THE EXTENDED PICTURE GROUP, WITH APPLICATIONS TO LINE ARRANGEMENT COMPLEMENTS

A Dissertation

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Abstract

We obtain the picture group as the quotient with a torsion subgroup, of an extended picture group, which is isomorphic to the kernel of a precrossed module homomorphism. In addition to expanding the notion of a picture group, the new formulation gives a natural way to construct homomorphisms between picture groups by describing deformations of one-vertex subpictures. The extended picture group thus provides a convenient way to describe generators for the second homotopy group of line arrangement complements as well as homomorphisms between these groups. In particular, we show that the homomorphisms relate to a lattice structure corresponding roughly to the condition of being more nearly in general position. Examples include generators for Falk's X_2 arrangement and for a generic section of braid arrangement A_3. Finally, we demonstrate that the $\mathbb{C}^3$ arrangement $C(5)$ is a $K(\pi, 1)$ space.

Chapter 1

Introduction

Igusa [12] first defined the picture group and established its relationship with the second homotopy group of a topological space. Fenn [10] and Loday [13] expanded this connection to include syzygies from the group ring homology and provided the necessary isomorphisms in terms of pictures. Topology certainly motivates the present study, but the results apply in a broader sense, starting with just a finite group presentation that meets certain conditions of minimality. We will show that the analysis first offered by Igusa can be enriched by way of defining an *extended picture group*, and that the familiar picture group emerges as the quotient with a torsion subgroup.

Topology also motivates interest in homomorphisms between picture groups as representing homomorphisms between second homotopy groups. In the context of the extended picture group, the well-definedness of the map between picture groups depends only upon a transformation of single-vertex subpictures, while the fact that the maps are homomorphisms depends only upon the bridge move, a process that does not involve vertices. Consequently, we can establish criteria for existence of homomorphisms between picture groups without considering topology.

In this paper, we present the extended picture group formulation and establish its relationship to the traditionally defined picture group. We then extend the crossed module analysis to include the extended picture group. The section on picture group homomorphisms contains the paper's main results, and we conclude with comments on the various picture groups as invariants.

Many studies have addressed the question of whether the complement of a hyperplane arrangement in $\mathbb{C}^l$ might be a $K(\pi, 1)$ space. The answer is affirmative in the case of the reflection arrangments [5] and negative for general position arrangements with sufficiently many hyperplanes [11]; [16] extended this last result to hypersolvable arrangements, and [6] specialized to line arrangement and $\mathbb{C}^3$ central arrangements. Other studies [7], [8], [9], examined the more general problem. While much is known about the $K(\pi,1)$ question, relatively little attention has been paid to producing specific generators for second homotopy groups. As we will show for the case of line arrangement complements (and hence $\mathbb{C}^3$ central arrangement complements), the technique of pictures can be applied to obtain an explicit description of these generators in the form of picture group generators. (In the case of line arrangement complements, the picture group and the second homotopy group of the space are isomorphic.) Furthermore, we show how picture group homomorphisms can be used to formalize the notion of a transformation of a line arrangement toward the condition of being in general position.

Chapter 2

Pictures and Picture Groups

Pictures were first introduced in 1979 by Rourke [19] to examine presentations of the trivial group, and in the same year by Igusa [12], who gave their collection a group structure. For thorough descriptions of pictures as graphs embedded in disks, see either Fenn [10] or Bogley and Pride [1].

Let G be a group with finite presentation $\langle x_1, \ldots, x_n | \, r_1, \ldots, r_m \rangle$. Let $P(G)$ denote the set of pictures, which are oriented plane graphs in which edges are labeled with generators of the group (though group words can be used to simplify complicated drawings) and each vertex corresponds to one of the group relations, or to its inverse. More precisely, near each vertex one designates a face to mark the starting point and proceeds counterclockwise around the vertex recording edge labels as they are encountered. Figure 2.1 illustrates such a vertex with an asterisk in the diagram to the right, corresponding to the relation $[x, y]$. An edge oriented toward the vertex corresponds to a $+1$ exponent in the word of the relation, with a -1 exponent corresponding to an orientation away from the vertex. If a vertex v is associated to some relation r, then we will refer to a vertex associated to the inverse of r as $-v$. We choose a point in the unbounded face of the graph as the *base point* for the picture. A *boundary word* of a subpicture is a sequence of edge labels encountered on traveling counterclockwise completely around the boundary of a subpicture. An inductive argument based upon the number of vertices in the subpicture establishes that the boundary word will always be trivial in the group G.

Here, we will consider only pictures in which all edges terminate in vertices or form vertex-free loops; so our pictures will be oriented graphs embedded in $\mathbb{R}^2$ with its usual counterclockwise orientation. If a simple closed curve embedded in a picture contains no vertices and encounters edges only transversely,

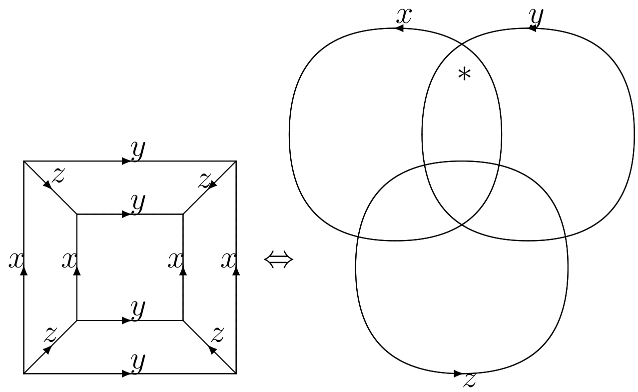

Figure 2.1: A homotopical syzygy (left) having a tricycle as its dual picture (right)

we call the interior of the curve a *subpicture*. Thus, our concept of subpicture agrees with the notion of a nonspherical picture in [1] or [10].

Traditionally, we obtain the group P_G from $P(G)$ as follows:

1. The binary operation is disjoint union.

2. Two pictures are equivalent if there is an isotopy in the plane taking one picture to the other.

3. Two pictures are equivalent if they differ by a finite number of *bridge moves*. A bridge move is a rearrangement of identically labeled edges having opposite orientation adjacent to a common face, as illustrated in figure 2.2.

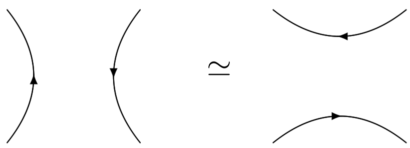

Figure 2.2: Bridge Move

4. Closed arcs having no vertices and having empty interior are trivial.

5. If a subpicture contains exactly two vertices that are the mirror image of each other, but with arrows reversed, and bridge moves can be found such that the vertices share a designated face, then we can replace the subpicture, as in figure 2.3, with one having the same boundary word but containing no vertices. The process can be reversed whenever the resulting vertices correspond to relations in the group presentation.

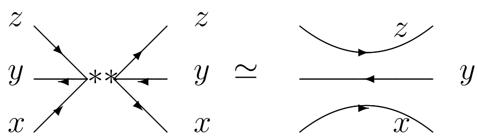

Figure 2.3: Vertex Creation/Deletion

Associativity and commutativity are clear. The identity element is the empty picture, and one can show by induction on the number of vertices that the inverse of a given picture can always be obtained by constructing its mirror image and then reversing the direction of all arrows. As an immediate consequence, a picture consisting of exactly two vertices that share the designated face must be trivial, since it can be replaced via the fourth condition with a set of concentric circles, trivial by repeated application of the third condition. The triviality of these two-vertex pictures, called *floating pairs*, is precisely what one needs to rid the picture group of constructions that relate to maps of positive-genus surfaces into a topological space.

In defining the extended picture group E_G, we still define the group operation to be disjoint union and keep isotopy and bridge moves as described above, but we replace rules (4) and (5) with a single rule that defines the inverse directly:

(4′) To each picture P we associate another picture $-P$ constructed by first forming the mirror image of P and then reversing all edge orientations. $P+Q$ is equivalent to the empty picture if Q is equivalent to $-P$ via bridge moves.

Since the fundamental group of a topological space acts upon the higher homotopy groups, there must be an action of the group G on pictures in E_G. We obtain this action by encircling a picture with an oriented closed arc bearing the label of a group generator. Note that the picture's base point must always be exterior to the closed arc. Note further that if a picture is acted upon by a sequence of closed arcs whose edge labels spell a word that is a consequence of the relations, then the sequence can be deleted, since the sequence represents action by the identity element.

Proposition 2.0.1. *E_G is an abelian group.*

Proof. Commutativity, associativity and the existence of an identity element (the empty picture) are clear. We see that the process of constructing a vertex in $-P$ (mirror image followed by reversing arrows) simply sends $a_1a_2\cdots a_n$ to $a_n^{-1}\cdots a_2^{-1}a_1^{-1}$, so relations are sent to their inverses or vice versa, and $-P$ must indeed be a legitimate picture. We need to see that $-P$ is an inverse for pictures that are equivalent to P via bridge moves.

If P and Q differ by a single bridge move, then the bridge move that takes Q back to P corresponds to a bridge move that takes $-Q$ to $-P$, perhaps after isotopy. Thus $-P$ and Q must be inverses. The result follows by observing that equivalent pictures differ by finitely many bridge moves. □

Now that we have established E_G as a group, we can explore its structure.

Lemma 2.0.2. *A circle with empty interior is trivial in the group E_G.*

Proof. Place two copies of an oriented, properly labeled circle in the same picture. Note first that this sum is equivalent to a single circle having the same label by performing a bridge move. Also note that forming the mirror image of a circle reverses its orientation, and so by then reversing the orientation on the mirror image we obtain a copy of the originally oriented circle. Thus the sum is also equivalent to the empty picture. □

Lemma 2.0.3. *A floating pair is an element of order two in E_G.*

Proof. Orient the floating pair with one vertex to the right of the other and then construct its mirror image across a vertical mirror. Noting that each vertex is a mirror image of the other with arrows reversed, the construction gives a picture containing two copies of the original picture. □

Lemma 2.0.4. *Suppose a picture in E_G has a face F that is incident to two vertices v and $-v$, and the two vertices can be positioned so that one is the mirror image of the other with arrows reversed. Then the picture is equivalent to one constructed by deleting the two vertices, connecting the corresponding edges and embedding in the face F a floating pair consisting of the two vertices v and $-v$.*

Proof. There may be several edges having both of the two vertices in question as ends. Orient the two vertices so that one is to the right of the other and note that there will be identically labeled edges with opposite orientation sharing a face. Conducting a bridge move will either result in a floating pair or will expose two more edges with the same label and opposite orientation in the same face. By repeated bridge moves, the floating pair becomes a separate component of the picture though embedded in a bounded face. We remove it to an unbounded face as follows.

From the base point of the picture, construct a path to the face that contains the floating pair by crossing edges only transversely and without intersecting any vertices. Working backward along this path beginning with the first edge encountered, we can execute a bridge move that causes the floating pair to be enclosed in an arc having the same label as that edge. After completing the operation, the encircled floating pair will be located in the next face traversed by the reverse path. Continue the process until the multiply encircled floating pair is in the exterior face.

□

Note that the process described in the foregoing lemma exactly mimics the fourth condition on the picture group, except that the resulting picture contains a floating pair in some face of the resulting picture.

Lemma 2.0.5. *E_G has a torsion subgroup generated by the collection of all floating pairs.*

Proof. Suppose a picture is equivalent to a sum of floating pairs. We can see that it has order two by writing the sum as the disjoint union of the floating pairs and then placing an exact copy into the same picture. We can then pair off the floating pairs, each pair being equivalent to the empty picture by the previous lemma.

We need to see that any element of E_G having order two can be constructed from floating pairs. Let v be a vertex incident to the unbounded face in a picture P having order 2. Since P is its own inverse, there is a corresponding vertex, which we will label $-v$ also incident to the unbounded face. We assume for the moment that these are not the same vertex. In this case, since the two vertices are in the same face of the picture, a sequence of bridge moves will uncouple the two vertices from the rest of the picture, producing a floating pair and a picture with two fewer vertices. By performing the corresponding operation in the inverse, we see that the resulting picture in fact consists of a floating pair and a component that has order two, but with two fewer vertices. By proceeding in this manner, we decompose the picture into the disjoint union of floating pairs, thus proving the lemma for pictures that have no vertex whose mirror image with arrows reversed is identical to itself.

Now suppose we did have a vertex w, as described above whose corresponding vertex is itself. Its word cannot be a single character, for then we would have a group presentation that lists a trivial character as a generator. Since the vertex must be incident to at least two edges, select two edges that follow in sequence and construct a drawing of w with the edges arranged to form a vee. Note that we cannot have both of the edges oriented the same, either both inward or both outward, for the process of constructing the mirror image and reversing arrows, the result of which we will call w', would have the corresponding edges with orientation opposite to that in w, contradicting the assumption that the two vertices are identical. Furthermore, the labels on the two edges must be identical since a sequence of ab in w will produce a sequence of ba in w', and the corresponding sequences must be identical. This means that any such vertex must correspond to a relation that is either not a reduced word in the free group of generators, or it is a conjugate of a reduced word, neither case being acceptable in the list of relations for G. □

Let T_G denote the subgroup of the proposition. Note that the group G acts on the groups E_G and T_G as well as the group P_G. Suppose P is a picture (in any of these groups), and g is a generator of G. Then the picture obtained by encircling P with an oriented closed arc labeled g is also a valid picture (in the same group as P) that differs from P.

Example 2.0.6. Let G be the group with two generators a and b and the commutator of a and b as the only relation. Then P_G is the trivial group, and E_G (isomorphic to T_G in this case) is an infinitely generated group whose generators can be described as the floating pair associated with the lone relation encircled by some sequence of arcs, each of which is labeled with either a or b.

We can now relate the group E_G to the picture group P_G.

Theorem 2.0.7. $E_G/T_G \cong P_G$.

Proof. By repeated application of lemma 2.0.4, we can convert any picture in E_G to one consisting of the disjoint union of encircled floating pairs along with a (possibly empty) picture that has no pairs of vertices in the same face corresponding respectively to a relation and its inverse. Consider the canonical projection homomorphism from E_G to P_G. Its kernel certainly contains T_G. We need to show that the kernel of the map is contained in T_G. A picture is trivial in P_G if all of its vertices can be deleted by application of the fourth condition on group followed by deletion of circles with empty interior. But by lemma 2.0.4, the process of deleting vertices in this way is equivalent to generating a floating pair and placing it into the unbounded face. Therefore a trivial picture in P_G corresponds precisely to a picture in T_G. □

Chapter 3

Pictures and Homological Syzygies

Let X be the set of generators and R the set of relations for a finitely presented group G. We begin a free resolution of the integers as $\mathbb{Z}G$ modules, where $\mathbb{Z}G$ is the group ring, via the following chain complex:

$$\bigoplus_{r\in R}\mathbb{Z}G \xrightarrow{\partial_2} \bigoplus_{x\in X}\mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where ∂_1 and ∂_2 are the Fox calculus maps and ϵ is the augmentation map. Loday [13] calls elements of $\ker\partial_2$ *homological syzygies*, and Brown [3] gives a procedure for obtaining a homological syzygy from a picture. To wit, for each vertex in the picture, construct an arc whose terminal points are the vertex in question and the picture's basepoint, such that:

1. The arc meets no other vertices;
2. The arc meets edges transversely; and
3. The arc meets no edge after entering the preferred face for the vertex.

This construction can and should be done so that the arcs do not meet pairwise except at the basepoint. Now for each arc, record the word w formed by the sequence of edge labels encountered on traveling from the basepoint to the vertex, applying an exponent of -1 whenever the oriented edge crosses the arc from left to right, and $+1$ otherwise. If the vertex corresponds to the relation r, then $+w[r]$ is a term in the homological syzygy, and if the vertex corresponds to r^{-1}, the the term is $-w[r]$.

Example 3.0.8. The picture in figure 2.1 corresponds to the homological syzygy

$$(y^{-1}x^{-1} - z^{-1}x^{-1}y^{-1})[r_1] + (z^{-1}y^{-1} - z^{-1}x^{-1}y^{-1})[r_2] + (z^{-1}x^{-1} - z^{-1}y^{-1}x^{-1})[r_3],$$

where $[r_1] = xyx^{-1}y^{-1}$, $[r_2] = yzy^{-1}z^{-1}$ and $[r_3] = zxz^{-1}x^{-1}$. This result can be simplified by multiplying by xyz and applying the group relations to produce $(z-1)[r_1] + (x-1)[r_2] + (y-1)[r_3]$.

Floating pairs always correspond to trivial homological syzygies, since it is always possible to obtain $w[r] - w[r]$ as the sum of the two terms obtained by the algorithm given above.

We now describe the relationship between the extended picture group and the kernel of a precrossed module homomorphism. Recall that a precrossed module consists of a group homomorphism $\phi : F \to X$ together with a group action $\Gamma : X \to \mathrm{Aut}(F)$, in which the group action is equivariant with the homomorphism, taking the action of X on itself to be by conjugation. We will write f^x to mean $\Gamma(x)(f)$ for $f \in F$ and $x \in X$. Also recall that an element of the form $f^{-1}g^{-1}fg^{\phi(f)}$ is called a *Peiffer element*, and that the set of all Peiffer elements generates a normal subgroup of F. If F is a precrossed module, then the quotient of F with this subgroup is a *crossed module*. We list some lemmas given in [3] and [12], all of which have elementary proofs:

1. X acts on $\ker\phi$.

2. $\ker\phi$ is in the center of F.

3. There is an induced action of cokerϕ on $\ker\phi$ given by $f^{[x]} = f^x$, where $f \in \ker\phi$ and $[x]$ is an equivalence class in cokerϕ with representative x.

Given the finitely presented group G, generated by $X = \{x_1, \ldots, x_n\}$, with relations $R = \{r_1, \ldots, r_m\}$, let F be the group freely generated by the elements of the form (x, r), with x in $\langle X \rangle$, the free group generated by X, and $r \in R$. Define $\phi : F \to \langle X \rangle$ by $\phi[(x, r)] = x^{-1}rx$, and define an action of $\langle X \rangle$ on F by $(x, r)^y = (xy, r)$. Note that $\phi[(x, r)^y] = \phi[(xy, r)] = y^{-1}x^{-1}rxy = y^{-1}\phi[(x, r)]y$. Thus ϕ is equivariant with the group action if $\langle X \rangle$ acts on itself by conjugation. Let T denote the normal closure in $\langle X \rangle$ of the subgroup generated by the Peiffer elements. We record the following observations:

1. coker$\phi \cong G$, since the image of ϕ is the subgroup of $\langle X \rangle$ generated by conjugates of relations of G.

2. $\phi[(x,r)^{-1}] = x^{-1}r^{-1}x$.

3. Note that $(x,r)^{\phi[(y,s)]} = (xy^{-1}sy, r)$, so that

$$\begin{aligned}\phi[(x,r)^{\phi[(y,s)]}] &= y^{-1}s^{-1}yx^{-1}rxy^{-1}sy \\ &= \phi[(x,s^{-1})]\phi[(x,r)]\phi[(y,s)] \\ &= \phi[(x,s)^{-1}]\phi[(x,r)]\phi[(y,s)].\end{aligned}$$

   Therefore, ϕ sends a Peiffer element to the identity in $\langle X \rangle$. Thus the image of T under ϕ is trivial, and so we get an induced map $\widetilde{\phi} : F/T \to X$. Consequently, $\mathrm{coker}\widetilde{\phi} \cong G$.

4. Elements of $\ker \phi$ and $\ker \widetilde{\phi}$ are called *identities among relations*.

Igusa [12] gives a procedure for constructing an identity among relations from the picture, which we describe in terms of the arcs just constructed. (See also [13] for its use in connecting homological and homotopical syzygies.) In addition to the arcs, we enclose each vertex in a closed arc containing only that vertex and meeting edges transversely. Now for each arc meeting the basepoint, travel from the basepoint to the preferred face, then counter-clockwise around the closed arc that encloses the vertex, then back to the basepoint, recording the word as above. We then concatenate these words starting with the word obtained from the right-most arc, proceding in sequence of arc to the left-most.

Loday [13] also gives a process for recovering a picture from an identity among relations. We will illustrate the process in the proof of Proposition 3.0.9. We can also recover a picture from a homological syzygy, though the process is a bit more tedious by the need to order the terms in the homological syzygy so as to produce an identity among relations. The process begins by a map that sends each term of the form $p[r]$ to the word $p^{-1}W(r)p$, where p is a word and $W(r)$ is the word that expresses the relation r. Under this map, the term $-p[r]$ goes to $p^{-1}[W(r)]^{-1}p$. After concatenating to form the identity among relations, a picture can be obtained as follows:

1. Above each relation, place a vertex and construct arcs terminating in the vertex and the symbol in the word of the relation (or its inverse), with orientation consistent with the rules for edges entering or leaving a vertex.

2. For the symbols that are not part of the word of a relation, construct an arc for each pair of symbols corresponding to the same generator, but with opposite exponents, giving each an orientation from x to x^{-1}. This must be done so that no two arcs meet.
3. Below, construct arcs as in the previous step, but with orientation from x^{-1} to x.

Proposition 3.0.9. *There are surjective homomorphisms $s' : T \to T_G$ and $s : \ker \phi \to E_G$.*

Proof. We define the map s and s' by following Loday's construction for obtaining a picture from an identity among relations. A typical Peiffer element $(x,r)^{-1}(y,s)^{-1}(x,r)(y,s)^{\phi[(x,r)]}$, correponding to the identity $x^{-1}r^{-1}xy^{-1}s^{-1}yx^{-1}rxx^{-1}r^{-1}xy^{-1}syx^{-1}rx$ in $\langle X \rangle$, maps to a picture containing three floating pairs, two of which are identical, as shown in figure 3.1. The two identical floating pairs sum to zero in T_G, and so the image of the map s' is a floating pair for any generator of T. Thus $s'(T) \subseteq T_G$. It is a homomorphism since the product of two identities corresponds to the disjoint union of the two pictures obtained from the respective identities. Surjectivity is clear. The map $s : \ker \phi \to E_G$, similarly constructed, is likewise a homomorphism, and surjectivity follows from a chase of the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & \ker \phi & \longrightarrow & \ker \widetilde{\phi} & \longrightarrow & 0 \\ & & {\scriptstyle s'}\big\downarrow & & {\scriptstyle s}\big\downarrow & & {\scriptstyle \cong}\big\downarrow & & \\ 0 & \longrightarrow & T_G & \longrightarrow & E_G & \longrightarrow & P_G & \longrightarrow & 0 \end{array}$$

in which the rows are exact. □

Note that we also get maps in the opposite direction, from Loday's algorithm for producing an identity among relations from a picture, but the maps fail to be homomorphims since the image of the sum of two identical pictures, equivalent to the empty picture in T_G, is not the identity in T.

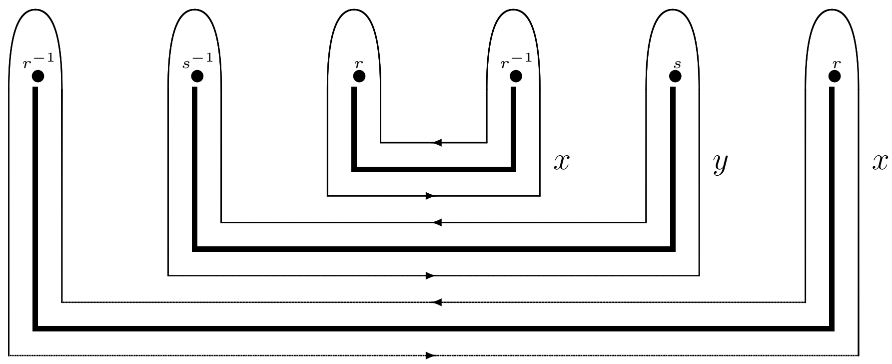

Figure 3.1: The image of a Peiffer element in T_G. Each of the broader lines represents the collection of all edges associated with a floating pair.

Chapter 4

Homomorphisms Between Picture Groups

Before stating the main theorem concerning homomorphisms between picture groups, we reiterate a general result about pictures (See for example [1],[3],[10].)

Lemma 4.0.10. *The boundary word of a subpicture is a product of conjugates of the relations associated to the vertices contained in the subpicture.*

Proof. Embed the subpicture in S^2, contract the boundary to a point and then project stereographically from a point not in an edge or a vertex back to $\mathbb{R}^2$ to obtain a (boundary-free) picture in the picture group for a new presentation of G obtained by adding the boundary word as a relation in the new presentation. By following Loday's construction [13], we identify the picture with an identity among relations in terms of the boundary word and the relations associated with vertices contained in the original subpicture. □

If $S \subset \langle X \rangle$, let $N(S)$ denote the least normal subgroup of F_X generated by the elements in S.

Theorem 4.0.11. *Let $G = \langle X|\ R \rangle$ and $G' = \langle X'|\ R' \rangle$ be finitely presented groups, and let $f : \langle X \rangle \to \langle X' \rangle$ be an injective homomorphism between the respective free groups on generators. If $f(r) \in NR'$ for each $r \in R$, then there exists a nontrivial homomorphism $\phi : P_G \to P_{G'}$.*

Proof. For each relation $r \in R$, we can construct a one-vertex subpicture S_r (that is, a picture with boundary) such that the boundary word is r, denoted by $w(S_r) = r$. We construct a set map ψ : $\{S_r|\ r \in R\} \to \{S'_r|\ r \in R\}$ from the set of all one-vertex subpictures associated with G to subpictures associated with G' as follows. Since we assume that $f(r)$ can be expressed in terms of the relations in R', we can choose a subpicture S'_r such that $w(S'_r) = f(r)$. Now extend to the full set of one-vertex subpictures by requiring that $S_{r^{-1}}$ maps to $-S_r$.

Given a picture $P \in E_G$, we define a picture $\widetilde{\phi}(P)$ as follows. Embed sufficiently many nonintersecting closed arcs in P so that each vertex is contained in a one-vertex subpicture in which every edge meets the vertex exactly once. Letting S denote the union of all these subpictures, form $P - S$. Thicken each edge in $P - S$ such that the thickened edges do not intersect, and then replace each thickened edge with the sequence of parallel edges mandated by applying f to the edge label for that edge, producing a diagram that is isomorphic (as a graph) to a picture in $E_{G''}$, where $G'' = \langle X' | \; f(r), r \in R \rangle$, the isomorphism being the contraction of each subpicture boundary to a point. Finally, replace each subpicture according to the choices made in defining ψ, matching edges between subpictures and $P - S$.

The map $\widetilde{\phi}$ is well defined on isotopy classes, but we need to see that it is well-defined on equivalence classes under bridge moves. This is most easily done by examining the diagram that defines a bridge move in figure 2.2. Fatten each each edge labeled x in the first diagram and replace the ribbon with the edges described by $f(x)$. Now successive bridge moves will be possible producing what the second diagram would be after a similar fattening and replacement. That the map is a homomorphism follows from observing that the group operation is disjoint union and that the various constructions are all local phenomena; so order of operation does not matter.

To obtain the induced homomorphism $\phi : P_G \to P_{G'}$, we need to see that the image of T_G under $\widetilde{\phi}$ must be contained in $T_{G'}$. Note that the two vertices in a floating pair get mapped to subpictures that are inverses of one another, so the end product must be identical to its mirror image. By lemma 2.0.5, the image of a floating pair must be a sum of floating pairs. □

Remark 4.0.12. In general, the map $\widetilde{\phi}$ need not be injective. Consider the groups $G = \langle a, b, c | \; abcabc = 1 \rangle$ and $G' = \langle a, b, c | \; abc = 1 \rangle$. In this case, $E_G \cong T_G$ is generated by the single floating pair, and one can easily construct a map that produces two copies of the floating pair generating $E_{G'} \cong T_{G'}$. However, with suitable restrictions on the presentations of the group G, we can assure injectivity of the map between picture groups. Also note that ϕ is not the homomorphism induced by the map f, since the image of ϕ must respect the choice of presentation for G'.

Corollary 4.0.13. *Let R be the set of relations for a group G and X the set of generators. If $r \notin N((R-r)$, and if no element in R can be written in the form $r = w^k$, or a conjugate thereof, with $k > 1$ and w a word in F_X, then the map given by the previous theorem is injective.*

Proof. Assume that $\phi : P_G \to P_{G'}$ is not injective, so $\phi(P) \in T_{G'}$ for some $P \in P_G$. Then $\phi(P) \sim -\phi(P)$, which means that by forming the mirror image of $\phi(P)$ and then reversing arrows, we obtain $\phi(P)$ again. Ignoring preferred faces of vertices for a moment, this means that $P\ P^{-1}$. This can be seen most easily by contracting the boundaries of $P - S$, where S is the collection of one-vertex subpictures described in the proof of the previous theorem. When a relation can be written as $r = w^k$, there are k choices for the preferred face, taking the choice to be the face just before the first symbol in w. By requiring that $k = 1$ and by restricting the designation of preferred face to the face incident to the first symbol of r, we guarantee that $P \sim -P$ as based pictures, as long as we do not permit both a word and its conjugate to be relations. □

Remark 4.0.14. The first condition imposed by the corollary is natural in the sense that we usually want presentation of G to meet the stronger condition of independence among relations. Because of the second condition, we may not get injectivity if a group G has torsion, as in the case when G is the fundamental group of a nonorientable surface.

For the extension of ϕ to all of E_G to be injective, we would need to see that its subsequent restriction to T_G is injective. In fact, the conditions imposed by the corollary are not sufficient to guarantee injectivity on the extended picture group. For example, if r is the relation $aba^{-1}b^{-1}$, and $f(a)$ and $f(b)$ are both relations in R', then the subpicture S_r can be replaced by a subpicture with four vertices, no two of which are incident to the same edge; all edges joint vertex to boundary. Thus, the floating pair associated with the relation r maps to a picture consisting of four floating pairs consisting of two pairs of identical component. So in this case the image of a floating pair can be trivial.

Chapter 5

Invariance of Picture Groups and Composition of Maps

We now examine how the various picture groups might depend upon the presentation of the group to which they are associated by considering the effect of Tietze transformations. Recall that Tietze transformations are of two types. Type I transformations either add or delete a generator. When a generator is added, we also add a relation that equates the new generator to a word expressed in terms of the other generators. A generator can be removed whenever there is a relation r that explicitly defines the generator to be removed in terms of the remaining generators; the process of removing the generator is completed by replacing all occurrences of that generator in the list of relation and then deleting the relation r. Type II transformations either add or delete relations that are consequences of other relations.

Proposition 5.0.15. *P_G is invariant under Type I transformations.*

Proof. Suppose G and G' are two presentations of the same group, differing by a type I Tietze transformation by which G' has an extra generator. Clearly $P_G \subseteq P_{G'}$, and so we need to establish the inclusion $P_{G'} \subseteq P_G$. Let $P \in P_{G'}$ and let $r' = zw(X)$ denote the added relation, where z is the new generator and $w(X)$ is some word in G. If no vertex in P is associated with r', then clearly $P \in P_G$, so suppose that some vertex in P is in fact associated with r'. The edge labeled z must have two ends in vertices, and so we can pair each occurrence of a vertex $v_{r'}$ with a vertex $-v_{r'}$ having a common edge labeled z incident to both. By following the sequence of edges around each vertex in the pairs, we see that we can isolate each pair by a sequence of bridge moves. Furthermore, the isolated pairs must be floating pairs, since the preferred faces for these vertices must occur in the same face of the picture. Thus, there is an equivalent picture in which each occurrence of an edge labeled z appears in a floating pair, and so P is equivalent to a picture devoid of vertices associated with the new relation; hence, $P \in P_G$. □

The extended picture group cannot be an invariant under type I Tietze transformations for the simple reason that each new generator produces a new generator for the torsion subgroup. More to the point is the question of whether two group presentations that are both minimal in terms of generators and differing by some sequence of type I Tietze transformations must have isomorphic extended picture groups. The answer depends upon what happens to T_G.

We define a *type I Tietze pair* as a sequence of two type I Tietze transformations in which the first adds a generator and the second deletes one. The relation associated with a type I Tietze pair must have the additional property that the added or deleted generator must appear in the relational word associated with the move exactly once with an exponent of ± 1.

Proposition 5.0.16. *If group presentations G and G' differ by a type I Tietze pair, then $T_G \cong T_{G'}$.*

Proof. The relation associated with the transformation pair defines a map between free groups of generators, and the map has an inverse. Starting with a floating pair, we replace one-vertex subpictures and adjust subpicture exteriors according to theorem 4.0.11, noting that we might produce nests of closed arcs in the process. Note also that each of the replacement subpictures contains exactly one vertex. Thus the process can be reversed, returning the original boundary word for each of the one-vertex subpictures. □

Example 5.0.17. We show that the presentations $G = \langle a, b, c |\ [a, b, c] \rangle$ and $G' = \langle x, y, z |\ [x, z], [y, z] \rangle$ are Tietze equivalent by the following sequence of type I Tietze transformations. Note that by $[x_1, \ldots, x_k]$ we mean a choice of $k-1$ independent equations from amongst those equating cyclic permutations of the word $x_1 x_2 \cdots x - k$.

$$\begin{aligned}
\langle a, b, c | \; [a, b, c] \rangle &\cong \langle a, b, c, x, y, z | \; [a, b, c], x = a, y = ab, z = abc \rangle \\
&\cong \langle b, c, x, y, z | \; [x, b, c], y = xb, z = xbc \rangle \\
&\cong \left\langle c, x, y, z | \; [x, x^{-1}y, c], z = yc \right\rangle \\
&\cong \left\langle x, y, z | \; [x, x^{-1}y, y^{-1}z] \right\rangle \\
&= \left\langle x, y, z | \; \begin{matrix} xx^{-1}yy^{-1}z = x^{-1}yy^{-1}zx, \\ xx^{-1}yy^{-1}z = y^{-1}zxx^{-1}y \end{matrix} \right\rangle \\
&= \left\langle x, y, z | \; z = x^{-1}zx, z = y^{-1}zy \right\rangle
\end{aligned}$$

Therefore, they have isomorphic picture groups. We could replace the sequence of type I Tietze transformations by the pairs:

- Use $x = a$ to replace a with x.
- Use $y = xb$ to replace b with $x^{-1}y$
- Use $z = yc$ to replace c with $y^{-1}z$

Thus $T_G \cong T_{G'}$.

Example 5.0.18. Suppose that we transform a group whose generators are $a, x_1, \ldots, x_n$ by adding a new generator a' along with the relation $a' = waw^{-1}$, where w is some word composed from the symbols $x_1, \ldots, x_n$ and their inverses. Then clearly we can remove the generator a by another type I Tietze transformation using the same relation. In general, when generating sets differ by conjugation as just described, we can demonstrate equivalence via a sequence of type I Tietze pairs.

We now turn our attention to type II Tietze transformations. These are characterized by the addition or deletion of a relation that can be expressed in terms of other relations or their inverses in the relation set for the group. Now adding a relation corresponds to adding a 2-cell to a CW complex, and so one cannot reasonably expect that P_G should be invariant under type II Tietze transformations. The next example illustrates the problem.

Example 5.0.19. Consider a group with presentations:

- $G = \langle a, b, c|\ abc = bca, bca = cab\rangle$, or

- $G' = \langle a, b, c|\ abc = bca, bca = cab, cab = abc\rangle$.

The presentations are clearly isomorphic since they differ by a type II Tietze transformation. The first group presentation has trivial picture group P_G, as will be shown in proposition 6.1.3, However, $P_{G'}$ is nontrivial, as shown in figure 5.1.

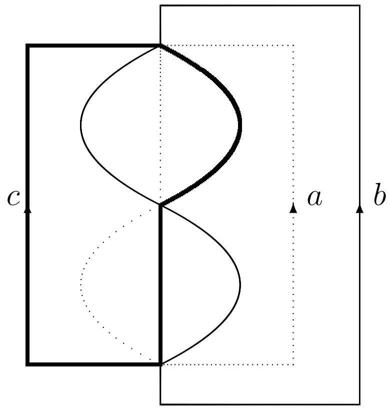

Figure 5.1: A nontrivial picture

Loday's result states that if a group presentation is minimal in the sense that no relation can be written as a consequence of the other relations in the group, then P_G is isomorphic to the second homotopy group of the minimal model 2-complex obtained from G, and hence of any topological space that is homotopy equivalent to it. We provide here a constructive means of implementing this isomorphism using the idea of a Tietze pair. We define a *type II Tietze pair* as a sequence of two transformations, in which the first adds a consequence of relations to the set of relations and the second deletes a relation that can be written in terms of the other relations in the set. Note that if the group presentation is minimal, then reversing the order makes no sense.

Proposition 5.0.20. *P_G, E_G and T_G are invariant under action of a type II Tietze pair.*

Proof. Let $G = \langle X|\ r_1, \ldots, r_m\rangle$ and $G' = G = \langle X|\ r_1, \ldots, \widehat{r_i}, \ldots, r_m, r\rangle$ be connected by a type II Tietze pair. Then there is an identity among relations in the group $G'' = \langle X|\ r_1, \ldots, r_m, r\rangle$ in which r and r_i each appears exactly once with exponent ± 1. Then there is a boundary-free picture associated with this identity in $P_{G''}$. Embed this picture on the 2-sphere and construct one-vertex subpictures to capture the vertices v_r and v_{r_i} (or there inverses, as the case might be). We use this picture together with its inverse to provide the isomorphism between the various sets of picture groups. In the forward direction (adding r and then deleting r_i), replace every subpicture associated with r_i or its inverse with the subpicture obtained by deleting one-vertex subpictures containing the vertices associated with r_i or its inverse in the pictures embedded on spheres. The reverse direction works the same way, but deleting the one-vertex subpictures associated with r or its inverse from the pictures embedded on the spheres.

We need to see that these operations are inverses of one another; that is, by operating in one direction and then in the other, that we return to an equivalent picture. So consider a subpicture whose boundary word is r_i containing in its interior a subpicture whose boundary word is r or r^{-1}, as is appropriate to the specific case, in turn containing in its interior a one-vertex subpicture whose boundary word is r_i. Embed this picture on a 2-sphere with the two vertices associated with r_1 at the poles, and the boundary of the subpicture whose boundary word is r (or r^{-1}) lying along the equator. Note that the northern hemisphere is equivalent to a mirror image of the southern hemisphere with arrows reversed, and so starting with vertices having edges that cross the equator, the entire picture can be decomposed into a collection of floating pairs. Thus the operation is an equivalence on P_G. If we start with a picture in T_G, the process produces matched pairs of floating pairs, and so the operation returns a picture in T_G to an equivalent picture in T_G. The argument operating in the other direction is the same. □

Example 5.0.21. Consider a group presentation in which all relations are commutators of the form $[w_1, \ldots, w_m]$, where each w_i is a word in the group. We claim that any two choices of a minimal set of $m-1$ of these relations are connected by a sequence of type II Tietze pairs. To show this, we must demonstrate that it is always possible to write a relation in such a way that the other relations appear at most once. Now each choice of relation r_i can be written in the form $u_i v_i^{-1}$, where the u's and v's are cyclic permutations of $w_1 \cdots w_m$. Suppose that the word $u_i v_i^{-1}$ appears more than once in an expression of say $u_k v_k^{-1}$. Then the word between two occurrences must be equivalent to $v_i u_i^{-1}$, and so one of the occurrences of $u_i v_i^{-1}$ can be removed.

Finally, we consider the question of when we can expect commutativity of a diagram involving homomorphisms between picture groups, as defined by Theorem 4.0.11. Since each map in principle involves a choice of subpictures in its construction, it is not reasonable to expect commutativity in every case. The following example serves as an illustration of the problem.

Example 5.0.22. Consider the following group presentations, all having the same generating sets:

- $G = \langle a, b, c, d |\ [a, b, c, d]\rangle$
- $G' = \langle a, b, c, d |\ [a, b, d], [a, c], [b, c], [c, d]\rangle$
- $G'' \cong \mathbb{Z}^4$

The diagram of subpictures in figure 5.2 shows a choice of maps in which commutativity fails.

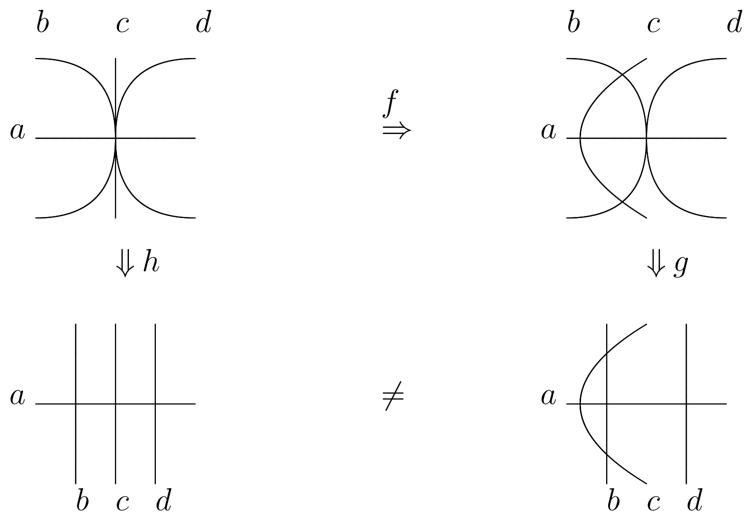

Figure 5.2: A noncommuting diagram of subpictures. All edges oriented either upward or to the right.

As the example illustrates, the noncanonical way in which homomorphisms arise confounds commutativity. In certain instances, however, one can do better. By *local abelianization* we mean the following process. Given a group presentation, delete a relation and add a commutator for each distinct pair of generators appearing in the relation's word. The term applies to the passage from G' to G'' in the previous example, but not to the passage from G to G'.

Proposition 5.0.23. *Let G be a finitely presented group and let S be the collection of groups obtained from G by a sequence of local abelianizations. Then the homomorphisms given by Theorem 4.0.11 between picture groups associated with the groups in S commute.*

Proof. Commutativity of homomorphisms follows by virtue of the fact that no vertex obtained from a replacement subpicture ever gets replaced in the process. □

Chapter 6

Application to Line Arrangements

By *line arrangement* we mean a set $\mathcal{A} = \{l_1, \ldots, l_n\}$ of finitely many one-dimensional affine subspaces of $\mathbb{C}^2$, each element of which is the zero set of some linear $f_i \in \mathbb{C}[x, y]$. The polynomial $Q = \prod_{i=1}^n f_i$ defines $\mathcal{A}$ as an affine algebraic set, and $M_{\mathcal{A}} = \mathbb{C}^2 - \cup_{i=1}^n l_i$ denotes its topological complement. Note that $M_{\mathcal{A}}$ has the homotopy type of a 2-dimensional CW complex [14]. We say that a point contained in exactly m lines of $\mathcal{A}$ has *multiplicity* m, calling it a *multipoint* if $m \geq 2$. We say $\mathcal{A}$ is *central* if $\cap_{i=1}^n l_i$ is nonempty; otherwise, it is called *affine*. If the lines in $\mathcal{A}$ intersect pairwise, but no point has multiplicity greater than 2, we say that $\mathcal{A}$ is in *general position*. We say that $\mathcal{A}'$ is a *subarrangement* of $\mathcal{A}$ if $\mathcal{A}' \subseteq \mathcal{A}$. If $\mathcal{A}'$ and $\mathcal{A}''$ are both subarrangements of $\mathcal{A}$, such that $\mathcal{A}' \cup \mathcal{A}'' = \mathcal{A}$ and $\mathcal{A}' \cap \mathcal{A}'' = \emptyset$ we say that $\mathcal{A}'$ and $\mathcal{A}''$ are *complementary* in $\mathcal{A}$.

For each line $l_i \in \mathcal{A}$, let $\phi_i : l_i \to \mathbb{C}^2$, be an affine transformation. Furthermore, let $\phi_i(t)$ denote a homotopy such that $\phi_i(0)l_i = l_i$ and $\phi_i(1)l_i = \phi_i l_i$. Then for each $t \in [0, 1]$, $\Phi_t(\mathcal{A}) = \{\phi_1(t)l_1, \ldots, \phi_n(t)l_n\}$ is a line arrangement having no more than n lines. We call Φ_t a *one-parameter family* of line arrangements [18]. We impose three *compatibility conditions*. For $t_2 > t_1$:

1. $\phi_i(t_2)l_i \cap \phi_j(t_2)l_j = \emptyset \Rightarrow \phi_i(t_1)l_i \cap \phi_j(t_1)l_j = \emptyset$.

2. $\phi_i(t_2)l_i \cap \phi_j(t_2)l_j \cap \phi_k(t_2)l_k \neq \emptyset \Rightarrow \phi_i(t_1)l_i \cap \phi_j(t_1)l_j \cap \phi_k(t_1)l_k \neq \emptyset$.

3. $|\Phi_{t_1}(\mathcal{A})| \geq |\Phi_{t_2}(\mathcal{A})|$.

Let $\mathfrak{L}_n$ denote the collection of all line arrangements having n lines, and let $\mathfrak{L} = \bigcup_{n \in \mathbb{N}} \mathfrak{L}_n$. We impose a partial order on $\mathfrak{L}$ as follows. Given line arrangements $\mathcal{A}$ and $\mathcal{B}$ in $\mathfrak{L}$, if there exists a one parameter family of line arrangements Φ_t satisfying the three compatibility conditions such that $\mathcal{B} = \Phi_1(\mathcal{A})$, then $\mathcal{A} > \mathcal{B}$. If $\mathcal{A} > \mathcal{B}$ and $\mathcal{B} > \mathcal{A}$, then we say that $\mathcal{A} \sim \mathcal{B}$, and note that $\mathcal{A}$ and $\mathcal{B}$ must be lattice isotopic [18].

Proposition 6.0.24. *The relation $>$ is a partial ordering of equivalence classes in $\mathfrak{L}$.*

Proof. Suppose that $\mathcal{A} \sim \mathcal{C}$, $\mathcal{B} \sim \mathcal{D}$, and $\mathcal{A} > \mathcal{B}$. Then transformations exist such that $\mathcal{B} = \Phi_t(\mathcal{A})$, $\mathcal{A} = \Phi'_{t'}(\mathcal{C})$ and $\mathcal{D} = \Phi''_{t''}(\mathcal{B})$. Hence, by concatenating the homotopies of line arrangement transformations at the coordinate level, we obtain $\mathcal{D} = [\Phi'' \circ \Phi \circ \Phi']_t(\mathcal{C})$, so $\mathcal{C} > \mathcal{D}$. Thus $>$ is well defined. Reflexivity comes from the identity transformation, antisymmetry by the definition of equivalence classes, and transitivity follows from composition of transformations. □

In general, the fundamental group of a line arrangement complement can be computed via a braided wiring diagram [4], simplifying to Randell's algorithm [17] when dealing with complexified real arrangements. Regardless of technique, finite presentations for fundamental groups of line arrangement complements are available having the following characteristics:

- There is a presentation having exactly one generator per line, and choices of generating sets are related by finite sequences of type I Tietze pairs. (Specifically, if x and y represent the same line, then we will be able to write $x = y^w$, where w is a word.)

- For every multipoint of multiplicity m in the arrangement, there are $m-1$ independent relations unique to that multipoint, each of the form $vw = wv$ (a commutator). Each of the $m-1$ relations will be chosen from a collection of $\frac{m(m-1)}{2}$ possible choices, and two such choices are related by a finite sequence of type II Tietze pairs, which can be understood as follows. We write the relations compactly as $[a_1, \ldots, a_m]$, but this notation does not indicate the choice of basis. Two standard choices are:

  1. $a_1 \cdots a_m = a_i \cdots a_m a_1 \cdots a_{i-1}$ for $i = 2, \ldots, m$, and

  2. $a_i \cdots a_m a_1 \cdots a_{i-1} = a_{i+1} \cdots a_m a_1 \cdots a_i$ for $i = 1, \ldots, m-1$.

  The second choice of basis is particularly convenient for the systematic construction of pictures.

We say that the fundamental group of a line arrangement is minimally presented when its presentation meets both of these criteria. From the discussion in examples 5.0.18 and 5.0.21, we see that two minimal presentations for fundamental group of a line arrangement complement are equivalent via some sequence of type I and type II Tietze transformations, and so:

Proposition 6.0.25. *If G is a minimal presentation of the fundamental group of a line arrangement complement, then E_G, T_G and P_G are invariants for the homotopy type of the space.*

6.1 Pictures and Line Arrangements

Much of what we say about pictures associated with the complement of a line arrangement applies more generally to pictures associated with any finitely presented commutator-relations group. Since generators and their inverses appear in pairs in a commutator, there is a corresponding pairing of identically labeled edges entering and leaving a vertex. Since pictures are finite graphs, we see that every edge in a picture is part of an oriented cycle, all of whose edges have the same label. We call such a cycle an *edge cycle*, or more specifically a *g-cycle*, where g is the generator that labels each edge in the cycle. Consequently, we can assemble pictures by constructing closed, oriented loops in the plane, allowing them to intersect in a way that is consistent with the information contained in the fundamental group. This paradigm shift away from assembling vertices (like pieces in a jigsaw puzzle) toward assembling edge cycles represents a simplification in constructing pictures.

A typical picture might be constructed as follows. First, produce a floating pair. Next, choose an edge cycle to divide the picture so that exactly one of the two vertices in the floating pair is in the interior of the edge cycle. In the process, we normally produce new vertices that do not comply with the rules for pictures, and so new edge cycles must be added in an attempt to bring the picture into compliance. With each additional edge cycle, we attempt to maximize efficiency by allowing the cycle to improve numerous vertices. We continue this process until the picture is complete. Unfortunately, this approach falls short of algorithmic status in several ways: there is no rule for chosing which edge cycle to add next, there is no rule in how to maximize efficiency in the process of adding a new edge cycle, and there is no guarantee that the process will terminate in a valid picture. Nevertheless, the process is of value.

A *tricycle*, shown in figure 2.1, is a picture consisting of exactly three edge cycles and six vertices, constructed as follows. Starting with two edge cycles having exactly two points of intersection, add a third edge cycle that intersects each of the first two edge cycles twice and has exactly one of the original vertices in its interior.

The following lemma, stated in a more general setting than its obvious application to line arrangements, will be useful in subsequent discussion.

Lemma 6.1.1. *If a finitely presented group G has three generators that commute pairwise, then P_G is nontrivial.*

Proof. To demonstrate that the tricycle is a nontrivial picture, construct its inverse to obtain another tricycle with the counterclockwise cyclic ordering of the edge labeled cycles altered. Then the picture cannot have order two, and so it does not belong to T_G. □

As an immediate consequence, we see that if a complexified real line arrangement contains a triangle, then Randell's algorithm provides three pairwise commuting generators for the fundamental group of its complement, and so the second homotopy group of the complement cannot be trivial. (This result has been given elsewhere [6].) This result suggests that knowledge about subarrangements can be translated into information about the original arrangement. The following proposition states sufficient conditions for this to be the case. We begin with a definition.

Let $\mathcal{A}'$ and $\mathcal{A}''$ be complementary subarrangements of a finite line arrangement $\mathcal{A}$. Let L' denote the set of intersections of lines in $\mathcal{A}'$ and L'' the set of intersections of lines in $\mathcal{A}''$. If $L' \cap \mathcal{A}''$ is empty and there exists a convex, open subspace $B \subset \mathbb{C}^2$ such that $L'' \cap \mathcal{A}' \cap B$ is empty, and the image of the subspace in a wiring diagram has no virtual vertices that involve both a line from $\mathcal{A}'$ and a line from $\mathcal{A}''$. In this case we say that $\mathcal{A}'$ is a *distinguished* subarrangement of $\mathcal{A}$.

Proposition 6.1.2. *If a line arrangement $\mathcal{A}$ has a distinguished subarrangement $\mathcal{A}'$, then the fundamental group of $M_{\mathcal{A}}$ has a presentation containing every generator and every relation in a presentation for the fundamental group of $M_{\mathcal{A}'}$. Consequently, $\pi_2(M_{\mathcal{A}'}) \neq 0 \Rightarrow \pi_2(M_{\mathcal{A}}) \neq 0$*

Proof. Note first that the condition $L' \cap \mathcal{A}'' = \emptyset$ guarantees that the relation corresponding to a particular multipoint in $\mathcal{A}'$ involves precisely the same lines as it does in $\mathcal{A}$. Next, note that the requirements of no virtual vertices and $L'' \cap \mathcal{A}' \cap B$ guarantee that at worst, each new intersections resulting from a mixing of $\mathcal{A}'$ and $\mathcal{A}''$ involves exactly two lines. Consequently, there are no differences owing to conjugation between the choices of relations for the respective fundamental groups. Finally, note that every nontrivial picture associated with $\mathcal{A}'$ is also a nontrivial picture associated with $\mathcal{A}$. □

Since we have as a goal to establish homomorphic maps between picture groups of related line arrangements, we begin with two well-known results for extreme cases, stated in terms of picture groups.

Proposition 6.1.3. *A central line arrangement has trivial picture group.*

Proof. This is an immediate consequence of the fact [6] that central line arrangement complements are $K(\pi, 1)$ spaces that are homotopy equivalent to 2-dimensional CW complexes. □

If a line arrangement with more than two lines is in general position, then it must have a nontrivial second homotopy group [11], and so the picture group must be nontrivial as well. We can give explicit description of its generators.

Proposition 6.1.4. *The picture group associated with an arrangement of n lines in general position is generated by the set of all tricycles.*

Proof. There are n generators for the fundamental group, and they commute pairwise. We can form $\binom{n}{3}$ tricycles, each of which is a nontrivial picture. We need to see that all pictures can be generated from this set. So let P be such a picture having an edge cycle labeled by a. If the edge cycle contains no vertices in its interior, then every internal edge must bound a bigon along with an edge in the a edge cycle. We can redraw each bigon to produce a picture in which the a edge cycle has empty interior, which means that it can be deleted from the picture.

Now suppose that the number of vertices in the interior of the a edge cycle is m. The following lemma establishes a technique whereby the picture can be modified by a tricycle in an exterior face of the a cycle and then applying bridge moves to produce a picture in which the a cycle contains $m - 1$ vertices in its interior. (Note that correct positioning of the tricycle might require addition of the tricycle acted upon by an appropriate sequence of closed arcs.) Once a picture has been obtained in which the a cycle has no vertices in its interior, we apply the process described in the preceeding paragraph. In this manner, a picture can be produced in which there are no a cycles. If other cycles remain, then we pick another label and proceed as we did with the a cycle. Since each picture is a finite graph, the process must terminate in a trivial picture. Thus, any picture in this group has as its inverse a picture equivalent to the disjoint union of a finite number of tricycles or their conjugates, and so the picture group must be generated by these objects. □

Remark 6.1.5. In the case of a line arrangement in general position, we can actually produce a presenatation for the fundamental group of the complement. Consider for example four lines in general position. We use the notation T_{abc} to indicate a tricycle with clockwise ordered edge cycles a, b and c, each oriented counter-clockwise. The the picture group associated with the complement has presentation:

$$P_G = \langle T_{abc}, T_{abd}, T_{acd}, T_{bcd} | (1-d)T_{abc} + (a-1)T_{bcd} + (c-1)T_{abd} + (1-b)T_{acd} = \emptyset \rangle \,.$$

To construct the relation, each occurence of a negative sign requires a transposition of the ordering of the cycles, and each occurence of a generator requires that the tricylce be encircled with an arc bearing that labeled, oriented counter-clockwise. The equivalence to the empty picture can be shown through a series of bridge moves accompanied by deletion of closed arcs with empty interior.

Lemma 6.1.6. *Suppose a picture in the picture group associated with a line arrangement in general position contains a face bounded by edges labeled $a, x_1, \ldots, x_n$, reading counterclockwise around the face from edge a, and this face is in the interior of the a edge cycle. Applying the tricycle consisting of edge cycles labeled a, x_1 and x_2, reading clockwise, to the face just exterior to the edge a in question, produces a picture in which the vertex formed by the x_1, x_2 intersection lies in the exterior region of the a edge cycle, and the face just internal to the a edge cycle is bounded by edges labeled $a, x_2, \ldots, x_n$.*

Proof. The entire proof is contained in the accompanying diagram, figure 6.1, which shows a region bounded by four edges; however, the edge labeled d could be replaced by a sequence of edges $d_1, \ldots, d_m$ without altering the accompanying argument.

Moving around the diagram clockwise, we pass from upper left to upper right by executing a bridge move using edges labeled a and then straightening the resulting a path. Moving to the lower right, we execute two bridge moves involving b edges, one to the left and one to the right of the a path, and one bridge move involving c edges, and then we straighten the b path. Note that there are two sets of vertices that can be converted to floating pairs, one in the upper part of the a path and the other in

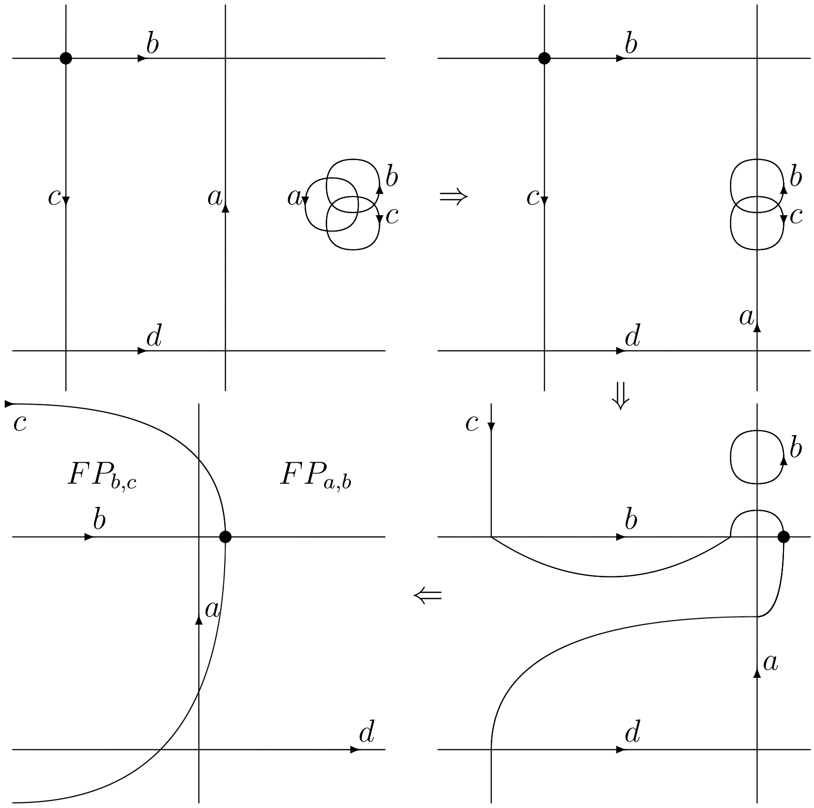

Figure 6.1: The vertex highlighted in the upper left diagram, taken to be in the interior of an edge cycle labeled a, moves to the exterior of the a cycle after a sequence of bridge moves.

the left part of the c path, and this we do in passing to the diagram in the lower left. □

Remark 6.1.7. Suppose that $\mathcal{A}$ is a near-pencil of n lines. An argument similar to that in lemma 6.1.6 shows that the picture group of $\mathcal{A}$ is generated by pictures constructed as follows. Let x_i be the fundamental group generator associated with line l_i for $i = 1, \ldots, n-1$, and let y correspond to the line in general position. Construct a floating pair corresponding to one of the relations given by $[x_1, \ldots, x_{n-1}]$. Each edge in the floating pair is subdivided by embedding a closed arc having exactly one of the vertices in its interior and then assigning the label y to each edge in this cycle, having points of intersection with the floating pair edges as ends. Since there are $n-2$ floating pairs for a choice of relations in the central subarrangment fundamental group the picture group for $\mathcal{A}$ must have $n-2$ generators as well.

6.2 Homomorphisms Between Picture Groups of Line Arrangements

We now specialize theorem 4.0.11 and its corollary to picture groups associated with the complements of line arrangements. Our main objective is to establish the following result.

Theorem 6.2.1. *If $\mathcal{A}$ and $\mathcal{B}$ are line arrangements with $\mathcal{A} > \mathcal{B}$, then there exists a nontrivial homomorphism between the respective extended picture groups. Furthermore, if $|\mathcal{A}| = |\mathcal{B}|$, then the homomorphism is injective.*

Proof. First observe that if two line arrangements are of the same equivalence class in $\mathfrak{L}$, then they are lattice isotopic, and so by a result of Randell's [18], their complements are diffeomorphic.

The following construction will be useful. For each multipoint x_i in $\mathcal{A}$ embed a 3-sphere $S^3_{\epsilon_i}$ centered at x_i of radius ϵ_i, where the ϵ_i's have been chosen so that the 3-spheres do not intersect pairwise. Also embed a 3-sphere S^3_∞ of sufficiently large diameter so that all of the embedded spheres just mentioned lie in its interior.

Let $\mathcal{A}_i$ denote the central subarrangement consisting of all lines in $\mathcal{A}$ containing x_i. If $\mathcal{B} = \Phi_t(\mathcal{A})$ has the property that for all $t \in [0, 1]$ and for each i, the multipoints of $\Phi_t(\mathcal{A}_i)$, all lie in the interior of the sphere $S^3_i(x_i)$, and any multipoints formed by intersection of lines initially parallel lie exterior to the sphere S^3_∞, then we say that $\mathcal{B}$ *lies close to* $\mathcal{A}$.

Brauner [2] showed that for each i, $S^3_{\epsilon_i} \cap M_\mathcal{A}$ is an n_i-component Hopf link, whose complement in $S^3_{\epsilon_i}$ has a fundamental group that is isomorphic to that of $M_{\mathcal{A}_i}$. The inclusions of these link complements in $M_\mathcal{B}$ provides the key to invoking theorem 4.0.11. Note that Brauner's result does not depend upon linearity of the affine subspaces, and so in the following lemma, we can, without loss of generality, argue in terms of pseudoline arrangments. Proof of Lemma 6.2.2 concludes the proof of this theorem. □

Lemma 6.2.2. *If $\mathcal{B}$ lies close to $\mathcal{A}$ with $\mathcal{A} > \mathcal{B}$, then there exists a nontrivial homomorphism between extended picture groups.*

Proof. If $\mathcal{A}$ is central, then the inclusion of the link complement of $\mathcal{A}$ into the complement of $\mathcal{B}$ induces a homomorphism between fundamental groups, and thus the desired homomorphism between extended picture groups.

Now suppose that $\mathcal{A}$ is not central. We proceed stepwise, invoking the privelege of passing through pseudoline arrangements on the way to our ultimate goal. Assume that the difference between $\mathcal{A}$ and $\mathcal{B}$ exists entirely within $S^3_{\epsilon_i}(x_i)$ for some i, and that ϵ_i is sufficiently small to assure that the image of $S^3_{\epsilon_i}(x_i)$ in a wiring diagram contains no virtual vertices. Let U be the subspace obtained by deleting the interior of $S^3_{\epsilon_i}(x_i)$ from $M_{\mathcal{A}}$, and let V be the complement of $\mathcal{A}_i$ in a ball of radius ϵ_i centered at x_i. Their intersection is the link complement in $S^3_{\epsilon_i}(x_i)$, and their union is all of $M_{\mathcal{A}}$. There is an obvious inclusion from U into $\mathcal{B}$, inducing a homomorphism, and a homomorphism from V to $M_{\mathcal{B}}$ given by the first part of the lemma. Since $M_{\mathcal{A}}$ is just $U \cup V$, the Seifert-Van Kampen theorem assures a homomorphism between fundamental groups. In the event that the difference between $\mathcal{A}$ and $\mathcal{B}$ lies in the exterior to S^3_{∞}, then let U be the closure of the exterior of S^3_{∞}, and V the closure of its interior, and argue as before. The desired homomorphism between fundamental groups arises by composing those obtained at each step in the sequence. □

6.3 Examples

Example 6.3.1. Let $\mathcal{A}$ be any arrangement of n lines and let $\mathcal{B}$ be the arrangement obtained by deleting one of the lines from $\mathcal{A}$. There is a homomorphism induced between fundamental groups by the inclusion of $M_{\mathcal{A}}$ in $M_{\mathcal{B}}$, and this provides a homomorphism between picture groups, realized by deleting every edge whose label corresponds to the discarded line.

Example 6.3.2. This example illustrates the general technique of constructing pictures by starting with a floating pair and adding edges. Consider the two arrangements in $\mathbb{C}^2$, $\mathcal{A}$, with defining polynomial $Q = xy(x-y)(x+y-1)(2x+y-1)(x+2y-1)$, and $\mathcal{A}'$, with defining polynomial $Q = xy(x-y)(x+y-1)(3x+y-1)(x+2y-1)$. Note that $\mathcal{A}$ is a generic section of the braid arrangement A_3. Shown in figure 6.2, they differ only in the positioning of the line labeled b, and $\mathcal{A} > \mathcal{A}'$ The respective fundamental groups have generators $X = X' = \{a, b, c, d, e, f\}$. We list their relations explicitly:

1. $R = \{cde = dec, dec = ecd, fea = eaf, eaf = afe, fdb = dbf, dbf = bfd, cba = bac, bac = acb, be = eb, cf = fc, ab^{-1}db = b^{-1}dba\}$

2. $R' = \{cde = dec, dec = ecd, fea = eaf, eaf = afe, fdb = dbf, dbf = bfd, cba = bac, bac =$

$acb, be = eb, cf = fc, ad = da, df = fd, bf = fb, bd = db\}$.

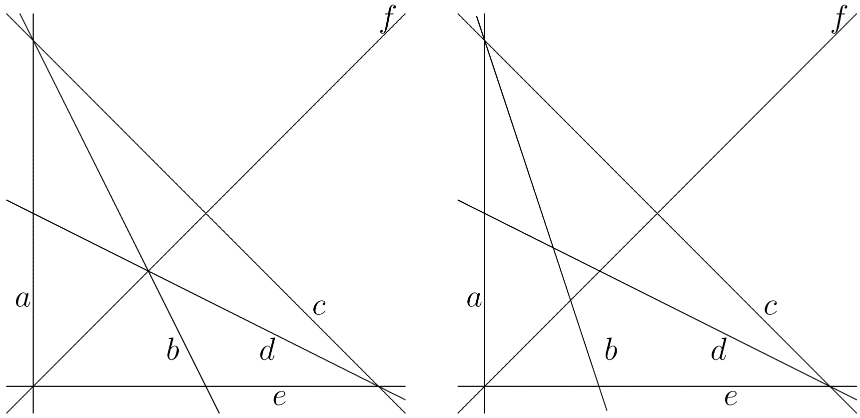

Figure 6.2: The line arrangements $\mathcal{A}$ and $\mathcal{A}'$

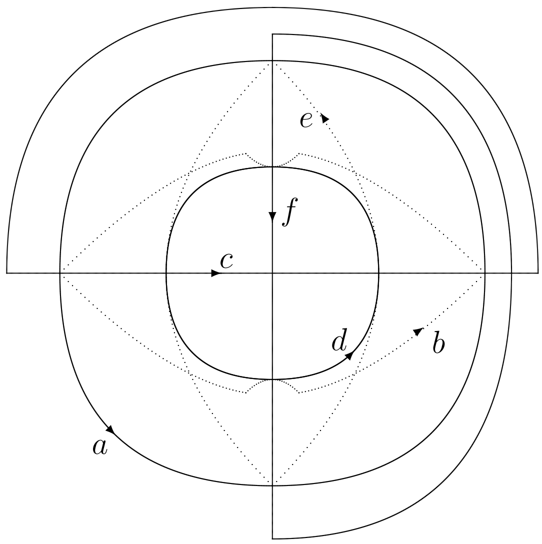

Figure 6.3: A picture in the picture group associated with arrangement $\mathcal{A}$

Figure 6.3 shows a nontrivial picture in $P_{G_{\mathcal{A}}}$, which we alter to obtain the nontrivial picture in $P_{G_{\mathcal{A}'}}$, shown in figure 6.4. Note that the two graphs have the same number of faces, but the graph in figure 6.4 has two extra edges and two extra vertices. In their duals, the dual associated with the picture in figure 6.4 has two extra faces and two extra edges compared to the dual of the picture in figure 6.3, and this modification comes about by subdividing the two hexagonal 2-cells in the dual for the picture in figure 6.3, thereby producing two pairs of quadrilateral 2-cells having a common edge in their boundary. If we had chosen to alter the b edge cycle by traveling clockwise around the df vertex, thereby creating four bd vertices in addition to the two bf vertices, we would have replaced the two hexagonal 2-cells by two sets of quadrilateral 2-cells, with four in each set.

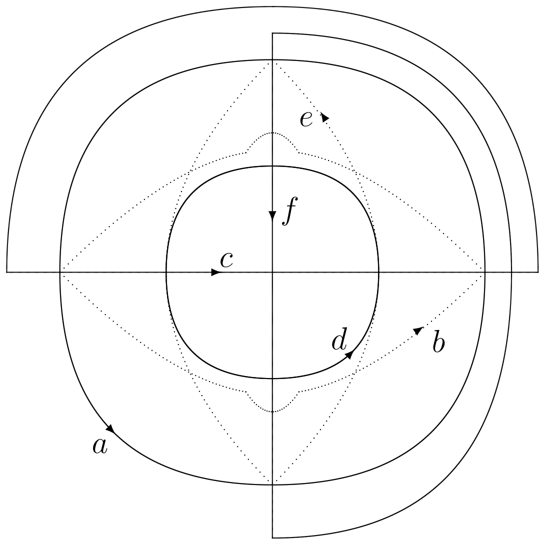

Figure 6.4: A picture in the picture group associated with arrangement $\mathcal{A}'$, obtained from the picture in figure 6.3 by resolving the intersection of the lines labeled b, d, f

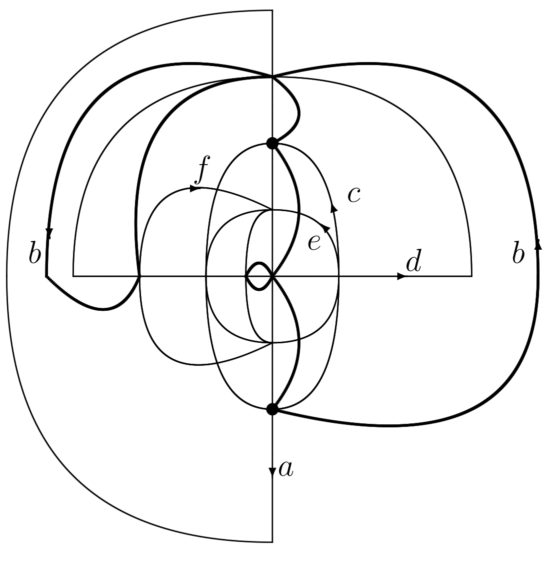

Figure 6.5: A second picture for arrangement $\mathcal{A}$

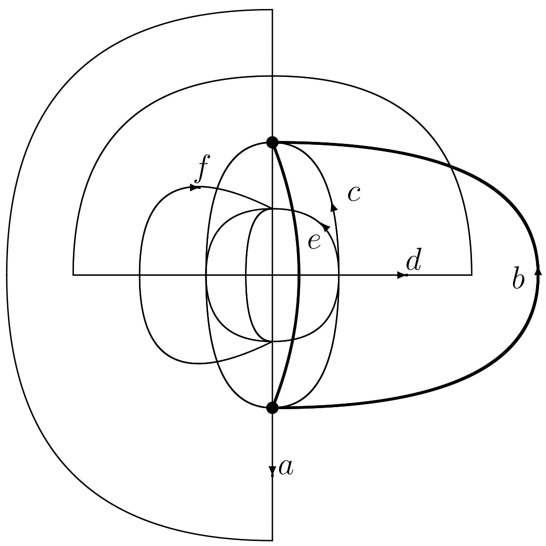

Figure 6.6: A second picture for arrangement $\mathcal{A}'$ obtained from the picture in figure 6.5 via homomorphism

Figures 6.5 and 6.6 show pictures for the respective picture groups, with the picture in figure 6.6 obtained by altering the three b edge cycles, drawn with the heavier line for clarity. The b edge cycle in the upper left portion of figure 6.5 becomes a floating pair after performing a bridge move. The small b edge cycle just to the right of center becomes a floating pair as well (after a bridge move). Note that the two vertices marked by the bullet do not change during the process. Both of these pictures are nontrivial in their respective picture groups, the picture in figure 6.6 being equivalent in its extended picture group to the same picture with two floating pairs. If we located these two floating pairs in the unbounded face, then one would be enclosed in a clockwise oriented arc labeled a and the other in five arcs, the innermost two labeled a and b, both with clockwise orientation, and the outer three labeled e, c and b, all with counterclockwise orientation.

Example 6.3.3. The line arrangement designated by Falk as X_2 has defining polynomial $Q = (x+1)(x-1)(y+1)(y-1)(x+y+2)(x+y-2)$, and consists of three sets of parallel lines with two points of multiplicity 4. The fundamental group of its complement has generators a, b, c, d, e, f, and the relations are:

$$r_1 = abca^{-1}c^{-1}b^{-1}, r_2 = bcab^{-1}a{-}1c{-}1, r_3 = defd^{-1}f^{-1}e^{-1},$$

$$r_4 = efde^{-1}d^{-1}f^{-1}, r_5 = ada^{-1}d^{-1}, r_6 = bdb^{-1}d^{-1},$$

$$r_7 = cec^{-1}e^{-1}, r_8 = aea^{-1}e^{-1}, r_9 = cfc^{-1}f^{-1}, r_{10} = bfb^{-1}f^{-1}.$$

The syzygy

$$s = [r_2](a^{-1}fdf^{-1}c^{-1}b^{-1} - a^{-1}fdc^{-1}b^{-1} + a^{-1}c^{-1}b^{-1}f - a^{-1}c^{-1}b^{-1})$$
$$+ [r_3](a^{-1}e^{-1} - e^{-1} + b^{-1}e^{-1} - b^{-1}e^{-1}a^{-1})$$
$$+ [r_5](a^{-1}e^{-1} - a^{-1}f + b^{-1}a^{-1}f - b^{-1}a^{-1}e^{-1})$$
$$+ [r_6](a^{-1}fab^{-1} - fb^{-1})$$
$$+ [r_8](a^{-1}e^{-1}d - a^{-1}e^{-1} + b^{-1}a^{-1}e^{-1} - b^{-1}a^{-1}e^{-1}d)$$
$$+ [r_9](a^{-1}c^{-1} - a^{-1}c^{-1}b^{-1} + fdf^{-1}a^{-1}c^{-1}b^{-1} - fdf^{-1}a^{-1}c^{-1})$$
$$+ [r_{10}](fdf^{-1}b^{-1} - b^{-1} + a^{-1}c^{-1}b^{-1} - fdf^{-1}a^{-1}c^{-1}b^{-1})$$

corresponds to the picture in figure 6.7, constructed to use relations r_2 and r_3. Isomorphic graphs can be constructed based upon the use of relation pairs r_1, r_3 and r_1, r_4, but the pair r_2, r_4 does not produce a corresponding picture since the corresponding to generators b and e are parallel.

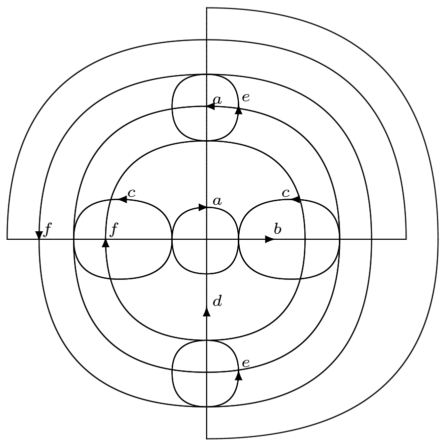

Figure 6.7: A nontrivial picture for the arrangement X_2.

Example 6.3.4. Consider the central $\mathbb{C}^3$ [15] with defining polynomial:

$$Q = xy(6x - y)(x - 4y)(x + y - z)(6x + y - 6z)(x - 2y - z)(x + 4y - 4z)(10x + 6y - 23z)(2x - y + z)$$

This is the arrangement C(5) described in [15], having a generic section that is the complexification of a real arrangement consisting of all lines through five generic points taken pairwise.

The affine part of the decone consists of nine lines as shown in figure 6.8.

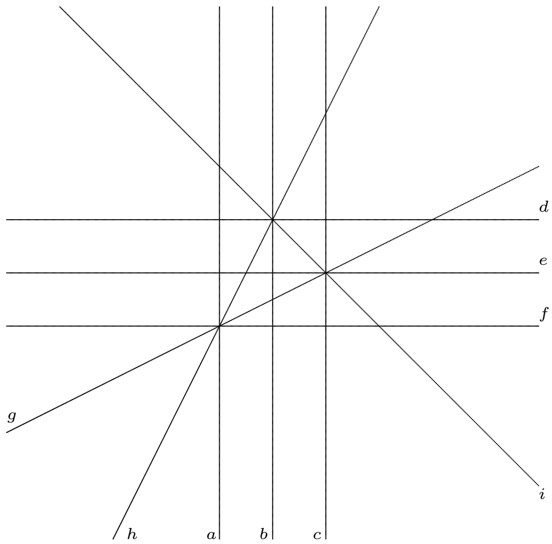

Figure 6.8: The affine portion of the decone for the $\mathbb{C}^3$ arrangement $C(5)$.

The complement of the line arrangement in figure 6.8 has a presentation for its fundamental group with relations:

$$[i,a],[i,f],[a,d],[a,e],[b,e],[b,g],[b,f],[c,h],[c,d],[c,f],[h,e],[d,g],$$

along with a choice of three relations from each of

$$[c,i,e,g],[h,d,i,b],[g,f,h,a].$$

We now show that the complement of the decone arrangement has trivial picture group, and consequently, the $\mathbb{C}^3$ arrangement defined by Q has $K(\pi,1)$ complement. The argument uses the facts that:

- Because the relations are commutator relations and all pictures are finite graphs, every edge x is contained in an x-edge cycle having well defined orientation, and hence there is an x-edge cycle having no edges labeled x in its interior.

- Vertices appear in pairs consisting of a vertex representing a relation and a vertex representing its inverse. For the picture to be nontrivial, one of these vertices must be in the interior of an edge cycle that does not involve the generators in the word of the relation.

- Generators representing parallel lines cannot meet at a vertex.
- We can describe a vertex in terms of the touching or crossing of edge cycles. For example the relation $hdbi = dbih$ correponds to a vertex in which the h-edge cycle is transverse to the other three. For this particular relation, we have four cases in which one edge cycle is transverse to the other three and two cases in which they are pairwise transverse.

We assume that there exists a nontrivial picture and show that this assumption leads to the conclusion that the graph must be infinite, contradicting the requirement that pictures be finite graphs.

1. There are no nontrivial pictures devoid of edges labeled g, h or i, for otherwise we could produce a fiber-type arrangement that is not $K(\pi, 1)$.

2. Edge cycles labeled g, h and i must all appear in any nontrivial picture. This can be seen as follows. If any two of these edge cycles meet, then the resulting vertex pair must be resolved with the remaining edge cycle from the three. Starting for example with the commutator $[d, g]$, we can attempt to resolve the associated floating pair with edge cycles labeled a, b or c. But using a also uses h and puts and f cycle interior to a d cycle, and the resulting $[g, f, h, a]$ must be resolved with i. We can argue in this way casewise. Finally, we conclude that vertices representing each of the multiplicity four vertices must be incorporated into a nontrivial picture.

3. Knowing that a nontrivial picture must have a vertex corresponding to the commutator $[h, d, i, b]$, we examine each of the six ways for constructing floating pairs. We can assume that the h cycle is inermost. Choose a basis having h, b or i transverse

   (a) h transverse: There is only one way to put in the required $[c, i, e, g]$ vertex pair, and interior to the c cycle is a vertex containing g and h. It requires an a-cycle, and the fact that this cycle must remain internal to the c-cycle implies that the h-cycle is not innermost or the picture is not finite.

   (b) In the case of b transverse, we cannot have the vertex containing g and i internal to the d-cycle. there are two ways to add an e- and an f cycle. When the f-cycle is inside the e

cycle, the h-cycle cannot be innermost; otherwise, the vertex with e and i requires g, leading to an infinite graph.

(c) With i transverse, the vertex with g and i internal to h requires an h edge internal the h-cycle.

This completes the demonstration.

Chapter 7

A Summary

As the final examples show, pictures can be used effectively in higher homotopy studies of line arrangement and central $\mathbb{C}^3$ arrangement complements. Along the way, we also gave the concept of pictures a deeper meaning in terms of new groups and maps between groups.

- We produced hitherto unknow generators for second homotopy groups associated with the arrangements A_3 and X_2.
- We demonstrated that the arrangement $C(5)$ has a complement which is a $K(\pi, 1)$ space.
- By redefining the picture group we defined two new groups E_G and T_G and showed their quotient to be equivalent to the group P_G.
- We defined certain homomorphisms between picture groups and applied them to a partial ordering of line arrangements, showing that the tendency toward general position corresponded with the existence of these maps.
- We gave a sufficient condition for demonstrating that nontriviality of second homotopy of subarrangement complements under certain conditions can imply nontriviality of second homotopy for the entire arrangement complement.
- We gave further definition to the concept of invariance of picture groups by analyzing them in the context of pairs of group presentation transformations that maintained the condition of minimality.

The problem of understanding the higher homotopy groups of hyperplane arrantements is by no means a closed question, and so there remain many questions that could be addressed by pictures. Can anything of substance be said about affine arrangements in $\mathbb{C}^3$? While we know that the reflection arrangements have $K(\pi, 1)$ complements, are there other classes of arrangements whose higher homotopy groups are trivial? Can line arrangements with parallel lines be classified in any meaningful way? Can the construction of pictures be mechanized?

Success in constructing pictures for line arrangements depended heavily upon the fact that the fundamental group relations involved commutators. Recongnizing that a picture group can be constructed corresponding to any finite group presentation, the question lingers about new techniques for constructing pictures. This thought leads natually to the possibility of finding spaces other than hyperplane arrangement complements that might yield information under a picture-based investigation.

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Vita

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