{\sim}$, and also that H is dense in the $L^2(G)^n$ topology in $H^{\sim}$. For $H^{\sim}$ is a closed subspace of $L^2(G)^n$ and if $H^{\sim} \ominus H$ would be nonempty, say $v \in H^{\sim} \ominus H$, then $v \in H^{\perp}$, and thus $v = \nabla p$ with $p \in H^1(G)$ and also $v \in H^{\sim}$, thus $\nabla \cdot (\nabla p) = \Delta p = 0$, $\gamma(u) = \frac{\partial p}{\partial n_G} = 0$. Thus, p must be constant on each connected component of G. Since H is closed $H = H^{\sim}$, we see that $u = 0$.

□

3.2 Introduction to the Stokes Equations

Definition 3.14. Let G be an open bounded set in $\mathbb{R}^n$. Let $f \in L^2(G)^n$. The *Stokes equations* for the vector $u = (u_1, \ldots, u_n)$ and the scalar f are:

$$\begin{cases} -\nu \Delta u + \nabla p = f, \quad \text{in } G \\ \text{div } u = \nabla \cdot u = 0, \quad \text{in } G \\ u = 0, \quad \text{on } \partial G \end{cases} \tag{3.6}$$

where $\nu > 0$ is a constant.

If u, p are smooth, then for all $v \in \mathcal{V}$,

$$\nu((u, v)) = (f, v).$$

To see this, we use integration by parts,

$$\int_G Du \cdot Dv dx = -\int_G v \Delta u dx + \int_{\partial G} \frac{\partial u}{\partial n} v d\mathcal{S}$$

and $(u, v) = -v\Delta u$. From now on, $((u, v))$ is the scalar product

$$((u, v)) = \sum_{j=1}^{n} (D_j u, D_j v).$$

Definition 3.15. We say that u is a *weak solution* of the Stokes equations (3.6), if

$$\begin{cases} u \in V, \quad \text{and} \\ \nu((u,v)) = (f,v), \ \text{for all } v \in \mathcal{V}, \end{cases}$$

Remark 3.16. By continuity, $\nu((u,v)) = (f,v)$, for all $v \in V$.

The following are important properties of Stokes equations. For more details of proofs, one can refer to Chapter 2 of Constantin and Foias [3].

Proposition 3.17. *Let G be open bounded and of class C^2. Then the following are equivalent:*

1. *u is a weak solution of the Stokes equations (3.6);*

2. *$u \in H_0^1(G)^n$ and satisfies: there exists $p \in L^2(G)$ such that*

$$\begin{cases} -\nu\Delta u + \nabla p = f, \quad \textit{in } \mathcal{D}'(G) \\ \nabla \cdot u = 0, \quad \textit{in } \mathcal{D}'(G) \\ \nu_0(u_j) = 0, \quad i = 1, \dots, n; \end{cases} \tag{3.7}$$

3. *$u \in V$ reaches the minumum of $\Phi(v) = \nu\|v\|^2 - 2(f,v)$ on V.*

Proof. To show that 2 implies 1: Since G is open bounded and of class C^2, it is locally Lipschitz. Thus $V = \{u \in H_0^1(G)^n | \nabla \cdot u = 0\}$. Therefore, if u is divergence free, it is in V.

To show that 1 implies 2: If u is a weak solution of the Stokes equations, then $-\nu\Delta u - f$ is a distribution in $H^{-1}(G)^n$, and also $\langle -\nu\Delta u - f, v\rangle = 0$ for all $v \in \mathcal{V}$. Then $-\nu\Delta u - f$ is the gradient of an $L^2(G)$ function. Thus 2 is true.

To show that 1 implies 3: If u is a weak solution of the Stokes equations, then

$$\begin{aligned}\Phi(u+w) &= \nu\|u+w\|^2 - 2(u+w,f)\\ &= \Phi(u) + \nu\|w\|^2\\ &\geq \Phi(u)\end{aligned}$$

for all $w \in V$.

To show that 3 implies 1: If 3 is true, then

$$\Phi(u+\lambda v) - \Phi(u) \geq 0$$

for all $v \in V$ and $\lambda \in \mathbb{R}$. Note that

$$\Phi(u+\lambda v) - \Phi(u) = \lambda^2\nu^2\|v\|^2 + 2\lambda[\nu((u,v)) - (f,v)]$$

and hence coefficient of λ has to vanish. Therefore $\nu((u,v)) = (f,v)$ and hence u is a weak solution.

□

Proposition 3.18. ***Lax-Milgram Theorem.*** *Let X be a separable Hilbert space and $\alpha : X \times X \to \mathbb{R}$ be a bilinear continuous coercive form; namely, if $\|\cdot\|_X$ denotes the norm in X, then*

1. $|\alpha(u,v)| \leq C\|u\|_X\|v\|_X$;
2. $\alpha(u,v) \geq \alpha\|u\|_X^2$;

Then for each l linear continuous form on X, there exists a unique element $u_l \in X$ such that

$$\alpha(u_l, v) = \langle l, v\rangle$$

for all $v \in X$.

Proof. $\alpha(\cdot,\cdot)$ is a scalar product in X. It induces a norm which is equivalent to the original norm. Then l is a linear continuous form on X with this scalar product. By the F. Riesz representation theorem, there exists a unique u_l such that $\alpha(u_l, v) = \langle l, v\rangle$ for all $v \in X$. The result is obtained.

□

Proposition 3.19. *Let G be open and bounded in some direction. Then for every $f \in L^2(G)^n$, $\nu > 0$, there exists a unique weak solution of the Stokes equations (3.6).*

Proof. By the Poincare inequality, $\alpha(\cdot,\cdot)$ is coercive on V. Then by the Lax-Milgram theorem, the result is obtained.

□

3.3 The Stokes Operator

Recall that $\mathcal{V} := \{u : u \in C_c^\infty(G) \text{ and } \nabla \cdot u = 0\}$. Let G be bounded, ∂G be of class C^2. Let π denote the Leray projection $\pi : L^2(G) \to H$. Let A stand for the operation $-\pi\Delta$. For any $u \in \mathcal{V}$ define $\|u\| := |Au|$.

We claim that this $\|\cdot\|$ defines a norm on $\mathcal{V}$. To see this, it suffices to show that $\|u\| = 0$ implies that $u = 0$. Let $u \in \mathcal{V}$ be such that $|Au| = 0$. Since

$$\|Au\| = \sup\{|(Au, v)| : v \in L^2(G) \text{ with } |v| = 1\},$$

it follows that $(Au, v) = 0$. Meanwhile.

$$\begin{aligned}(Au, v) &= -(\Delta u, \pi u)\\ &= -(\Delta u, u)\\ &= (\nabla u, \nabla u)\\ &= |\nabla u|^2\end{aligned}$$

Thus, $\nabla u = 0$ in G. Therefore, u is a constant in G. Note that $u = 0$ on the boundary ∂G, we obtain that $u \equiv 0$.

The completion of $\mathcal{V}$ in this norm is denoted by H^2. The operation A can be naturally extended to a map from H^2 to H. In fact, if $u \in H^2$, then consider a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $\mathcal{V}$ converging to u in H^2. Thus $\{u_j\}$ is Cauchy in H^2. This implies that $\{Au_j\}$ is Cauchy in H. Define Au to be the limit of this Cauchy sequence. Clearly, this definition does not depend on the choice of the above sequence $\{u_j\}$. Therefore, the following operator is well defined.

Definition 3.20. The Stokes operator is defined by:

$$A : \mathcal{D}(A) \subset H \to H,\ A = -\pi\Delta,\ \mathcal{D}(A) = H^2(G) \cap \mathcal{V}.$$

Proposition 3.21. *The Stokes operator A is symmetric; that is, for all u, v in $\mathcal{D}(A)$,*

$$(Au, v) = (u, Av). \tag{3.8}$$

Proof. If u, v are in $C_c^\infty(G)^n$ and they are divergence free, that is, $u, v \in \mathcal{V}$, then since π is the Leray projection, $\pi u = u$ and $\pi v = v$, we note that (3.8) is equivalent to

$$-\int_G (\Delta u_j) v_j dx = \int_G \frac{\partial u_j}{\partial x_j} \frac{\partial v_j}{\partial x_j} dx. \tag{3.9}$$

If u, v are in $\mathcal{D}(A)$ and arbitrary, we can approximate them in $H^1(G)^n$ by functions in $\mathcal{V}$. If $u \in \mathcal{D}(A)$ and $v \in \mathcal{V}$, then it is easy to see that (3.9) is true. Also, by passing to the limit in the v's in $H^1(G)$, we get that (3.9) holds for arbitrary $u \in \mathcal{D}(A)$ and $v \in V$. In particular, (3.9) means

$$(Au, v) = ((u, v)) \quad \text{for all } u, v \in \mathcal{D}(A). \tag{3.10}$$

Since the right hand side of (3.10) is symmetric, (3.8) is proven.

Moreover, we note that (3.10) is true for $u \in \mathcal{D}(A)$ and $v \in V$.

$\square$

Lemma 3.22. *(See Theorem 3.11 in [3]) Let G be open bounded of class C^2. Let $f \in L^2(G)^n$. There exists a unique pair of solutions, $u \in H^2(G) \cap V$ and $p \in H^1(G)$, of the Stokes equations (3.6). Moreover,*

$$\|u\|_{H^2(G)} + \|p\|_{H^1(G)} \leq c\|f\|_{L^2(G)}.$$

Theorem 3.23. *(See Theorem 4.3 in [3]) The Stokes operator A is self-adjoint.*

Proof. Let $u \in \mathcal{D}(A^*)$. By definition of A^*, there exists some $f \in H$ such that

$$(Av, u) = (v, f) \ \text{ for all } v \in \mathcal{D}(A).$$

Since $f \in H \subset L^2(G)^n$, by Lemma (3.22), there exist some $\tilde{u} \in \mathcal{D}(A)$ such that $A\tilde{u} = f$. We need to show that $u = \tilde{u}$. To prove this, let $g \in H$. By Lemma (3.22), there exists some $v \in \mathcal{D}(A)$ such that $Av = g$. Thus,

$$\begin{aligned}(g, u - \tilde{u}) &= (Av, u) - (Av, \tilde{u}) \\ &= (v, f) - (v, A\tilde{u}) \\ &= (v, f) - (v, f) \\ &= 0.\end{aligned}$$

Note that $g \in H$ is arbitrary, so $u = \tilde{u}$ and hence $u \in \mathcal{D}(A)$ and $f = Au$.

$\square$

Lemma 3.24. ***Rellich-Kondrachov Compactness Theorem.*** *Assume G is a bounded open subset of $\mathbb{R}^n$, and ∂G is C^1. Suppose $1 \leq p < n$. Then*

$$W^{1,p}(G) \subset\subset L^q(G)$$

for each $1 \leq q < q^*$.

See Evans [7].

Theorem 3.25. *(See Theorem 4.4 in [3]) The inverse of the Stokes operator, A^{-1}, is compact in H.*

Proof. For $f \in H$, $A^{-1}f = u$, where u is the unique solution of the Stokes equations and $u \in \mathcal{D}(A) = H^2(G) \cap V$. By Lemma (3.22),

$$\|u\|_{H^2(G)} + \|p\|_{H^1(G)} \leq c\|f\|_{L^2(G)}$$

for some $p \in H^1(G)$. Thus, $A^{-1} : H \to V$ is bounded. By Lemma (3.24), we have that the inclusion $V \subset H$ is compact.

□

So far, we have seen that A^{-1} is self-adjoint, injective and compact. Then by a well-known theorem of Hilbert, there exists a sequence of positive numbers $\mu_j > 0$, $\mu_{j+1} \leq \mu_j$, and an orthonormal basis of H, (e_j) such that $A^{-1}e_j = \mu_j e_j$. Denote $\lambda_j = \mu_j^{-1}$. Note that A^{-1} has range in $\mathcal{D}(A)$ we obtain that

$$Ae_j = \lambda_j e_j, \quad e_j \in \mathcal{D}(A) \tag{3.11}$$

$$0 < \lambda_1 < \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots \tag{3.12}$$

$$\lim_{j\to\infty} \lambda_j = \infty \tag{3.13}$$

$$(e_j)_{j=1,\ldots} \quad \text{are orthonormal basis of } H. \tag{3.14}$$

Proposition 3.26. *If G is bounded of class C^{l+2}, $l \geq 0$, then $e_j \in H^{l+2}(G)^n$.*

Let $\alpha > 0$ be a real number. We define the operator A^α by

$$A^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha u_j e_j$$

for $u = \sum_{j=1}^\infty u_j e_j$, $u \in \mathcal{D}(A^\alpha)$, where

$$\mathcal{D}(A^\alpha) = \{u \in H | u = \sum_{j=1}^{\infty} u_j e_j,\ \sum_{j=1}^{\infty} \lambda_j^{2\alpha}|u_j|^2 < \infty,\ u_j \in \mathbb{R}\}.$$

The spaces $\mathcal{D}(A^\alpha)$ are equipped with a natural scalar product; namely, for $u = \sum_{j=1}^\infty u_j e_j$ and $v = \sum_{j=1}^\infty v_j e_j$,

$$\langle u, v\rangle_\alpha = \sum_{j=1}^\infty \lambda_j^{2\alpha} u_j v_j.$$

For this scalar product, the vectors $\lambda_j^{-\alpha} e_j$, $j = 1, \ldots,$ form a complete orthonormal system.

For $\alpha = \frac{1}{2}$, we have $\mathcal{D}(A^{1/2}) = V$ and $\langle u, v\rangle_1 = ((u, v))$. In fact, the vectors $\lambda^{-1/2} e_j$ are in V and also

$$\begin{aligned}\langle \lambda_j^{-1/2} e_j, \lambda_k^{-1/2} e_k\rangle_{1/2} &= \delta_{jk} \lambda_j^1 \cdot \lambda_k^{-1} \\ &= \delta_{jk} \\ &= (A(\lambda_j^{-1/2} e_j), \lambda_k^{-1/2} e_j) \\ &= ((\lambda_j^{-1/2}, \lambda_k^{-1/2} e_k)).\end{aligned}$$

Thus, $\mathcal{D}(A^\alpha) \subset V$ and moreover, it is closed in V. In particular, if $v \in V$ and v is orthogonal to $\mathcal{D}(A^\alpha)$, then $((v, e_j)) = 0$ for all j. Note that $v \in V$, $e_j \in \mathcal{D}(A)$, and $Ae_j = \lambda_j e_j$ for all $e_j \in \mathcal{D}(A)$, we see that

$$\begin{aligned}0 &= ((v, e_j)) \\ &= (v, Ae_j) \\ &= \lambda_j (v, e_j).\end{aligned}$$

Since $\lambda_j > 0$, we obtain that

$$(v, e_j) = 0$$

for all j. Therefore, $v = 0$.

3.4 The Stokes Semigroup Generated by the Stokes Operator

For each $\lambda \in [0,\infty)$, let E_λ be a *projection operator* which projects H onto a subspace $D_\lambda \subset H$. We call $\{E_\lambda; \lambda \geq 0\}$ *a family of projections*. Let $0 \leq \lambda_0 \leq \infty$. Then we write

$$E_{\lambda_0} = \lim_{\lambda \to \lambda_0} E_\lambda$$

if and only if

$$E_{\lambda_0} v = \lim_{\lambda \to \lambda_0} E_\lambda v$$

holds for all $v \in H$ (strong convergence of operators).

Definition 3.27. A family of projections $\{E_\lambda; \lambda \geq 0\}$ is called a *resolution of the identity* I on $[0,\infty)$, if the following properties are satisfied:

$$\begin{cases} E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda, \quad 0 \leq \lambda \leq \mu < \infty \\ E_\lambda = \lim_{\mu \to \lambda} = E_\lambda, \quad 0 < \mu < \lambda < \infty \\ E_0 = 0, \quad \lim_{\lambda \to \infty} E_\lambda = I. \end{cases}$$

Definition 3.28. Let $B : \mathcal{D}(B) \to H$ be any positive self-adjoint operator with (dense) domain $\mathcal{D}(B) \subset H$. Then there exists a uniquely determined resolution

$$\{E_\lambda; \lambda \geq 0\}$$

of identity such that

$$B = \int_0^\infty \lambda dE_\lambda,$$

with $\mathcal{D}(B) = \{v \in H; \int_0^\infty \lambda^2 d\|E_\lambda v\|^2 < \infty\}$. This is called the *spectral representation* of B.

See these definition and more properties [17], II.

We use the spectral representation to define for each $t \geq 0$ the operator

$$S(t) := e^{-tA} := \int_0^\infty e^{-t\lambda} dE_\lambda.$$

Since $\lambda \mapsto e^{-t\lambda}$, $\lambda \geq 0$, is a bounded positive function defined on $[0, \infty)$, each $S(t)$ is a bounded everywhere defined and positive self-adjoint operator in the Hilbert space $L^2(G)$. It is not hard to see that

$$\|S(t)\| \leq \sup_{\lambda \geq 0} e^{-t\lambda} \leq 1,$$

and the above definition of $S(t)$ yields that

$$\begin{aligned} S(t_1)S(t_2) &= \int_0^\infty e^{-t_1\lambda} e^{-t_2\lambda} dE_\lambda \\ &= \int_0^\infty e^{-(t_1+t_2)\lambda} dE_\lambda \end{aligned}$$

and hence

$$S(t_1)S(t_2) = S(t_1 + t_2)$$

for all $t_1, t_2 \geq 0$. Furthermore, we have

$$S(0) = \int_0^\infty dE_\lambda = I$$

where I means the identity. See [17], IV.

The operator family $\{S(t); t \geq 0\}$ is called the *Stokes semigroup* of G concerning semigroups.

Chapter 4

The Stochastic Navier-Stokes Equations (SNSE) with fBm

In this chapter, we introduce the stochastic Navier-Stokes equations (SNSE for short) and the SNSE with fractional Brownian motion (fBm for short). The main result is the existence and uniqueness of the solutions when a non-linear term is involved under certain conditions.

4.1 Introduction to the SNSE

Let G be a bounded open subset of $\mathbb{R}^2$ with a smooth boundary and let $x \in G$. The *Navier-Stokes system* will be cast as a stochastic evolution equation. The *Navier-Stokes system* is as below:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + B(u) = -\nabla p + f(t) \\ \nabla \cdot u = 0 \end{cases} \tag{4.1}$$

where $p(t,x)$ is the scalar-valued *pressure* of at time t and at position x, $\nabla p = (\frac{\partial p}{\partial x_j})_{j=1}^2$, $\nu > 0$ is the *viscosity coefficient*, and $u(t,x)$ is the *velocity* at time t and at space x. We also have the *initial condition*

$$u(t,x) = u_0(x),$$

and the *boundary condition*

$$u(t,x) = 0, \text{ for } x \in \partial G, \ \forall t \geq 0.$$

The $\mathbb{R}^2$-valued function $f(t,x)$ is called *external force* or *body force*, and

$$B(u) = b(u,u) = \Big(\sum_{i=1}^{2} u_i \frac{\partial u_j}{\partial u_i}\Big)_{j=1}^{2}$$

is known as the *inertial term*, which is nonlinear, non-Lipschitz and unbounded. We want to solve for $u(t)$ and p.

Recall that

$$\nabla \cdot u = \mathrm{Div}(u) = \sum_{j=1}^{2} \frac{\partial u_j}{\partial x_j}$$

and $\nabla \cdot u = 0$ means *"incompressibility" of fluid*, also known as *conservation of mass*. The volume occupied by the fluid remains the same at all times, i.e. u is solenoidal. Equation (4.1) is the balance of momentum equation. In 2-D, (4.1) is the same as

$$\begin{cases} \frac{\partial u_j}{\partial t} - \nu \Delta u_j + \sum_{i=1}^{2} u_i \frac{\partial u_j}{\partial x_i} = \frac{\partial p}{\partial x_j} + f_j(t), \forall j = 1, 2 \\ \nabla \cdot u = 0 \end{cases}$$

The boundary condition is the *no-slip* condition.

Let $T > 0$ be given. We want a solution $u(t,x)$ in $[0,T] \times G$. In 2-dimension case, we again use the same notations for the spaces:

$$H := \{u \in L^2(G)^2 : \nabla u = 0, u \cdot \vec{n} = 0\}$$

where $\vec{n}$ is the exterior normal vector to G, and

$$V := \{u \in W^{1,2}(G)^2 : \nabla \cdot u = 0, u|_{\partial G} = 0\}$$

Then $V \hookrightarrow H$. Let $|\cdot|$ denote the H-norm given by the usual L^2-norm, that is,

$$|u| = \Big(\int_G (u_1^2 + u_2^2) dx\Big)^{\frac{1}{2}},$$

and $(\cdot,\cdot)$ denote the H-inner product. Let $\|\cdot\|$ denote the V-norm given by the H^1-norm where $H^1 = W^{1,2}(G)$.

Proposition 4.1. *With the boundary condition, the* V*-norm is equivalent to* $|\nabla u|^2$.

Proof. Clearly,

$$\begin{aligned}\|u\|^2 &= \int_G |u|^2 dx + \int_G |\nabla u|^2 dx \\ &\geq \int_G |\nabla u|^2 dx.\end{aligned}$$

Conversely, we only need to show that

$$\int_G |u|^2 dx \leq c \int_G |\nabla u|^2 dx$$

for some constant c. This result is immediate due to the Poincaré inequality from the previous chapter.

□

Thus H and V are Hilbert spaces, and moreover,

$$V \hookrightarrow H \hookrightarrow V'$$

where the inclusions are dense, continuous and compact. The embedding of V in H is compact since G is bounded. Let $\langle \cdot, \cdot \rangle_{V',V}$ denote the dual pairing.

The Hodge-Leray (also known as Helmholtz) decomposition says $L^2(G) = H \oplus H^\perp$ where $H^\perp = \{\nabla g : g \in H^1\}$. Let $\pi : L^2(G) \to H$ be the Leray projection on H. Then apply π to equation (4.1) and get

$$\begin{cases} \frac{\partial(\pi u)}{\partial t} - \nu\pi\Delta u + \pi B(u) = \pi f(t) \\ \nabla \cdot u = 0 \end{cases}$$

Therefore, we may take $u \in H$, then $\pi u = u$. Rewrite πB as B and πf as f. Again, let $A : \mathcal{D}(A) \to H$, $A = -\pi\Delta$, be the Stokes operator, then the original system (4.1) becomes

$$\begin{cases} \frac{\partial u}{\partial t} + \nu A u + B(u) = f(t) \\ \nabla \cdot u = 0 \end{cases} \tag{4.2}$$

where $\nabla \cdot u = 0$ is automatically satisfied since $u \in H$. Again, $f \in L^2(0, T; V')$.

Note that the operator A is self-adjoint and positive definite. Let $e_1, e_2, \cdots$ be the eigenfunctions. Let $0 < \lambda_1 < \lambda_2 < \cdots$ be the eigenvalues, such that $Ae_j = \lambda_j e_j$. Also, $e_j \in V$. Then $\{e_j\}$ form a complete orthonormal basis in H.

Define in general for $u, v, w \in \mathbb{C}_c^\infty(G)$ which are divergence free,

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_G u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

Then

$$(B(u), w) = b(u, u, w) = \sum_{i,j=1}^{2} \int_G u_i \frac{\partial u_j}{\partial x_i} w_j dx.$$

Proposition 4.2. *The operator b satisfies $b(u, v, w) = -b(u, w, v)$ and thus, in particular, $b(u, v, v) = 0$.*

Proof. It is easy to see that

$$\begin{aligned} b(u, v, w) &= -\sum_{i,j} \int_G v_j \frac{\partial (u_i w_j)}{\partial x_i} dx \\ &= -\sum_{i,j} \int_G v_j (w_j \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial w_j}{\partial x_i}) dx \\ &= \sum_j \int_G v_j w_j \mathrm{Div}(u) dx - \sum_{i,j} \int_G v_j u_i \frac{\partial w_j}{\partial x_i} dx \\ &= -b(u, w, v) \end{aligned}$$

This finishes the proof.

$\square$

Since b is a trilinear function, it gives rise to a bilinear operator B on $V \times V$ that maps to V', i.e. if $u, v \in V$ then $B(u, v) \in V'$, and $\langle B(u, v), w \rangle_{V',V} = b(u, v, w)$. Let $B(u)$ denote $B(u, u)$. Then we only need to solve $u \in H$ for equation (4.2) with the given initial condition $u(0) = u_0$.

The following lemma is quite useful in this study.

Lemma 4.3. *For any real-valued smooth functions ϕ and ψ with compact support in $\mathbb{R}^2$, the following inequalities hold:*

1. $|\phi^2\psi^2|_{L^1} \leq |\phi|_{L^2}|\psi|_{L^2}|\nabla\phi|_{L^2}|\nabla\psi|_{L^2}$

2. $|\phi|_{L^4}^4 \leq \frac{1}{2}|\phi|_{L^2}^2|\nabla\phi|_{L^2}^2$

Proof. For any (x, y), by the fundamental theorem of calculus,

$$\phi(x,y) = \int_{-\infty}^{x} \partial_1\phi(s,y)ds = -\int_{x}^{\infty} \partial_1\phi(s,y)ds,$$

and

$$\psi(x,y) = \int_{-\infty}^{y} \partial_2\psi(x,t)dt = -\int_{y}^{\infty} \partial_2\psi(x,t)dt.$$

Then

$$|\phi(x,y)| = \frac{1}{2}|\int_{-\infty}^{x} \partial_1\phi(s,y)ds + \int_{x}^{\infty} -\partial_1\phi(s,y)ds| \leq \frac{1}{2}\int_{-\infty}^{\infty} |\partial_1\phi(s,y)|ds,$$

and

$$|\psi(x,y)| = \frac{1}{2}|\int_{-\infty}^{y} \partial_2\psi(x,t)dt + \int_{y}^{\infty} -\partial_2\psi(x,t)dt| \leq \frac{1}{2}\int_{-\infty}^{\infty} |\partial_2\phi(x,t)|dt.$$

Using the above two inequalities, and integrating with respect to x and y,

$$|\phi\psi|_{L^1} \leq \frac{1}{4}|\partial_1\phi|_{L^1}|\partial_2\psi|_{L^1}.$$

Replacing ϕ by ϕ^2 in the above, and using Cauchy-Schwarz inequality twice, we see that

$$\begin{aligned}
|\phi^2\psi^2|_{L^1} &\leq |\phi\partial_1\phi|_{L^1}|\psi\partial_2\psi|_{L^1} \\
&\leq |\phi|_{L^2}|\psi|_{L^2}|\partial_1\phi|_{L^2}|\partial_2\psi|_{L^2} \\
&\leq |\phi|_{L^2}|\psi|_{L^2}|\nabla\phi|_{L^2}|\nabla\psi|_{L^2}
\end{aligned}$$

Thus the first inequality of this lemma is proved.

The second inequality can be proved by replacing ϕ by ψ in the last inequality above and applying the following *Young's inequality.*

□

Lemma 4.4. *For any nonnegative real numbers a, b, and positive real numbers p, q satisfying $\frac{1}{p} + \frac{1}{q} = 1$,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The equality holds if and only if $a^p = b^q$.

4.2 The Weak Solutions of the SNSE

The existence of the weak solutions of the Navier-Stokes system will be proved using the Galerkin approximation and energy estimates. We consider the first m eigenfunctions of the Stokes operator A, $e_1, \ldots, e_m$, where m is a positive number. Denote

$$H_m = \mathrm{Span}\{e_1, \ldots, e_m\}.$$

Consider the projectors $P_m : H \to H$ onto the space H_m. Applying P_m to equation (4.2), it yields:

$$\frac{d}{dt} P_m u + \nu A(P_m u) + P_m B(u) = P_m f.$$

The Galerkin system of order m is the system

$$\begin{cases} \frac{du_m}{dt} + \nu A(u_m) + P_m B(u_m) = f_m \\ u_m(0) = u_m^0 \end{cases} \tag{4.3}$$

where the functions $u_m(t)$'s are in H_m'.

We hope to have enough control on the solutions of (4.3) so that we can obtain a solution to (4.2) as m tends to infinity.

Let ξ_j and η_j be the j-th components of u_m and f_m, respectively; that is,

$$\xi_j(t) = (u_m(t), e_j),$$

and

$$\eta_j(t) = (f_m(t), e_j).$$

Then the first equation of (4.3) is equivalent to

$$\frac{d\xi_j}{dt} + \nu\lambda_j\xi_j + \sum_{k,l=1}^{m} b(e_k, e_l, e_j)\xi_j\xi_l = \eta_j, \quad j = 1, \ldots, m. \tag{4.4}$$

Also, since $(u_m^0, e_j) = \xi_j^0$, the initial condition in (4.3) is equivalent to

$$\xi_j(0) = \xi_j^0, \quad j = 1, \ldots, m. \tag{4.5}$$

Let $T > 0$ be arbitrary. Assume that the function $g_m(t)$ is continuous on $[0, T]$ taking values in V', where V' is again the dual space of V. Since $e_j \in V$, the function

$$\eta : [0, T] \to \mathbb{R}^m$$

is continuous. From the ordinary differential equations theory, we know that the system of (4.4) and (4.5) has a unique solution on $[0, \tau]$, say $\xi(t)$, defined for t in a neighborhood of $t = 0$. In fact, τ can be ∞.

Recall that P_m is the projection from H to H_m. Note that $H_m = H'_m = V_m = V'_m$ being all n-dimention spaces. Moreover, for $g \in H_m$,

$$\begin{aligned} g &= \sum_{j=1}^{m} \xi_j e_j \\ &= \sum_{j=1}^{m} \xi_j \lambda_j^{1/2} \cdot \frac{e_j}{\lambda_j^{1/2}} \in V_m \end{aligned}$$

Denote $h_j = \frac{e_j}{\lambda_j^{1/2}}$, then $\{h_j\}_{j=1}^{\infty}$ form a complete orthonormal basis in V.

In equation (4.3), each $u_m = \sum_{j=1}^{m}(u_m, e_j)_H e_j = \sum_{j=1}^{m} \xi_j e_j$. Thus, equation (4.3) is equivalent to:

$$\sum_{j=1}^{m} e_j \cdot \frac{d\xi_j(t)}{dt} + \nu A(\sum_{j=1}^{m} \xi_j e_j) + B(\sum_{j=1}^{m} \xi_j e_j) = \sum_{j=1}^{m} \eta_j e_j,$$

which is

$$\sum_{j=1}^{m} e_j \cdot \frac{d\xi_j(t)}{dt} + \nu \sum_{j=1}^{m} \xi_j \lambda_j e_j + \sum_{j,k=1}^{m} \xi_j \xi_k B(e_j, e_k) = \sum_{j=1}^{m} \eta_j e_j.$$

Then the system (4.3) yields

$$\begin{aligned} &u_m(t) + \nu \int_0^t P_m A u_m(s) ds + \int_0^t P_m B(u_m(s)) ds \\ &= u_m(0) + \int_0^t f_m(s) ds. \end{aligned}$$

By the energy equality, one obtains

$$\begin{aligned} &|u_m(t)|^2 + 2\nu \int_0^t \langle P_m A u_m(s), u_m(s)\rangle ds + 2 \int_0^t \langle P_m B(u_m(s)), u_m(s)\rangle ds \\ &= |u_m(0)|^2 + 2 \int_0^t \langle f_m(s), u_m(s)\rangle ds. \end{aligned}$$

Note that the scalar product in the third term on the left hand side of the above equality vanishes since

$$B(u_m(s), u_m(s)) = 0.$$

We thus have the following:

$$\begin{aligned} &|u_m(t)|^2 + 2\nu \int_0^t \langle P_m A u_m(s), u_m(s)\rangle ds \\ &= |u_m(0)|^2 + 2 \int_0^t \langle f_m(s), u_m(s)\rangle ds, \end{aligned}$$

that is,

$$
\begin{aligned}
|u_m(t)|^2 + 2\nu \int_0^t \|u_m(s)\|_V^2 ds &= |u_m(0)|^2 + 2\int_0^t \langle f_m(s), u_m(s)\rangle ds \\
&\le |u_m(0)|^2 + 2\int_0^t |f_m(s)|_{V'} \cdot \|u_m(s)\|_V ds
\end{aligned}
\tag{4.6}
$$

Using the Young's inequality (Lemma 4.4),

$$
\begin{aligned}
2|f_m(s)|_{V'} \cdot \|u_m(s)\|_V &= 2\frac{|f_m(s)|_{V'}}{\nu^{1/2}} \cdot \nu^{1/2}\|u_m(s)\|_V \\
&\le \frac{|f_m(s)|_{V'}^2}{\nu} + \nu\|u_m(s)\|_V^2
\end{aligned}
$$

Applying the above inequality to (4.6) we get that

$$
\begin{aligned}
|u_m(t)|^2 + \nu \int_0^t \|u_m(s)\|_V^2 ds &\le |u_m(0)|^2 + \frac{1}{\nu}\int_0^t |f_m(s)|_{V'} ds \\
&\le |u(0)|^2 + \frac{1}{\nu}\int_0^t |f_m(s)|_{V'} ds \\
&= \text{a constant, say } C.
\end{aligned}
$$

Thus,

$$
\sup_{0\le t\le T} |u_m(t)|^2 \le C,
$$

and

$$
\int_0^t \|u_m(s)|_V^2 ds \le C.
$$

Note that m is arbitrary, which is the above constant C is independent of m. Therefore, $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^\infty(0,T;H)$ and in $L^2(0,T;V)$.

Therefore, there exists a subsequence of $\{u_m\}_{m=1}^{\infty}$, say $\{u_{m_k}\}_{k=1}^{\infty}$ such that $\{u_{m_k}\}$ converges weakly in $L^\infty(0,T;H)$, and also converges weakly in $L^2(0,T;H)$.

Definition 4.5. A *weak solution* of the Navier-Stokes equations (4.2) is a function $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ satisfying $\frac{du}{dt} \in L^1_{loc}(0,T;V')$ and the following system:

$$\begin{cases} \langle \frac{du}{dt}, v\rangle + \nu((u,v)) + B(u) = \langle f, v\rangle, \quad \text{a.e. in } t \text{ for all } v \in V, \\ u(0) = u_0. \end{cases}$$

Theorem 4.6. ***Existence.*** ***(Leray)*** *There exists at least a weak solution of the Navier-Stokes system (4.2) for every $u_0 \in H$ and $f \in L^2(0,T;V')$. Moreover, $\frac{du}{dt} \in L^2(0,T;V')$ and the energy inequality*

$$\frac{1}{2}|u(t)|^2 + \nu \int_{t_0}^{t} \|u(s)\|^2 ds \leq \frac{1}{2}|u(t_0)|^2 + \int_{t_0}^{t} \langle f(s), u(s)\rangle ds$$

holds for all $0 \leq t_0 \leq t \leq T, t_0$ a.e. in $[0,T]$.

4.3 The SNSE with fBm and Some Results

Now we consider the Navier-Stokes system with fBm. Let W^H be a fBm. First consider, for $0 \leq t \leq T$, the simplest system:

$$\begin{cases} \frac{\partial u}{\partial t} - \pi \Delta u = \Phi \frac{dW^H}{dt} \\ \nabla \cdot u = 0 \end{cases} \tag{4.7}$$

where $u(0) = u_0 \in H$. Then

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\Phi dW_s^H$$

where formally $S(t) = e^{-tA}$ is the *semigroup* generated by $A = -\pi\Delta$. First let $H \in (\frac{1}{2}, 1)$, for the simple case.

Suppose W^H is a cylindrical fBm. Then replacing the right side of (4.7) by $\Phi \frac{dW^H}{dt}$ where $\Phi \in L_2(H,H)$, we consider

$$\frac{\partial u}{\partial t} + Au = \Phi \frac{dW^H}{dt} \tag{4.8}$$

Then the *mild solution* is given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\Phi dW_s^H. \tag{4.9}$$

Denote

$$\begin{aligned} z(t) &:= \int_0^t S(t-s)\Phi dW_s^H \\ &= \sum_{n=1}^{\infty} \int_0^t S(t-s)\Phi e_n d\beta_n^H(s) \\ &= \int_0^t \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} (S(t-s)\Phi e_n, e_j) d\beta_n^H(s) \cdot e_j \\ &= \int_0^t \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} e^{-(t-s)\lambda_j}(\Phi e_n, e_j) d\beta_n^H(s) \cdot e_j \end{aligned}$$

Then $\{z(t)\}$ is the solution to (4.8) with $z(0) = 0$.

We notice that $z(t)$ is a.s. H-valued. To see this,

$$\begin{aligned} &E|z(t)|_H^2 \\ =&E[\sum_n \sum_j \int_0^t e^{-(t-s)\lambda_j}(\Phi e_n, e_j) d\beta_n^H(s)]^2 \\ =&\sum_n E[\sum_j \int_0^t e^{-(t-s)\lambda_j}(\Phi e_n, e_j) d\beta_n^H(s)]^2 \\ =&\sum_j \sum_n C \int_0^t \int_0^t e^{-(t-u)\lambda_j}(\Phi e_n, e_j) e^{-(t-v)\lambda_j}(\Phi e_n, e_j)|u-v|^{2H-2} dudv \\ =&2C \sum_j \sum_n (\Phi e_n, e_j)^2 \int_0^t e^{-(t-u)\lambda_j}[\int_0^u e^{-(t-v)\lambda_j}(u-v)^{2H-2} dv] du \\ \leq&2C \sum_j \sum_n |\Phi e_n|_H^2 \int_0^t e^{-(t-u)\lambda_j}[\int_0^u e^{-(t-v)\lambda_j}(u-v)^{2H-2} dv] du \\ \leq&2C \sum_j \sum_n |\Phi e_n|_H^2 \cdot \lambda_j^{-2H} \\ =&2C \|\Phi\|_{HS}^2 \sum_j \lambda_j^{-2H} \end{aligned}$$

Since $H > \frac{1}{2}$, $\sum_j \lambda_j^{-2H} \le C \sum_j j^{-2H} < \infty$. Thus $E\|z(t)\|_H^2 < \infty$.

We will also show that $z(t) \in L^4([0,T] \times G)$. In fact, we will prove a stronger statement that $E\int_0^T \int_G z^4(t,x)dxdt < \infty$ which will show that $z(t) \in L^4([0,T]\times G)$ almost surely.

Theorem 4.7. *Let the Hurst parameter* $H > \frac{1}{2}$. *Let*

$$\sum_n (\sum_j \lambda_j^{1/4-H} |(\Phi e_n, e_j)|)^2 < \infty.$$

Then

$$E\int_0^T \int_G z^4(t,x)dxdt < \infty.$$

Proof.

$$\begin{aligned}
&E\int_0^T \int_G z^4(t,x)dxdt \\
=&E\int_0^T \int_G [\sum_n \sum_j \int_0^t e^{-(t-s)\lambda_j} (\Phi e_n, e_j) e_j(x) d\beta_n^H(s)]^4 dxdt
\end{aligned}$$

Denoting $p_s^t(n,j) = e^{-(t-s)\lambda_j}(\Phi e_n, e_j)e_j(x)$, we have that

$$\begin{aligned}
&E\int_0^T \int_G z^4(t,x)dxdt \\
=&E\int_0^T \int_G [\sum_n \sum_j \int_0^t p_s^t(n,j) d\beta_n^H(s)]^4 dxdt \\
=&T_1 + T_2,
\end{aligned}$$

where

$$T_1 = \sum_n \int_0^T \int_G E[\sum_j \int_0^t p_s^t(n,j) d\beta_n^H(s)]^4 dxdt$$

and

$$T_2 = 3\sum_{n\neq m}\int_0^T\int_G E[(\sum_j\int_0^t p_s^t(n,j)d\beta_n^H(s))^2$$
$$\cdot(\sum_i\int_0^t p_s^t(m,i)d\beta_m^H(s))^2]dxdt.$$

For the first term T_1, since the integrand is deterministic with respect to x, we have that

$$\begin{aligned}T_1 =&\sum_n\int_0^T\int_G 3[E(\sum_j\int_0^t p_s^t(n,j)d\beta_n^H(s))^2]^2dxdt\\ =&3\sum_n\int_0^T\int_G[\sum_j E(\int_0^t p_s^t(n,j)d\beta_n^H(s))^2+\\ &\sum_{j\neq k}E(\int_0^t p_s^t(n,j)d\beta_n^H(s)\int_0^t p_s^t(n,k)d\beta_n^H(s))]^2dxdt\end{aligned}$$

Recall that K_t^* is an isometry between $\mathcal{H}$ and $L^2(T)$, the first summand in T_1 becomes

$$\begin{aligned}&E(\int_0^t e^{-(t-s)\lambda_j}(\Phi e_n,e_j)e_jd\beta_n^H(s))^2\\ =&|K_t^*(e^{-(t-s)\lambda_j}(\Phi e_n,e_j)e_j)|_{L^2}^2\\ =&|e^{-(t-s)\lambda_j}(\Phi e_n,e_j)e_j|_{\mathcal{H}}^2\\ =&H(2H-1)\int_0^t\int_0^t p_u^t(n,j)\cdot p_v^t(n,j)|u-v|^{2H-2}dvdu\\ =&H(2H-1)(\Phi e_n,e_j)^2e_j^2e^{-2t\lambda_j}\int_0^t\int_0^t e^{(u+v)\lambda_j}|u-v|^{2H-2}dvdu\end{aligned} \tag{4.10}$$

Note that

$$\begin{aligned}&\int_0^t\int_0^t e^{(u+v)\lambda_j}|u-v|^{2H-2}dvdu\\ =&2\int_0^t\int_0^u e^{(u+v)\lambda_j}(u-v)^{2H-2}dvdu\\ =&2\int_0^t e^{2u\lambda_j}\int_0^u e^{-(u-v)\lambda_j}(u-v)^{2H-2}dvdu\end{aligned} \tag{4.11}$$

Using the change of variables $u - v = \frac{w}{\lambda_j}$, the above (4.11) becomes

$$
\begin{aligned}
&=2\int_0^t e^{2u\lambda_j}\int_{u\lambda_j}^0 e^{-w}(\frac{w}{\lambda_j})^{2H-2}(-\frac{1}{\lambda_j})dwdu\\
&=2\lambda_j^{1-2H}\int_0^t e^{2u\lambda_j}\int_0^{u\lambda_j} e^{-w}w^{2H-2}dwdu\\
&=2\lambda_j^{1-2H}\int_0^{t\lambda_j} e^{-w}w^{2H-2}\int_{\frac{w}{\lambda_j}}^t e^{2u\lambda_j}dudw\\
&=2\lambda_j^{1-2H}\int_0^{t\lambda_j} e^{-w}w^{2H-2}(\frac{e^{2t\lambda_j}-e^{2w}}{2\lambda_j})dw\\
&=\lambda_j^{-2H}\int_0^{t\lambda_j} e^{-w}w^{2H-2}(e^{2t\lambda_j}-e^{2w})dw
\end{aligned}
$$

then (4.10) becomes

$$
\begin{aligned}
&H(2H-1)(\Phi e_n, e_j)^2 e_j^2 e^{-2t\lambda_j}\lambda_j^{-2H}\int_0^{t\lambda_j} e^{-w}w^{2H-2}(e^{2t\lambda_j}-e^{2w})dw\\
=&C(H)(\Phi e_n, e_j)^2 e_j^2\lambda_j^{-2H}\int_0^{t\lambda_j} e^{-w}w^{2H-2}(\frac{e^{2t\lambda_j}-e^{2w}}{e^{2t\lambda_j}})dw\\
\leq&C(H)(\Phi e_n, e_j)^2 e_j^2\lambda_j^{-2H}\int_0^{\infty} e^{-w}w^{2H-2}dw\\
\leq&C(H)(\Phi e_n, e_j)^2 e_j^2\lambda_j^{-2H}
\end{aligned}
$$

The second summand inside T_1 is

$$
\begin{aligned}
&\sum_{j\neq k} E(\int_0^t p_s^t(n,j)d\beta_n^H(s)\int_0^t p_s^t(n,k)d\beta_n^H)\\
=&\sum_{j\neq k}[E(\int_0^t p_s^t(n,j)d\beta_n^H(s))^2\cdot E(\int_0^t p_s^t(n,k)d\beta_n^H)^2]^{\frac{1}{2}}\\
\leq&\sum_j\sqrt{E(\int_0^t p_s^t(n,j)d\beta_n^H(s))^2}\cdot\sum_{k\neq j}\sqrt{E(\int_0^t p_s^t(n,k)d\beta_n^H)^2} \qquad (4.12)\\
\leq&\sum_j\sqrt{C(H)(\Phi e_n, e_j)^2 e_j^2\lambda_j^{-2H}}\cdot\sum_{k\neq j}\sqrt{C(H)(\Phi e_n, e_k)^2 e_k^2\lambda_k^{-2H}}\\
\leq&C(H)(\sum_j|(\Phi e_n, e_j)|\cdot|e_j|\cdot\lambda_j^{-H})^2
\end{aligned}
$$

Thus T_1 can be bounded by

$$\begin{aligned} &C\sum_n \int_0^T \int_G [\sum_j (\Phi e_n, e_j)^2 e_j^2 \lambda_j^{-2H} + (\sum_j |(\Phi e_n, e_j)| \cdot |e_j| \cdot \lambda_j^{-H})^2]^2 dxdt \\ \leq &2C\sum_n \int_0^T \int_G (\sum_j (\Phi e_n, e_j)^2 e_j^2 \lambda_j^{-2H})^2 + (\sum_j |(\Phi e_n, e_j)| \cdot |e_j| \cdot \lambda_j^{-H})^4 dxdt \end{aligned} \tag{4.13}$$

The first summand of (4.13) is

$$\begin{aligned} &2TC\sum_n \int_G (\sum_j (\Phi e_n, e_j)^2 e_j^2 \lambda_j^{-2H})^2 dx \\ =&2TC_1 \sum_n \sum_{i,j} \lambda_i^{-2H} \lambda_j^{-2H} (\Phi e_n, e_i)^2 (\Phi e_n, e_j)^2 \int_G e_i^2\, e_j^2 dx \\ \leq &2TC_1 \sum_n \sum_{i,j} \lambda_i^{1/2-2H} \lambda_j^{1/2-2H} (\Phi e_n, e_i)^2 (\Phi e_n, e_j)^2 \end{aligned} \tag{4.14}$$

Since

$$\begin{aligned} |e_j|_{L^4} &\leq \frac{1}{2} |e_j|_{L^2}^{\frac{1}{2}} |\nabla e_j|_{L^2}^{\frac{1}{2}} \\ &= \frac{1}{2} |\nabla e_j|_{L^2}^{\frac{1}{2}} \\ &= C|e_j|_{W^{1,2}}^{\frac{1}{2}} \\ &= C(\sqrt{\lambda_j})^{\frac{1}{2}}, \end{aligned}$$

the equation (4.14) becomes

$$\leq 2TC_1 \sum_n (\sum_j \lambda_j^{1/2-2H} (\Phi e_n, e_j)^2)^2 < \infty,$$

by using the assumption on Φ.

Meanwhile, the second summand of (4.13) is

$$\begin{aligned} &2TC\sum_n \int_G (\sum_j |(\Phi e_n, e_j)| \cdot |e_j| \cdot \lambda_j^{-H})^4 dx \\ \leq &2TC\sum_n \int_G \sum_{j_k} \prod_{k=1}^4 \lambda_{j_k}^{-H} |(\Phi e_n, e_{j_k})|\, |e_{j_k}|\, dx \\ \leq &2TC\sum_n (\sum_j \lambda_j^{1/4-H} |(\Phi e_n, e_j)|)^4 < \infty, \end{aligned}$$

by the hypothesis on Φ.

Thus, $T_1 < \infty$.

For the second term T_2, using the independence, we get that

$$T_2 = \sum_{n \neq m} \int_0^T \int_G E[\sum_j \int_0^t p_s^t(n,j) d\beta_n^H(s)]^2 \cdot E[\sum_i \int_0^t p_s^t(m,i) d\beta_m^H(s)]^2 dxdt$$

$$\leq \int_0^T \int_G \{\sum_n E[\sum_j \int_0^t p_s^t(n,j) d\beta_n^H(s)]^2\}^2 dxdt$$

$$\leq \int_0^T \int_G \{\sum_n (\sum_j |(\Phi e_n, e_j)||e_j|\lambda_j^{-H})^2\}^2 dxdt$$

$$= \int_0^T \int_G \sum_{m,n} \sum_{i_1,i_2,j_1,j_2} \prod_{k=1}^2 |(\Phi e_m, e_{j_k})|e_{j_k}|\lambda_{j_k}^{-H}|(\Phi e_m, e_{i_k})|e_{i_k}|\lambda_{i_k}^{-H}$$

$$\leq \int_0^T (\sum_n \sum_{j_1,j_2} \prod k = 1^2 \, |(\Phi e_n, e_{j_k})|\lambda_{j_k}^{1/4-H})^2$$

$$= \int_0^T (\sum_n (\sum_j |(\Phi e_n, e_j)|^2 \lambda_j^{1/4-H})^2)^2$$

$$< \infty$$

by the hypothesis on Φ.

Therefore, $T_2 < \infty$.

Thus we proved that

$$E \int_0^T \int_G z^4(t,x) dxdt < \infty,$$

that is, $z(t,x) \in L^4([0,T] \times G)$ a.s., for the case $H > \frac{1}{2}$.

□

Next we consider the case for $H \in (0, \frac{1}{2})$. A useful inequality will be applied to prove the following lemma.

Minkowski's Inequality for Integrals. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let f be an $\mathcal{M} \otimes \mathcal{N}$-measurable function on $X \times Y$. If

$f \geq 0$ and $1 \leq p < \infty$, then

$$[\int(\int f(x,y)d\nu(y))^p d\mu(x)]^{1/p} \leq \int[\int f(x,y)^p d\mu(x)]^{1/p} d\nu(y).$$

Lemma 4.8. *For* $0 < H < \frac{1}{2}$, $\int_0^t e^{-2x}(\int_0^x y^{H-\frac{3}{2}}(e^y - 1)dy)^2 dx$ *is bounded.*

Proof. Define two measures on $[0,\infty)$, $d\nu(y) = y^{H-\frac{3}{2}}(e^y - 1)dy$ and $d\mu(x) = e^{-2x}dx$, then both ν and μ are σ-finite. To see this, it suffices to show that for fixed $n = 0, 1, 2, \cdots$, $\nu([n, n+1))$ and $\mu([n, n+1))$ are finite. Clearly, $\mu([n, n+1)) < \infty$. For ν,

$$\begin{aligned} \nu([0,1)) &= \int_0^1 y^{H-\frac{3}{2}}(e^y - 1)dy \\ &= \lim_{\epsilon \to 0} \int_\epsilon^1 y^{H-\frac{3}{2}}(e^y - 1)dy, \end{aligned} \tag{4.15}$$

By integration by parts, (4.15) becomes

$$\begin{aligned} &= \frac{e-1}{H-\frac{1}{2}} - \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{y^{H-\frac{1}{2}}e^y}{H-\frac{1}{2}}dy \\ &= \frac{e-1}{H-\frac{1}{2}} - \frac{1}{H-\frac{1}{2}}(\frac{e}{H+\frac{1}{2}} - \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{y^{H+\frac{1}{2}}e^y}{H+\frac{1}{2}}dy) \\ &= -\frac{e-1}{\frac{1}{2}-H} + \frac{e}{(\frac{1}{2}-H)(\frac{1}{2}+H)} - \frac{1}{\frac{1}{2}-H}\lim_{\epsilon \to 0} \int_\epsilon^1 \frac{y^{H+\frac{1}{2}}e^y}{H+\frac{1}{2}}dy) \\ &\leq -\frac{e-1}{\frac{1}{2}-H} + \frac{e}{(\frac{1}{2}-H)(\frac{1}{2}+H)} \\ &= \frac{e}{\frac{1}{2}+H} + \frac{1}{\frac{1}{2}-H} \\ &< \infty. \end{aligned}$$

For $n \geq 1$,

$$\begin{aligned} \nu([n, n+1)) &= \int_n^{n+1} y^{H-\frac{3}{2}}(e^y - 1)dy \\ &\leq \int_n^{n+1} n^{H-\frac{3}{2}}(e^y - 1)dy \\ &= n^{H-\frac{3}{2}}(e^{n+1} - e^n - 1) \\ &< \infty. \end{aligned}$$

Therefore, by the Minkowski's inequality for integrals,

$$\begin{aligned}
&[\int_0^t e^{-2x}(\int_0^t 1_{(0,x)}(y)y^{H-\frac{3}{2}}(e^y-1)dy)^2dx]^{\frac{1}{2}}\\
&\leq \int_0^t [\int_0^t 1_{(0,x)}(y)e^{-2x}dx]^{\frac{1}{2}}y^{H-\frac{3}{2}}(e^y-1)dy\\
&\leq \int_0^t e^{-y}y^{H-\frac{3}{2}}(e^y-1)dy\\
&< \infty.
\end{aligned}$$

So the result is obtained. The bound is independent of t.

□

Theorem 4.9. *Let the Hurst parameter be as* $\frac{1}{8} < H < \frac{1}{2}$. *If*

$$\sum_n(\sum_j \lambda_j^{1/4}|(\Phi e_n, e_j)|)^2 < \infty,$$

it follows that

$$E\int_0^T\int_G z^4(t,x)dxdt < \infty.$$

Proof.

$$\begin{aligned}
E\int_0^T\int_G z^4(t,x)dxdt =&E\int_0^T\int_G(\int_0^t e^{-(t-s)A}\Phi dW^H(s))^4dxdt\\
=&E\int_0^T\int_G(\sum_n\int_0^t e^{-(t-s)A}\Phi e_n d\beta_n^H(s))^4dxdt\\
=&E\int_0^T\int_G(\sum_n\int_0^t a_s^t(n)+b_s^t(n)d\beta_n(s))^4dxdt,
\end{aligned}$$

where

$$a_s^t(n) = K(t,s)e^{-(t-s)A}\Phi e_n$$

and

$$b_s^t(n) = \int_s^t (e^{-(t-r)A} - e^{-(t-s)A})\Phi e_n \frac{\partial K(r,s)}{\partial r}dr.$$

Then

$$
\begin{aligned}
&E\int_0^T\int_G z^4(t,x)dxdt\\
=&E\int_0^T\int_G[\sum_n(\int_0^t a_s^t(n)d\beta_n(s))^4+\sum_n(\int_0^t b_s^t(n)d\beta_n(s))^4\\
&+3\sum_{n\neq m}(\int_0^t a_s^t(n)d\beta_n(s))^2(\int_0^t b_s^t(m)d\beta_m(s))^2]dxdt
\end{aligned}
\tag{4.16}
$$

The first term of (4.16) is

$$
\begin{aligned}
&\int_0^T\int_G\sum_n E(\int_0^t K(t,s)e^{-(t-s)A}\Phi e_n d\beta_n(s))^4dxdt\\
=&\int_0^T\int_G\sum_n 3[E(\int_0^t K(t,s)e^{-(t-s)A}\Phi e_n d\beta_n(s))^2]^2dxdt
\end{aligned}
$$

Since $K(t,s)\leq c(H)(t-s)^{H-\frac{1}{2}}s^{H-\frac{1}{2}}$, see Decreusefond and Ustunel [5], we get

$$
\begin{aligned}
&E(\int_0^t K(t,s)e^{-(t-s)A}\Phi e_n d\beta_n(s))^2\\
\leq&\int_0^t(\sum_j(\Phi e_n,e_j)e_j(x)(t-s)^{H-1/2}s^{H-1/2}e^{-(t-s)\lambda_j})^2ds\\
\leq&\int_0^t\sum_{i,j}|(\Phi e_n,e_i)||(\Phi e_n,e_j)||e_i(x)||e_j(x)|\\
&\qquad\cdot(t-s)^{2H-1}s^{2H-1}e^{-(t-s)(\lambda_i+\lambda_j)}ds\\
\leq&\sum_{i,j}|(\Phi e_n,e_i)||(\Phi e_n,e_j)||e_i(x)||e_j(x)|\\
&\qquad\cdot(\int_0^t(t-s)^{2H-1}s^{2H-1}e^{-2(t-s)\lambda_i}ds)^{1/2}\\
&\qquad\cdot(\int_0^t(t-s)^{2H-1}s^{2H-1}e^{-2(t-s)\lambda_j}ds)^{1/2}
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_0^t (t-s)^{2H-1} s^{2H-1} e^{-2(t-s)\lambda_j} ds \\
=&\int_{2t\lambda_j}^0 (\frac{w}{2\lambda_j})^{2H-1} (t-\frac{w}{2\lambda_j})^{2H-1} e^{-w} (-\frac{1}{2\lambda_j}) dw,
\end{aligned}
$$

by changing variables $(t-s)=\frac{w}{2\lambda_j}$. Then the above becomes

$$
\begin{aligned}
=&(2\lambda_j)^{-2H} \int_0^{2t\lambda_j} e^{-w} w^{2H-1} (t-\frac{w}{2\lambda_j})^{2H-1} dw \\
=&(2\lambda_j)^{-2H} (\int_0^{\lambda_j t} + \int_{\lambda_j t}^{2\lambda_j t}) e^{-w} w^{2H-1} (t-\frac{w}{2\lambda_j})^{2H-1} dw
\end{aligned}
$$

Since $2H-1<0$, the above term is

$$
\begin{aligned}
\leq&(2\lambda_j)^{-2H}[(\frac{t}{2})^{2H-1}\int_0^{\lambda_j t} e^{-w} w^{2H-1} dw + \\
&\qquad (\lambda_j t)^{2H-1} \int_{\lambda_j t}^{2\lambda_j t} e^{-w} (t-\frac{w}{2\lambda_j})^{2H-1} dw] \\
\leq&(2\lambda_j)^{-2H}[(\frac{t}{2})^{2H-1}\int_0^{\lambda_j t} w^{2H-1} dw + \\
&\qquad (\lambda_j t)^{2H-1} \int_{t/2}^{0} e^{-2(t-s)\lambda_j} s^{2H-1} (-2\lambda_j) ds] \\
=&(2\lambda_j)^{-2H}[(\frac{t}{2})^{2H-1} \cdot \frac{(\lambda_j t)^{2H}}{2H} + (\lambda_j t)^{2H-1}(2\lambda_j)\int_0^{\frac{t}{2}} e^{-2(t-s)\lambda_j} s^{2H-1} ds] \\
\leq&(2\lambda_j)^{-2H}[(\frac{t}{2})^{2H-1} \cdot \frac{(\lambda_j t)^{2H}}{2H} + (\lambda_j t)^{2H-1}(2\lambda_j)(e^{-t\lambda_j}\frac{(\frac{t}{2})^{2H}}{2H})] \\
=&\frac{2^{-4H}}{H}(1+e^{-\lambda_j t}) t^{4H-1}
\end{aligned}
$$

Let $C(H)$ again be a generic constant. By the above calculation,

$$
\begin{aligned}
&E(\int_0^t K(t,s) e^{-(t-s)A} \Phi e_n d\beta_n(s))^2 \\
\leq&C(H)\sum_{i,j} |(\Phi e_n, e_i)||(\Phi e_n, e_j)||e_i(x)||e_j(x)| t^{4H-1}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\int_0^T \int_G \sum_n [E(K(t,s)e^{-(t-s)A}\Phi e_n \, d\beta_n(s))^2]^2 dx dt \\
\le &\int_0^T \int_G \sum_n (\sum_{i,j} |(\Phi e_n, e_i)||(\Phi e_n, e_j)||e_i(x)||e_j(x)|t^{4H-1})^2 dx dt \\
\le &\int_G \sum_n (\sum_{i,j} |(\Phi e_n, e_i)||(\Phi e_n, e_j)||e_i(x)||e_j(x)|)^2 \frac{T^{8H-1}}{8H-1} dx \\
= &C \sum_n \sum_{i_1,i_2,j_1,j_2} |\prod_{k=1}^2 (\Phi e_n, e_{i_k})||(\Phi e_n, e_{j_k})||e_{i_k}(x)||e_{j_k}(x)| dx \\
\le &C \sum_n (\sum_i |(\Phi e_n, e_i)|\lambda_i^{1/4})^4 \\
< &\infty
\end{aligned}$$

by the assumption on Φ.

Thus the first term of (4.16) is finite.

Now let us denote

$$b_s^t(n,j) = \int_s^t (e^{-(t-r)\lambda_j} - e^{-(t-s)\lambda_j})(\Phi e_n, e_j) e_j \frac{\partial K(r,s)}{\partial r} dr,$$

then the second term of (4.16) is

$$\begin{aligned}
&3\int_0^T \int_G \sum_n [E(\sum_j \int_0^t b_s^t(n,j) d\beta_n(s))^2]^2 dx dt \\
= &3\int_0^T \int_G \sum_n I^2 dx dt, \qquad \text{say.}
\end{aligned}$$

Since

$$|\frac{\partial K(r,s)}{\partial r}| \le C(H)(r-s)^{H-\frac{3}{2}},$$

we see that

$$
\begin{aligned}
I \leq & E \sum_{i,j} \int_0^t b_s^t(n,i) b_s^t(n,j) ds \\
= & C \sum_{i,j} \int_0^t \Big(\int_s^t (e^{-(t-r_1)\lambda_i} - e^{-(t-s)\lambda_i}) |(\Phi e_n, e_i)| |e_i(x)| (r-s)^{H-\frac{3}{2}} dr_1 \Big) \\
& \cdot \int_s^t (e^{-(t-r_2)\lambda_j} - e^{-(t-s)\lambda_j}) |(\Phi e_n, e_j)| |e_j(x)| (r-s)^{H-\frac{3}{2}} dr_2 \Big) ds \\
= & C \sum_{i,j} |(\Phi e_n, e_i)| |e_i(x)| |(\Phi e_n, e_j)| |e_j(x)| \\
& \cdot \int_0^t ds \int_s^t dr_1 \int_s^t dr_2 (r_1 - s)^{H-3/2} (r_2 - s)^{H-3/2} \\
& \cdot (e^{-(t-r_1)\lambda_i} - e^{-(t-s)\lambda_i})(e^{-(t-r_2)\lambda_j} - e^{-(t-s)\lambda_j})
\end{aligned}
$$

Using the change of variables

$$u = t - s,\ v_1 = t - r_1,\ v_2 = t - r_2,$$

the above expression becomes:

$$
\begin{aligned}
= & C \sum_{i,j} |(\Phi e_n, e_i)| |e_i(x)| |(\Phi e_n, e_j)| |e_j(x)| \qquad (4.17) \\
& \cdot \int_0^t du \int_0^u dv_1 \int_0^u dv_2 (u - v_1)^{H-3/2} (u - v_2)^{H-3/2} \\
& \cdot (e^{-v_1\lambda_i} - e^{-u\lambda_i})(e^{-v_2\lambda_j} - e^{-u\lambda_j}) \\
= & C \sum_{i,j} |(\Phi e_n, e_i)| |e_i(x)| |(\Phi e_n, e_j)| |e_j(x)| \qquad (4.18) \\
& \cdot \int_0^t du \Big(\int_0^u (u - v_1)^{H-3/2} (e^{-v_1\lambda_i} - e^{-u\lambda_i}) dv_1 \Big) \\
& \cdot \Big(\int_0^u (u - v_2)^{H-3/2} (e^{-v_2\lambda_j} - e^{-u\lambda_j}) dv_2 \Big). \qquad (4.19)
\end{aligned}
$$

Now Consider

$$\int_0^u (u - v_1)^{H-3/2} (e^{-v_1\lambda_i} - e^{-u\lambda_i}) dv_1.$$

Putting $r = u - v_1$, it is equal to

$$e^{-\lambda_i u} \int_0^u r^{H-3/2} (e^{\lambda_i r} - 1) dr.$$

Using this twice, we get that the expression on the right side of (4.19) is

$$
\begin{aligned}
=&C\sum_{i,j}|(\Phi e_n,e_i)||e_i(x)||(\Phi e_n,e_j)||e_j(x)|\int_0^t e^{-(\lambda_i+\lambda_j)u} \\
&\cdot(\int_0^u r_1^{H-3/2}(e^{\lambda_i r_1}-1)dr_1)(\int_0^u r_2^{H-3/2}(e^{\lambda_j r_2}-1)dr_2)du \\
\leq&K_H\sum_{i,j}|(\Phi e_n,e_i)||e_i(x)||(\Phi e_n,e_j)||e_j(x)|\lambda_i^{-H}\lambda_j^{-H},
\end{aligned}
$$

where we have used Lemma 2 in Tindel, Tudor and Viens [21].

Thus the square of the expression in (4.19) can be bounded above by

$$
K_H^2\sum_{i_1,i_2,j_1,j_2}|\prod_{k=1}^{2}|(\Phi e_n,e_{i_k})||e_{i_k}(x)||(\Phi e_n,e_{j_k})||e_{j_k}(x)|\lambda_{i_k}^{-H}\lambda_{j_k}^{-H}.
$$

Therefore,

$$
\int_0^T\int_G\sum_n I^2dxdt\leq\sum_n(\sum_i|(\Phi e_n,e_i)|\lambda_i^{1/4-H})^4 \tag{4.20}
$$

which is finite by the hypothesis on Φ.

Finally, the third term of (4.16) is

$$
\begin{aligned}
=&3E\int_0^T\int_G\sum_{m,n,m\neq n}(\int_0^t a_s^t(n)d\beta_n(s))^2(\int_0^t b_s^t(m)d\beta_m(s))^2dxdt \\
\leq&C(H)\int_T\int_G\sum_{m,n}E(\int_0^t a_s^t(n)d\beta_n(s))^2\cdot E(\int_0^t b_s^t(m)d\beta_m(s))^2dxdt.
\end{aligned}
$$

We use the basic inequality

$$
ab\leq\frac{a^2}{2}+\frac{b^2}{2}
$$

on the right side of the above expression.

Consider

$$
\begin{aligned}
&\int_T \int_G (\sum_n E(\int_0^t a_s^t(n) d\beta_n(s))^2)^2 \\
=&\int_T \int_G \sum_{m,n} \sum i_1, i_2, j_1, j_2 \prod_{k=1}^{2} |(\Phi e_n, e_{i_k})||e_{i_k}(x)||(\Phi e_n, e_{j_k})||e_{j_k}(x)| t^{8H-2} \\
\leq&(\sum_n \sum_{j_k} \prod k = 1^2 |(\Phi e_n, e_{j_k})| \lambda_{j_k}^{1/4})^2 \frac{T^{8H-1}}{8H-1} \\
<&\infty.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\int_T \int_G (\sum_m E(\int_0^t b_s^t(m) d\beta_m(s))^2)^2 \\
\leq &\int_T \int_G (\sum_{m,n} \sum_{i_k, j_k} \prod |(\Phi e_n, e_{j_k})| \, |(\Phi e_m, e_{i_k})| \, |e_{j_k}(x)| \, |e_{i_k}(x)| \, \lambda_{j_k}^{-H} \, \lambda_{i_k}^{-H} \\
\leq &C(\sum_n (\sum_j |(\Phi e_n, e_j)| \lambda_j^{1/4-H})^2)^2 \\
<&\infty.
\end{aligned}
$$

Therefore, the third term of (4.16) is finite.

□

4.4 The Existence and Uniqueness of the Solutions of the SNSE with fBm

Now we consider the stochastic Navier-Stokes system with fBm with non-linear term.

$$
\begin{cases}
\frac{\partial u}{\partial t} + Au + B(u(t)) = \Phi \frac{dW^H}{dt} \\
\nabla \cdot u = 0
\end{cases}
\tag{4.21}
$$

with the initial condition $u(0,x) = u_0(x) \in H$, and the boundary condition $u(t,x) = 0$ for $x \in \partial G$, $\forall t \geq 0$; $A = -\pi\Delta$ being the Stokes operator, and B

being the non-linear term. From the previous theorems, we know that $z(t)$ satisfies

$$\frac{\partial z}{\partial t} + Az = \Phi\frac{dW^H}{dt},$$

with $z(0) = 0$. Denote $v := u - z$. Notice that

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial u}{\partial t} - \frac{\partial z}{\partial t} \\
&= (-Au - B(u,u) + \Phi\frac{dW^H}{dt}) - (-Az + \Phi\frac{dW^H}{dt}) \\
&= -A(u-z) - B(u,u) \\
&= -Av - B(v+z, v+z)
\end{aligned}$$

So solving u for (4.21) is equivalent to solving v for

$$\frac{\partial v}{\partial t} + Av + B(v+z, v+z) = 0$$

with $v(0) = u_0 \in H$. The mild solution v should satisfy

$$v(t) = S(t)u_0 - \int_0^t S(t-s)B(v(s)+z(s), v(s)+z(s))ds \tag{4.22}$$

Recall from Section (4.1) that

$$\langle B(u), w\rangle = \langle \pi((u\cdot\nabla)u), w\rangle,$$

where π is the Hodge-Leray Projection. B is a bilinear operator from $V \times V$ to V'; That is, if $u \in V$ then $u, v \in W^{1,2}(G)$, $B(u,v) \in V'$, and

$$\langle B(u,v), w\rangle_{V'\times V} = b(u,v,w) = -b(u,w,v).$$

The Sobolev embedding gives

$$H^{\frac{1}{2}} = W^{\frac{1}{2},2}(G) \hookrightarrow L^4(G),$$

so

$$\|u\|_{L^4} \leq C\|u\|_{W^{\frac{1}{2},2}(G)}.$$

Then for $u \in L^4(G)$,

$$\begin{aligned}
|\langle B(u), w\rangle| &= |b(u,w,u)| \\
&= |\sum_{i=1}^{2} \int_G u_i \frac{dw_j}{dx_i} u_j dx| \\
&\leq |u|_{L^4} |\nabla w|_{L^2(G)} |u|_{L^4} \\
&\leq C|u|_{W^{\frac{1}{2},2}} |\nabla w|_H |u|_{W^{\frac{1}{2},2}} \\
&= C|u|_{\frac{1}{2}} |w|_V |u|_{\frac{1}{2}} \\
&\leq C|u|_H^{\frac{1}{2}} |u|_V^{\frac{1}{2}} |w|_V |u|_H^{\frac{1}{2}} |u|_V^{\frac{1}{2}} \\
&\leq C|u|_H |u|_V |w|_V
\end{aligned}$$

Then

$$\sup_{w \in V, |w|_V \neq 0} \frac{|\langle B(u), w\rangle|}{|w|_V} \leq C|u|_H |u|_V,$$

and thus

$$\|B(u)\|_{V'} \leq C|u|_H |u|_V.$$

Now define

$$h(t) = -\int_0^t S(t-s) B(y(s), y(s)) ds,$$

where $S(t) = e^{-tA}$ is again the semigroup generated by the operator A, and

$$y \in L^4([0,T]) \times G.$$

Then $h(0) = 0$. Besides, by the energy equality,

$$\begin{aligned}
|h(t)|_{L^2}^2 =& -2\int_0^t |\nabla h|_{L^2}^2 ds - 2\int_0^t \langle B(y(s)), h(s)\rangle_{V' \times V} ds \\
\leq& -2\int_0^t |h|_V^2 ds + 2\int_0^t |B(y(s))|_{V'} \cdot |h(s)|_V ds \\
\leq& -2\int_0^t |h|_V^2 ds + \int_0^t |B(y(s))|_{V'}^2 ds + \int_0^t |h(s)|_V^2 ds
\end{aligned}$$

So

$$|h|_H^2 + \int_0^t |h(s)|_V^2 ds \leq \int_0^t |B(y)|_{V'}^2 ds,$$

and thus

$$\sup_{0\leq t\leq T} |h|_H^2 + \int_0^t |h(s)|_V^2 ds \leq 2\int_0^t |B(y)|_{V'}^2 ds,$$

which is bounded.

Therefore, $h(t) \in L^\infty(0,T;H) \cap L^2(0,T;V)$.

Now consider the map

$$R(y) := -\int_0^t S(t-s)B(y(s))ds,$$

for $y \in L^4([0,T]\times G)$. Since

$$\begin{aligned}
& |\langle B(u_1) - B(u_2), \psi\rangle| \\
= \; & |b(u_1,u_1,\psi) - b(u_2,u_2,\psi)| \\
\leq \; & |b(u_1-u_2,u_1,\psi)| + |b(u_2,u_1-u_2,\psi)| \\
= \; & |b(u_1-u_2,\psi,u_1)| + |b(u_2,\psi,u_1-u_2)| \\
\leq \; & |u_1-u_2|_{L^4}|\nabla\psi|_H|u_1|_{L^4} + |u_2|_{L^4}|\nabla\psi|_H|u_1-u_2|_{L^4} \\
= \; & |u_1-u_2|_{L^4}|\psi|_V(|u_1|_{L^4} + |u_2|_{L^4}),
\end{aligned}$$

which means

$$|B(u_1) - B(u_2)|_{L^2(0,T;V')} \leq C|u_1-u_2|_{L^4}(|u_1|_{L^4} + |u_2|_{L^4})$$

and thus

$$|B(v_1+z) - B(v_2+z)|_{L^2(0,T;V')} \leq C|v_1-v_2|_{L^4}(|v_1+z|_{L^4} + |v_2+z|_{L^4}).$$

Then

$$\begin{aligned}
|R(y_1) - R(y_2)|_{L^4} &\leq C|R(y_1) - R(y_2)|_{W^{\frac{1}{2},2}} \\
&\leq C|R(y_1) - R(y_2)|_H^{\frac{1}{2}} \cdot |R(y_1) - R(y_2)|_V^{\frac{1}{2}}
\end{aligned}$$

Then

$$
\begin{aligned}
&|R(y_1) - R(y_2)|_{L^4(0,T;L^4(G))} \\
=&(\int_0^T |R(y_1) - R(y_2)|_{L^4}^4)^{\frac{1}{4}} \\
\leq &C(|R(y_1) - R(y_2)|_H^2 \cdot |R(y_1) - R(y_2)|_V^2 dt)^{\frac{1}{4}} \\
\leq &C(\sup_{0\leq t\leq T} |R(y_1) - R(y_2)|_H^2 \cdot \int_0^T |R(y_1) - R(y_2)|_V^2 dt)^{\frac{1}{4}} \\
\leq &C[(\int_0^T |B(y_1) - B(y_2)|_{V'}^2 dt)^2]^{\frac{1}{4}} \\
\leq &C|y_1 - y_2|_{L^4([0,T]\times G)}(|y_1|_{L^4([0,T]\times G)} + |y_2|_{L^4([0,T]\times G)}).
\end{aligned}
$$

Now define

$$
\begin{aligned}
L(y) :=&S(t)v_0 - \int_0^t S(t-s)B(y(s) + z(s))ds \\
=&Sv_0 + R(y+z)
\end{aligned}
$$

for $Sv_0, y, z \in L^4$. Then L maps from $L^4(0,T;L^4(G))$ to itself, and

$$
\begin{aligned}
&|L(y_1) - L(y_2)|_{L^4(0,T;L^4(G))} \\
&\leq C|y_1 - y_2|_{L^4([0,T]\times G)}(|y_1 + z|_{L^4([0,T]\times G)} + |y_2 + z|_{L^4([0,T]\times G)}).
\end{aligned}
$$

Then choose T_1 such that

$$
y_i \in B_{\frac{1}{2C}}(-z), \quad i = 1, 2,
$$

where B denotes the ball in $L^4([0,T]\times G)$, with center $-z$ and radius $\frac{1}{2C}$. This implies

$$
C(|y_1 + z|_{L^4([0,T]\times G)} + |y_2 + z|_{L^4([0,T]\times G)}) < 1,
$$

for all $y_1, y_2 \in B$.

Therefore, L forms a contraction on the local time interval $[0, T_1]$. Then by the following fixed point theorem, the unique solution exists, say v_1.

Theorem 4.10. ***The fixed point theorem.*** *Let L be a map of the complete metric space X into itself. If R is a contraction, then there exists a unique point z such that*

$$Lz = z.$$

Now considering v_1 as the initial condition, one can find $T_2 > T_1$ such that the time interval can be extended to $[0, T_2]$ on which the unique solution exists, say v_2.

Continuing this way, suppose τ is the maximum time up to which the unique solution exists, and suppose $\tau < T$, then we can extend $\tau_1 > \tau$ such that the unique solution exists on $[0, \tau_1]$, since the $L^\infty(0, T; H) \cap L^2(0, T; V)$ bounds hold for any initial condition that lies in H. Therefore, $\tau = T$.

Therefore, the unique solution exists on the entire time interval $[0, T]$.

The above work is summarized in the main theorem given below:

Theorem 4.11. *Let W_t^H be a cylindrical fBm. Let $\Phi \in L(H, H)$. Then there exists a unique mild solution of the stochastic Navier-Stokes system (4.21):*

$$\begin{cases} \frac{\partial u}{\partial t} + Au + B(u(t)) = \Phi \frac{dW^H}{dt} \\ \nabla \cdot u = 0 \end{cases}$$

under the following conditions on Φ:

1. *For the case $\frac{1}{2} < H < 1$: $\sum_n (\sum_j \lambda_j^{1/4-H} |\Phi e_n, e_j|)^2 < \infty$;*

2. *For the case $\frac{1}{8} < H < \frac{1}{2}$: $\sum_n (\sum_j \lambda_j^{1/4} |(\Phi e_n, e_j)|)^2 < \infty$.*

References

[1] R. B. Ash and C. A. Doléans-Dade, *Probability and Measure Theory*, Academic Press, San Diego, second ed., 2000.

[2] P. Billingsley, *Probability and Measure*, Wiley Series in Probability and Mathematical Statistics, John Wiley and Sons, Inc., New York, third ed., 1995.

[3] P. Constantin and C. Foias, *Navier-Stokes Equations*, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, 1989.

[4] G. DaPrato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications Series, Cambridge University Press, New York, 1992.

[5] L. Decreusefond and A. S. Ustunel, *Stochastic analysis of the fractional brownian motion*.

[6] R. Durrett, *Stochastic Calculus: A Practical Introduction*, Probability and Stochastics Series, CRC Press, Boca Raton, 1996.

[7] L. C. Evans, *Partial Differential Equations*, Graduate studies in mathematics, AMS, Province Rhode Island, 1998.

[8] F. Flandoli and D. Gatarek, *Martingale and stationary solutions for stochastic navier-stokes equations*, Probability Theory and Related Fields, (1995), pp. 367 – 391.

[9] I. Gihman and A. V. Skorohod, *Stochastic Differential Equations*, Springer-Verlag, Berlin, 1972.

[10] A. Gut, *Probability: A Graduate Course*, Springer Texts in Statistics, Springer, New York, 2005.

[11] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics, Springer, New York, second ed., 1991.

[12] H.-H. Kuo, *Introduction to Stochastic Integration*, Universitext, Springer, New York, 2005.

[13] D. Nualart, *Stochastic integration with respect to fractional brownian motion and applications*, Contemporary Mathematics 336, (2003), pp. 1 – 14.

[14] S. C. Port, *Theoretical Probability for Applications*, Wiley Series in Probability and Mathematical Statistics, John Wiley and Sons, New York, 1994.

[15] P. E. Protter, *Stochastic Integration and Differential Equations*, Stochastic Modelling and Applied Probability No. 21, Springer, Berlin, second ed., 2005.

[16] A. N. Shiryaev, *Probability*, Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 1996.

[17] H. Sohr, *The Navier-Stokes Equations*, Birkhauser advanced texts, Birkhauser Verlag, Germany, 2001.

[18] S. S. Sritharan and P. Sundar, *The stochastic magneto-hydrodynamic system*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, World Scientific Publishing Company, New Jersey, 1997.

[19] W. A. Strauss, *On continuity of functions with values in various banach spaces*, Pacific Journal Of Mathematics, (1966), pp. 543 – 545.

[20] R. Teman, *Navier-Stokes Equations*, Studies in mathematics and its applications, Elsevier Science Publishers B. V., New York, second ed., 1979.

[21] S. Tindel, C. A. Tudor, and F. Viens, *Stochastic evolution equations with fractional brownian motion*, Probability Theory and Related Fields, (2003), pp. 1 – 18.

Vita

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