OPTIMIZATION OF NEW CHINESE REMAINDER THEOREMS USING SPECIAL MODULI SETS

A Thesis

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TABLE OF CONTENTS

Acknowledgements ........................................................................................................... ii

List of Figures .................................................................................................................... v

Abstract ............................................................................................................................. vi

1. Introduction To Residue Number Systems .................................................................. 1
   1.1 Introduction .............................................................................................................. 1
   1.2 Fundamentals of Residue Number System ............................................................. 1
       1.2.1 Weighted to RNS Conversion ................................................................. 1
       1.2.2 Arithmetic Operations in RNS Domain .......................................................... 2
       1.2.3 Dynamic Range of the RNS ............................................................................ 2
       1.2.4 RNS to Weighted Conversion ................................................................. 3
           1.2.4.1 Chinese Remainder Theorems (CRT) ..................................................... 3
           1.2.4.2 Mixed Radix Conversion (MRC) .......................................................... 4

2. New Chinese Remainder Theorems ............................................................................ 7
   2.1 New Chinese Remainder Theorem I (CRT I) ........................................................ 7
       2.1.1 Traditional Hardware Implementation for CRT I ........................................ 10
   2.2 New Chinese Remainder Theorem II (CRT II) ...................................................... 10
       2.2.1 Traditional Hardware Implementation for CRT II ....................................... 14
   2.3 New Chinese Remainder Theorem III (CRT III) .................................................... 15
       2.3.1 Requirements for the Efficient Implementation of CRT III ......................... 17
       2.3.2 Decoding Conjugate Pairs of Moduli ........................................................... 18
           2.3.2.1 Decoding Two Conjugate Pair Moduli ................................................... 18
           2.3.2.2 Decoding Three Conjugate Pair Moduli ................................................. 19
       2.3.3 Traditional Hardware Implementation for CRT III ...................................... 20

3. Optimizing New Chinese Remainder Theorems ......................................................... 22
   3.1 Shortfalls of New Chinese Remainder Theorems .................................................. 22
   3.2 Basic Lemmas Used for the Optimization ............................................................ 30
   3.3 Optimizing the Base Theorem for New Chinese Remainder Theorem I ............... 31
   3.4 Optimizing New Chinese Remainder Theorem II ............................................... 33
   3.5 Optimizing New Chinese Remainder Theorem III .............................................. 35
       3.5.1 Extending the Optimized Four Moduli Set to A Five Moduli Set ................. 38

4. Optimized Hardware Implementation of New Chinese Remainder Theorems .......... 41
   4.1 Introduction ............................................................................................................ 41
   4.2 Optimized Hardware Implementation of the Base Theorem for New
       Chinese Remainder Theorem I ............................................................................ 41
   4.3 Optimized Hardware Implementation of New Chinese Remainder Theorem II ..... 43
   4.4 Optimized Hardware Implementation of New Chinese Remainder Theorem III .... 46
   4.5 Comparison between Optimized and Non-Optimized CRT II Implementations .... 47
   4.6 Comparison between Optimized and Non-Optimized CRT III Implementations .... 48
5. Conclusion and Future Work ........................................................................................................50

References........................................................................................................................................51

Appendix A: Inverse Modulus Operation.........................................................................................53
   A.1. Methods to Calculate Inverse Modulus Values.................................................................53
   A.2. Code in C to Calculate Multiplicative Inverse.................................................................58

Appendix B: Proofs for the Sets to Be Pair Wise Relatively Prime.............................................60
   B.1. The Set \( \{2^{n-1}+1, 2^{n-1}, 2^n, 2^n+1\} \), where \( n = 2k + 1, k = 1, 2, 3 \ldots \) Is Pair Wise Relatively Prime.................................................................60
   B.2. The Set \( \{2^{n+2}+3, 2^{n+1}+1, 2^n+1, 2\} \), where \( n = 1, 2, 3 \ldots \) Is Pair Wise Relatively Prime........................................................................................................60

Vita..................................................................................................................................................62
LIST OF FIGURES

Fig.2.1. Hardware assembly for calculating X using CRT I ........................................11
Fig.2.2. Implementation of findno procedure.............................................................12
Fig.2.3. Hardware assembly for calculating N₁ and N₂ using CRT II ..........................15
Fig.2.4. Hardware assembly for calculating X using CRT II.......................................16
Fig.2.5. Traditional hardware implementation of CRT III..........................................21
Fig.4.1.a. Hardware implementation of Q₁ by CRT I without using multipliers..............42
Fig.4.1.b. Hardware implementation of Q₂ by CRT I without using the multipliers.........42
Fig.4.1.c. Hardware implementation of Q₃ by CRT I without using the multipliers.........43
Fig.4.1.d. Hardware implementation of X by CRT I without using the multipliers..........43
Fig.4.2.a. Hardware implementation of N₁ by CRT II without using multipliers ............45
Fig.4.2.b. Hardware implementation of N₂ by CRT II without using multipliers ..........45
Fig.4.3. Hardware implementation of X by CRT II without using multipliers.................46
Fig.4.4. Hardware implementation of X by CRT III without using multipliers..............48
ABSTRACT

The residue number system (RNS) is an integer number representation system, which is capable of supporting parallel and high-speed arithmetic. This system also offers some useful properties for error detection, error correction and fault tolerance. It has numerous applications in computation-intensive digital signal processing (DSP) operations, like digital filtering, convolution, correlation, Discrete Fourier Transform, Fast Fourier Transform, direct digital frequency synthesis, etc.

The residue to binary conversion is based on Chinese Remainder Theorem (CRT) and Mixed Radix Conversion (MRC). However, the CRT requires a slow large modulo operation while the MRC requires finding the mixed radix digits which is a slow process. The new Chinese Remainder Theorems (CRT I, CRT II and CRT III) make the computations faster and efficient without any extra overheads. But, New CRTs are hardware intensive as they require many inverse modulus operators, modulus operators, multipliers and dividers. Dividers and inverse modulus operators in turn needs many half and full adders and subtractors. So, some kind of optimization is necessary to implement these theorems practically.

In this research, for the optimization, new both co-prime and non co-prime multi modulus sets are proposed that simplify the new Chinese Remainder theorems by eliminating the huge summations, inverse modulo operators, and dividers. Furthermore, the proposed hardware optimization removes the multiplication terms in the theorems, which further simplifies the implementation.
1. INTRODUCTION TO RESIDUE NUMBER SYSTEMS

1.1. Introduction:

Residue number system is an integer system which is capable of supporting parallel, carry-free and high speed arithmetic. This system also offers some useful properties for error detection, error correction and fault tolerance in digital systems. It is very efficient in carrying out arithmetic operations like additions, subtractions and multiplications. The speed of the arithmetic operations relies on the size of the numbers involved; smaller numbers imply faster operations. Since, the numbers used in this system are smaller, it is known for faster implementation of arithmetic operations, and hence it is very attractive. This system is applied in the fields of Digital Signal Processing (DSP) intensive computations like digital filtering, convolutions, correlations, Discrete Fourier Transform (DFT) computations, Fast Fourier Transform (FFT) computations and direct digital frequency synthesis.

1.2. Fundamentals of Residue Number System:

A Residue Number System is defined by a set \( S = \{m_1, m_2, m_3, \ldots, m_L\} \) and this set is called the moduli set, where \( m_1, m_2, m_3, \ldots, m_L \) are positive integers. Earlier, the important criteria of moduli set is that the elements are pair wise relatively prime,

\[
S = \{m_1, m_2, m_3 \ldots m_L\}, \quad (m_i, m_j) = 1 \quad \text{for} \quad i \neq j
\]  

(1.1)

where \((m_i, m_j)\) is the greatest common divisors of \(m_i\) and \(m_j\). The works of [2] and [4] clearly show that the moduli set need not be pair wise relatively prime. This gives rise to wide choices of moduli sets and makes the RNS system efficient and attractive.

1.2.1. Weighted to RNS Conversion:

Let \( Z_M \) be \( \{0, 1, 2 \ldots M-1\} \), where \( M \) is the product of all moduli in the set \( S \). Any integer \( X \) belonging to \( Z_M \) has a unique representation,
\[ X^{\text{RNS}} \rightarrow (X_1, X_2, X_3, ..., X_L) \]  

(1.2)

where,

\[ X_i = X \mod m_i , \quad i = 1, 2, ..., L \]  

(1.3)

The above equations (1.2) and (1.3) show the weighted to RNS conversion.

1.2.2. Arithmetic Operations in RNS Domain:

Let \( X \) and \( Y \) belong to \( Z_M \), where \( Z_M = \{0, 1, 2... M-1\} \), while \( M \) is the product of all moduli in the set \( S \). Let \( X \) and \( Y \) have RNS representations as follows,

\[ X^{\text{RNS}} \rightarrow (X_1, X_2, X_3... X_L) \]  

(1.4)

\[ Y^{\text{RNS}} \rightarrow (Y_1, Y_2, Y_3... Y_L) \]  

(1.5)

Then \( Z = X \circ Y \) is given in the RNS as follows:

\[ Z^{\text{RNS}} = X \circ Y \rightarrow ((X_1 \circ Y_1) \mod m_1 , (X_1 \circ Y_1) \mod m_2 ,..., (X_1 \circ Y_1) \mod m_L) \)  

(1.6)

where \( \circ \) denotes addition, subtraction or multiplication. Equation (1.6) clearly shows that the nature of the implementations of RNS operations is carry free and parallel.

1.2.3. Dynamic Range of the RNS:

Let \( M \) be the product of all the moduli in the moduli set. Then if the system supports only unsigned numbers, the dynamic range is

\[ DR = [0 \quad M-1] \]  

(1.7)

If the system supports signed numbers then the dynamic range is

\[ DR = [-M/2 \quad (M/2) - 1], \text{ if } M \text{ is even} \]  

(1.8)

\[ DR = [- (M-1)/2 \quad (M-1)/2], \text{ if } M \text{ is odd} \]  

(1.9)
1.2.4. **RNS to Weighted Conversion:**

Traditionally, there are two basic techniques for converting the Residue Numbers System into the weighted system. The two techniques are:

1) **Chinese Remainder Theorem (CRT)**

2) **Mixed Radix Conversion (MRC)**

1.2.4.1. **Chinese Remainder Theorem (CRT):**

Consider the moduli set \( S = \{m_1, m_2, m_3, \ldots m_L\} \), and let the RNS representation of an integer \( X \) be \((X_1, X_2, X_3, \ldots X_L)\).

Then the Chinese Remainder Theorem reconstructs \( X \) from its residues as follows,

\[
X = (X_1 M_1 N_1 + X_2 M_2 N_2 + \ldots + X_L M_L N_L) \mod M
\]  
(1.10)

Alternately, we can have

\[
X = \left( \sum_{i=1}^{L} X_i N_i M_i \right) \mod M
\]  
(1.11)

where,

\[
M = \prod_{i=1}^{L} m_i
\]  
(1.12)

\[
M_i = M / m_i \quad \text{and} \quad N_i = (M_i^{-1}) \mod m_i
\]  
(1.13)

**Illustration 1:**

Consider an RNS system defined by the set \( S = \{m_1, m_2, m_3\} = \{15, 16, 17\} \). Here the value of \( M = (m_1 \cdot m_2 \cdot m_3) = 4080 \).

For an unsigned system, the dynamic range is \([0 \, M-1]\), and hence here dynamic range is \([0 \, 4079]\).

For instance, let the two weighted numbers be \( X = 55 \) and \( Y = 58 \). Then the multiplication \( (Z = X \cdot Y) \) is carried out as follows,
From (1.2) and (1.3), $X = (X_1, X_2, X_3) = (10, 7, 4)$ and $Y = (Y_1, Y_2, Y_3) = (13, 10, 7)$.

From (1.6), $Z = X \times Y = ( (X_1 \times Y_1) \mod m_1, (X_1 \times Y_1) \mod m_2, (X_1 \times Y_1) \mod m_3)$

$= (10, 6, 11)$

Conversion of $Z = (10, 6, 11)$ into the weighted system using the CRT is as follows:

From (1.12) and (1.13), $M = 4080$, $M_1 = 272$, $N_1 = 8$

$M_2 = 255$, $N_2 = 15$

and $M_3 = 240$, $N_3 = 9$

Using the CRT equation,

$$Z = \left( \sum_{i=1}^{3} (Z_i N_i M_i) \mod M \right)$$

(1.14)

$$Z = 3190$$

For verification, $Z = X \times Y = (55 \times 58) = 3190$.

**1.2.4.2. Mixed Radix Conversion (MRC):**

For the moduli set $S = \{m_1, m_2, m_3..., m_L\}$, $(m_i, m_j) = 1 \text{ for } i \neq j$, and the RNS representation of $X \longrightarrow (X_1, X_2, X_3..., X_L)$, $X$ can be calculated from its residues using the Mixed Radix Conversion formula as shown below,

$$X = X_1' + m_1 X_2' + m_1 m_2 X_3' + m_1 m_2 m_3 X_4' + \ldots + m_1 m_2 m_3 m_4...m_{L-1} X_L'$$

(1.15)

where $X_1'$, $X_2'$, $X_3'$, ..., $X_L'$ are called the mixed radix digits and $X_i'$ belongs to $Z_{m_i} = 0, 1, 2, \ldots, m_i - 1$. The mixed radix digits $X_1'$, $X_2'$, $X_3'$, ..., $X_L'$ can be represented as functions of the residues $X_1$, $X_2$, $X_3$, ..., $X_L$ and the moduli $\{m_1, m_2, m_3, ..., m_L\}$. These mixed radix digits have respective weights associated with them. The weight associated with $X_1'$ is 1,
$X_2'$ is $m_1$, $X_3'$ is $m_1m_2$, and similarly the weight associated with $X_L'$ is $m_1m_2m_3...m_{L-1}$. These mixed radix digits can be calculated as shown.

Consider a 4–moduli system, which can be easily generalized to an arbitrary L–moduli system. Then the MRC formula is

$$X = X_1' + m_1X_2' + m_1m_2X_3' + m_1m_2m_3X_4'$$  \hspace{1cm} (1.16)

Taking (mod $m_1$) on both sides of the above equation yields,

$$X \mod m_1 = X_1' \mod m_1$$  \hspace{1cm} (1.17)

We know, $X_1' \in \mathbb{Z}_{m_1}$, therefore $X_1' \mod m_1 = X_1'$  \hspace{1cm} (1.18)

Hence $X_1' = X_1$ (Since $X \mod m_1 = X$)  \hspace{1cm} (1.19)

For the calculation of $X_2'$, consider (1.16) again, we get

$$X - X_1' = m_1X_2' + m_1m_2X_3' + m_1m_2m_3X_4'$$

Taking inverse modulus of $m_1$ on both sides, we get

$$m_1^{-1}(X - X_1') = X_2' + m_2X_3' + m_2m_3X_4'$$

Taking both sides mod $m_2$, we get,

$$[m_1^{-1}(X - X_1')] \mod m_2 = X_2' \mod m_2 + (m_2X_3' + m_2m_3X_4') \mod m_2$$

Hence $X_2' = [m_1^{-1}(X - X_1')] \mod m_2$  \hspace{1cm} (1.20)

For the calculation of $X_3'$, consider (1.16) again. We get

$$X - X_1' - m_1X_2' = m_1m_2X_3' + m_1m_2m_3X_4'$$

Taking inverse modulus of $m_1m_2$ on both sides, and applying mod $m_3$, we get,
\[
[(m_1m_2)^{-1}(X - X'_1 - m_1X'_2)] \mod m_3 = [X'_3 \mod m_3 + (m_1m_2m_3X'_4)] \mod m_3 \tag{1.21}
\]

Hence \(X'_3 = ((m_1m_2)^{-1}(X_3 - X'_1 - m_1X'_2)) \mod m_3\)

Similarly \(X'_4\) can be calculated as \(((m_1m_2m_3)^{-1}(X_4 - X'_1 - m_1X'_2 - m_1m_2X'_3)) \mod m_4\) \tag{1.22}

**Illustration 2:**

Consider an RNS system defined by the moduli set \(S = \{7, 9, 11, 13\}\). Let the weighted number \(X\) be 1478.

The RNS representation of \(X\) is \(X = (X_1, X_2, X_3, X_4) = (1, 2, 4, 9)\). Using the Mixed Radix Conversion technique, the above RNS can be converted into weighted number. From the equations (1.19), (1.20), (1.21) and (1.22), the values of \(X'_1\), \(X'_2\), \(X'_3\) and \(X'_4\) can be calculated as 1, 4, 1, 2 respectively. Finally, the MRC equation (1.16) gives the value of \(X\) as 1478.
2. NEW CHINESE REMAINDER THEOREMS

The speed of the arithmetic operations is based on the numbers involved in the operations. The size of the numbers is directly proportional to the delay of the operations, and therefore smaller numbers imply faster operations. However, the Chinese Remainder Theorem requires a slow large modulo operation while the Mixed Radix Conversion requires finding the mixed radix digits which is a slow process. New Chinese Remainder Theorems were designed to make the computations faster and efficient without any overheads.

2.1. New Chinese Remainder Theorem I (CRT I):

New Chinese Remainder theorem I (CRT I) is a modified version of the traditional Chinese Remainder Theorem [1]. In this conversion process, the weighted number can be retrieved faster because the operations are done in parallel, without depending on other results.

Some propositions proposed in [1] are necessary for this new conversion and are as follows,

Proposition 1:

\[ a = 1 \mod (m_1 m_2) \implies a = 1 \mod m_1 \text{ and } 1 \mod m_2 \]  \hspace{1cm} (2.1)

The above proposition is obtained from the corollary,

\[ a = 1 \mod (m_1 m_2 \ldots m_k) \implies a = 1 \mod m_1, a = 1 \mod m_2 \ldots, a = 1 \mod m_k \]  \hspace{1cm} (2.2)

Proposition 2:

\[ [am_1] \mod (m_1 m_2) = [a] \mod m_2 \cdot m_1 \]  \hspace{1cm} (2.3)

Proposition 3:

For any \( y \) belonging to \([0, M-1]\), where \( M = m_1 \cdot m_2, \ldots, m_{n-1} \cdot m_n \), there is unique mixed radix representation as follows, where \( y_i \) satisfies the condition \( 0 \leq y_i \leq m_i \cdot m_{i+1} \).
\[ y = y_0 + y_1 m_1 + y_2 m_1 m_2 + \ldots + y_{n-1} m_1 m_2 \ldots m_{n-1} \]  \hspace{1cm} (2.4)

In the above equation, \( y_{n-1} \) is the most significant digit and \( y_0 \) is the least significant digit.

Given the residue numbers \((X_1, X_2, X_3, \ldots, X_n)\), the corresponding weighted number \(X\) can be computed using the following equation.

\[ X = \left[ X_1 + k_1 m_1 (X_2 - X_1) + k_2 m_1 m_2 (X_3 - X_2) + \ldots + k_{n-1} m_1 m_2 \ldots m_{n-1} (X_n - X_{n-1}) \right] \mod m_1 m_2 \ldots m_{n-1} m_n \]  \hspace{1cm} (2.5)

where \( k_1 = (m_1)^{-1} \mod m_2 \), \( k_2 = (m_1 m_2)^{-1} \mod m_3 m_4 \ldots m_n \) and similarly

\[ k_{(n-1)} = (m_1 m_2 \ldots m_{n-1})^{-1} \mod m_n \]

It is to be noted that, the above equation is different from the traditional CRT and MRC equations.

**Proof for the above equation:**

From the proposition 1, we know

\[ k_1 = (m_1)^{-1} \mod m_2 \]

\[ k_2 = (m_1 m_2)^{-1} \mod m_3 \]

Extending the above to the larger dimensions, we get

\[ k_1 = (m_1)^{-1} \mod m_2, \ k_2 = (m_1 m_2)^{-1} \mod m_3, \ldots, k_{(n-1)} = (m_1 m_2 \ldots m_{n-1})^{-1} \mod m_n \]

Henceforth, \( X = X_1 \mod m_1 \)

\[ X = \{ X_1 \mod m_2 + (k_1 m_1) \mod m_2 \times (X_2 - X_1) \} = X_2 \mod m_2 \]
\[ X = \{X_1 \mod m_1 + (k_1m_1) \mod m_2 \mod (X_2-X_1) + (k_2m_1m_2) \mod m_3 \mod (X_3-X_2)\} \]

\[ = X_3 \mod m_2 \]

Similarly, extending the above to the larger dimensions,

\[ X = \{X_1 \mod m_n + (k_1m_1) \mod m_n \mod (X_2-X_1) + (k_2m_1m_2) \mod m_n \mod (X_3-X_2) + \ldots \]

\[ + (k_{n-1}m_{1}m_{2} \ldots m_{n-1}) \mod m_n \mod (X_n-X_{n-1})\} \mod m_n \]

\[ = (X_1 - (X_2 - X_1) + (X_3 - X_2) + \ldots + (X_n - X_{n-1}) ) \mod m_n \]

\[ = X_n \]

This conversion process enables the parallel processing of the mixed radix digits because the digits are given based on the co-ordinates \( X_i \) as in the CRT. Hence this conversion overcomes the bottleneck of using traditional CRT and MRC techniques.

**Illustration 1:**

Consider a weighted number \( X = 2011 \) and the moduli set of the form \( \{2^{n-1}+1, 2^n-1, 2^n, 2^n+1\} \) where \( n=2k+1, \ k = 1,2,3, \ldots \) Taking the value of \( n = 3 \), we get the moduli set \( m = (m_1, m_2, m_3, m_4) = (5, 7, 8, 9) \).

The proof for the set \( \{2^{n-1}+1, 2^n-1, 2^n, 2^n+1\} \) to be pair wise relatively prime is dealt in the appendix B (Section B.1).

RNS representation of \( X = (X_1, X_2, X_3, X_4) \)

\[ = (X \mod m_1, X \mod m_2, X \mod m_3, X \mod m_4) \]

\[ = (1, 2, 3, 4) \]
\[k_1 = (m_1)^{-1} \mod (m_2m_3m_4) = 5^{-1} \mod (7 \times 8 \times 9) = 101\]

\[k_2 = (m_1m_2)^{-1} \mod (m_3m_4) = (5 \times 7)^{-1} \mod (8 \times 9) = 35\]

\[k_3 = (m_1m_2m_3)^{-1} \mod m_4 = (5 \times 7 \times 8)^{-1} \mod 9 = 1\]

\[X = [X_1 + k_1m_1(X_2 - X_1) + k_2m_1m_2(X_3 - X_2) + k_3m_1m_2m_3(X_4 - X_3)] \mod (m_1m_2m_3m_4)\]

\[= [1 + 505 + 1225 + 280] \mod 2520\]

\[X = 2011\]

2.1.1. Traditional Hardware Implementation for CRT I:

In the traditional method, CRT I is implemented using adders, subtractors, multipliers, modulus operators and inverse modulus operators. Multipliers need several full and half adders, which increases the delay and cost involvement in the operations. This part is explained in Chapter 4 in detail. For 4–moduli set, hardware requirements are as follows:

1) Three Subtractors
2) Nine multipliers
3) Three adders
4) One modulus operator
5) Three Inverse Modulus operators

Figure 2.1 describes the hardware assembly for implementing CRT I using a 4-moduli set.

2.2. New Chinese Remainder Theorem II (CRT II):

The converter based on new Chinese remainder theorem II does not require any big size modulo adders for any moduli in the conversion process. This approach is applied only for co-prime moduli sets, and the converter algorithm is based on divide and conquer approach.
For instance, consider a 2-moduli set $S = \{m_1, m_2\}$ where $m_1 < m_2$ and the RNS representation

![Diagram](image)

**Fig.2.1. Hardware assembly for calculating X using CRT I**

of a weighted number, $X$ be $(X_1, X_2)$. From the residue numbers $X_1$ and $X_2$, we want to calculate $X$, such that $X = X_1 \mod m_1$ and $X = X_2 \mod m_2$. The following derivation [1] shows how to retrieve the weighted number $X$ from its residues.

**Derivation:**

Given that,

$$X = X_2 \mod m_2 \text{ and } X = X_1 \mod m_1$$
We know,  \( M_2 = \prod_{i=t+1}^{n} m_i \), \( X = X_2 + k * m_2 \), for some integer \( k \).

\[
X_2 + k * m_2 = X_1 \mod m_1
\]

\[
k * m_2 = X_1 - X_2 \mod m_1
\]

Alternately, \( k = (m_2)^{-1} \mod m_1 \)

\[
k * (X_1 - X_2) * m_2 = X_1 - X_2 \mod m_1
\]

Therefore we have, \( X = [X_2 + k (X_1 - X_2) \cdot m_2] \mod (m_1 * m_2) \)

\[
X = X_2 + [(k (X_1 - X_2) \cdot m_2) \mod m_1] \cdot m_2
\]

(2.6)

The above derivation shows the decoding of two moduli sets in which the big sized modulo \( M = m_1 \cdot m_2 \) is not required. The algorithm for finding the weighted number \( X \) from the residues \( X_1 \) and \( X_2 \) is done using two procedures. They are \textit{findno} and \textit{translate}, and they are described as follows,

\textit{Procedure: findno}(\( X_1, X_2, m_1, m_2, X \)):

1) Find the value of \( k \), where \( k = (m_2)^{-1} \mod m_1 \)

2) \( X = X_2 + [(k (X_1 - X_2) \cdot m_2) \mod m_1] \cdot m_2 \)

The figures below show the diagrammatic representation of this procedure.

\[\text{Fig.2.2. Implementation of findno procedure}\]
Similarly, this derivation can be extended for bigger moduli sets. The procedure used is

\textit{translate} \((X_1, X_2, X_3 \ldots X_n, X)\) and is as follows,

Step 1) Consider an \(n\) moduli set and the residues of a number \(X = (X_1, X_2, X_3 \ldots X_n)\).

Consider

\[ t = \left\lfloor \frac{n}{2} \right\rfloor \]

The procedure \textit{translate} is defined as follows,

\textit{translate} \((X_1, X_2, \ldots, X_t), N_1)\)

\[ M_1 = \prod_{i=1}^{t} m_i \quad (2.7) \]

\textit{translate} \((X_{t+1}, X_{t+2}, \ldots, X_n), N_2)\)

\[ M_2 = \prod_{i=t+1}^{n} m_i \quad (2.8) \]

Step 2) Follow the procedure \textit{findno} \((N_1, N_2, M_1, M_2, X)\)

This retrieves the correct decimal value of \(X\) from \((X_1, X_2, X_3 \ldots X_n)\). The correctness of the above algorithm is proved in [1] by the induction method.

\textit{Illustration 2:}\n
Consider a weighted number \(X = 2011\) and a moduli set of the form \(\{2^{n-1}+1, 2^n-1, 2^n, 2^{n+1}\}\)

where \(n = 2k+1, k = 1, 2, 3\ldots\)

The proof for the set \(\{2^{n-1}+1, 2^n-1, 2^n, 2^{n+1}\}\) to be pair wise relatively prime is dealt in the appendix B (Section B.1). For this illustration let us take the value of \(n\) to be 3. So we get the moduli set as \(\{5, 7, 8, 9\}\). The RNS representation of \(X = (X_1, X_2, X_3, X_4)\)
\[ X_1=2011 \mod 5 = 1; \ X_2=2011 \mod 7 = 2; \ X_3= 2011 \mod 8 = 3; \ X_4=2011 \mod 9 = 4 \]

Therefore \( X = (1, 2, 3, 4) \).

\[ k_1= (m_2)^{-1} \mod m_1 = (7)^{-1} \mod 5 = 3 \]

\[ k_2= (m_4)^{-1} \mod m_3 = (9)^{-1} \mod 8 = 1 \]

\[ k_3= (m_3m_4)^{-1} \mod (m_1m_2) = (72)^{-1} \mod 35 = 18 \]

\[ N_1 = X_2 + [k_1 \times (X_1 - X_2) \mod m_1] \times m_2 \]

\[ = 2 + \{[3 \times (-1)] \mod 5\} \times 7 \]

\[ = 16 \]

\[ N_2 = X_4 + [k_2 \times (X_3 - X_4) \mod m_3] \times m_4 \]

\[ = 4 + \{[1 \times (-1) \mod 8] \times 9 \}

\[ = 67 \]

\[ X = N_2 + [ k_3 \times (N_1 - N_2) \mod (m_1m_2)] \times (m_3m_4) \]

\[ = 67 + \{[18 \times (-51)] \mod 35\} \times 72 \]

\[ = 2011 \]

2.2.1. Traditional Hardware Implementation for CRT II:

In the traditional method, CRT II is implemented using subtractors, multipliers, dividers, modulus operators and inverse modulus operators. Multipliers and dividers need several full and half adders, which increases the delay and cost involvement of the operations. For 4–moduli set,
hardware requirements are as follows:

1) Three Subtractors
2) Four multipliers
3) Three adders
4) Three modulus operator
5) Three Inverse Modulus operator

Figures 2.3 and 2.4 describe the hardware assembly for implementing CRT II using 4-moduli set.

Fig.2.3. Hardware assembly for calculating $N_1$ and $N_2$ using CRT II

2.3. New Chinese Remainder Theorem III (CRT III):

CRT I and CRT II are designed for co-prime moduli sets, whereas CRT III is designed for non co-prime moduli sets. That is, for a moduli set $S = \{m_1, m_2, m_3... m_k\}$ the non co-prime
criteria is \((m_i, m_j) \neq 1, i \neq j\) and \(k \geq 2\). The dynamic range for this non co-prime moduli set is the least common multiple of the moduli in the set \(S\), that is

\[
M = \text{lcm} (m_1, m_2, m_3... m_k)
\]  

(2.9)

Let \(t\) be the floor of \((k/2)\).

\[
t = \left\lfloor \frac{k}{2} \right\rfloor
\]  

(2.10)

\[\text{Fig.2.4. Hardware assembly for calculating X using CRT II}\]

Splitting the moduli set \(S\) into two subsets as \(\{m_1, m_2, m_3... m_t\}\) and \(\{m_{t+1}, m_{t+2}, m_{t+3}... m_k\}\). \(M_1\) and \(M_2\) is defined as follows,

\[
M_j = \text{lcm} (m_1, m_2, m_3... m_t)
\]  

(2.11)
\[ M_2 = \text{lcm}(m_{t+1}, m_{t+2}, m_{t+3}, \ldots m_k) \]  

(2.12)

The weighted number \( X \) has the following RNS representation,

\[ X \rightarrow (X_1, X_2) \]  

(2.13)

where \( X_1 = X \mod M_1 \) and \( X_2 = X \mod M_2 \).

The reconstruction of \( X \) from \( X_1 \) and \( X_2 \) is shown in equation (2.14).

\[
X = X_1 + M_1 \left[ \left( \frac{M_1}{d} \right)^{-1} \left( \frac{X_2 - X_1}{d} \right) \right] \mod \left( \frac{M_2}{d} \right)
\]

(2.14)

where \( d = (M_1, M_2) \). In the above formula, it is to be noticed that if \( d = 1 \), then it yields the traditional Mixed Radix Conversion formula.

### 2.3.1. Requirements for the Efficient Implementation of CRT III:

Consider the equation (2.14), in which the constants \( M_1 \) and \( M_2 \) are involved. The moduli set should be chosen in such a way that \( M_1 > M_2 \). If this condition is satisfied, then the hardware requirements will be less because \( \text{mod}(M_2/d) \) hardware is required instead of \( \text{mod}(M_1/d) \) hardware. Also, moduli sets should be chosen such that the least common multiple of the moduli involved is as large as possible, so that bigger dynamic range is obtained. As the division operation can slow down the process, it should be avoided if possible. In equation (2.14), we have the term \((X_2 - X_1)/d\). This division can be eliminated if \((d^{-1}) \mod (M_2/d)\) exists. In this case, the formula simplifies to

\[
X = X_1 + M_1 \left[ \left( \frac{M_1}{d} \right)^{-1} [X_1 - X_2](d^{-1}) \right] \mod \left( \frac{M_2}{d} \right)
\]

(2.15)
All these issues are addressed, if the chosen moduli sets have conjugate pairs of moduli. Refer to [2] for the details of the conjugate pair of moduli. In the later section, special moduli sets are used to handle all these issues and some other special issues for the efficient and simplified implementation of CRT III.

2.3.2. Decoding Conjugate Pairs of Moduli:

As conjugate pairs of moduli offer larger dynamic ranges, hardware efficient implementation and fast and balanced RNS arithmetic, conjugate pair moduli is one of the best choices.

2.3.2.1. Decoding Two Conjugate Pair Moduli:

Consider the 4-moduli set \( S \) consisting of two pairs of conjugate moduli [2],

\[
S = m_1^*, m_1, m_2^*, m_2 = 2^n + 1, 2^n - 1, 2^{n_2} + 1, 2^{n_2} - 1
\]  

(2.16)

where \( 2^n + 1 \) and \( 2^n - 1 \) is a conjugate pair and \( 2^{n_2} + 1 \) and \( 2^{n_2} - 1 \) is another conjugate pair. The dynamic range of the above set is the least common multiple of \( \{m_1^*, m_1, m_2^*, m_2\} \). Let the weighted number be \( X \) and its RNS representation be

\[
X \xrightarrow{\text{RNS}} (X_1^*, X_1, X_2^*, X_2)
\]

(2.17)

\[
X_i^* = X \mod m_i^* \quad \text{and} \quad X_i = X \mod m_i \quad \text{where } i = 1, 2.
\]

(2.18)

For better understanding, the decoding is done in two stages. In the first stage, moduli \( \{m_1^*, m_1\} \) and \( \{m_2^*, m_2\} \) are used separately in the conversion process. As \( m_1^* \) and \( m_1 \) are coprime, CRT III becomes MRC and let the variable \( Z_i \) be described by MRC formula as follows,

\[
Z_i = X_i^* + m_1^* [(m_1^*)^{-1} (X_i - X_i^*)] \mod m_1
\]

(2.19)
Using the moduli from (2.16) and substituting in (2.19), we get,

\[ Z_i = X_i^* + (2^n + 1)[(2^n - 1)^{-1}(X_1 - X_i^*)] \mod (2^n - 1) \]  

(2.20)

Similarly using \( m_2^* \) and \( m_2 \) and, let the variable \( Z_2 \) be described by MRC formula as follows,

\[ Z_2 = X_2^* + m_2^* \left[ (m_2^*)^{-1} (X_2 - X_2^*) \right] \mod m_2 \]  

(2.21)

Using the moduli from (2.16) and substituting in (2.21), we get

\[ Z_2 = X_2^* + (2^{n_2} + 1)[(2^{n_2} - 1)^{-1}[X_2 - X_2^*]] \mod (2^{n_2} - 1) \]  

(2.22)

In the second stage, using \( Z_1, Z_2, M_1 = \text{lcm} (m_1^*, m_1) \text{ and } M_2 = \text{lcm} (m_2^*, m_2) \) in the equation (2.14), we get

\[ X = Z_1 + (2^{2n_1} - 1) \left[ \left( \frac{2^{2n_1} - 1}{d} \right)^{-1} \left( \frac{Z_2 - Z_1}{d} \right) \right] \mod \left( \frac{2^{2n_2} - 1}{d} \right) \]  

(2.23)

where \( d = \gcd \left( 2^{2n_1} - 1, 2^{2n_2} - 1 \right) \)

Depending on the value of \( n_1 \) and \( n_2 \), the value of \( d \) differs, and since \( d \) differs, the calculation of \( X \) from the equation (2.15) also differs. If \( (d^{-1}) \mod 2^{n_2} - 1 / d \) exists, then the calculation of \( X \) varies and the equation (2.23) becomes of the form (2.15).

Hence, the weighted number is retrieved back from its residues using CRT III.

**2.3.2.2. Decoding Three Conjugate Pair Moduli:**

The sets with three pairs of conjugate moduli achieve large dynamic ranges and become more balanced. According to [2], the largest possible dynamic range and the most balanced RNS arithmetic is achieved by the following moduli set,
\[ S = \{ m_1^*, m_1, m_2^*, m_2, m_3^*, m_3 \} = \{ 2^{n+2}+1, 2^{n+2}-1, 2^{n+1}+1, 2^{n+1}-1, 2^n+1, 2^n-1 \} \] (2.24)

where \( n = \text{odd} \).

The dynamic range for the above set is given by [2] as,

\[ M = [(2^{2n} - 1) (2^{2n+2} - 1) (2^{2n+4} - 1) / 3^2] \] (2.25)

The decoding technique for this is same as the technique used in the previous section for decoding two pairs of conjugate moduli [2], but the difference is that decoding three pairs is done in three stages. In the first stage, \( Z_1 \) is calculated using \( m_1^* \) and \( m_1 \), \( Z_2 \) is calculated using \( m_2^* \) and \( m_2 \), and \( Z_3 \) is calculated using \( m_3^* \) and \( m_3 \). In the second stage, \( Z_4 \) is calculated using equation (2.14) replacing \( Z_1 \), \( Z_2 \) in the place of \( X_1 \) and \( X_2 \) respectively. In the third stage, \( X \) is calculated using (2.14), replacing \( Z_4 \) and \( Z_3 \) in the place of \( X_1 \) and \( X_2 \) respectively and replacing \( M_4 \) and \( M_3 \) in place of \( M_1 \) and \( M_2 \), where \( M_4 = \text{lcm}(M_1, M_2) \) and \( M_3 = \text{lcm}\{m_3^*, m_3\} \). Hence the weighted number \( X \) is retrieved back from its residues.

2.3.3. Traditional Hardware Implementation for CRT III:

In the traditional method, CRT III is implemented using subtractors, multipliers, dividers, modulus operators and inverse modulus operators. Multipliers and dividers need several full and half adders, which increases the delay and cost involvement in the operations. For 4–moduli set, hardware requirements are as follows:

1) One Subtractor
2) Four multipliers
3) Three dividers
4) One Modulus operator
5) One Inverse Modulus operator

Figure 2.5 describes the hardware assembly for implementing CRT III using 4-moduli set.

Fig.2.5. Traditional hardware implementation of CRT III
3. OPTIMIZING NEW CHINESE REMAINDER THEOREMS

3.1. Shortfalls of New Chinese Remainder Theorems:

New Chinese Remainder theorems overcome the shortfalls of traditional CRT and MRC techniques by improving parallelization. However, New CRTs are hardware intensive as the equations involve many inverse modulus operators, modulus operators, multipliers and dividers. Dividers and inverse modulus operators in turn needs many half and full adders and subtractors. This can be explained by the following illustrations.

Illustration 1:

For instance, consider the generalized application of CRT II using a moduli set from [4],

\[ M = \{2^{n-1}+1, 2^n-1, 2^n, 2^n+1\} \] where \( n = 2k +1, k = 1, 2, 3, 4 \ldots \)

The proof for the set \( \{2^{n-1}+1, 2^n-1, 2^n, 2^n+1\} \) to be pair wise relatively prime is dealt in the appendix B (Section B.1).

Therefore \( n = 3, 5, 7, 9 \ldots \)

Let \( X = \{x_1, x_2, x_3, x_4\} \)

\[ k_1 = (m_2)^{-1} \mod m_1 \]

\[ = (2^n-1)^{-1} \mod (2^n+1) \]

Let us calculate the values of \( k_1 \) for the different values of \( n \)

For \( n = 3, k_1 = (7)^{-1} \mod 5 = 3 \]

\[ = 011 \text{ (in binary)} \]
\[ k_1 = 2^{l+1} \]
\[ = 2^{n-2} + 1 \]

For \( n = 5 \), \( k_1 = (31)^{\frac{1}{l}} \mod (17) = 11 \)
\[ = 1011 \text{ (in binary)} \]
\[ = 2^3 + 2^{l+1} \]
\[ = 2^{n-2} + 2^{n-4} + 1 \]

For \( n = 7 \), \( k_1 = (127)^{\frac{1}{l}} \mod (65) = 43 \)
\[ = 101011 \text{ (in binary)} \]
\[ = 2^5 + 2^3 + 2^{l+1} \]
\[ = 2^{n-2} + 2^{n-4} + 2^{n-6} + 1 \]

For \( n = 9 \), \( k_1 = (511)^{\frac{1}{l}} \mod (257) = 171 \)
\[ = 10101011 \text{ (in binary)} \]
\[ = 2^7 + 2^5 + 2^3 + 2^{l+1} \]
\[ = 2^{n-2} + 2^{n-4} + 2^{n-6} + 2^{n-8} + 1 \]

Thus the above values of \( k_1 \) can be generalized as follows,

\[ k_1 = \sum_{i=1}^{n-2} 2^i + 1 \text{ where } l \text{ is odd.} \]

Let us calculate the values of \( k_2 \) for the different values of \( n \)
\[ k_2 = (m_4)^{-1}m_3 \]
\[ = (2^n + 1)^{-1} \mod 2^n \]

\[(m_4)^{-1}m_3 = (m_4 \mod m_3)^{-1} \mod m_3 \]
\[ = [(2^n + 1) \mod 2^n]^{-1} \mod 2^n \]

Since \((2^n + 1) \mod 2^n = 1\)
\[ k_2 = 1^{-1} \mod 2^n \]
\[ k_2 = 1 \]

For example,

For \(n = 3\), \[k_2 = (9)^{-1} \mod 8 = 1\]

For \(n = 5\), \[k_2 = (33)^{-1} \mod 32 = 1\]

For \(n = 7\), \[k_2 = (129)^{-1} \mod 128 = 1\]

For \(n = 9\), \[k_2 = (513)^{-1} \mod 512 = 1\]

Similarly, let us calculate the values for \(k_3\) by substituting different values of \(n\).

\[ k_3 = [m_4m_3]^{-1} \mod (m_1m_2) \]
\[ k_3 = [(2^n)(2^n + 1)]^{-1} \mod [(2^{n-1} + 1)(2^n - 1)] \]
\[ k_3 = [(2^n)(2^n + 1)]^{-1} \mod [(2^{n-1} + 1)(2^n - 1)] \]

Using the property, \(a^{-1} \mod b = x\), where \([a \times x] \mod b = 1\)
\[ [(2^n)(2^n + 1) \times x] \mod [(2^{n-1} + 1)(2^n - 1)] = 1 \]
Therefore \(x = 2^{n-2} + 2^{2(n-1)}\)
\[ k_3 = 2^{n-2} + 2^{2(n-1)} \]
For example,

For $n = 3$, $72^{-1} \mod 35 = 18$

$$= 1 0 0 1 0 \ (\text{in binary})$$

$$= 2^4 + 2^4$$

For $n = 5$, $1056^{-1} \mod 527 = 264$

$$= 1 0 0 0 0 1 0 0 0 \ (\text{in binary})$$

$$= 2^3 + 2^8$$

For $n = 7$, $16512^{-1} \mod 8255 = 4128$

$$= 1 0 0 0 0 0 0 1 0 0 0 0 \ (\text{in binary})$$

$$= 2^5 + 2^{12}$$

For $n = 9$, $262656^{-1} \mod 131327 = 65664$

$$= 2^7 + 2^{16}$$

Hence $N_1$ and $N_2$ are calculated as follows,

$$N_1 = x_2 + \left[ \sum_{i=1}^{n-2} (2^i + 1)(x_i - x_2) \mod m_1 \right] * m_2 \ (3.1)$$

$$N_2 = x_4 + \left[ (x_3 - x_4) \mod m_3 \right] * m_4 \ (3.2)$$

Hence $X$ can be calculated as,

$$X = N_2 + \left[ (2^{n-2} + 2^{2(n-1)}) (N_1 - N_2) \mod m_1 m_2 \right] \ast m_3 m_4 \ (3.3)$$
Illustration 2:

Consider a generalized application of CRT III using a non co-prime moduli set from [4],

\[ M = \{2^{n-2} + 1, 2^{n-2} - 1, 2^n + 1, 2^n - 1\} \] where \( n = 2k + 1, k = 3, 4, 5 \).

Therefore \( n = 7, 9, 11, 13, 15 \ldots \)

\[ X = \{x_1^*, x_1, x_2^*, x_2\} \]

\[ Z_1 = x_1^* + m_1^* \left[ (m_1^*)^{-1} (x_1 - x_1^*) \right] \mod m_1 \]

\[ Z_1 = x_1^* + 2^{n-2} + 1 \left[ (2^{n-2} + 1)^{-1} (x_1 - x_1^*) \right] \mod (2^{n-2} - 1) \quad (3.4) \]

Let us calculate the value of \( (2^{n-2} + 1)^{-1} \mod (2^{n-2} - 1) \).

\[ (2^{n-2} + 1)^{-1} \mod (2^{n-2} - 1) = (2^{n-2} + 1) \mod (2^{n-2} - 1) \mod (2^{n-2} - 1) \]

\[ = 2^1 \mod (2^{n-2} - 1) \]

Using the property, \( a^{-1} \mod b = x, \) where \( [a * x] \mod b = 1 \)

\[ [2 * x] \mod (2^{n-2} - 1) = 1 \]

\[ x = 2^{n-3} \]

Therefore, \( (2^{n-2} + 1)^{-1} \mod (2^{n-2} - 1) = 2^{n-3} \)

For example,

For \( n = 7 \), \( (33)^{-1} \mod 31 = 16 = 2^4 \)

For \( n = 9 \), \( (129)^{-1} \mod 127 = 64 = 2^6 \)

For \( n = 11 \), \( (513)^{-1} \mod 511 = 256 = 2^8 \)

For \( n = 13 \), \( (2049)^{-1} \mod 2047 = 1024 = 2^{10} \)

For \( n = 15 \), \( (8193)^{-1} \mod 8191 = 4096 = 2^{12} \)
\[ Z_2 = x_2^* + (2^n + 1) [(2^n + 1)^-l (x_2 - x_2^*)] \mod (2^n - 1) \] (3.5)

Let us calculate the value of \([(2^n + 1)^-l \mod (2^n - 1)]\).

\[
(2^n + 1)^-1 \mod (2^n - 1) = [(2^n + 1) \mod (2^n - 1)]^-1 \mod (2^n - 1)
\]

\[
= 2^{-1} \mod (2^n - 1)
\]

Using the property, \(a^{-1} \mod b = x\), where \([a \times x] \mod b = 1\)

\[
[2 \times x] \mod (2^n - 1) = 1
\]

\[
x = 2^{n-1}
\]

Therefore, \([(2^n + 1)^{-1} \mod (2^n - 1)] = 2^{n-1}\)

For example,

For \(n = 7\), \((129)^{-1} \mod 127 = 64 = 2^6\)

For \(n = 9\), \((513)^{-1} \mod 511 = 256 = 2^8\)

For \(n = 11\), \((2049)^{-1} \mod 2047 = 1024 = 2^{10}\)

For \(n = 13\), \((8193)^{-1} \mod 8191 = 4096 = 2^{12}\)

For \(n = 15\), \((32769)^{-1} \mod 32767 = 16384 = 2^{14}\)

\[ M_1 = \text{lcm} (m_1^*, m_1) = \text{lcm} (2^{n-2} + 1, 2^{n-2} - 1) \]

\[ = 2^{2(n-2)} - 1 \]

\[ M_2 = \text{lcm} (m_2^*, m_2) = \text{lcm} (2^n + 1, 2^n - 1) \]

\[ = 2^{2n} - 1 \]

\[ d = (2^{2(n-2)} - 1, 2^{2n} - 1) = 3 \]
\[
X = Z_1 + (2^{2(n-2)} - 1)[(\frac{2^{2(n-2)} - 1}{3})^{-1}(Z_2 - Z_3)] \mod (\frac{2^{2n} - 1}{3})
\] (3.6)

The value of \( \left( \frac{2^{2(n-2)} - 1}{3} \right)^{-1} \mod \left( \frac{2^{2n} - 1}{3} \right) \) can be calculated for different values of \( n \) as follows,

For \( n = 7 \), \( (341)^{-1} \mod 5461 = 1089 \)

\[
= 1 0 0 0 1 0 0 0 0 0 1 \text{ (in binary)}
\]

\[
= 2^{10} + 2^{6} + 1
\]

\[
= 2^{n+3} + 2^{n-1} + 1
\]

For \( n = 9 \), \( (5461)^{-1} \mod 87381 = 17473 \)

\[
= 1 0 0 0 1 0 0 0 1 0 0 0 0 0 1 \text{ (in binary)}
\]

\[
= 2^{14} + 2^{10} + 2^{6} + 1
\]

\[
= 2^{n+5} + 2^{n+1} + 2^{n-3} + 1
\]

For \( n = 11 \), \( (87381)^{-1} \mod 1398101 = 279617 \)

\[
= 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 1 \text{ (in binary)}
\]

\[
= 2^{18} + 2^{14} + 2^{10} + 2^{6} + 1
\]

\[
= 2^{n+7} + 2^{n+3} + 2^{n-1} + 2^{n-5} + 1
\]

For \( n = 13 \), \( (87381)^{-1} \mod 1398101 = 4473921 \)

\[
= 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 1 \text{ (in binary)}
\]

\[
= 2^{22} + 2^{18} + 2^{14} + 2^{10} + 2^{6} + 1
\]

28
\[= 2^{n+9} + 2^{n+5} + 2^{n+1} + 2^{n-3} + 2^{n-7} + 1\]

This can be generalized as follows,

\[
\left(\frac{2^{2(n-2)} - 1}{3}\right)^{-1} \mod \left(\frac{2^{2n} - 1}{3}\right) = \sum_{k=6-n}^{n-4} 2^{n+k} + 1
\]  

(3.7)

which can be used in (3.6).

Therefore, (3.6) becomes as follows,

\[
X = Z_1 + (2^{2(n-2)} - 1)[\sum_{k=6-n}^{n-4} 2^{n+k} + 1(\frac{Z_2 - Z_1}{3})] \mod \left(\frac{2^{2n} - 1}{3}\right)
\]  

(3.8)

Now let us observe the hardware implementation of the above two illustrations for calculating \(X\) from its residues.

First, let us implement CRT II to calculate \(X\) using the equations (3.1), (3.2) and (3.3) from illustration 1. In equation (3.1), to calculate \(N_1\), \((n-2)\) adders are required for evaluating the summation term in the equation, where \(n = 3, 5, 7, 9 \ldots\) Apart from this, one full adder, one subtractor, one multiplier and one modulus operator are also needed. In equation (3.2), to calculate \(N_2\), one full adder, one subtractor, one multiplier and one modulus operator are required. Similarly, to calculate \(X\) from equation (3.3), two adders, one subtractor, two multipliers and a modulus operator are needed.

Now, let us implement CRT III to calculate \(X\) using the equations (3.4), (3.5) and (3.6). To calculate \(Z_1\), two subtractors, three adders, one modulus operator and one multiplier are needed. Similarly for \(Z_2\), three adders, two subtractors, one modulus operator and one multiplier are required. To calculate \(X\) from equation (3.6), we need \((2n-9)\) adders for implementing the
summation term in the equation, where \( n = 7, 9, 11, 13, 15 \ldots \) Apart from this, we also need three subtractors, two adders, two dividers, one multiplier and one modulus operator.

These two illustrations clearly show that the New Chinese Remainder Theorems are hardware intensive to implement them practically. So, these theorems have to be optimized for the pragmatic implementations in the field of Digital Signal Processing intensive computations such as digital filtering, convolutions, correlations, discrete Fourier transforms and fast Fourier transform computations and direct digital frequency synthesis.

3.2. Basic Lemmas Used for the Optimization:

The main drawback in using New Chinese Remainder theorems is hardware implementation, because the inverse modulo operation requires large number of full and half adders. So, it was observed that, if the inverse modulo operations are eliminated from those theorems, then the new CRTs are practically feasible to implement. In order to eliminate inverse modulo operations, optimized moduli sets have to be chosen based on the following lemmas.

Theorem: Let the co-prime moduli set be \( M = \{m_1, m_2, m_3, m_4\} \). For co-prime numbers, if \((m_1 \mod m_2) = 1\), then \((m_1)^{-1} \mod m_2\) is also equal to 1.

Proof: By the property of inverse modulo,

\[
(m_1)^{-1} \mod m_2 = [(m_1 \mod m_2)^{-1}] \mod m_2
\]

Substituting, \(m_1 \mod m_2 = 1\) in the above,

\[
(m_1)^{-1} \mod m_2 = 1 \mod m_2
\]

\[
= 1
\]

This theorem gives rise to 3 lemmas. They are as follows,

Lemma 1: If \(m_3 \mod m_4 = 1\), then \((m_3)^{-1} \mod m_4 = 1\)
Lemma 2: If \((m_1 \cdot m_2) \mod (m_3 \cdot m_4) = 1\), then \((m_1 \cdot m_2)^{-1} \mod (m_3 \cdot m_4) = 1\)

Lemma 3: If \((m_1 \cdot m_2 \cdot m_3) \mod m_4 = 1\) then, and \((m_1 \cdot m_2 \cdot m_3)^{-1} \mod m_4 = 1\) respectively.

3.3. Optimizing the Base Theorem for New Chinese Remainder Theorem I:

Consider the theorem which is a base for constructing the new Chinese Remainder Theorem I. For a four moduli set, the equation is

\[
X = X_1 + k_1 m_1 (X_2 - X_1) + k_2 m_1 m_2 (X_3 - X_2) + k_3 m_1 m_2 m_3 (X_4 - X_3) \mod (m_1 m_2 m_3 m_4) \tag{3.9}
\]

where, \(k_1 = (m_1)^{-1} \mod (m_2 m_3 m_4)\)

\[k_2 = (m_1 m_2)^{-1} \mod (m_3 m_4)\]

\[k_3 = (m_1 m_2 m_3)^{-1} \mod m_4\]

To implement the above equation, the requirement is 3 inverse modulo operations which may have huge summation terms, 9 multipliers, 3 adders, 3 subtractors and 1 modulo operator. Hence, it is necessary to optimize this theorem so that it can be implemented in hardware. Let us optimize the theorem using 4-moduli set.

Consider a moduli set \(M = \{2^{n+2}+3, 2^{n+1}+1, 2^n+1, 2\}\) where \(n = 1, 2, 3\ldots\). Plugging the values for \(n\), we get sample moduli sets as follows:

For \(n = 1\): \(M = \{11, 5, 3, 2\}\)

For \(n = 2\): \(M = \{19, 9, 5, 2\}\)

For \(n = 3\): \(M = \{35, 17, 9, 2\}\) and so on.

The proof for the set \(M = \{2^{n+2}+3, 2^{n+1}+1, 2^n+1, 2\}\) where \(n = 1, 2, 3\ldots\), to consist of pair wise relatively prime moduli is dealt in Appendix B(Section B.2).
Consider, $k_1 = (m_1)^{l_1} \mod (m_2m_3m_4)$

$= (2^{n+2}+3)^{-1} \mod [(2^{n+1}+1) \ast (2^{n}+1) \ast (2)]$

$= [(2^{n+2}+3) \mod (2^{n+1}+1) \ast (2^{n}+1) \ast (2)]^{-1} \mod [(2^{n+1}+1) \ast (2^{n}+1) \ast (2)]$

Using the property, $a^{-1} \mod b = x$, where $[a \ast x] \mod b = 1$

$[((2^{n+2}+3) \ast x) \mod [(2^{n+1}+1) \ast (2^{n}+1) \ast (2)] = 1$

$x = (2^{n+2}+3)$, which is same as $m_1$

The value of $(m_1)^{-l_1} \mod (m_2m_3m_4) = m_1$ and this can be verified by plugging different values for $n$.

Consider $k_2 = (m_1m_2)^{-l_1} \mod (m_3m_4)$

$= [(2^{n+2}+3) \ast (2^{n+1}+1)]^{-1} \mod [(2^{n}+1) \ast (2)]$

$= [(2^{n+2}+3) \ast (2^{n+1}+1)] \mod [(2^{n}+1) \ast (2)]^{-1} \mod [(2^{n}+1) \ast (2)]$

$= 1^{-1} \mod [(2^{n}+1) \ast (2)]$

$= 1$

This can be verified by plugging the different values for $n$.

Consider $k_3 = (m_1m_2m_3)^{-l_1} \mod m_4$

$= [(2^{n+2}+3) \ast (2^{n+1}+1) \ast (2^{n}+1)]^{-1} \mod 2$

$= [((2^{n+2}+3) \ast (2^{n+1}+1) \ast (2^{n}+1)) \mod 2]^{-1} \mod 2$

$= 1^{-1} \mod 2$

$= 1$

This can be verified by plugging the different values for $n$.

Thus this moduli set reduces the equation (3.9) to,

$$X = [X_1 + m_1^2(X_2 - X_1) + m_1m_2(X_3 - X_2) + m_1m_2m_3(X_4 - X_3)] \mod (m_1m_2m_3m_4) \quad (3.10)$$

The above does not require any inverse modulo operators, which means it does not require any adders to implement huge summation terms, and the number of multipliers and adders also gets reduced. The hardware optimization is separately dealt in the next chapter.
Illustration 3:

Consider a weighted number \( X = 75 \) and moduli set \( M = \{11, 5, 3, 2\} \).

Residue numbers for \( X \) are, \( X = \{X_1, X_2, X_3, X_4\} = \{9, 0, 0, 1\} \)

Using equation (3.10), we get,

\[
X = [9 + 121(-9) + 55(0) + 165(1)] \mod 330
\]

\[
= [-915] \mod 330
\]

\[
= 75
\]

3.4. Optimizing New Chinese Remainder Theorem II:

Earlier, illustration 1 in this chapter proved that new CRT II is hardware intensive and it requires eliminating some of the terms that it has, so that it is feasible to implement in real time.

Let us focus our attention to a four moduli set to optimize the theorem.

Consider a 4- moduli set \( M = \{2, 2^n+1, 2^{n+1}+1, 2^{n+2}+3\} \) where \( n = 1, 2, 3 \ldots \). Sample moduli sets are as follows,

\( n=1: M = \{2, 3, 5, 11\} \), \( n=2: M= \{2, 5, 9, 19\} \), \( n = 3: M= \{2, 9, 17, 35\} \) and so on.

The proof for the set \( M= \{2, 2^n+1, 2^{n+1}+1, 2^{n+2}+3\} \) where \( n = 1, 2, 3 \ldots \) to consist of pair wise relatively prime moduli is dealt in Appendix B (Section B.2).

Consider \( k_1 = (m_2)^{-l} \mod m_1 \)

\[
= (2^n+1)^l \mod 2
\]

\[
= [(2^n+1) \mod 2]^l \mod 2
\]
\[= 1^{-1} \mod 2\]
\[= 1\]

This can be verified by plugging different values for \(n\) in \((2^n + 1)^{-1} \mod 2\)

Consider \(k_2 = (m_4)^{-1} \mod m_3\)
\[= (2^{n+2} + 3)^{-1} \mod 2^{n+1} + 1\]
\[= [(2^{n+2} + 3) \mod 2^{n+1} + 1]^{-1} \mod 2^{n+1} + 1\]
\[= 1^{-1} \mod 2^{n+1} + 1\]
\[= 1\]

This can be verified by plugging different values for \(n\) in \((2^{n+2} + 3)^{-1} \mod 2^{n+1} + 1\)

Consider, \(k_3 = (m_4m_3)^{-1} \mod (m_1m_2)\)
\[= [(2^{n+2} + 3) \ast (2^{n+1} + 1)]^{-1} \mod [(2^n + 1)^2]\]
\[= [(2^{n+2} + 3) \ast (2^{n+1} + 1)] \mod [(2^n + 1)^2]^{-1} \mod [(2^n + 1)^2]\]
\[= 1^{-1} \mod [(2^n + 1)^2]\]
\[= 1\]

This can be verified by plugging different values for \(n\) in \([(2^{n+2} + 3) \ast (2^{n+1} + 1)]^{-1} \mod [(2^n + 1)^2]\)

Hence the CRT II equations become as follows,

\[N_1 = x_2 + ([x_1 - x_2] \mod m_1) \ast m_2\]
\[N_2 = x_4 + ([x_3 - x_4] \mod m_3) \ast m_4\]
\[X = N_2 + [N_1 - N_2] \mod (m_1m_2) \ast (m_3m_4)\] \hspace{1cm} (3.11)

Equation (3.11) is a fully optimized CRT II equation, where it does not have any big size terms and inverse modulo terms. Hence, this equation is easy to implement in hardware for practical purposes and the hardware implementation is shown in the next section.
Illustration 4:

Consider a weighted number \( X = 200 \) and a moduli set \( M = \{2, 3, 5, 11\} \).

Residue numbers for \( X \) are, \( X = \{0, 2, 0, 2\} \)

\[
N_1 = x_2 + ([x_1 - x_2] \mod m_1) \times m_2
\]

\[
= 2 + (-2 \mod 2) \times 3
\]

\[
= 2
\]

\[
N_2 = x_4 + ([x_3 - x_4] \mod m_3) \times m_4
\]

\[
= 2 + (-2 \mod 5) \times 11
\]

\[
= 35
\]

\[
X = N_2 + [N_1 - N_2] \mod (m_1 m_2)(m_3 m_4)
\]

\[
= 35 + [-33 \mod 6] \times 55
\]

\[
= 200
\]

3.5. Optimizing New Chinese Remainder Theorem III:

Consider a moduli set,

\[
M = \{2^{n+2}+1, 2^{n+2}, 2^{n+1}+1, 2\}, \text{ where } n = 1, 2, 3 \ldots
\] (3.12)

When \( n = 1 \), \( M = \{9, 8, 5, 2\} \), \( n = 2 \), \( M = \{17, 16, 9, 2\} \) \( n = 3 \), \( M = \{33, 32, 17, 2\} \) and so on.

For the above moduli set in (3.12),

\[
M_l = (2^{n+2}+1)(2^{n+2})
\] (3.13)
\[ M_2 = (2^{n+1} + 1) \quad (2) \quad (3.14) \]

\[ d = \gcd(M_1, M_2) \]

\[ = 2 \]

\[
\left( \frac{M_1}{d} \right)^{-1} \mod \left( \frac{M_2}{d} \right) = \left[ \frac{(2^{n+2} + 1) \cdot (2^{n+1})}{2} \right]^{-1} \mod \left[ \frac{(2^{n+1} + 1)(2)}{2} \right] \quad (3.15)
\]

\[
= [(2^{n+2} + 1) \cdot (2^{n+1})]^{-1} \mod (2^{n+1} + 1)
\]

\[
= [(2^{n+2} + 1) \cdot (2^{n+1})] \mod (2^{n+1} + 1) - 1 \mod (2^{n+1} + 1)
\]

\[ = 1 \mod (2^{n+1} + 1) \]

\[ = 1 \]

For example,

For \( n = 1 \), \((36)^{-1} \mod 5 = 1.\)

For \( n = 2 \), \((136)^{-1} \mod 9 = 1.\)

Similarly, for \( n = 3 \), \((528)^{-1} \mod 17 = 1.\)

This series goes on and thus,

\[
\left( \frac{M_1}{d} \right)^{-1} \mod \left( \frac{M_2}{d} \right) = 1 \quad (3.16)
\]

Substituting the above in the original CRT III equation (2.14), we get

\[
X = X_1 + M_1 \left[ \left( \frac{X_2 - X_1}{d} \right) \mod \left( \frac{M_2}{d} \right) \right] \quad (3.17)
\]

Also,
\[
\frac{M_2}{d} = \left\lfloor \frac{(2^{n+1}+1)(2)}{2} \right\rfloor = (2^{n+1} + 1) = m_3 \quad (3.18)
\]

Substituting (3.18) in (3.17), we get

\[
X = X_1 + M_1 \left[ \left( \frac{X_2 - X_1}{d} \right) \right] \mod m_3 \quad (3.19)
\]

The division term in the above equation can be eliminated as follows,

\[
X = X_1 + M_1 \left[ (d)^{-1} (X_2 - X_1) \right] \mod m_3 \quad (3.20)
\]

\[
(d)^{-1} \mod m_3 = (2)^{-1} \mod (2^{n+1} + 1)
\]

\[
= 2^n + 1 \quad (3.21)
\]

Substituting (3.21) in (3.20), we get

\[
X = X_1 + M_1 \left( (2^n + 1) (X_2 - X_1) \right) \mod m_3 \quad (3.22)
\]

Equation (3.22) is a completely optimized new Chinese Remainder Theorem III, which has no big size summation terms and the most important is that it does not have any division terms and inverse modulo terms. To implement this equation in hardware is not a resource intensive and further the hardware optimization is done in the next chapter.

**Illustration 5:**

Let us consider a weighted number \( X = 75 \), and the moduli set be \( M = \{9, 8, 5, 2\} \).

\[ M_1 = \text{lcm}(m_1, m_2) = (9, 8) = 72 \]

\[ M_2 = \text{lcm}(m_3, m_4) = (5, 2) = 10 \]
\[ d = \text{gcd} (M_1, M_2) = 2 \]

\[ X_1 = X \mod M_1 \]
\[ = 75 \mod 72 = 3 \]

\[ X_2 = X \mod M_2 \]
\[ = 75 \mod 10 = 5 \]

Using the equation (3.22), we calculate the value of \( X \) as follows,

\[ X = X_1 + M_1 [(2^n + 1) (X_2 - X_1)] \mod m_3 \]
\[ = 3 + 72 [(3) (5-3)] \mod 5 \]
\[ = 3 + 72 [6] \mod 5 \]
\[ = 75 \]

3.5.1. Extending the Optimized Four Moduli Set to A Five Moduli Set:

Consider the following moduli set,

\[ M = \{2^{n+3}+2, 2^{n+2}+1, 2^{n+2}, 2^{n+1}+1, 2\}, \text{where } n = 1, 2, 3 \ldots \]  

\( M_1 = (2^{n+3}+2)*(2^{n+2}+1) \)

\( M_2 = (2^{n+2})*(2^{n+1}+1)*(2) \)

\[ d = \text{gcd} (M_1, M_2) = 2 \]

\[ \left(\frac{M_1}{d}\right)^{-1} \mod \left(\frac{M_2}{d}\right) = \left[\frac{(2^{n+3} + 2)(2^{n+2} + 1)}{2}\right]^{-1} \mod \left[\frac{(2^{n+2})(2^{n+1}+1)(2)}{2}\right] \]
\[ = [(2^{n+2}+1)^2 \mod (2^{n+2})*(2^{n+1}+1)]^{-1} \mod (2^{n+2})*(2^{n+1}+1) \]

\[ = 1 \mod (2^{n+2})*(2^{n+1}+1) \]

\[ \left( \frac{M_1}{d} \right)^{-1} \mod \left( \frac{M_2}{d} \right) = 1 \]

Thus the CRT III equation becomes as follows:

\[ X = X_1 + M_1 \left\lceil \left( \frac{X_2 - X_1}{d} \right) \right\rceil \mod \left( \frac{M_2}{d} \right) \]

Also,

\[ \frac{M_3}{d} = \left\lceil \frac{(2^{n+2})*(2^{n+1}+1)}{2} \right\rceil = m_3 \cdot m_4 \]

Here \((d^I) \mod (m_3 \cdot m_4)\) does not exist because both \(d\) and \((m_3 \cdot m_4)\) are even.

Hence the optimized equation is,

\[ X = X_1 + M_1 \left\lceil \left( \frac{X_2 - X_1}{d} \right) \right\rceil \mod (m_3 \cdot m_4) \quad (3.24) \]

**Illustration 6:**

Consider a weighted number \(X = 221\) and the moduli set \(M = \{18, 9, 8, 5, 2\}\)

\[ M_1 = \text{lcm} (m_1, m_2) = (18, 9) = 162 \]

\[ M_2 = \text{lcm} (m_3, m_4) = (8, 5, 2) = 80 \]

\[ d = \text{gcd} (M_1, M_2) = 2 \]

\[ X_1 = X \mod M_1 \]
Using the equation (3.24), we calculate the value of $X$ as follows:

$$X = X_0 + p \left( \frac{X_r - X_0}{m} \right) \mod m$$

where $\text{num} = 221 \mod 162 = 61$.

So, $X = 61 + 221 = 282 \mod 162 = 59$. 

$$X = 221 \mod M_2$$
4. OPTIMIZED HARDWARE IMPLEMENTATION OF NEW CHINESE REMAINDER THEOREMS

4.1. Introduction:

In the previous chapter, the new Chinese Remainder theorems were optimized by removing inverse modulo and division operations which reduced the overhead of using big size summation terms in the equations. But, those equations carried multiplication terms which requires additional full and half adders, and this is a barrier for hardware implementation. This chapter deals with the removal of those multiplication terms which means the usage of additional full and half adders can be eliminated. The optimization is carried out by replacing multipliers by left shift registers, which not only avoids the usage of adders but also avoids the time delay in carrying out the multiplication operations.

4.2. Optimized Hardware Implementation of the Base Theorem for New Chinese Remainder Theorem I:

Using the moduli set from section 3.3, \( M = \{2^{n+2}+3, 2^{n+1}+1, 2^n+1, 2\} \) where \( n = 1, 2, 3... \) and the optimized equation from the previous chapter, we get

\[
X = [X_1 + m_1^2(X_2 - X_1) + m_1m_2(X_3 - X_2) + m_1m_2m_3(X_4 - X_3)] \mod (m_1m_2m_3m_4)
\]

Substituting the above moduli set values in the equation, we get

\[
X = [X_1 + Q_1 + Q_2 + Q_3] \mod (m_1m_2m_3m_4) \quad (4.1)
\]

Where

\[
Q_1 = (2^{2n+4} + 2^{n+4} + 2^{n+3} + 2^3 + 1)(X_2 - X_1)
\]

\[
Q_2 = (2^{2n+3} + 2^{n+3} + 2^n + 2^1 + 1)(X_3 - X_2)
\]

\[
Q_3 = [(2^{3n+3} + 2^{2n+3} + 2^{n+1} + 2^n) + (2^{2n+3} + 2^{n+3} + 2^n + 2^1 + 1)] \ast (X_4 - X_3)
\]
The hardware implementation of equation (4.1) without using multipliers is shown in the figures Fig. 4.1.a, Fig. 4.1.b, Fig. 4.1.c and Fig. 4.1.d.

Fig. 4.1.a Hardware implementation of $Q_1$ by CRT I without using multipliers.
Fig. 4.1.b Hardware implementation of $Q_2$ by CRT I without using multipliers.

The total hardware requirements for the above implementations are

1) 3 subtractors
2) 1 modulus operators
3) 16 adders
4) 2 (n+4)-bit, 3 (n+3)-bit, 2 (2n+3)-bit, 2 (n+1)-bit, 2 1-bit, 1 (n+3)-bit, and 1 3-bit left shift registers.
4.3. Optimized Hardware Implementation Of New Chinese Remainder Theorem II:

Using the moduli set from the section 3.4, $M = \{2, 2^n+1, 2^{n+1}+1, 2^{n+2}+3\}$ where $n = 1, 2, 3, \ldots$, and considering the optimized CRT II equation from chapter 3,

$$N_1 = x_2 + ([x_1 - x_2] mod m_1) * m_2$$

$$= x_2 + ([x_1 - x_2] mod 2) * (2^n + 1)$$
\[ N_1 = x_2 + ([x_1 - x_2] \mod m_1) * 2^n + ([x_1 - x_2] \mod m_1) \]  
(4.2)

\[ N_2 = x_4 + ([x_3 - x_4] \mod m_3) * m_4 \]
\[ = x_4 + ([x_3 - x_4] \mod m_3) * (2^{n+2}+3) \]
\[ = x_4 + ([x_3 - x_4] \mod m_3) * (2^{n+2}+2^1+1) \]

\[ N_2 = x_4 + ([x_3 - x_4] \mod m_3) * (2^{n+2}) + ([x_3 - x_4] \mod m_3) * (2^1) + ([x_3 - x_4] \mod m_3) \]  
(4.3)

\[ X = N_2 + [N_1 - N_2] \mod (m_1m_2) * (m_3m_4) \]
\[ = N_2 + [N_1 - N_2] \mod (m_1m_2) * [(2^{n+1}+1), (2^{n+2}+3)] \]

\[ X = N_2 + [N_1 - N_2] \mod (m_1m_2) * [2^{2n+3} + 2^{n+3} + 2^{n+1} + 2^2] \]  
(4.4)

Equations (4.2), (4.3) and (4.4) are implemented in hardware without using multipliers as shown in the figures 4.2.a, 4.2.b and 4.3.

The total hardware requirements for the above implementations are

1) 3 subtractors
2) 3 modulus operators
3) 9 adders
4) 1 n-bit, 1 (n+1)-bit, 1 1-bit, 1 (2n+3)-bit, 1 (n+3)-bit, 1 (n+1)-bit and 1 2-bit left shift registers.
Fig.4.2.a Hardware implementation of $N_1$ by CRT II without using multipliers.

Fig.4.2.b Hardware implementation of $N_2$ by CRT II without using multipliers.
Fig. 4.3. Hardware implementation of X by CRT II without using multipliers

4.4. Optimized Hardware Implementation of New Chinese Remainder Theorem III:

Using the moduli set from the section 3.5, \( M = \{2^{n+2}+1, 2^{n+2}, 2^{n+1}+1, 2\} \), where \( n = 1, 2, \ldots \), and considering the optimized CRT III equation from the chapter 3,

\[
X = X_1 + M_1 [(2^n + 1) (X_2 - X_1)] \mod m_3
\]
where \( M_1 = (2^{n+2} + I) \cdot 2^{n+2} = (2^{2(n+2)} + 2^{n+2}) \)

\[
X = X_I + (2^{2(n+2)} + 2^{n+2}) \cdot [(2^n + I) (X_2 - X_1)] \mod m_3
\]

\[
X = X_I + (2^{2(n+2)}) \cdot [(2^n + I) (X_2 - X_1)] \mod m_3 + (2^{n+2}) \cdot [(2^n + I) (X_2 - X_1)] \mod m_3 \quad (4.5)
\]

The hardware implementation of the above equation without using multipliers is shown in the figure (4.4).

The total hardware requirements for the above implementation are as follows:

1) 1 subtractor
2) 1 modulus operator
3) 3 adders
4) 1 n-bit, 2(n+2)-bit left shift registers.

4.5. Comparison between Optimized and Non-Optimized CRT II Implementation:

Consider a moduli set \( C = \{r, r^{n-3} + I, r^{n-1} + I, r^n + I\} \) where \( r = 2 \) and \( n = 2k+1, k=2, 3, 4,... \) from [5] and the moduli \( M = \{2, 2^n+I, 2^{n+1}+I, 2^{n+2}+3\} \) where \( n = 1, 2, 3 \) from this paper for comparison, as the dynamic ranges of both sets is almost same. To implement CRT II using the moduli set \( C \) requires 3 subtractor units, 4 multiplier units, 3 adder units, 3 modulus operators and 3 inverse modulus operators. Similarly to implement CRT II using the optimized moduli set \( M \), 3 subtractor units, 3 modulus operators, 9 adder units and 1 n-bit, 1 (n+1)-bit, 1 1-bit, 1 (2n+3)-bit, 1 (n+3)-bit, 1 (n+1)-bit and 1 2-bit left shift registers. The shift registers do not offer any time delay, and hence the optimized implementation of CRT II is faster and utilizes less hardware.
4.6. Comparison between Optimized and Non-Optimized CRT III Implementation:

Consider a non co-prime moduli set $G = \{2^{n+2}+1, 2^{n+2}-1, 2^{n+1}+1, 2^{n+1}-1, 2^n+1, 2^n-1\}$ where $n = \text{odd}$, from [2] and the moduli set $N = \{2^{n+2}I, 2^{n+2}, 2^{n+1}I, 2I\}$, where $n = 1, 2, 3...$ for...
comparison. To implement CRT III using the set G requires 1 subtractor, 4 multipliers, 3 dividers, 1 modulus operator and 1 inverse modulus operator. But the optimized implementation of CRT III using set N requires only 1 subtractor, 1 modulus operator, 3 adders and 1 n-bit, 2(n+2)-bit left shift registers. Hence, this comparison shows the optimized implementation of CRT III using set N offers faster computations than the non-optimized implementation.
5. CONCLUSION AND FUTURE WORK

The main goal of this research is to simplify the New Chinese Remainder Theorems, for their practical implementations in the real world. In this research, new 4 and 5 moduli sets are proposed that eliminate huge summation terms, inverse modulo terms and division terms. Furthermore, the hardware optimization using shift registers removes the usage of multipliers which in turn needs many half and full adders. The comparisons with other modulus sets in earlier papers clearly prove that the proposed modulus sets reduce area and time delay in the implementations. Therefore, the introduced CRT optimizations facilitate the efficient and cost effective hardware implementations of the New CRTs for various applications.

The future work that can come out of this research is that the proposed modulus sets can be extended to 8, 12 and 16 moduli sets with both co-prime and non co-prime moduli, so that they will simplify the new CRTs. Further, there can be radical improvements in the hardware implementations as well, by using CSAs, CPAs, and more VLSI designs.
REFERENCES


[7] Mohan, P.V.A. and Premkumar, A.B., “RNS-to-Binary Converters for Two Four-Moduli Sets \{2^n-1,2^{n-1}, 2^n+1, 2^n+1-1\} and \{2^n -1,2^n, 2^n+1, 2^n+1+1\},” *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol.54, no.6, pp.1245-1254, June 2007.


APPENDIX A: INVERSE MODULUS OPERATION

This appendix deals with the description and calculation of inverse modulus values, which are used in most of the RNS to Binary conversions. This is an intensive operation because the hardware requirements to implement this are high and expensive. But, it is used in all traditional and new Chinese Remainder Theorems and Mixed Radix Conversions. There are many theorems to calculate multiplicative inverse modulus values, and some familiar ones are Euclidean algorithm, Euler Totient function, Euler and Fermat’s theorems. Now let us deal with the basics and calculation of inverse modulus functions.

The multiplicative inverse of a number \( a \mod b \) is denoted as \( a^{-1} \mod b \). The value of \( a^{-1} \mod b \) exists if and only if \( a \) and \( b \) are co-prime to each other, that is, greatest common divisor of \( a \) and \( b \) is 1.

A.1. Methods to Calculate Inverse Modulus Values:

There are many ways to calculate multiplicative inverse values, out of which some are trial and error methods and some converge closely to the trial and error value. The following theorem mentioned in the third chapter 3 (repeated here for convenience) is very useful in many cases.

*Theorem:*

For co-prime numbers \( m_i \), if \( (m_1 \mod m_2) = 1 \), then \( (m_1)^{-1} \mod m_2 = 1 \).

Refer to chapter 3 for the proof. The above theorem has the following 3 *lemmas* that refer to the moduli of a co-prime four-modulus set \( M = \{m_1, m_2, m_3, m_4\} \):

*Lemma 1:* If \( (m_3 \mod m_4) = 1 \), then \( (m_3)^{-1} \mod m_4 = 1 \).
Lemma 2: If \((m_1, m_2) \mod (m_3, m_4) = 1\), then \((m_1, m_2)^{-1} \mod (m_3, m_4) = 1\).

Lemma 3: If \((m_1, m_2, m_3) \mod (m_4) = 1\), then \((m_1, m_2, m_3)^{-1} \mod (m_4) = 1\).

This theorem eliminates the chances of calculating inverse modulo values between two numbers, if their modulus value is 1.

The following property is also useful in some cases, where the numbers are huge. It helps in breaking the big numbers to smaller numbers.

Property: \(a \mod b = [a \mod b]^{-1} \mod b\)

Example 1: Find the value of \(17^{-1} \mod 11\).

\[
17^{-1} \mod 11 = [17 \mod 11]^{-1} \mod 11
\]

\[
= 6^{-1} \mod 11
\]

\[
= 2
\]

Method 1:

Let \(x\) be the value of the \(a \mod b\). The value of \(x\) is such that,

\[
[a \times x] \mod b = 1 \tag{A.1}
\]

The smallest value of \(x\) is the inverse modulus value.

Example 2: Find the inverse modulus value of 15 mod 11.

Using the equation (1), \([15 \times x] \mod 11 = 1\)

\[
x = 3
\]

Example 3: Find the inverse modulus value of 13 mod 7.
Using the equation (A.1), \( [13 \times x] \mod 7 = 1 \)

\[ x = 6 \]

**Method 2:**

This method is based on the Extended Euclidean algorithm. Extended Euclidean algorithm is the extension of Euclidean algorithm, which is mainly used when two numbers \( a \) and \( b \) are co-prime to each other. It states that,

\[ ax + by = \gcd(a, b) \]  \hspace{1cm} (A.2)

where \( a, b \) are integers.

By the definition of inverse modulus, \( ax \) is congruent to \((1 \mod b)\)

Hence, \( b \) is the divisor of \( ax – 1 \), and it gives the following,

\[ ax – by = 1 \]  \hspace{1cm} (A.3)

In the above equation, \( a \) and \( b \) are the numbers whose inverse modulo value is to be found out, \( x \) is the inverse modulus value and \( y \) is an integer and it can be ignored.

**Example 4:** Find the value of \( 23^{-1} \mod 7 \).

Using the equation A.3, and approaching trial and error method, it is found out that \( x = 4 \) and \( y = 13 \). Ignoring the value of \( y \), the value of \( 23^{-1} \mod 7 \) is 4.

**Example 5:** Find the value of \( 14^{-1} \mod 3 \).

Using the equation A.3, and approaching trial and error method, it is found out that \( x = 2 \) and \( y = 9 \). Ignoring the value of \( y \), the value of \( 14^{-1} \mod 3 \) is 2.

**Euler’s Totient Function (Phi function):**

Consider a positive integer \( m \), and let \( \varphi(m) \) be the number of positive integers that are smaller than \( m \) and also co-prime to \( m \). \( \varphi(m) \) is called Euler’s totient function or phi function. Then,

- If \( P \) is a positive prime then \( \varphi(P) = P-1 \).
• If \( m = P^e \), where \( m > 0 \) and \( P \) is prime then \( \varphi (P^e) = P^{e-1} (P-1) \).

• If \( (a, b) = 1 \), then \( \varphi (a b) = \varphi (a) \varphi (b) \).

• If \( m \) is composite and \( m = P_1^{e_1} P_2^{e_2} \cdots P_L^{e_L} \)

where \( P_1, P_2 \ldots P_L \) are primes and \( e_1, e_2 \ldots e_L \) are integers then,

\[
\varphi (m) = \varphi (P_1^{e_1}) \varphi (P_2^{e_2}) \cdots \varphi (P_L^{e_L}) = P_1^{e_1-1} P_1 - 1 \cdot P_2^{e_2-1} P_2 - 1 \cdots P_L^{e_L-1} P_L - 1
\]

Euler’s Theorem:

If \( (a, m) = 1 \) (\( a \) and \( m \) are relatively prime), \( m > 0 \), then,

\[
[a^{\varphi (m)}] \mod m = 1
\]

Fermat’s Theorem:

This is a special case of Euler’s theorem. It states that if \( P \) is prime and \( (a, P) = 1 \), then

\[
[a^{P-1}] \mod P = 1
\]

Usefulness of Euler’s and Fermat’s Theorems:

Euler’s and Fermat’s theorems are useful since they provide closed solution for computing the multiplicative inverse. Recalling Euler’s theorem, \( (a, m) = 1 \), then \( [a^{\varphi (m)}] \mod m = 1 \).

Thus, \( a^{-1} \mod m = [a^{\varphi (m)-1}] \mod m \)

Fermat’s theorem states that if \( P \) is prime and \( (a, P) = 1 \), then \( [a^{P-1}] \mod P = 1 \), thus,

\[
a^{-1} \mod P = a^{P-2} \mod P
\]

Example 6: Find the value of \( 6^{-1} \mod 11 \) using the above theorems.

Here \( \gcd(6, 11) = 1 \) and thus \( 6^{-1} \mod 11 \) exists. The number \( P=11 \) is prime and therefore using Fermat’s theorem

\[
6^{-1} \mod 11 = 6^{(11-2)} \mod 11 = 6^9 \mod 11
\]
\[
= (6^2)^d \cdot 6 \mod 11
= (3^d \cdot 6) \mod 11
= (81 \cdot 6) \mod 11
= 2
\]

**Example 7:** Find the value of \(5^1 \mod 36\) using the above theorems.

Here \(\gcd(5, 36) = 1\) and thus \(5^1 \mod 36\) exists. The number 36 is composite and hence using Euler’s theorem,

\[
5^1 \mod 36 = (5^{\phi(36)-1}) \mod 36
\]

The value of \(\phi(m)\) where \(m = 36\) can be found as follows,

\[
m = 36 = 2^2 \cdot 3^2; (2 \text{ and } 3 \text{ are primes})
\]

Therefore \(\phi(36) = \phi(2^2 \cdot 3^2) = \phi(2^2) \cdot \phi(3^2)\)

\[
= 2^{2-1} \cdot (2-1) \cdot 3^{2-1} \cdot (3-1)
\]

\[
= 12
\]

Thus there are 12 numbers smaller than 36, which are relatively prime to 36. They are 1, 4, 7, 11, 13, 17, 19, 23, 25, 29, 31, and 35. Therefore,

\[
(5^{\phi(36)-1}) \mod 36 = (5^{12-1}) \mod 36
\]

\[
= ((5^3)^3 \cdot 5^2) \mod 36
\]

\[
= (125 \mod 36)^3 \cdot 5^2 \mod 36
\]

\[
= (17^3 \cdot 5^2) \mod 36
\]

\[
= (17^2 \cdot 17 \cdot 5^2) \mod 36
\]

\[
= (289 \cdot 17 \cdot 5^2) \mod 36
\]

\[
= (17 \cdot 5 \cdot 5) \mod 36 = 29
\]
A.2. Code in C to Calculate Multiplicative Inverse:

The following code in C is used to simulate the series of multiplicative inverse values, and is based on Extended Euclidean algorithm.

Code:
/* To calculate multiplicative inverse series */
#include<stdio.h>
main()
{
    system("clear");
    int Inverse(int a[], int n[])
    {
        int k, i[5], v[5], d[5], t[5], x[5];
        for(k=0;k<5;k++)
        {
            i[k] = n[k], v[k] = 0, d[k] = 1;
            while (a[k]>0)
            {
                t[k] = i[k]/a[k], x[k] = a[k];
                a[k] = i[k] % x[k];
                i[k] = x[k];
                x[k] = d[k];
                d[k] = v[k] - t[k]*x[k];
                v[k] = x[k];
            }
            v[k] %= n[k];
            if (v[k]<0) v[k] = (v[k]+n[k])%n[k];
            printf("%d \n",v[k]);
        }
        printf("n\n");
    }
    int x[10], y[10];
printf("Enter the value of x:\n");
for(j=0;j<5;j++)
{
    scanf("%d",&x[j]);
}
printf("Enter the value of y:\n");
for(j=0;j<5;j++)
{
    scanf("%d",&y[j]);
}
printf("The inverse modulo values are:\n");
Inverse(x,y);
}

Result:

Enter the value of x:
141
5461
87381
1598101
2129
Enter the value of y:
141
5461
87381
1598101
2129
The inverse modulo values are:
1009
17473
279637
4473021
64
APPENDIX B: PROOFS FOR THE SETS TO BE PAIR WISE RELATIVELY PRIME

B.1. The Set \{2^{n-1}+1, 2^n-1, 2^n, 2^{n+1}\}, \text{where } n= 2k + 1, k = 1, 2, 3\ldots \text{ is Pair Wise Relatively Prime:}

Proof:

It is known that \(2^{n-1}, 2^n, 2^{n+1}\) are relatively prime to each other.

Since \(n\) is odd, \(2^n + 1 \in S_1\) of [7]. And, since \(n-1\) is even, \(2^{n-1}+1 \not\in S_1\) of [7]. Therefore, according to theorem 1 of [7], \((2^{n-1}+1, 2^n+1) = 1\).

Since \(n\) is odd, using theorem 3 of [7], we get \((2^{n-1}+1, 2^n-1) = 1\).

Therefore the set \{\(2^{n-1}+1, 2^n-1, 2^n, 2^{n+1}\)\} is pair wise co-prime.

B.2 The Set \{\(2^{n+2}+3, 2^{n+1}+1, 2^n+1, 2\), \text{where } n =1, 2, 3\ldots\text{is Pair Wise Relatively Prime:}

Proof:

If \(n = \text{odd}, 2^n + 1 \in S_1\), and \(n+1\) is even, \(2^{n+1}+1 \not\in S_1\) [7].

If \(n = \text{even}, 2^n+1 \not\in S_1\) and \(n+1\) is odd, \(2^{n+1}+1 \in S_1\) [7].

Therefore according to theorem 1 of [7], \(2^n+1\) and \(2^{n+1}+1\) are relatively prime.

Let \(a/b\) mean “\(a\) divides \(b\).”

It is true that, if \(a/b\) and \(a/c\), then \(a/(k_1b \pm k_2c)\). Let \(d\) be a common divisor of \(2^{n+2}+3\) and \(2^{n+1}+1\). Then \(d/2^{n+2}+3\) and \(d/2^{n+1}+1\) implies \(d/ [ (2^{n+2}+3) – 2(2^{n+1}+1)]\) and in turn implies \(d/1\). Therefore \(d=1\).

Thus the only common divisor of \(2^{n+2}+3\) and \(2^{n+1}+1\) is \(1\) and therefore \(2^{n+2}+3\) and \(2^{n+1}+1\) are relatively prime.
At last, let $d'$ be a common divisor of $2^{n+2}+3$ and $2^n+1$. Then $d'/2^{n+2}+3$ and $d'/2^n+1$ implies $d'/ [2^2(2^n+1)−(2^{n+2}+3)]$ and in turn implies $d'/1$. Therefore $d' = 1$.

Thus the only common divisor of $2^{n+2}+3$ and $2^n+1$ is 1 and therefore $2^{n+2}+3$ and $2^n+1$ are relatively prime.

Therefore the set $\{2^{n+2}+3, 2^n+1, 2^n+1, 2\}$ is a co-prime set.
VITA

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