A UNIFIED METHODOLOGY OF MAINTENANCE MANAGEMENT FOR REPAIRABLE SYSTEMS BASED ON OPTIMAL STOPPING THEORY

A Dissertation

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOTATIONS

t	Calendar time
i	Integer variable
Δt	Fixed time interval between two specified epoch
x	Remaining maintenance service time
v	Virtual age
V	Virtual age limit
n	the repaired times of the system since last replacement
(n, v, x)	The state of system which could be $(\bullet, v, x)$, $(\bullet, v, \bullet)$, $(n, v, \bullet)$.
W	Length of maintenance horizon. (finite $W < \infty$; infinite $W = \infty$)
$h(n, v)$	Failure rate of a system which could be $h(\bullet, v)$ or $h(v)$.
$F_v(t)$	Distribution of residual lifetime from age v, i.e., $F_v(t) = \frac{F(t+v)-F(v)}{1-F(v)}$
A_m	Minimal repair
A_g	general repair
A_r	Failure replacement
A_p	Preventive replacement
C_m	Cost of A_m
C_f	Cost of failure lost
C_r	Cost of A_r
C_p	Cost of A_p
C_g	Deterministic cost of A_g
$G(c)$	Cumulative distribution function of $C(\bullet, v)$
$G_n(c)$	Distribution of random repair cost $C(n, v)$
θ	Repair degree $0 \leq \theta \leq 1$
$C(n, v)$	Random cost of repair with (n, v)
$C(\bullet, v)$	Random cost of repair with v only
$RCL(n, v, x)$	repair-cost-limit function for failure replacement with (n, v, x)
$APL(n, x)$	Age-limit-function for preventive replacement with (n, x)
$W^n(x, v)$	Optimal cost for working system starting at state (x, n, v) after nth repair
$Z^n(x, v)$	Optimal cost for failed system before state (x, n, v) after nth repair done
$Z_m^n(x, v)$	Minimal expected total cost in $[W - x, W]$ given the product just failed at age v after nth repair done and mth repair left.
$W_m^n(x, v)$	Minimal expected total cost in $[W - x, W]$ given the product is working at age v after nth repair done and mth repair left.
$[RCL^S(v), APL^S(v)]$	Generic form of repair-cost-limit policy and age-limit policy

ABSTRACT

This dissertation focuses on the study of maintenance management for repairable systems based on optimal stopping theory. From reliability engineering's point of view, all systems are subject to deterioration with age and usage. System deterioration can take various forms, including wear, fatigue, fracture, cracking, breaking, corrosion, erosion and instability, any of which may ultimately cause the system to fail to perform its required function. Consequently, controlling system deterioration through maintenance and thus controlling the risk of system failure becomes beneficial or even necessary. Decision makers constantly face two fundamental problems with respect to system maintenance. One is whether or when preventive maintenance should be performed in order to avoid costly failures. The other problem is how to make the choice among different maintenance actions in response to a system failure.

The whole purpose of maintenance management is to keep the system in good working condition at a reasonably low cost, thus the tradeoff between cost and condition plays a central role in the study of maintenance management, which demands rigorous optimization.

The agenda of this research is to develop a unified methodology for modeling and optimization of maintenance systems. A general modeling framework with six classifying criteria is to be developed to formulate and analyze a wide range of maintenance systems which include many existing models in the literature. A unified optimization procedure is developed based on optimal stopping, semi-martingale, and λ-maximization techniques to solve these models contained in the framework. A comprehensive model is proposed and solved in this general framework using the developed procedure which incorporates many other models as special cases. Policy comparison and policy optimality are studied to offer further insights.

Along the theoretical development, numerical examples are provided to illustrate the applicability of the methodology.

The main contribution of this research is that the unified modeling framework and systematic optimization procedure structurize the pool of models and policies, weed out non-optimal policies, and establish a theoretical foundation for further development.

CHAPTER 1
INTRODUCTION

In this chapter, we will introduce the background of maintenance management, describe its fundamental problems, present the rationale of this research, clarify our research goal and objectives, and indicate the approaches of solution. A brief outline of the dissertation organization is also included.

1.1 Background

For the last six decades, interest in the study of maintenance management has been pervasive throughout the fields of industrial, civil, chemical, mechanical, and electronic engineering. Widespread mechanization and automation have increased the fraction of employees working in the maintenance area as well as the fraction of total costs spent on maintenance activities, which makes maintenance and reliability engineering a significant component of modern engineering practice.

From reliability engineering's point of view, all systems are subject to deterioration with age and usage. System deterioration can take various forms, including wear, fatigue, fracture, cracking, breaking, corrosion, erosion, and instability, any of which may ultimately cause the system to fail to perform its required function. Consequently, controlling system deterioration through maintenance and thus controlling the risk of system failure becomes beneficial or even necessary.

According to Jardine and Buzacott (1985) and Dekker (1996), maintenance can be defined as "the combination of all technical and associated administrative actions intended to retain an item or system in, or restore it to, a state in which it can perform its required function." Decision makers constantly face two fundamental problems with respect to system maintenance. One is whether or when preventive maintenance should be performed in order to avoid costly failures. The other problem is how to make the choice among different maintenance actions in response to a system failure. Normally, decision makers can have two basic choices with respect to these problems. One is to schedule a preventive maintenance on a working system at the high failure risk situation. The other is to choose an action between repair and replacement after the system runs to failure, which is referred to as corrective maintenance. In practice, common maintenance actions include repair of the failed parts, substitution of the worn parts, major overhaul of the equipment, and so on. Abstractly, these maintenance actions can be characterized as minimal repair, general repair, and replacement according to the effects against system failure risks. Figure 1.1 (a)-(d) illustrates the effects of different maintenance actions on system condition respectively.

In particular, the vertical axis in each panel depicts the failure risk to which a system is subject. The horizontal axis represents the calendar time of this system. The solid curves represent the risk of system failure. At the beginning, a new system with age 0 is installed. As the system operates, its failure risk increases in time. When the system fails, maintenance decision makers may choose one of the following three maintenance actions: minimal repair, general repair, and failure replacement. Minimal repair in Panel (a) does not change the risk level of the repaired system, i.e., the failure risk of the system is restored to exactly the same level as just prior to the failure. Failure replacement in Panel (c) resets the risk level of the system to the one of a new system of age zero. Therefore, the solid curve after failure replacement is identical with the solid curve of a new system. General repair in Panel (b) brings the risk level to somewhere between that of the new system and the risk level prior to failure. As a result, the trajectory of the system failure risk becomes non-monotone and non-deterministic. When a system is operating properly as shown in Panel (d) of Figure 1.1, the maintenance decision makers may take a preventive replacement which resets the risk level of the current system to that of a new system in order to avoid a high risk of failure.

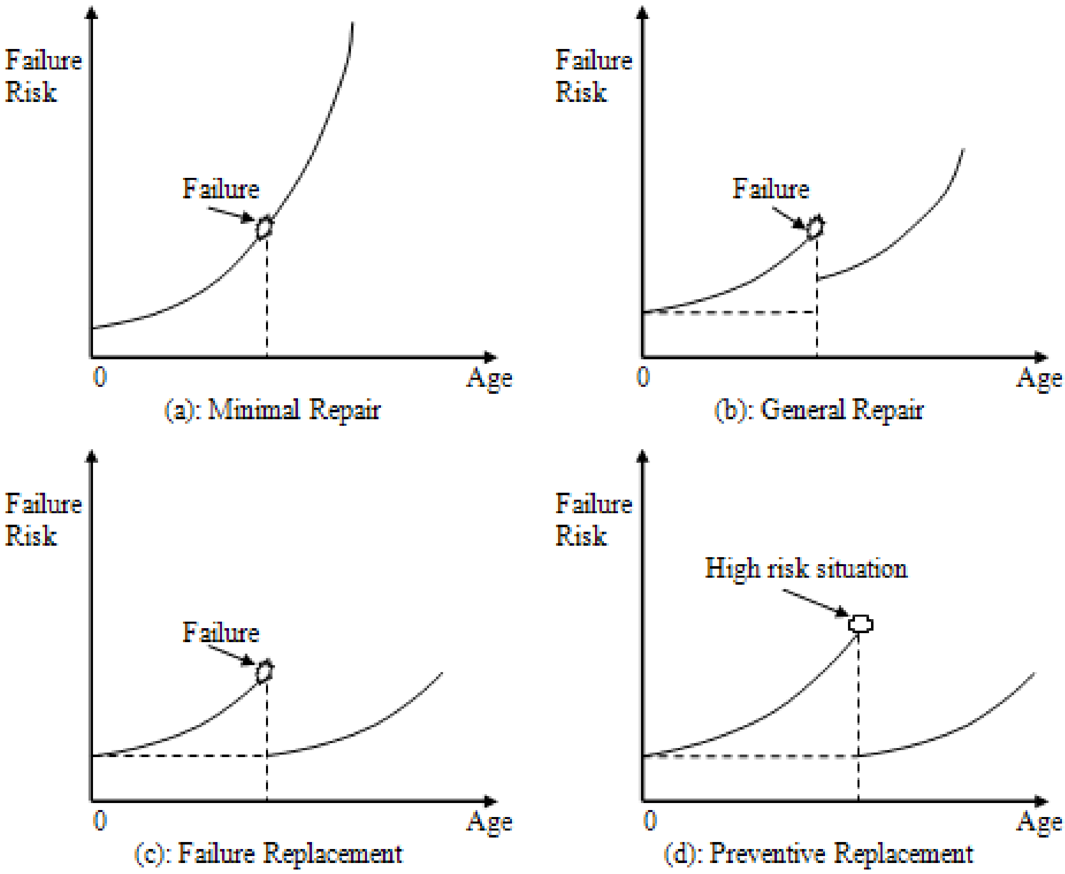

Figure 1.1: Effects of Maintenance Actions on System Failure Risks

In brief, a minimal repair makes the system as bad as old; a replacement makes the system as good as new; and a general repair returns the system to a condition between old and new. The whole idea of performing maintenance actions is to keep the system in good working condition at a reasonably low cost, and the tradeoff between cost and condition demands rigorous optimization, which plays a central role in the study of maintenance management.

1.2 Problem Statement and Rationale

Essentially, the problem of maintenance optimization can be described as follows. No systems can operate forever. Instead of running the system to failure, one can preventively replace the system at high risk situations to avoid costly failures. The key issue is how to determine the preventive replacement epoch. This epoch should be not too soon, nor too late. If it is taken too soon, the costs needed for too many preventive replacements are large. If preventive replacements are taken too late, the costs of system fails can be excessive. Therefore, the basic problem is how to arrange preventive replacement optimally, based on available information of the health condition of a working system.

Also, one may have the opportunity at the failure epochs to decide whether to repair the system or to replace it by a new one. In general, the corrective maintenance actions include minimal repair, general repair, and failure replacement. Therefore, the second basic problem is how to make a choice between repair and replacement optimally based on the available information at the system's failure epochs.

Researchers and engineers have never stopped their efforts in studying maintenance problems in order to improve system reliability, to reduce the risk of unexpected system failures, and to reduce maintenance costs. Such efforts have resulted in thousands of maintenance models and policies in the literature. As most of these models and policies were developed in response to domain-specific applications, they normally have rather limited application scopes. Many of these models were not thoroughly analyzed, and the proposed policies were often based on heuristics, which often led to non-optimality. From a practical perspective, a large pool of un-organized models without sufficient methodological support severely limits the impact of these theoretical works on maintenance practice. It is clear that a unified modeling framework and optimization procedures would significantly alter this status and possess appealing applicability.

1.3 Research Goal and Objectives

The goal of this research is to develop a unified methodology for modeling and optimization of maintenance systems. The above goal will be achieved through the following specific objectives:

First, a general modeling framework with six classifying criteria is to be developed to formulate and analyze a wide range of maintenance systems which include many existing models in the literature.

Second, a unified optimization procedure will be developed based on optimal stopping, semi-martingale, and λ-maximization techniques.

Third, a comprehensive model will be proposed and solved in this general framework using the developed procedure which incorporates many other models as special cases.

Fourth, policy comparison and policy optimality will be studied to offer further insights.

Along the theoretical development, numerical examples will also be provided to illustrate the applicability of the methodology.

1.4 Solution Approach

We have described fundamental problems of maintenance management and the goal of maintenance management. We identify the following relationship between maintenance terminologies and mathematical concepts, such as information level, maintenance policy, replacement time, and optimal policy.

Information Level ⟺ Filtration

Maintenance Policy ⟺ Stopping Time

Replacement Epoch ⟺ Realization of a Stopping Time

Optimal Policy ⟺ Optimal Stopping Time

These correspondences lay the foundation of our mathematical modeling and optimization for maintenance systems. Theories of stochastic processes and optimal stopping provide a rigorous language as well as tools for maintenance modeling and optimization. A general modeling framework, together with a set of mathematical techniques including λ-maximization, semi-martingale decomposition, and optimal stopping, will grasp the essence of maintenance management.

1.5 Summary and Dissertation Outline

In this chapter, we have introduced the background of maintenance management, basic problems faced by researchers, rationale of current studies and our research goals. The remaining part of the dissertation is organized as follows. Chapter 2 is literature review. In Chapter 3, a unified maintenance modeling framework and optimization methodology based on the modeling framework is developed. In Chapter 4, a general model is proposed and solved thoroughly using the methodology developed in Chapter 3. Chapter 5 presents the policies optimality verification and a comparison of the policies. Chapter 6 presents conclusions and future research directions.

CHAPTER 2
LITERATURE REVIEW

Interests in maintenance management are pervasive throughout the industrial, civil, mechanical, chemical, and electronic engineering communities.

The main question faced by maintenance management is how maintenance costs and the reliability of systems are quantified and optimally balanced. The studies on this topic in the early sixties by researchers like Barlow, Proschan, McCall *et al.* initiated a large area in operations research, which is maintenance optimization. Since this area was introduced, it has undergone explosive growth. Thousands of models and policies have been proposed and reviewed in the literature, including those by McCall (1965), Pierskalla and Voelker (1976), Sherif and Smith (1981), Jardine and Buzacott (1985), Valdez-Flores and Feldman (1989), Dekker (1996), Pham and Wang (1996), Moraru and Sisak (1998), Wang (2002), and Nakagawa and Mizutani (2007). McCall (1965) summarized and reviewed maintenance policies for stochastically failing equipment for studies in the early sixties. Pierskalla and Voelker (1976), Sherif and Smith (1981), and Valdez-Flores and Feldman (1989) updated the survey in 1976, 1981 and 1989, respectively. In addition, Pham and Wang (1996) reviewed imperfect maintenance in 1996, Dekker (1996) reviewed and analyzed maintenance optimization models, Moraru and Sisak (1998) reviewed preventive maintenance policies particularly for digital systems in 1998, Wang (2002) surveyed maintenance policies of deteriorating systems, and Nakagawa and Mizutani (2007) summarized maintenance policies for a finite interval. In brief, the literature and its survey are abundant and the attempt for a new and thorough survey is going to be very difficult if it is at all possible. Instead, we review here this research field by presenting the key methodologies through some most significant models and policies.

In order to provide sufficient background for further discussion, Section 2.1 is devoted to an introduction to maintenance management and a summary of typical maintenance models and policies. Following this, typical research methods are reviewed in Section 2.2 and a brief discussion on the gaps in the current research is presented in Section 2.3.

2.1 Overview of Maintenance Optimization

As indicated earlier in Chapter 1, maintenance can be considered as actions which control the deterioration process of a system, or restore the system to its operational state from a failure state.

Systems perform some mission and consist of several units, where unit means item, component, part, device, subsystem, equipment, circuit, material, structure, or machine. Such systems cover a very wide class from simple parts to large-scale space systems. System reliability can be evaluated by unit reliability and system configuration, and can be improved by adopting some appropriate maintenance actions (Nakagawa (2005)).

In particular, maintenance actions can be classified as two types: corrective maintenance and preventive maintenance. Corrective maintenance is adopted in the case where units can be fixed. If units fail, then they may begin to be repaired immediately or may be scrapped. After repair completion, units can operate again. According to different ways of performing corrective maintenance, maintenance was defined in the literature as minimal repair, general repair, imperfect repair, and so on. Preventive maintenance is adopted in the case where maintenance of units after failure may be costly, and sometimes may require a long time to rectify the failed units. The most important problem is to determine when and how to maintain units to prevent failure. However, it is not wise to maintain units with unnecessarily high frequency. From this viewpoint, the object of maintenance optimization problems is to determine the frequency and timing of preventive maintenance according to costs and effects.

In each type of maintenance, various formats of maintenance actions have been defined. In this section, we summarize some typical formats for preventive maintenance and corrective maintenance, respectively.

In the case of corrective maintenance, the following formats can be identified in the literature:

(i) Failure replacement (Good as new)

(ii) Minimal repair (Bad as old)

(iii) General repair and/or Imperfect repair (Some improvement of system condition, better than the old but worse than a new).

In the case of preventive maintenance, the following policy forms can be found in the literature:

(i) Block-based maintenance

(ii) Age-based maintenance

(iii) Condition-based maintenance.

More details on corrective maintenance and preventive maintenance are presented below.

2.1.1 Corrective Maintenance

In the very earlier studies, corrective maintenance included only replacement, as in Barlow *et al.* (1963). Later, systems under investigation are extended to include both non-repairable cases and repairable cases. For a repairable system, different types of maintenance actions can be taken into account, which include minimal repair, failure replacement, imperfect repair, general repair, and so on. In this section, we review maintenance studies according to different corrective maintenance actions.

In the literature related to repairable systems, special attention has been paid to minimal repair models. See for example Aven (1983), Block *et al.* (1985), Block *et al.* (1988), Stadje and Zuckerman (1991), Finkelstein (1992), and Beichelt (1993). Aven (2000) clarified the general definition of minimal repair given system failure intensity and illustrated it with a number of examples. Basically, a minimal repair rectifies the failure and brings the system condition back to the same condition as just prior to the failure, which is the so-called "as bad as old" repair. On the contrary, a replacement is the so-called "as good as new" repair. For an extension of minimal repair and replacement, Brown and Proschan (1983) proposed a so-called (p,q) model as a type of imperfect repair. Shown in Figure 2.1, system failures are rectified by minimal repair with probability p or by replacement with probability q, where $p + q = 1$. Related research can be found also in Nakagawa (1979) and Whitaker and Samaniego (1989), among other works, which have stimulated more research in the notion of imperfect repair by many later researchers.

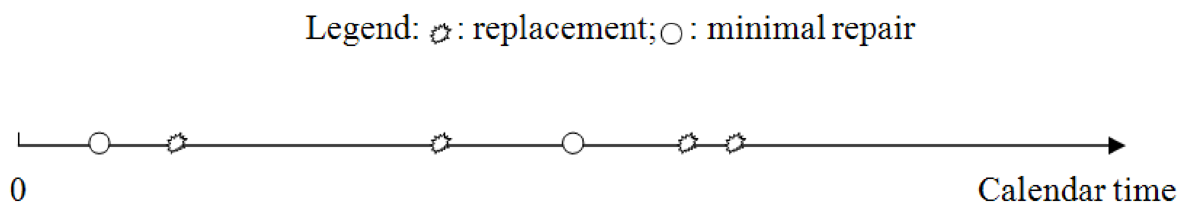

Figure 2.1: (p, q) Imperfect Maintenance Model

The well-known T-Policy or Age-Replacement Policy offers an optimal solution to combine minimal repair and failure replacement. Research related to T-Policy can be found, for example, in Aven (1983). Figure 2.2 illustrates this policy. The failure in a system is rectified by minimal repair if the age of the system is less than a specified T; otherwise, the failure is rectified by a replacement. Parameter T is subject to optimization.

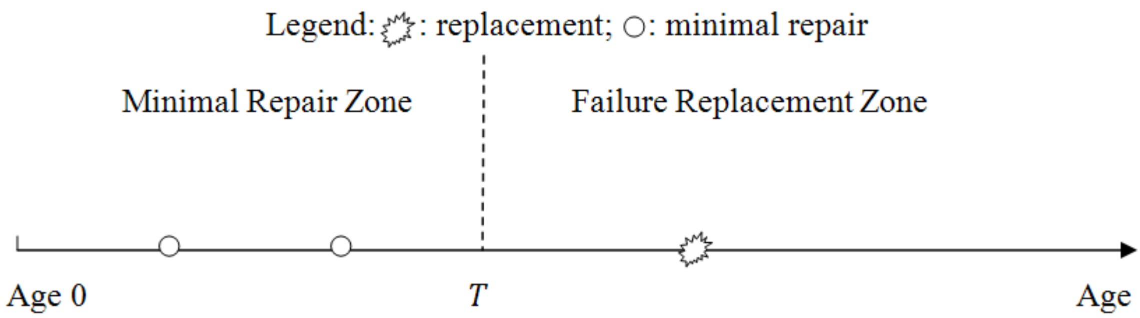

Figure 2.2: T- Policy for Failure Repair and Replacement

Similarly, the so-called N-Policy offers a natural alternative solution to repair/replacement arrangement which can be described as follows: The first $(N-1)$ failures are rectified by minimal repairs, whereas the N-th failure is rectified by a replacement. Parameter N is subject to optimization.

Maintenance models can include random repair costs as the information additional to the age information. Those models offer a nontrivial generalization to classic maintenance models with deterministic repair cost. As economic considerations are important for maintenance practice, this type of generalization is appealing in practice. Under certain monotonicity assumptions representing the deterioration of the system, it can be proved that the repair-cost-limit policy is optimal, i.e., when the repair cost exceeds an age-dependent limit, then replace the system, otherwise, repair it. See Jiang *et al.* (1998, 2001) and Makis *et al.* (2000) for detailed analysis. The so-called repair-cost-limit policy, as shown in Figure 2.3, is proved to be optimal, which suggests that a repair action should be taken if the cost is lower than a repair-cost-limit function, $RCL(v)$, which can be interpreted as the residual value of the system once it is repaired. In addition, a preventive replacement action is taken as soon as the residual value reaches 0. Some other works related to random repair cost can be found in Glasser (1967), Scheaffer (1971), Cleroux and Hanscom (1974), Subramaian and Wolff (1976), Cleroux *et al.* (1979), Nguyen and

Murthy (1986), Block *et al.* (1988), Beichelt (1993), Makis and Jardine (1993), Sheu (1994, 1999), and Sheu *et al.* (1995).

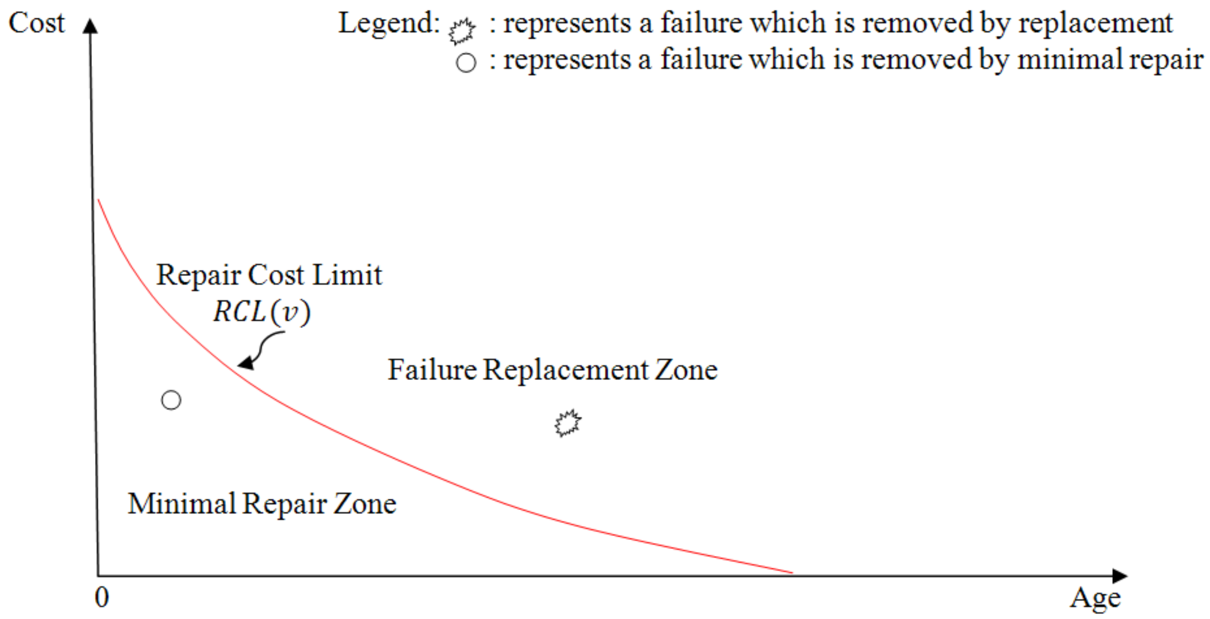

Figure 2.3: Repair-Cost-Limit Policy for Model with Random Repair Cost

The studies of maintenance models on a finite horizon are gaining popularity as they are appropriate for applications such as product warranty, maintenance service contracts, and outsourcing.

A model on the finite time horizon is shown in Figure 2.4; see Nguyen and Murthy (1981, 1986) for further reference. Assume a system is failing at age v, and the remaining warranty (or service) period is x units of time; a minimal repair action will be taken if the age is lower than an age limit $T(x)$, and otherwise, a replacement action will be taken. Clearly it is an extension of the T-Policy what parameterizes the age limit with the remaining service time x. More warranty - related works can be found in Jack *et al.* (2000), Chattopadhyay and Murthy (2000), Marquez and Heguedas (2002), Chen and Popova (2002), Iskandar and Murthy (2003), Dimitrov *et al.* (2004), Kim *et al.* (2004), Chukova *et al.* (2004), Huang and Zhuo (2004), Yeh *et al.* (2005)

Pascual and Ortega (2006), Jiang *et al.* (2006), Lugtigheid *et al.* (2007), Manna *et al.* (2007) and Yun *et al.* (2007).

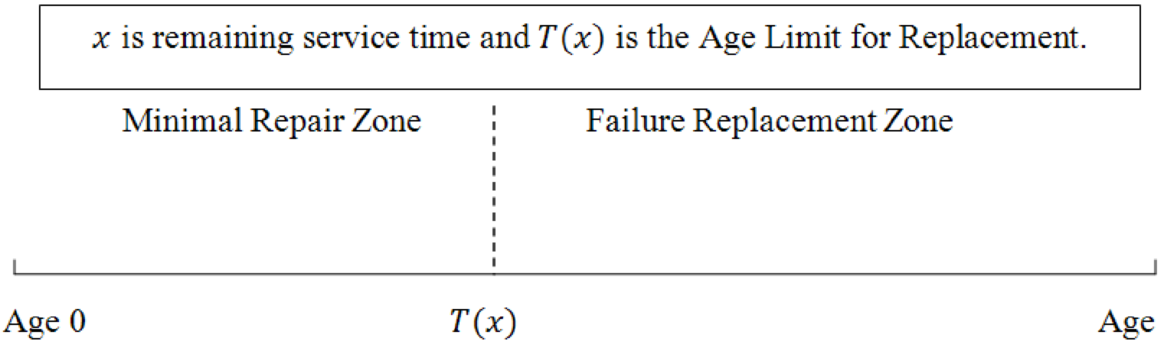

Figure 2.4: Age-based Maintenance Policy on a Finite Horizon

Among age-based maintenance models, one extension is particularly appealing. The essence of this type of model is to generalize the notion of age to a new one - the virtual age, and consequently to characterize the system condition and maintenance performance based on virtual age. Kijima *et al.* (1988) and Kijima (1989) proposed two types of maintenance models which are now know as Kijima Type I and Type II general repairs, in which, the concept of virtual age and repair degree are used to describe the condition of the system and the effect of maintenance actions.

In particular, the virtual ages are characterized by the following equations

Type I: $V_n = V_{n-1} + \theta_n X_n$, for $n \geq 0$

Type II: $V_n = \theta_n (V_{n-1} + X_n)$, for $n \geq 0$

where V_n is the virtual age after the n-th repair; $V_0 \equiv 0$; i.e., the initial age is 0; θ_n is the repair degree; X_n is the n-th run time, i.e., the failure-free operating period between the (n-1)-th and n-th failures. Normally θ_n chosen between 0 and 1 indicates the improvement of the system's virtual age. Both types of general repairs degenerate to minimal repair and replacement if $\theta_n \equiv 1$

and $\theta_n \equiv 0$, respectively. Thus, the concept of general repair generalizes minimal repair and replacement.

According to different repair degrees, the maintenance actions are defined as minimal repair, general repair and failure replacement, which are adopted in numerous studies of maintenance modeling. The major difference between age and virtual age is that the virtual age is no longer monotone, nor deterministic. See Kijima *et al.* (1988) and Kijima (1989) as examples.

It is worth noting that general repair represents just one of many possible ways that have been proposed to describe the effect of imperfect repair. Other models include the proportional age reduction model proposed by Malik (1979), proportional (hazards) intensity reduction model proposed by Chan and Shaw (1993), the shock models (see e.g., Kijima and Nakagawa (1991)), and the so-called (α, β) – rule model proposed by Wang and Pham (1996). The so-called (α, β) – rule model is based on Quasi Renewal Processes, which essentially reduces the lifetime of a system to a fraction α of the one immediately preceding it. Some of these models are mathematically overlapping or even equivalent. For instance, the proportional age reduction (PAR) models proposed by Malik (1979) are in fact mathematically equivalent to Kijima Type I General Repair models.

Further extensions of maintenance models can be done by taking preventive maintenance into account. Since a system which is repaired after failure may require much time and high cost, preventive maintenance is needed to prevent failures. The advent of preventive maintenance in maintenance modeling research is a significant milestone, and we focus on this field in Section 2.1.2 below.

2.1.2 Preventive Maintenance

Preventive maintenance strategy problems have been extensively studied, and many maintenance models have been developed in response to the need to improve system reliability, preventing system failures and reducing maintenance costs. A good survey of applied preventive maintenance models was given in Scarf (1997). In this section, four representative maintenance models and policies are presented to give a quick exposure to this subject matter.

One of the earliest preventive maintenance models is the block-replacement model found in Savits (1988) and Block *et al.* (1990) among many other papers. In this model, a preventive replacement action is taken periodically over a fixed time interval Δ, i.e. at calendar times $i\Delta$, $i = 1, 2, 3, \ldots$, and each failure is removed immediately by replacement. In essence, the block-replacement policy is proposed for a calendar-time based maintenance model. Figure 2.5 illustrates the block-replacement model.

Various modifications have been made to the basic block-replacement model by, for example, engaging minimal repair in the model, or imposing more general cost structures. Exemplary works of this kind study include Tilquin and Cleroux (1975), Boland (1982), Boland and Proschan (1982), Aven (1983), Markis and Jardine (1991, 1993), Bagai and Jain (1994) and Chen and Feldman (1997). The comparison between the block-replacement policy and other replacement policies can also be found in Langberg (1988) and Belzunce *et al.* (2006).

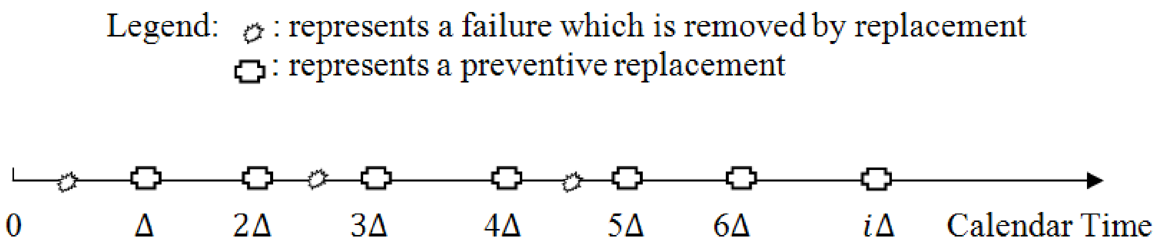

Figure 2.5: Block-Replacement Model

Another large family of maintenance models and policies alternative to the block-replacement models is the age-based maintenance models. The basic age-based maintenance model can be described as follows. The system is preventively replaced as soon as it reaches a pre-specified age t; failures prior to age t are rectified by repair or replacement which is subject to further modeling specifications. Figure 2.6 shows a basic age-based maintenance model.

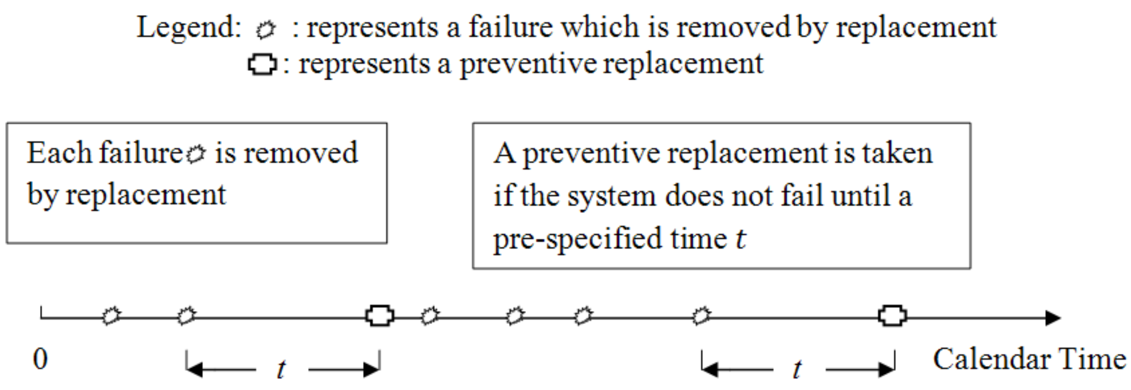

Figure 2.6: Age-Replacement Model

The basic idea behind the age-based maintenance models is to describe the system's deterioration by a single index, the age. This quantity possesses some nice analytical properties, such as being deterministic, one-dimensional, monotone, etc., which make the analysis of this class of models relatively more convenient than other alternatives, for example, those models involving virtual age as discussed in the context of imperfect repair. Numerous works on the age-based maintenance models exist in the literature. For a taste of some most representative works, see Bergman (1980), Staje *et al.* (1991), Zhang (1994), Davis and Karatzas (1994), Kao and Smith (1996), Frenk *et al.* (1997), Dagpunar (1997,1998), Limnios and Oprisan (1999), Jhang and Sheu (1999), Zhang and Love (2000), Satow *et al.* (2000), Segawa and Ohnishi (2000), Lai *et al.* (2000), Kim and Rao (2000), Park *et al.* (2000), Crocker and Kumar (2000), Yeh and Lo (2001), Chiang and Yuan (2001), Bruns (2002), Jiang and Ji (2002), Bloch-Mercier (2002), Iskandar and Murthy (2003), Seo and Bai (2004), Moustafa *et al.* (2004), Sheu *et al.* (2005), Wu and Clements-Croome (2005a,b), Wang and Zhang (2006), Lam (2006), Panangiotidou and Tagaras (2007), Shirmohammadi *et al.* (2007), and Guo *et al.* (2007).

A natural extension to the basic age-based preventive replacement model is to combine the decisions on corrective maintenance and preventive maintenance. A classic age-based model of this nature is shown in Figure 2.7 and can be described as follows: Preventive replacement action is taken as soon as the system reaches age APL; at failure epochs, a replacement is to be chosen if the current age of the failing system is beyond an age limit ARL. All other failures (at ages younger than ARL) are rectified by minimal repairs.

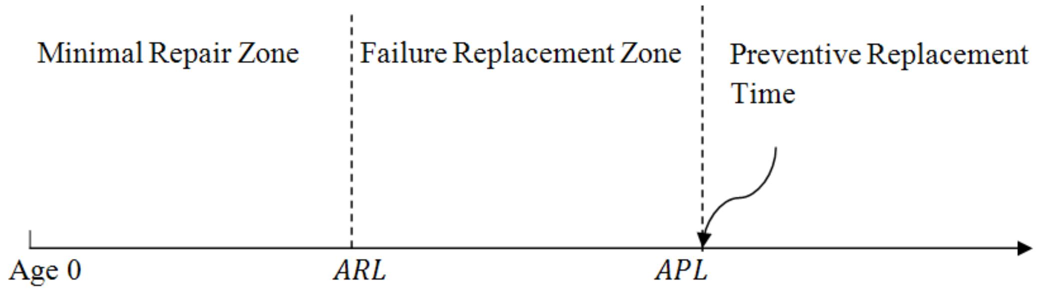

Figure 2.7: Classic Age-Based Maintenance Model

The minimal repair/replacement model with a preventive replacement in a finite time horizon is examined in Jack and Van (2000), where a conjecture on the form of optimal policy, as shown Figure 2.8, is proposed, which is later confirmed by Jiang *et al.* (2006). Moreover, the model is further extended in Jiang *et al.* (2006) to include random repair costs.

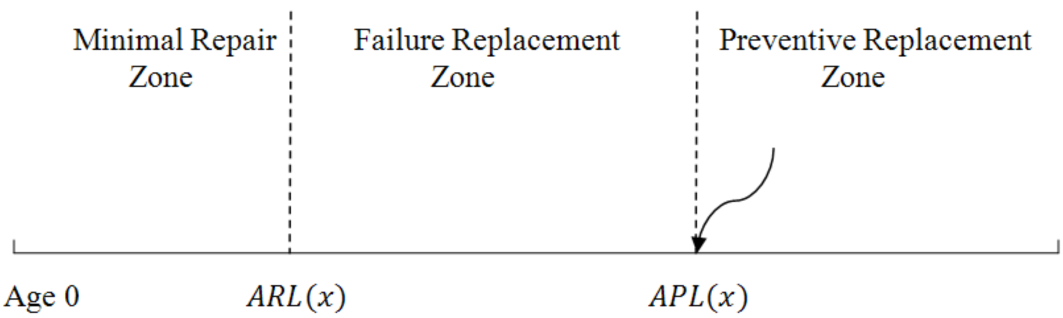

Figure 2.8: Age-Based Model with Preventive Replacement and Remaining Service Time

The incorporation of random repair cost into age-based maintenance models reflects a recent trend in maintenance practice which is generally referred to as condition-based maintenance (CBM). In contrast to block-based maintenance and age-based maintenance models, CBM models enable the utilization of information, in addition to system age, which is mostly obtained through condition monitoring. Typical information for machinery includes vibration signals and spectrometric analysis of engine oil (Mitchell (1981)).

Proportional Hazards Model (PHM) is a widely accepted CBM model which incorporates both age information and condition information (covariates) in a natural manner. A few maintenance policies have been developed for these kinds of models, among which Makis and Jardine (1992) derived the optimal replacement policy for a model with equal inspection intervals. The hidden Markov Model has also been applied to CBM (see Makis *et al.* (1998)) to emphasize the fact that condition monitoring techniques, no matter how sophisticated, represent only partial information on a system's true condition. Other types of CBM models include a state space model considered by Christer and Wang *et al.* (1997) for furnace erosion prediction and replacement and a counting process model by Aven (1996). Other studies of CBM maintenance systems can also be found in Stadje and Zuckerman (1991), Makis and Jardine (1993), Liao *et al.* (2006), and Zhou *et al.* (2007).

The family of shock models (see Chien and Sheu (2006), Chien *et al.* (2006)) also shares some similar ideas with the CBM. The basic idea of the shock model can be interpreted as follows. Shocks occur randomly in time as a stochastic process and post a certain amount of damage to a system. This damage accumulates and gradually weakens the system. A system fails when the total damage has exceeded a failure level. Suppose that a system is replaced at failure by a new one. It may be wise to exchange a system before failure at a smaller cost. Qian *et al.*

(2003) derived the optimal control limit policies where a system is replaced before failure when the total damage has exceeded a threshold level. Recently, Jiang *et al.* (1998), Sheu (1998), Sheu and Griffith (2002), Lam and Zhang (2004), and Sheu and Chien (2004) studied some replacement models of a system subject to shocks.

2.2 Classification of Optimization Methods

Thousands of maintenance models have been proposed in the literature, which are accompanied by more maintenance policies for the proposed models. Generally speaking, these maintenance policies were proposed by using one of the following three approaches: the heuristic approach, the Markov approach, and the optimal stopping approach. Instead of presenting a long list of specific models, these three approaches which cover most of the existing models are presented in this subsection. Some typical maintenance models with policies will be described in the next subsection.

The heuristic approach is often based on strong application experience and bears intuitive meanings. The basic procedure of this approach is to first propose a specific policy class in which each policy can be determined by one or two parameters. Once an objective criterion is selected and an objective function is calculated, optimization is performed by determining the optimal parameters within the pre-specified policy class. For a typical application of this approach, see Beichelt (1993) and Valdes-Flores *et al.* (1989). A nice feature of the heuristic approach is that its analysis is relatively straightforward. This approach made significant contributions in the early era of maintenance field.

The Markov method is widely used to solve failure count-based or age-based maintenance models (see Berg (1976) and Zheng *et al.* (2006)). The basic idea behind this method is to describe a system's deterioration dynamics by using Markov processes.

Consequently, one needs only to take into account the system's current condition for optimal maintenance decision making. Variations of basic Markov models, including semi-Markov, Markov renewal, and hidden Markov models, further enhance the modeling capability of the Markov approach. Markov decision processes, controlled Markov processes and dynamic programming, offer a variety of effective tools for the control and optimization of systems with partial information. The related examples of their applications can be found in Boukas and Haurie (1990, 1991) and Yeh (1991).

The optimal stopping approach is a powerful addition to the previous two approaches. In fact, the optimal stopping approach (see Chow (1971) and Shiryayev (1978)) and its application to reliability and maintenance (see Bergman (1978) and Aven and Bergman (1986)) present a fundamentally different perspective. Based on the general theory of stochastic processes, especially the martingale theory, the optimal stopping approach offers a rigorous modeling language and a set of sophisticated techniques for optimization. In optimal stopping terminology, the information available to decision makers corresponds to the notion of filtration, and a replacement policy is characterized by the concept of stopping time. The introduction of stopping time allows decision makers to utilize the whole history of available information for decision making. Thus, the optimal stopping policy is in principle superior to Markov policies, which use the system's current condition only. As a consequence, the optimal stopping approach is particularly suitable for the investigation on the optimality possessed by specific maintenance policies, which further enables policy comparison according to the strength of their optimality. Attempts on policy comparison based on optimal stopping can be found in Jiang *et al.* (1995).

2.3 Gaps in the Past Research

The number of papers in maintenance optimization has been overwhelming. It has been observed that many models and policies that have been proposed are based on existing practice, heuristics, or insufficient analysis, which are often settled with non-optimality and contain little or no methodological merits. From a practical perspective, a large pool of ill-organized models without adequate methodological support offer rather limited help for real world applications, sometimes, it may even reinforce the label of "Garbage In Garbage Out" that has been unfairly attached to maintenance management and reliability engineering. While it is impossible to construct a universal model to fit all possible cases, it is still possible to identify a small number of methodologies and techniques that are critical to most of the models. This research is aimed at developing a unified modeling framework and systematic optimization procedure that structurize the pool of models and policies, weed out non-optimal policies, and establish a theoretical foundation for further development.

CHAPTER 3
THE UNIFIED MODELING FRAMEWORK AND METHODOLOGY BASED ON OPTIMAL STOPPING

In this chapter, a unified maintenance modeling framework with six classifying factors is developed for formulation and analysis of a wide range of maintenance systems, under which many existing models in literature will be formulated as optimal stopping models; a systematic optimization methodology will be developed based on optimal stopping, semi-martingale, and λ-maximization techniques; a concrete model will be presented and solved as an example to illustrate the proposed modeling and optimizing methodology. The model to be studied in the next chapter is generated under this modeling framework, which contains many other models as its degenerated forms.

3.1 Maintenance Modeling Framework

As has been discussed in the previous chapters, the diversity and complexity of maintenance problems from various practical situations have led to numerous concrete models and policies. While it is not possible to come up with a generic model that is capable of describing all interesting situations and scenarios, it is feasible to generalize and reorganize the conceptual models under a unified maintenance modeling framework based on the essence of these problems.

We consider a maintenance system that consists of stochastically identical and independent units which are put into use sequentially in time. The unified maintenance modeling framework proposed for such a system includes six factors:

(i) Maintenance Horizon: it describes the time interval on which all the maintenance activities take place.

(ii) System Deterioration Dynamics: it describes how the system becomes more and more prone to failure.

(iii) Maintenance Actions: it describes various preventive measures and ways to rectify system failures.

(iv) Cost Structures: it describes the costs that are related to maintenance actions and the loss due to system failure.

(v) Information Level: it describes information on system condition that is available for decision making.

(vi) Objective Criterion: it describes the objective of maintenance management, which could be minimization of cost or maximization of benefits.

By specifying each of the six modeling factors, we can systematically construct a large number of maintenance models with mathematical rigor. The models consequently govern the form of optimal maintenance policies. In Chapter 5, we identify 38 policies which possess optimality for 258 models generated from this modeling framework.

In this study, we focus our attention on the following class of models that can be generated from the framework. While we do not attempt to cover all possible models, we expect this class contains many well-known and interesting models and policies that have been scattered in the literature.

In particular,

(i) We consider two basic scenarios for the maintenance horizon: an infinite maintenance horizon, denoted by $W = \infty$; a finite maintenance horizon, denoted by $W < \infty$.

(ii) System deterioration dynamics is described by time to failure and the repair cost associated with each failure. Time to failure is assumed to be random variables that can be characterized by their associated hazards function. The hazards function can be assumed to be depending on failure count n and failure age v, or a subset of these two factors. The repair cost is also assumed to depend on a subset of n and v.

(iii) Maintenance Action is specified by a (sub-) set of three possible maintenance actions including general repair A_g and failure replacement A_r at failure epochs and preventive replacement A_p when the system is up and running. We focus on the two Kijima-types of general repairs. In particular, for a given repair degree, θ: the virtual ages associated with these two types of A_g are defined as follows:

Type I: $V_n = V_{n-1} + \theta X_n$;

Type II: $V_n = \theta(V_{n-1} + X_n)$.

In addition, a special case of A_g at $\theta \equiv 1$, which is generally referred to as minimal repair, A_m, has a unique historical, analytical, and conceptual significance. We thus make it a stand-alone maintenance action that can substitute A_g in the specification of concrete models. Also, when $\theta \equiv 0$, A_g degenerates to a replacement, A_r as $V_n \equiv 0$ for both Type I and Type II cases.

(iv) Cost structure is specified by various sources of cost, including repair cost, replacement cost, and cost due to failures.

(v) Information level is to be rigorously characterized in terms of filtrations for the underlying stochastic processes. Three different information levels are considered: Level 1 is denoted by (Count), representing the knowledge on failure counts; Level 2 is denoted by (Count, Age), representing knowledge on failure counts as well as the time at which each failure occurs. Level 3, denoted by (Count, Age, Cost), contains complete knowledge of the system dynamics that includes information on repair costs, failure times (ages), and failure counts.

(vi) We consider the two most popular objective criteria: Criterion 1 is the expected total discounted costs. In particular, the cost incurred at calendar t time is subject to a discount of $e^{-\alpha t}$ with $\alpha > 0$. Criterion 2 considers no discount, which corresponds to the expected average cost criterion on $W = \infty$, and expected total non-discounted cost criterion on $W < \infty$. To simplify presentation, we denote it as $e^{-\alpha t}$ with $\alpha = 0$.

Figure 3.1 gives a flavor of the model construction process by representing a subset of possible models in a modeling tree, where only two modeling factors, Maintenance Horizon and Maintenance Actions, are considered. When more factors are incorporated, the combination of the alternative specifications leads to a larger number of correlated models.

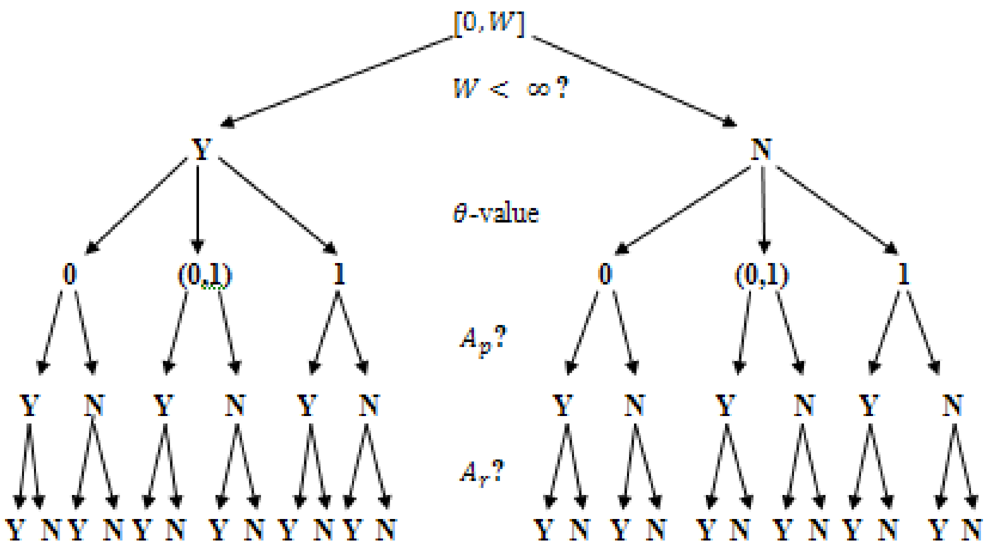

Figure 3.1: Modeling Tree with Two Specification Factors

Table 3.1 indicates 258 models obtained under this modeling framework, and we have not attempted to try to make it as long as possible. The complete list of these 258 maintenance models, together with their corresponding optimal policies, can be found in Appendix A. The notations used in Table 3.1 is listed as follows:

1. $\mathrm{M_i}$ is the ID of each of specified maintenance models.
2. NR stands for non-repairable and R stands for repairable.
3. Criterion 1 is the expected total discounted criterion and Criterion 2 is the non-discounted one.
4. $W < \infty$ represents a finite maintenance horizon and $W = \infty$ represents an infinite maintenance horizon.
5. Information Level: $c, v,$ and n correspond, respectively, to information on repair costs, age at failure times, and failure counts.

Table 3.1: Model Construction under the Modeling Framework

Model #[1]	System[2]	Objective[3]	Maintenance Horizon[4]	Failure Rate	Information Level[5]	Maintenance Actions	Optimal Policy
M_1	NR	Criterion 2	$W = \infty$	$h(n, v)$	(n)	A_r	P_{39}
..........omitted..........							
M_5	NR	Criterion 1	$W < \infty$	$h(n, v)$	(n, v)	A_r	P_{39}
.... omitted							
M_{11}	NR	Criterion 1	$W < \infty$	$h(v)$	(n, v)	(A_r, A_p)	P_{35}
.... omitted							
M_{18}	NR	Criterion 2	$W = \infty$	$h(n, v)$	(n, v)	(A_r, A_p)	P_{36}
.... omitted							
M_{20}	R	Criterion 1	$W = \infty$	$h(n, v), C(n)$	(n)	A_g I	P_{38}
.... omitted							
M_{24}	R	Criterion 1	$W = \infty$	$h(n, v), C(n)$	(n)	$(A_r, A_g$ II$)$	P_{29}
....							
M_{48}	R	Criterion 1	$W = \infty$	$h(n, v), C(n)$	(n, c)	$(A_r, A_g$ II$)$	P_{30}
.... omitted							
M_{70}	R	Criterion 1	$W = \infty$	$h(n, v), C(n, v)$	(n, v)	(A_r, A_m)	P_{24}
M_{220}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	(A_m, A_p, A_r)	P_2
M_{221}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_g$I$, A_p, A_r)$	P_1
M_{222}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_g$II$, A_p, A_r)$	P_2
.... omitted							
M_{258}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	$(A_g$II$, A_p, A_r)$	P_6

- There are two kinds of maintenance horizons:

  (1) $W < \infty$
  (2) $W = \infty$

- There are six classes of maintenance actions:

  (1) A_r
  (2) A_m
  A_g type I
  A_g type II
  (3) (A_r, A_m)
  $(A_r, A_g$ type I)
  $(A_r, A_g$ type II)
  (4) (A_r, A_p)
  (5) (A_m, A_p)
  (A_g type I, A_p)
  (A_g type II, A_p)
  (6) (A_m, A_p, A_r)
  (A_g type I, A_p, A_r)
  (A_g type II, A_p, A_r)

- There are two kinds of deterioration dynamics:

  (1) $h(n, v)$, $c(n, v)$
  (2) $h(v)$, $c(v)$

- There are three kinds of information levels:

  (1) Level 1: (Count), i.e., (n)
  (2) Level 2: (Age, Count), i.e., (n, v)
  (3) Level 3: (Age, Count, Cost), i.e., (n, v, c)

- There are two kinds of objective criteria ($e^{-\alpha t}$):

  (1) Criterion 1: $\alpha > 0$
  (2) Criterion 2: $\alpha = 0$

Total Combinations: $2 \times 14 \times 2 \times 3 \times 2 = 672$

Figure 3.2: Possible Combinations of Modeling Factors

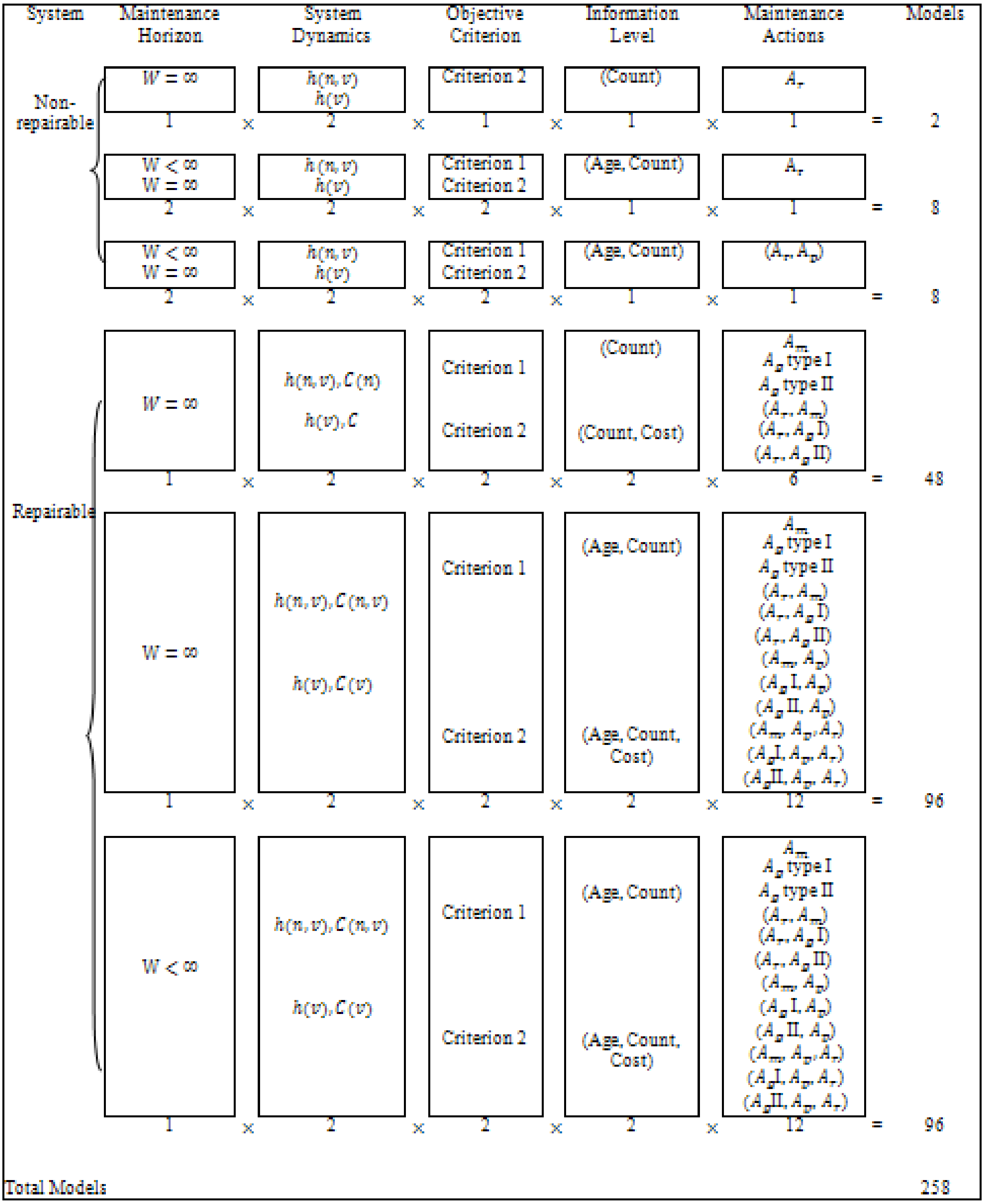

Figure 3.3: Number of Models of Interests

Figure 3.2 presents a detailed specification scheme of the six modeling factors. Figure 3.3 summarizes a total of 258 models that have been selected from the pool of 672 candidate models that have been generated from the full combinations.

In principle, for any given model constructed under this proposed framework, there is a corresponding optimal maintenance policy. Figure 3.4 illustrates such a relationship. The optimization methodology to be presented in the next section will not only provide a standard optimization procedure for any of the models in this framework, but also enables the comparison of policies through a comparison of their associated models.

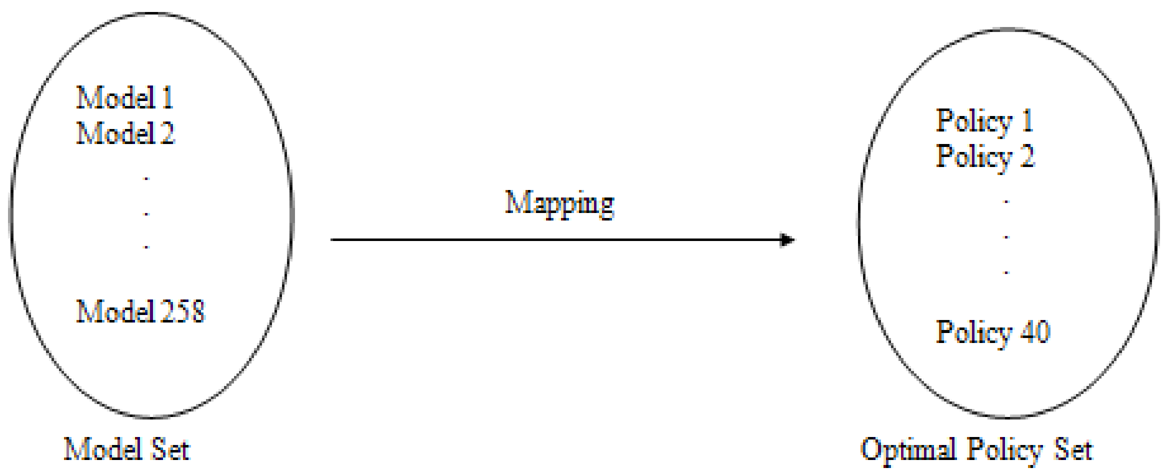

Figure 3.4: Mapping of Models and Policies through Optimization

As an illustration, let us consider a maintenance model involving random repair cost where the so-called Repair-Cost-limit (RCL) policy is known to be optimal (see Beichelt (1993), Jiang and Cheng (1995), Jiang *et al.* (1998) and Jiang *et al.* (2001)). In this model, the repair cost limit function $RCL(v)$ can be intuitively understood as the residual value of the system at (virtual) age v. When the system fails, the repair option will be taken if and only if the repair cost is lower than the residual value. When the residual value reaches 0, a preventive replacement is to be taken because the system has no residual value left. Figure 2.3 in chapter two depicts such a policy. In particular, when the repair cost degenerates to a deterministic function of age v, the RCL-policy degenerates to the well-known ARL Age-Limit Policy. Accordingly, any failure should be removed by a repair if it occurs before age ARL and by a replacement if after ARL. With preventive replacement considered, the preventive replacement should be scheduled at APL if the replacement has not been performed. In this way, the (ARL, APL)-policy can be interpreted as a special case of the (RCL, APL)-policy, and it will be clearer later in Chapter 5 that it can also be understood as a projection of the $RCL-$policy from the full information filtration (c, n, v) onto a sub-filtration containing only partial information on (n, v). See also Figure 5.1 for additional details.

After constructing concrete models in the modeling framework, an immediate task is to look for a unified treatment so that these models can be solved and their associated optimal policies can be derived in a standard procedure. Using the unified treatment presented in the following sections, we can convert the conceptual models to mathematical models and derive the optimal maintenance policies.

3.2 Optimization Methodology

The mathematical foundation for solving models arising from the proposed modeling framework is the theory of optimal stopping and martingale analysis. For detailed presentations, see Chow *et al.* (1971), Shiryayev (1978), Jensen (1989), and Jiang *et al.* (2001).

3.2.1 Optimal Stopping Formulation

The maintenance optimization problems for the models constructed under the proposed framework are naturally formulated as optimal stopping problems as described below.

Let (Ω, F, P) be a complete probability space and (F_t), $t \in R_+$, be a right continuous complete filtration, i.e., a monotone increasing family of sub-σ-fields of F with $F_t = F_{t+}$ containing all P-negligible sets of F. Let $Y = (Y_t)$, $t \epsilon R_+$ be a real right-continuous stochastic process adapted to (F_t), i.e., Y_t is F_t-measurable for all $t \epsilon R_+$. Admissible stopping time τ of (F_t) is a random variable τ: $\Omega \rightarrow R_+$ with $\{\tau \leq t\} \in F_t$ for all $t \epsilon R_+$. Then the stopping problem is to find a stopping time $\tau^* \in C$ with $EY_{\tau^*} = \inf\{EY_\tau | \tau \epsilon C\}$, where

$C = \{\tau | \tau(F_t) - \text{stopping time}, EY_\tau^- < \infty\}$. Intuitively, filtration (F_t) stores all the historical information up to time *t* that is available to the decision maker.

The process *Y* is further called Smooth Semi-martingale (SSM) if it has a representation $Y_t = Y_0 + \int_0^t f_t \ dx + M_t$, where $E|Y_0| < \infty$, (f_t) is a progressively measurable progress, $E(\int_0^t |f_t| dx) < \infty$, $\forall\, t \epsilon R_+$, and $M = (M_t) \epsilon M_0$. In short, $Y = (f, M)$. Essentially, this representation separates the informative trend of deterioration (f_t) with random fluctuation (M_t).

A combination of the following properties is called the monotone condition:

i). $\{f_t \leq 0\} \subset \{f_{t+x} \leq 0\}$, $\forall t, x \epsilon\ R_+$; and

ii). $\bigcup_{t \epsilon R_+} \{f_t \leq 0\} = \Omega$.

Stopping time $\sigma = \inf\{t | f_t \leq 0\}$ is called the infinitesimal-look-ahead (ILA) stopping time. Under some mild integrability conditions that are usually satisfied in maintenance systems, the ILA-stopping time gives rise to the simplest form of maintenance policy in the control-limit form. Similarly, for the discrete time system, the so-called One-step-Look-Ahead (OLA) stopping time is optimal under the corresponding monotone condition.

When the monotone condition does not hold, a weaker sense of mononoticity, the conditional monotone condition (see Jensen and Hsu (1993)) together with an additional Markov property which usually holds for age-based maintenance models, often yields the control-limit form of optimal maintenance policies.

3.2.2 Optimization Procedure

The optimal stopping formulation ensures the optimality in the most general sense. Yet the explicit solution, in particular, the control-limit form solution, relies on the aforementioned monotone condition or a combination of the conditional monotone and the Markov property. A suite of supporting techniques is called upon to simplify the general formulation step by step until reaching the nice-structured optimal policy.

Step 1: λ-maximization technique. This technique is introduced for the following two purposes:

(i) It transforms the original objective function to an additive function, where optimal stopping theory can be readily applied.

(ii) It absorbs a substantial amount of formulating and computational complexity into parameter λ. More insights on intrinsic properties such as monotonicity and convexity can be gained, and less burden of numerical computation can be achieved.

Step 2: Characterization of stopping time for jump process. Noting the system failure dynamics forms a jump process; thus the explicit characterization of stopping time for the jump process in continuous time developed in Makis *et al.* (2000) leads to a critical simplification: each optimal stopping time can be decomposed into two simpler ones, one for failure replacement and the other for preventive replacement. With the decomposition, the two simpler stopping times can be obtained in sequence without loss of optimality.

Step 3: Smooth semi-martingale decomposition. This technique is essentially to separate the deterioration trend (the progressive part), with the noninformative randomness (the martingale part), and allow one to consider only the trend part without loss of the optimality (guaranteed by the Optional Stopping Theorem). Critical properties such as monotonicity and Markov can be identified in this step.

Step 4: Dynamical programming. With the Markov property being identified, dynamic programming can be applied to yield the dynamical equation of the value function. Further analysis of the value function reveals the nice-structure of the optimal policy.

Note that even if the optimal policy obtained after Step 4 turns out to be a Markov policy, i.e., it uses only the current information for decision making, it is still optimal in the optimal stopping sense.

In the next section, we will show how to use the optimal framework for the formulation and analysis of a wide class of maintenance decision problems. The optimization methodology is illustrated by analyzing a concrete repair/replacement model. The model, which includes some models previously studied in the literature, is a special case of the model to be studied in Chapter 4. The structure of the optimal policy is obtained under the expected total discounted cost criterion.

3.3 An Illustrative Model

We consider jointly a repair/replacement problem at failure times and a preventive replacement problem in continuous time. The former is a discrete-time stopping problem and the latter is a continuous-time one. A concrete maintenance model with solution is presented in this section. In Subsection 3.3.1, we summarize the relevant results from the optimal stopping. We will show how an originally fractional optimization problem can be transformed to a parametric optimization problem with an additive objective function using the λ-maximization technique (e.g., Aven and Bergman (1986)), and the objective function can be further simplified by removing its martingale part without the loss of optimality. The methodology presented in Sections 3.2 and 3.3.1 is applied in Subsection 3.3.2 to a general repair/replacement model, and the form of optimal policy is derived.

3.3.1 λ-Maximization, Semi-Martingale Decomposition, and Optimal Stopping on Discrete Time

In this subsection we provide the necessary results from the optimal stopping theory and introduce the λ-maximization technique and the semi-martingale decomposition for the discrete time case.

Denote $N = \{1,2, \dots\}$. Let (Ω, F, P) be a probability space and $\{F_n, n \in N\}$, be a complete filtration, i.e., $\{F_n\}$ is a nondecreasing sequence of sub-σ algebras of F such that F_n contains all P-null sets of the complete σ algebras F. Note that $\{F_n\}$ can be considered as a description of the history of some stochastic processes in discrete time and F_n then represents the σ-field of events prior to time n.

A stopping time τ is a random variable $\tau: \Omega \to N \cup \{+\infty\}$ such that $\{\tau = n\} \in F_n$ for all n. Let $\{Y_n, n \in N\}$ be a sequence of random variables such that $\{Y_n\}$ is adapted to $\{F_n\}$, i.e., $\{Y_n\}$ is $F_n -$ measurable for each n. Define

$$Y_\tau = \sum_{n=1}^{+\infty} I_{\{\tau=n\}} Y_n = \begin{cases} Y_n & on\ \{\tau = n\} \\ \limsup_{n\to+\infty} Y_n, & on\ \{\tau = +\infty\} \end{cases} \tag{3.1}$$

where I is the set indicator function, and $I_A(w) = 1$ if $w \in A$ and zero otherwise. The optimal stopping problem is formulated as follows. Find a stopping time τ^*, if it exists, such that

$$EY_{\tau^*} = \sup\{EY_\tau : \tau \in C'\}, \tag{3.2}$$

where C' is the class of stopping times such that EY_τ exists.

It can be shown that it is sufficient to consider the following class of stopping times:

$$C = \{\tau | \tau \text{ is an } (F_n) - \text{stopping time}, EY_\tau^- < +\infty\}$$

i.e., $\sup_{C'} EY_\tau = \sup_C EY_\tau$.

Let $C_n = \{\max(\tau, n) : \tau \in C\}$ and define

$$r_n = \text{ess sup}_{C_n} E(Y_\tau | F_n), \tag{3.3}$$

$$\sigma_n = \begin{cases} \text{first } i \geq n & \text{such that } Y_i = r_i \\ \infty & \text{if no such } n \text{ exists} \end{cases} \tag{3.4}$$

$$\sigma \equiv \sigma_1$$

Note that r_n is essential supremum of $\{E(Y_\tau|F_n), \tau \in C_n\}$, if $P\big(r_n \geq E(Y_\tau|F_n)\big) = 1$ for every $\tau \in C_n$ and if $r_n^{'}$ is any random variable such that $P\left(r_n^{'} \geq E(Y_\tau|F_n)\right) = 1$ for every $\tau \in C_n$, then $P(r_n^{'} \geq r_n) = 1$. We have the following result.

Theorem 3.1. [Chow *et al.* (1971), Theorem 4.5', P.82] If $E(\sup Y_n^+) < \infty$, then σ is optimal in C.

For the Markov case, the optimal stopping time has a more specific form.

Theorem 3.2. [Corollary 2.4, Jiang *et al.* (1998)] Let $\{X_n\}$ be a homogeneous Markov chain and $Y_n = \sum_{k=1}^{n-1} \theta_k(X_k) + \varphi_k(X_n)$. Define

$$V_n(x) = \sup_\tau E^x\Big(\sum_{k=1}^{n-1} \theta_k(X_k) + \varphi_k(X_n)\Big), n = 1,2, \ldots \tag{3.5}$$

Then the optimal stopping time σ for sequence $\{Y_n\}$ has the form:

$$\sigma = \begin{cases} \text{first } n \geq 1 & \varphi_n(X_n) \geq V_n(X_n) + \theta_n(X_n) \\ \infty & \text{if no such } n \text{ exists} \end{cases} \tag{3.6}$$

Next, we will introduce the λ-maximization technique (see Aven and Bergman (1986)) to solve the following minimization problem.

Let $A_n = \sum_{i=0}^{n} a_i \geq 0$, $B_\mathrm{n} = \sum_{i=0}^{n} b_i \geq 0$. To find

$$\lambda^* = \inf_\tau \frac{EA_\tau}{EB_\tau} \tag{3.7}$$

We can solve the following parametric optimal stopping problem (with parameter λ).

Find

$$V(\lambda) = \sup_{\tau \in D} E(\lambda B_\tau - A_\tau), \tag{3.8}$$

where *D* is the class of $\{F_n\}$-stopping times such that $E(A_\tau) < \infty$ and $E(B_\tau) < \infty$. One can show that it is sufficient to consider class *D* of stopping times.

The optimal value λ can be obtained as

$$\lambda^* = \sup\{\lambda : V(\lambda) < 0\}. \tag{3.9}$$

If there is a λ such that $V(\lambda) = 0$, then $\lambda = \lambda^*$.

Assume that $V(\lambda^*) = 0$ and let τ^* be the stopping time maximizing the right-hand side of (3.8) for $\lambda = \lambda^*$. Then we have from (3.8),

$0 = E(\lambda^* B_{\tau^*} - A_{\tau^*}) = \lambda^* E(B_{\tau^*}) - E(A_{\tau^*})$, So that $\lambda^* = E(A_{\tau^*})/E(B_{\tau^*})$.

On the other hand, since $V(\lambda^*)$ is the supremum, we have from (3.8). That for any $\tau \in D$, $0 \geq E(\lambda^* B_\tau - A_\tau)$ or, $\lambda^* \leq E(A_\tau)/E(B_\tau)$ so that λ^* is the infimum defined by (3.7) and τ^* is the optimal stopping time minimizing $E(A_\tau)/E(B_\tau)$.

As noted earlier, this technique transforms the original fractional optimization problem (3.7) (which is difficult to analyze) into a parametric optimization problem with an additive objective function.

The problem can be further simplified by applying the semi-martingale decomposition that will remove the martingale part. Semi-martingale decomposition in discrete time can be described as follows.

Assume that $Y_n = \sum_{i=0}^{n} f_i$ is (F_n) −adapted and integrable for each n.

Then it can be decomposed into two parts: $Y_n = \bar{Y}_n + M_n$, where M_n is (F_n) −martingale, and $(\bar{Y}_n)$ is (F_{n-1}) adapted. This decomposition is unique, $\bar{Y}_n = \sum_{i=0}^{n} E(f_i | F_{i-1})$, and $M_n = \sum_{i=0}^{n}(f_i - E(f_i | F_{i-1}))$, where F_{-1} is the trivial $\sigma -$algebra and $F_0 \subset F_1$. Note that (M_n) is (F_n) −martingale if $E(M_m | F_n) = M_n$ for $m > n$.

Obviously, $(\bar{Y}_n)$ is (F_{n-1}) –measurable so$(\bar{Y}_n)$ is (F_{n-1}) − adapted.

Next, we show that (M_n) is (F_n) −martingale.

We have for any $m > n$,

$$E(M_m | F_n) = E\left(\sum_{i=0}^{m}(f_i - E(f_i | F_{i-1})) | F_n\right) \tag{3.10}$$

$= E(\sum_{i=0}^{m} f_i \,|F_n) - \sum_{i=0}^{n} E(f_i \,|F_{i-1}) - \sum_{i=n+1}^{m} E(f_i \,|F_n)$

(because $E(f_i|F_{i-1})$ is F_n- measurable for $i \leq n$)

$= E(\sum_{i=0}^{n} f_i \,|F_n) - \sum_{i=0}^{n} E(f_i \,|F_{i-1}) = M_n$

(because $\sum_{i=0}^{n} f_i$ is F_n- measurable).

Denote $Y_n(\lambda) = \sum_{i=0}^{n}(\lambda b_i - a_i) \equiv \sum_{i=0}^{n} f_i$ and $\bar{Y}_n(\lambda) = \sum_{i=0}^{n} E(f_i|F_{i-1})$, where a_i and b_i are non-negative.

It can be shown that the optimal stopping problems for $(Y_n(\lambda))$ and $(\bar{Y}_n(\lambda))$ are equivalent, i.e.,

$$\sup_{\tau\in D} E\big(Y_\tau(\lambda)\big) = \sup_{\tau\in D} E(\bar{Y}_\tau(\lambda)). \tag{3.11}$$

The process presented in this subsection will be applied in the next subsection to a maintenance model with Kijima Type II general repair.

3.3.2 A Repair / Replacement Model and Its Optimal Policy

In this subsection, we consider a repairable system subject to random failure. First, we provide a detailed description of the model and then formulate a repair/replacement problem in the optimal stopping framework. We will make the following assumptions.

(i) Maintenance Horizon: The maintenance service period is infinite

(ii) Deterioration Dynamics: The time to the *n*-th failure is a generally distributed random variable with distribution function$F_n(t)$, density$f_n(t)$, and hazards rate$h_n(t) = f_n(t)/(1 - F_n(t))$, which is a continuous and non-decreasing function of t.

(iii) Maintenance Actions: Three types of maintenance actions are available: general repair, failure replacement and preventive replacement. All maintenance actions take negligible time.

(iv) Cost Structure: At failure time, the system can be either generally repaired at cost $C(n, v)$, where n is the number of repairs since last replacement and v is the virtual age of the system, or replaced at a cost of $C_r = C_f + C_p$. The preventive replacement can be performed at any time prior to failure with a cost of $C_p < C_r$. We assume that C_p and $C_r = C_f + C_p$ are constants and $C(n, v)$ are random variables stochastically increasing in n and v. Further, we assume that $EC(n, v)$ is continuous in n and v, and that $C_f \leq C(n, v) \leq C_r$ without loss of generality.

(v) Information Level: Full information is available.

(vi) Objective Criterion: The objective is to find the repair/replacement policy that minimizes the expected total discounted cost.

Consider the repair/replacement policy determined by a sequence of stopping times $\pi = \{\tau_i, i = 1,2, \dots\}$, where τ_i is the replacement time of the i-th unit. Then the total discounted cost over an infinite time horizon has the form

$$K_\alpha(\pi) = \sum_{i=1}^{\infty} TC_\alpha(\tau_i) \exp\left(-\alpha \sum_{j=1}^{i-1} \tau_j\right) . \tag{3.12}$$

It is intuitive and not difficult to see that it is sufficient to consider the stationary policy, i.e., $\tau_i \equiv \tau$ for all i. The original optimization problem can then be reduced to the following optimal stopping problem with a fractional objective function: Find

$$\lambda^* = \inf_\tau \frac{E(TC_\alpha(\tau))}{E(1-e^{-\alpha\tau})} , \tag{3.13}$$

and the optimally stopping time τ_α^* (if it exists), minimizing the expression in (3.13).

Using the notation in the above section, the total discounted cost $TC_\alpha(\tau)$ has the form

$$\begin{aligned} TC_\alpha(\tau) &= \sum_{i=0}^{\sigma-1} \left((C(i, V_i) - C_f) e^{-\alpha S_i} + C_f e^{-\alpha S_{i+1}} I_{\{S_{i+1} \leq T_i\}} \right) I_{\{S_i \neq T_i\}} \prod_{j=0}^{i-1} I_{\{S_{j+1} \leq T_j\}} \\ &\equiv TC_\alpha(\sigma, \{T_i\}) , \end{aligned} \tag{3.14}$$

where $C(0,0) \equiv C_r$, S_i is the calendar time of the *i*-th failure, $S_0 = 0$, $N(t) = \sum_{i=1}^{\infty} I_{\{S_i \le t\}}$ is the number of failures before time t, where I is the set indicator function. V_i is the virtual age of the system just prior to the *i*-th failure. By definition, $V_i = \theta(V_{i-1} + S_i - S_{i-1})$.

Next, for the denominator in (3.13)

$$(1 - e^{-\alpha\tau})/\alpha \quad = \int_0^{\infty} e^{-\alpha t}\, I_{(\tau > t)} dt \tag{3.15}$$

$$= \sum_{i=0}^{\sigma-1} \int_{S_i}^{T_i} e^{-\alpha t}\, I_{(S_{i+1} \ge t)} dt \prod_{j=0}^{i-1} I_{\{S_{j+1} \le T_j\}}$$

$$\equiv \tau_\alpha(\sigma, \{T_i\}).$$

Applying the λ-maximization technique and the semi-martingale decomposition, we obtain the formulas for $Y_\lambda^\alpha(\sigma, \{T_i\})$ and $\bar{Y}_\lambda^\alpha(\sigma, \{T_i\})$, corresponding to $Y_\lambda(\sigma, \{T_i\})$ and $\bar{Y}_\lambda(\sigma, \{T_i\})$ in the previous section:

$$Y_\lambda^\alpha(\sigma, \{T_i\}) = \quad \sum_{i=0}^{\sigma-1} (\lambda \int_{S_i}^{T_i} e^{-\alpha t}\, I_{\{S_{i+1} \ge t\}} dt - \big(C(i, V_i) - C_f\big) e^{-\alpha S_i} - \tag{3.16}$$

$$C_f e^{-\alpha S_{i+1}} I_{\{S_{i+1} \le T_i\}}) I_{\{S_i \ne T_i\}} \prod_{j=0}^{i-1} I_{\{S_{j+1} \le T_j\}}, \text{ and}$$

$$\bar{Y}_\lambda^\alpha(\sigma, \{T_i\}) = \quad \sum_{i=0}^{\sigma-1} (-\big(C(i, V_i) - C_f\big) e^{-\alpha S_i} + \int_{S_i}^{T_i} e^{-\alpha + V_i - S_i t}\, \bar{F}_{i,V_i}(t + V_i - \tag{3.17}$$

$$S_i) \Big(\lambda - C_f h_i(t + V_i - S_i)\Big) dt) \prod_{j=0}^{i-1} I_{\{S_{j+1} \le T_j\}}.$$

Thus, the original optimal stopping problem is translated as follows:

$$\inf_\pi E\big(K_\alpha(\pi)\big) = \lambda_\alpha^*/\alpha, \tag{3.18}$$

where

$$\lambda_\alpha^* = sup\{\lambda : V^\alpha(\lambda) < 0\}, \tag{3.19}$$

and

$$\mathrm{V}^\alpha(\lambda) = sup_\sigma(sup_{\{T_i\}} E\big(\bar{Y}_\lambda^\alpha(\sigma, \{T_i\})\big)). \tag{3.20}$$

Noticing that $\bar{Y}_\lambda^\alpha$ has become additive with respective to stages indexed by i, and each of the summands is governed by the entering state, (i, V_i), the optimization problems (3.20) parameterized by λ now falls into the regime of Markov decision problems, for which the dynamical programming method is readily applied and ultimately yields the solution, which is summarized as follows.

Theorem 3.3. The optimal policy exists, and it has a form of repair-cost-limit policy for failure replacement (RCL) and an age-limit policy (ARL) for preventive replacement, i.e., the optimal replacement time τ is determined by the first S_n such that $C(n, V_n) \geq RCL(n, \theta V_n)$.

$$\tau = \begin{cases} S_n \\ S_{n-1} + (APL(n) - V_{n-1}) \end{cases}$$

where S_n is the n-th failure time, V_n is the virtual age just before the n-th failure, and $C_n \triangleq C(n, V_n)$ is the random repair cost at S_n. The repair-cost-limit RCL is calculated from

$$RCL(n, v) = \quad \frac{1}{\bar{F}_n(v)} \int_v^{APL(n)} e^{-\alpha(u-v)} \bar{F}_n(u) \{\lambda + h_n(\mathrm{u})(C_f - g(n, \theta u))\} du\,, \tag{3.21}$$

$$g(n, v) = \quad E^{C\left(n, \frac{v}{\theta}\right)}[(RCL(n, v) - C\left(n, \frac{v}{\theta}\right))^+] \tag{3.22}$$

$$APL(n) = \quad inf\{u\colon \lambda \leq h_n(u)(C_f - g(n, \theta u))\} \tag{3.23}$$

The optimal total discounted cost is λ_α^*/α, where λ_α^* corresponds to the λ that satisfies a boundary condition: $RCL(0,0) = C_p$. Intuitively this condition means that the residual value of a new system is identical to the cost of a preventive replacement.

3.4 Summary

In this chapter, we have proposed a unified maintenance modeling framework and constructed 258 maintenance models under the unified modeling framework. A unified optimization methodology for these models is developed based on optimal stopping theory. An illustrative model is constructed under this modeling framework and solved by following the proposed optimization procedure.

A non-trivial extension to this model is to be analyzed in greater depth in the next chapter, which switches from a model with an infinite horizon to one on a finite horizon. While the finiteness of the model horizon endows natural additivity to its objective function, which consequently spares the need of the λ-maximization step, the newly introduced constraints on the remaining service period make overall analysis in particular, analysis of monotonicity, harder. The solution to the current model, especially the conversion from the original problem to a dynamical programming problem, allows us to start the analysis with the dynamical programming formulation without loss of generality.

CHAPTER 4
A COMPREHENSIVE MAINTENANCE MODEL

In this chapter, a comprehensive maintenance model is proposed and solved. This model is in fact an extension of the model discussed in Section 3.3 by further restricting its horizon to a finite one. The model is in fact one of the two most comprehensive models among those 258 models in the sense that to specify each of the six model characteristics, we always choose the least restrictive alternatives.

It is shown that a form of repair-cost-limit and age-limit policy, denoted as $(RCL(n,v,x), APL(n,x))$, is optimal. A synopsis of this chapter is as follows: In Section 4.1, model description and main result are presented, and optimality of the repair-cost-limit and age-limit policy is proved in Section 4.2. In Section 4.3, a computational algorithm is provided and numerical examples are given in Section 4.4. The chapter is concluded with a brief summary in Section 4.5.

4.1 Model Description and the Main Results

A rigorous description of the mathematical model and its optimal policy are presented in this section. We will first introduce the model assumptions.

4.1.1 Model Descriptions

We now describe the model as follows.

(i) **Maintenance Horizon**: The maintenance service period is $[0, W]$, where $W < \infty$.

(ii) **System Deterioration Dynamics**: failure rate $h(n, v)$ is increasing in both n and v.

(iii) **Maintenance Actions**: Three types of maintenance actions are available: Type II general repair (A_g), failure replacement (A_r), and preventive replacement (A_p). All actions take negligible time.

(iv) **Cost Structure**: random repair cost $C(n, v)$ is stochastically increasing in both n and x. The repair cost $C(n, v)$ is a random variable with distribution $G_n(c)$. The repair costs of different failures are mutually independent. The cost of preventive replacements (A_p) and failure replacement (A_r) are denoted by C_p and $C_{r,}$ respectively. $C_f = C_r - C_p$ is denoted as the loss due to a failure event.

(v) **Information Level**: Full information on age, count, and cost is available.

(vi) **Objective Criterion:** The objective is to minimize the expected total discounted costs. The discount factor is $e^{-\alpha t}$.

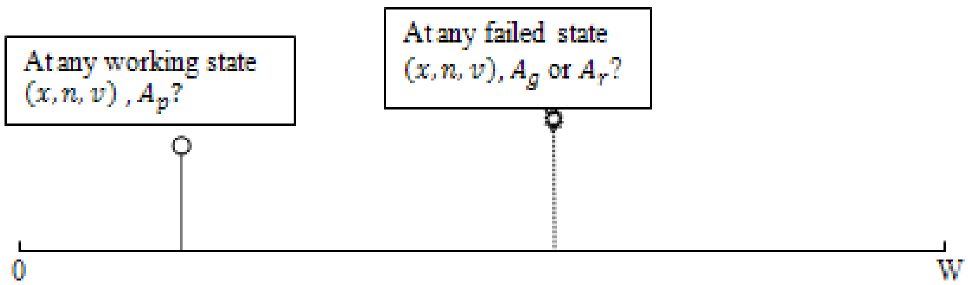

Figure 4.1: Fundamental Maintenance Problems

It is noticed that at any point in time, whether the system is working or not, there always exist two maintenance options. When the system fails, then repair (A_g) or replacement (A_r) will be performed on the failed system. When the system is working, one can choose to leave it as is without any intervention or carry out a preventive replacement with a cost of C_p. Therefore, the two fundamental maintenance problems can be described as follows: 1) whether and when a preventive replacement A_p should be taken when the system is in working condition; 2) how to choose between repair A_g and replacement A_r when the system is in failed condition. Figure 4.1 illustrates the two fundamental questions.

4.1.2 Main Results

The main results of this chapter are summarized in the following theorem which characterizes the optimal strategy and the optimal cost functions.

Theorem 4.1: For the model with Type II general repair on a finite horizon, the optimal policy is of $(RCL(n,v,x), APL(n,x))$ form, i.e.,

(i) For preventive replacement A_p, there exists a control limit, $APL(n,x)$, such that for the system resuming its service from state (n,v,x), where n is the number of repairs (or failures) since last replacement, x is the remaining service time, and v is the virtual age. A preventive replacement A_p is taken as soon as the system's virtual age reaches $APL(n,x+v)$.

(ii) Failure replacement A_r is chosen for the failed system from state (n,v,x) if the repair cost $C \geq RCL(n,\theta v,x)$. Here, n is the number of failures since last replacement, x is the remaining service time, and v is the virtual age prior to failure.

(iii) The optimal cost for working system starting at state (n,v,x), denoted by $W^n(x,v)$, increases in n, v, and x.

(iv) $RCL(n,v,x)$ decreases in n and v, and $APL(n,x)$ decreases in n.

Figure 4.2 illustrates the random cost limit $RCL(n,v,x)$ with respect to a fixed remaining maintenance service time x_0. The vertical axis depicts the maintenance costs. The horizontal axis represents the age of this system. The solid curves represent the repair cost limits which decrease in age v and in failure count n. Similarly, the age-limit function $APL(n,x)$ decreases in n as illustrated in Figure 4.3.

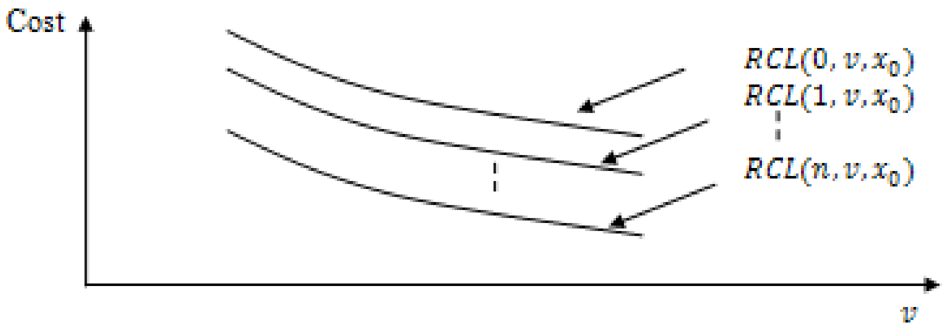

Figure 4.2: Random Cost Limit

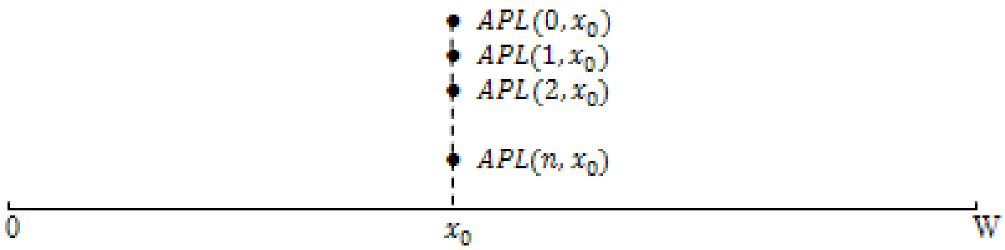

Figure 4.3: Age Limit

Figure 4.4 depicts $RCL(n,v,x)$ function as a series of surfaces indexed by n where each surface is parameterized v and x. A sample path of system dynamics is also depicted.

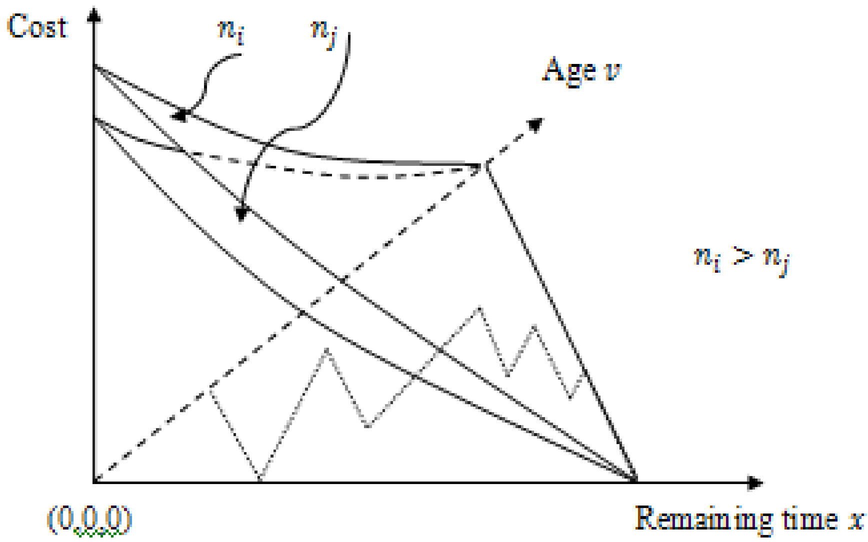

Figure 4.4: Repair-Cost-Limit Function and a Sample Path of the System

4.2 Optimality of Repair-Cost-Limit & Age-Limit Policy

Taking the experience gained from Section 3.3, we can formulate the optimization problem as a Markov decision problem without loss of optimality.

Let $W^n(x,v)$ be the optimal cost for the system that is currently in working condition at state (x,n,v), where n is the number of repairs that the system has received since last replacement, x is the remaining service time, and v is the virtual age.

Let $Z^n(x,v)$ be the optimal cost for the system that is currently in failed condition at state (x,n,v), where n is the number of repairs that the system has had done since last replacement and before current failure, x is the remaining service time, and v is the virtual age right before current failure.

At the failure epoch, $Z^n(x,v)$ satisfies the following equation:

$$\begin{aligned} Z^n(x,v) &= E^{C(n,v)}\left[\left(C(n,v)+W^n(x,\theta v)\right)\wedge\left(C_p+W^0(x,0)\right)\right] \\ &= \int_0^{RCL(x,n,v)}(C+W^n(x,\theta v))dG^{(n,v)}(r)+\bar{G}^{(n,v)}\left(RCL(x,n,v)\right)(C_p+W^0(x,0)), \end{aligned} \tag{4.1}$$

where

$$RCL(x,n,v) = C_p+W^0(x,0)-W^n(x,\theta v) \tag{4.2}$$

is the repair cost limit function, $G^{(n,v)}$ and $\bar{G}^{(n,v)}$ are the distribution and survival functions of repair cost of the n-th failure at virtual age v, respectively.

Equation (4.1) means that during the failure epoch, the system will be either generally repaired with the cost of $(C(n,v)+W^n(x,\theta v))$ or replaced with the cost of $(C_p+W^0(x,0))$ at the n-th failure epoch. Thus, a decision on failure replacement is made if and only if $C(n,v)+W^n(x,\theta v)\geq(C_p+W^0(x,0))$, i.e., $C(n,v)\geq RCL(x,n,v)$ for RCL function satisfied (4.2).

Let $F_{n,v}(u)=\frac{F_n(u)}{\bar{F}_n(v)}\triangleq P_r(\xi_n<u|\xi_n>v)$, which is the conditional distribution of the n-th system failure time, ξ_n, given it has survived till v.

When the system is in the working state, $W^n(x,v)$ satisfies the following equation (4.3)

$$W^n(x,v) = \inf_{v_1\in[v,x+v]}\left[\int_v^{v_1}\left(Z^{n+1}(x+v-u,u)+C_f\right)e^{-\alpha(u-v)}\,dF_{n,v}(u) + \bar{F}_{n,v}(v_1)e^{-\alpha(v_1-v)}(C_p + W^0(\,x+v-v_1,0))\right] \wedge \int_v^{x+v}(Z^{n+1}(x+v-u,u) + C_f)e^{-\alpha(u-v)}dF_{n,v}(u). \tag{4.3}$$

In the dynamical equation (4.3), $e^{-\alpha(u-v)}$ is the discounted factor, $\int_v^{v_1}(\,Z^{n+1}(x+v-u,u)+C_f)e^{-\alpha(u-v)}dF_{n,v}(u)$ is the cost when the system fails before preventive replacement at virtual age v_1. $\bar{F}_{n,v}(v_1)e^{-\alpha(v_1-v)}(C_p+W^0(\,x+v-v_1,0))$ corresponds to the cost if no failure occurs before virtual age v_1 and a preventive replacement is taken at virtual age v_1. $\int_v^{x+v}(\,Z^{n+1}(x+v-u,u)+C_f)e^{-\alpha(u-v)}dF_v(u)$ represents the cost under the strategy that no preventive replacement will be taken.

Now, define

$$v(n,x) \quad \triangleq inf\{v|W^n(x,v) = C_p + W^0(x,0)\}\,, \tag{4.4}$$

and

$$APL(n,x) \triangleq inf\{v|v \geq v(n,x-v)\}. \tag{4.5}$$

If we can prove that $W^n(x,v)$ increases in v and x, then it is clear that for all $v \geq v(n,x)$, $W^n(x,v) = C_p + W^0(x,0)$. Otherwise, $W^n(x,v) < C_p + W^0(x,0)$. Thus, the preventive replacement policy exhibits an age limit form, i.e., for any given x, preventive replacement needs to be taken as soon as $v \geq v(n,x)$. As the system state (n,v,x) evolves along the trajectory $(n,v+u,x-u)$ for $u \geq 0$, thus the first u that hits the preventive replacement limits $v(n,x+v-u)$ becomes the optimal preventive replacement time for the system starting from state (n,v,x).

We now list and prove the desired monotonicity of $W^n(x,v)$ in Lemma 4.1,

Lemma 4.1: $W^n(v,x)$ increases in n, v, and x.

Proof. According to Equation (4.1), $Z^n(x,v)$ will be increasing in (x,n,v) as long as $W^n(x,v)$ increases in (x,n,v). We now prove the monotonicity of $W^n(x,v)$ through mathematical induction. The basic idea to be applied is described as follows: Let $W_m^n(x,v)$ be the optimal cost for the working system starting from condition (n,v), having a remaining service time x, and is allowed for up to m maintenance actions. If we can show that $W_m^n(x,v)$ is monotomically increasing in x, n, v, and m, the monotomicity of $W^n(x,v)$ is implied in the convergence of $W_m^n(x,v)$ to $W^n(x,v)$.

We apply mathematical induction to prove the monotonicity of $W_m^n(v,x)$ in n,m, v and x.

(i). Let $W_0^n(v,x) \equiv 0, \forall\, n \leq N$.

Here consider first the case, where $C(n,v) \equiv C_p$, for all $n \geq N < \infty$. i.e., when n is sufficiently large, the system become non-repairable. This restriction is not essential because we have $C(n,v) \xrightarrow{n\to\infty} C(\infty,v)$. We can consider approximation $\forall\, n \geq N, C(n,v) = C(\infty,v)$, and argument is going to be similar.

So, $W_0^n(v,x)$ increasing with (n,v,x) holds true.

(ii). Let $W_i^j(v,x)$ increase with i,j,v, and x for all $v,x \geq 0, i < m$.

(iii). If we can confirm the following three claims (A), (B) and (C), the proof is done.

(A) $W_{m-1}^{n-1}(v,x) \leq W_m^{n-1}(v,x) \leq W_m^n(v,x)$

(B) $W_m^n(v,x) \leq W_m^n(V,x)$ if $v \leq V$

(C) $W_m^n(v,x) \leq W_m^n(v,X)$ if $x \leq X$

Proof of (A):

Assume it is true for $j \geq n$, $W_m^j(v,x) \leq W_m^{j+1}(v,x)$,

$$W_m^{n-1}(v,x) = \quad inf_{v_1 \in [v,x+v]}[\int_v^{v_1} (Z_m^n(u,x+v-u) + C_f) e^{-\alpha(u-v)} dF_{n-1,v}(u) +$$

$\bar{F}_{n-1,v}(v_1)e^{-\alpha(v_1-v)}(C_p + W_{m-1}^0(0, x + v - v_1))] \ \wedge$

$\int_v^{x+v}(Z_m^n(u, x + v - u) + C_f)e^{-\alpha(u-v)}dF_{n-1,v}(u)$

Noticing

$Z_m^n(v,x)= E^{C(n,v)}\big[(C(n,v) + W_{m-1}^n(v,x)) \wedge \ (C_p + W_{m-1}^0(0,x))\big]$

$\leq E^{C(n+1,v)}\big[(C(n+1,v) + W_{m-1}^{n+1}(v,x)) \wedge \ (C_p + W_{m-1}^0(0,x))\big]$

$= Z_m^{n+1}(v,x)$

Therefore, $W_m^{n-1}(v,x) \leq W_m^n(v,x)$ for all n.

Similarly,

$Z_{m-1}^n(v,x) \leq Z_m^n(v,x)$, $W_{m-1}^0(0,x) \leq W_m^0(0,x)$ and $\Rightarrow W_{m-1}^{n-1}(v,x) \leq W_m^{n-1}(v,x)$.

So, (A) is approved.

Proof of (B):

Given (m,n) and $v < V$, we need only to prove

(B-1): For any strategy with a preventive time V_1, for the system starting from state (n,V,x), there exists a strategy for the younger system (n,v,x) such that its cost is lower than the former.

i.e., for all $V_1 > V$, there exists a v_1 such that

$\int_V^{V_1}(Z_m^{n+1}(u, x + V - u) + C_f)e^{-\alpha(u-V)}dF_{n,V}(u) + \bar{F}_{n,V}(V_1)e^{-\alpha(V_1-V)}(C_p +$ (4.6)

$W_{m-1}^0(0, x + V - V_1)$

$\geq \int_v^{v_1}(V_m^{n+1}(u, x + v - u) + C_f)e^{-\alpha(u-v)}dF_{n,v}(u) + \bar{F}_{n,v}(v_1)e^{-\alpha(v_1-v)}(C_p +$

$W_{m-1}^0(0, x + v - v_1))$

(B-2): For any strategy without using preventive replacement,

$\int_V^{x+V}(Z_m^{n+1}(u, x + V - u) + C_f)e^{-\alpha(u-V)}dF_{n,V}(u)$ (4.7)

$$\geq \int_{v}^{x+v} (Z_m^{n+1}(u, x+v-u) + C_f) e^{-\alpha(u-v)} dF_{n,v}(u)$$

Proof of (B-1):

Define v_1 satisfied $\frac{\bar{F}_n(v_1)}{\bar{F}_n(v)} = \frac{\bar{F}_n(V_1)}{\bar{F}_n(V)} \Leftrightarrow \int_{v}^{v_1} h_n(u)du = \int_{v}^{V_1} h_n(u)du$, where h_n is the corresponding hazards function. It is easy to see $v_1 \leq V_1$ and $v_1 - v \geq V_1 - V$.

The proof of (B-1) proceeds with the following two cases:

Case (B-1-1): $v_1 - v < x$

Case (B-1-2): $v_1 - v \geq x$

Proof for Case (B-1-1): We prove that the two terms at both sides of (4.6) satisfy the desired inequality. For the second term, we need to prove

$$\bar{F}_{n,V}(V_1) e^{-\alpha(V_1-V)} (C_p + W_{m-1}^0(0, x+V-V_1))$$

$$\geq \bar{F}_{n,v}(v_1) e^{-\alpha(v_1-v)} (C_p + W_{m-1}^0(0, x+v-v_1))$$

By the definition of v_1, we have

$$\bar{F}_{n,v}(v_1) = \frac{\bar{F}_n(v_1)}{\bar{F}_n(v)} = \frac{\bar{F}_n(V_1)}{\bar{F}_n(V)} = \bar{F}_{n,V}(V_1).$$

Also by $v_1 - v \geq V_1 - V$, we have

$$e^{-\alpha(V_1-V)} \geq e^{-\alpha(v_1-v)},$$

And $x + V - V_1 > x + v - v_1 \Rightarrow W_{m-1}^0(0, x+V-V_1) \geq W_{m-1}^0(0, x+v-v_1)$ by induction assumption. So the desired inequality of the second term in (4.6) is proved.

For the first term, we needed

$$\int_{V}^{V_1} (Z_m^{n+1}(u, x+V-u) + C_f) e^{-\alpha(u-V)} dF_{n,V}(u)$$

$$\geq \int_{v}^{v_1} (Z_m^{n+1}(u, x+v-u) + C_f) e^{-\alpha(u-v)} dF_{n,v}(u).$$

Define $\delta = \int_{v}^{a} \frac{f_n(u)}{\bar{F}_n(v)} du = G_{\mathrm{A}}(a) = \int_{V}^{b} dF_V(u) = \int_{V}^{b} \frac{f_n(u)}{\bar{F}_n(V)} du = G_B(b)$. Then

$a = G_A^{-1}(\delta) \leq G_B^{-1}(\delta) = b$. In particular, let $\delta_1 = G_A(v_1) = G_B(V_1)$, and define

$$I(V) \triangleq \int_V^{V_1} (Z_m^{n+1}(u, x+V-u) + C_f) e^{-\alpha(u-V)} dF_{n,V}(u)$$

$$= \int_0^{\delta_1} (Z_m^{n+1}\big(G_B^{-1}(\delta), x+V-G_B^{-1}(\delta)\big) + C_f)\, e^{-\alpha(G_B^{-1}(\delta)-V)} d\delta.$$

As we have known that for all $\delta \leq 1$, $G_A^{-1}(\delta) \leq G_B^{-1}(\delta)$, and

$$v - G_A^{-1}(\delta) \leq V - G_B^{-1}(\delta) \Leftrightarrow G_A^{-1}(\delta) - v \geq G_B^{-1}(\delta) - V$$

Thus, $I(V) \geq I(v)$ and the inequality between the first term on both sides of (4.7) is proved, which completes the proof of Case (B-1-1).

Proof of Case (B-1-2): We need to show

$$\int_V^{V_1} (Z_m^{n+1}(u, x+V-u) + C_f) e^{-\alpha(u-V)} dF_{n,V}(u)$$

$$+\bar{F}_{n,V}(V_1) e^{-\alpha(V_1-V)} (C_p + W_{m-1}^0(0, x+V-V_1))$$

$$\geq \int_v^{x+v} (Z_m^{n+1}(u, x+v-u) + C_f) e^{-\alpha(u-v)} dF_{n,v}(u).$$

Recall

$$I(V) = \int_V^{V_1} (Z_m^{n+1}(u, x+V-u) + C_f) e^{-\alpha(u-V)} dF_V(u)$$

$$= \int_0^{1-\bar{F}_{n,V}(V_1)} (Z_m^{n+1}\big(G_B^{-1}(\delta), x+V-G_B^{-1}(\delta)\big) + C_f) e^{-\alpha(G_B^{-1}(\delta)-V)} d\delta$$

$$= \int_0^{1-\bar{F}_{n,v}(v_1)} (Z_m^{n+1}\big(G_A^{-1}(\delta), x+v-G_A^{-1}(\delta)\big) + C_f) e^{-\alpha(G_A^{-1}(\delta)-v)} d\delta$$

$$\geq \int_0^{1-\bar{F}_{n,v}(x+v)} \big(Z_m^{n+1}\big(G_A^{-1}(\delta), x+v-G_A^{-1}(\delta)\big) + C_f\big)\, e^{-\alpha(G_A^{-1}(\delta)-v)} d\delta$$

$$= \int_v^{x+v} (Z_m^{n+1}(u, x+v-u) + C_f) e^{-\alpha(u-v)} dF_{n,v}(u).$$

which proves case (B-1-2) and verifies Claim (B-1).

Proof of (B-2): It follows the same argument used in the proof of (B-1) Case (B-1-2).

In summary, Statement (B) is approved.

Proof of (C): We need to prove $W_m^n(v,x) \le W_m^n(v,X)$ for any $x \le X$. For the system with a longer remaining service time, X, assume the preventive replacement age is v_1. The proof of case when there is no preventive replacement follows an identical procedure. The proof proceeds with the following two cases

(C-1): $v_1 - v < x$

(C-2): $v_1 - v \ge x$

Proof of (C-1): Using the same preventive replacement policy for the system with a shorter remaining service period, we have

$$\int_v^{v_1} (Z_m^{n+1}(u, X+v-u) + C_f) e^{-\alpha(u-v)} dF_{n,v}(u)$$

$$+\bar{F}_{n,v}(v_1) e^{-\alpha(v_1-v)} (C_p + W_{m-1}^0(0, X+v-v_1))$$

$$\ge \int_v^{v_1} (Z_m^{n+1}(u, x+v-u) + C_f) e^{-\alpha(u-v)} dF_{n,v}(u)$$

$$+\bar{F}_{n,v}(v_1) e^{-\alpha(v_1-v)} (C_p + W_{m-1}^0(0, x+v-v_1)),$$

and the statement of (C-1) follows immediately from the monotonicity of $Z_m^{n+1}(u,x)$ and $W_{m-1}^0(0,x)$ in x.

Proof of (C-2): Taking no preventive replacement for the system with a shorter remaining service time, we have

$$\int_v^{v_1} (Z_m^{n+1}(u, X+v-u) + C_f) e^{-\alpha(u-v)} dF_{n,v}(u)$$

$$+\bar{F}_{n,v}(v_1) e^{-\alpha(v_1-v)} (C_p + W_{m-1}^0(0, X+v-v_1))$$

$$\ge \int_v^{v_1} (Z_m^{n+1}(u, X+v-u) + C_f) e^{-\alpha(u-v)} dF_{n,v}(u)$$

$$\ge \int_v^{x+v} (Z_m^{n+1}(u, X+v-u) + C_f) e^{-\alpha(u-v)} dF_{n,v}(u)$$

$$\ge \int_v^{x+v} (Z_m^{n+1}(u, x+v-u) + C_f) e^{-\alpha(u-v)} dF_v(u).$$

Now we have shown that $W_i^j(v,x)$ increases in i, j, v, and x and $W_i^j(v,x) \le W^j(v,x)$.

Therefore, for $i \to \infty$, $W_i^j(v,x) \to W^j(v,x)$ and $W^j(v,x)$ increases in $j, v,$ and x, which complete the proof of Lemma 4.1 and Theorem 4.1 follow immediately.

The same argument applies to the counterpart model with Kijima Type I general repair. We now list the main result without proof.

Theorem 4.2: For the model with Type I General repair on a finite horizon, the optimal policy is $(RCL(n,v,x), APL(n,v,x))$ form. i.e.,

(i) For preventive replacement A_p, there exists a control limit, $APL(n,v,x)$, such that for the system resuming its service from state (n,v,x), a preventive replacement A_p is to be scheduled at virtual age $APL(n,v,x)$.

(ii) Failure replacement A_r is chosen for the failed system from state (n,v,x) if the repair cost $C(n,v_-,x) \geq RCL(n,v_+,x)$, where v_+ denotes the virtual age after repair, and v_- the virtual age just prior to the failure.

(iii) The optimal cost for a working system starting at state (n,v,x), denoted by $W^n(x,v)$, increases in n, v, and x.

(iv) $RCL(n,v,x)$ and $APL(n,v,x)$ decrease in n and v.

4.3 Computational Algorithm

While it is possible to compute $Z^n(x,v)$ and $W^n(x,v)$ directly from Equations (4.1) and (4.3), the structural result on the form of optimal policy offers further opportunities to develop more efficient computational algorithms. In this section, we present the computational algorithm that is used to develop numerical examples in later sections.

Recall in Section 4.2, we used Equations (4.1) and (4.3), which are

$Z^n(x,v) = E^{C(n,v)}\left[(C(n,v) + W^n(x,\theta v)) \wedge (C_p + W^0(x,0))\right]$ and

$$W^n(x,v) = \inf_{v_1\in[v,x+v]}[\int_v^{v_1}(Z^{n+1}(x+v-u,u)+C_f)e^{-\alpha(u-v)}dF_{n,v}(u) + \bar{F}_{n,v}(v_1)e^{-\alpha(v_1-v)}(C_p+W^0(x+v-v_1,0))] \wedge \int_v^{x+v}(Z^{n+1}(x+v-u,u)+C_f)e^{-\alpha(u-v)}dF_{n,v}(u),$$

which becomes

$$W^n(x,v) = \inf_{v_1\in[v,x+v]}[\int_v^{v_1}(E^{C(n+1,u)}\left[\left(C(n+1,u)+W^{n+1}(x+v-u,\theta u)\right)\wedge\left(C_p+W^0(x+v-u,0)\right)\right] + C_f)e^{-\alpha(u-v)}dF_{n,v}(u) + \bar{F}_{n,v}(v_1)e^{-\alpha(v_1-v)}(C_p+W^0(x+v-v_1,0))] \wedge \int_v^{x+v}(E^{C(n+1,u)}\left[\left(C(n+1,u)+W^{n+1}(x+v-u,\theta u)\right)\wedge\left(C_p+W^0(x+v-u,0)\right)\right]+C_f)e^{-\alpha(u-v)}dF_{n,v}(u). \tag{4.8}$$

For each state (n,v,x), we introduce a new parameter, an offset k, such that $k \triangleq x+v$. The optimal preventive replacement age, according to Theorem 4.1 becomes $APL(n,k)$. To simplify the presentation, we denote $T(k) \triangleq APL(n,k)$. Then Equation (4.8) can be rewritten as Equation (4.9)

$$W^n(k-v,v) = [\int_v^{T(k)}(E^{C(n+1,u)}\left[\left(C(n+1,u)+W^{n+1}(k-u,\theta u)\right)\wedge\left(C_p+W^0(k-u,0)\right)\right]+C_f)e^{-\alpha(u-v)}dF_{n,v}(u) + \bar{F}_{n,v}(v_1)e^{-\alpha(v_1-v)}(C_p+W^0(k-T(k),0))] \wedge \int_v^k(E^{C(n+1,u)}[(C(n+1,u)+W^{n+1}(k-u,\theta u))\wedge(C_p+W^0(k-u,0))]+C_f)e^{-\alpha(u-v)}dF_{n,v}(u) \tag{4.9}$$

Notice that the value of v takes no part in the integrands in Equation (4.9). We will make use of this property to develop an efficient computational algorithm. First, for $v=0$, then we have for $W^n(x,0)$ the following equation

$$W^n(k,0) = [\int_0^{T(k)}(E^{C(n+1,u)}\left[\left(C(n+1,u)+W^{n+1}(k-u,\theta u)\right)\wedge\left(C_p+W^0(k-u,0)\right)\right]+C_f)e^{-\alpha u}dF_n(u) + \bar{F}(v_1)e^{-\alpha v_1}(C_p+W^0(k-v_1,0))] \wedge \int_0^k(E^{C(n+1,u)}\left[\left(C(n+1,u)+W^{n+1}(k-u,\theta u)\right)\wedge\left(C_p+W^0(k-u,0)\right)\right]+C_f)e^{-\alpha u}dF_n(u). \tag{4.10}$$

It is easy to see that in the equations all integrals have a shared integrand $E^{C(n+1,u)}\left[\left(C(n+1,u)+W^{n+1}(k-u,\theta u)\right)\wedge\left(C_p+W^0(k-u,0)\right)\right]+C_f)e^{-\alpha u}$, so we can define a template of integrand as follows:

$$I^n(k,v)= \int_0^v E^{C(n+1,u)}\left[\left(C(n+1,u)+W^{n+1}(k-u,\theta u)\right)\wedge\left(C_p+W^0(k-u,0)\right)\right]+C_f)e^{-\alpha u}\,dF_n(u)$$
$$= \int_0^v(\int_0^{RCL(k-u,n+1,u)}(c+W^n(k-u,\theta u))dG^{(n,u)}(c)+\bar{G}^{(n+1,u)}\big(RCL(k-u,n+1,u)\big)(C_p+W^0(k-u,0))+C_f)e^{-\alpha u}dF_n(u) \quad (4.11)$$

We also define $I_m^n(k,v)$ by substituting W^n in (4.11) with W_{m-1}^n, i.e.,

$$I_m^n(k,v)= \int_0^v(\int_0^{RCL(k-u,n+1,u)}(c+W_{m-1}^n(k-u,\theta u))dG^{(n,u)}(c)+\bar{G}^{(n+1,u)}\big(RCL(k-u,n+1,u)\big)(C_p+W_{m-1}^0(k-u,0))+C_f)e^{-\alpha u}dF_n(u) \quad (4.12)$$

Equations (4.10) and (4.9) can be rewritten as (4.13) and (4.14), respectively, as follows

$$W^n(k,0)= [I^n(k,T(k))+\bar{F}_n(T(k))e^{-\alpha T(k)}(C_p+W^0(k-T(k),0))]\ \wedge\ I^n(k,k)\,. \quad (4.13)$$

$$W^n(k-v,v)=\frac{1}{\bar{F}_n(v)}([\int_v^{T(k)}(E^{C(n+1,u)}\left[\left(C(n+1,u)+W^{n+1}(k-u,\theta u)\right)\wedge\left(C_p+W^0(k-u,0)\right)\right]+C_f)e^{-\alpha(u-v)}dF_n(u)+\bar{F}_n(T(k))e^{-\alpha(T(k)-v)}(C_p+W^0(k-v_1,0))]\ \wedge\ \int_v^k(E^{C(n+1,u)}\left[\left(C(n+1,u)+W^{n+1}(k-u,\theta u)\right)\wedge\left(C_p+W^0(k-u,0)\right)\right]+C_f)e^{-\alpha(u-v)}dF_n(u))\,. \quad (4.14)$$

$$=\frac{e^{\alpha v}}{\bar{F}_n(v)}(\big[I^n(k,T(k))e^{-\alpha T(k)}-I^n(k,v)+\bar{F}_n(T(k))e^{-\alpha T(k)}(C_p+W^0(k-v_1,0))\big]\wedge I^n(k,k)-I^n(k,v)))$$
$$=\frac{e^{\alpha v}}{\bar{F}_n(v)}[W^n(k,0)-I^n(k,v)]$$

Similarly, we have

$$W_m^n(k,0)= [I_m^n(k,T(k))+\bar{F}_n(T(k))e^{-\alpha T(k)}(C_p+W_{m-1}^0(k-T(k),0))]\ \wedge\ I_m^n(k,k)\,, \quad (4.15)$$

and

$$W_m^n(k-v,v) = \frac{e^{\alpha v}}{\bar{F}_n(v)}[W_m^n(k,0) - I_m^n(k,v)]. \tag{4.16}$$

In particular, the preventive replacement age $T(k)$ can be computed from $W^n(k,0)$ through

$$T(k) = \operatorname{argmin}\{s \in [v,k] | \; I^n(k,s) + \bar{F}_n(s)e^{-\alpha s}(C_p + W^0(k-s,0))\}, \tag{4.17}$$

and for $v \leq T(k)$, $T(k)$ is still the optimal preventive time for system operates along the path of $(n, v, k-v)$.

4.3.1 Computing Procedure

The key quantity to compute is $W^n(k-v,v)$. Due to the induction argument presented in the previous section, we know that $W_m^n(k-v,v)$ converges to $W^n(k-v,v)$ as m goes to infinity, and $W^n(k-v,v)$ converges to $W^\infty(k-v,v)$, where W^∞ is the cost function for the system with $C(n,v) \equiv C_p$. The following algorithm essentially computes W_m^n from W_{m-1}^{n+1}, $n<N$ and $m<M$ for sufficiently large M and N.

Step 1: $m=0$, $n \leq N$

For $m=0$, $W_m^n(k-v,v) = 0$ for all $n \leq N$.

Step 2: For $m>0$,

Compute $W_{m+1}^n(k,0)$ with (4.15) and (4.12) for $n < N$;

Compute $W_{m+1}^n(k,0)$ with (4.13) and (4.11) for $n = N$.

Step 3: For $n \leq N$

Compute $W_{m+1}^n(k-v,v)$ with (4.16).

Go to Step 2 if $m+1 < M$; otherwise go to Step 4.

Step 4: $n \leq N$

i) Compute the repair-cost-limit functions

$$RCL(n,v,x) = C_p + W_M^0(x,0) - W_M^n(x,\theta v)$$

ii) Compute the optimal preventive replacement ages.

$$APL(n,k) = \quad \text{argmin}\{s \in [v,k] | \; I_M^n(k,s) + \bar{F}_n(s)e^{-\alpha s}(C_p + W_M^0(\,k-s,0))\}.$$

Step 5: Stop.

This computational procedure is implemented in MATLAB. The numerical examples and the analysis to be in the next section will provide some interesting insights and confirm theoretical results on monotonicity.

4.4 Numerical Examples

The following numerical example illustrates the optimal policy as well as the optimal cost functions that have been studied in previous sections. Assume the lifetime of the system is Weibull distributed with probability distribution function

$$F(t) = 1 - e^{-t^{\beta}},$$

where β =2. In addition, let $C(n,t)$ be unifomly distributed on interval [0,1], $C_p = 2, C_f = 4$ and $W = 3$, repair degree $\theta = 0.9$, and the discount factor α = 0.2.

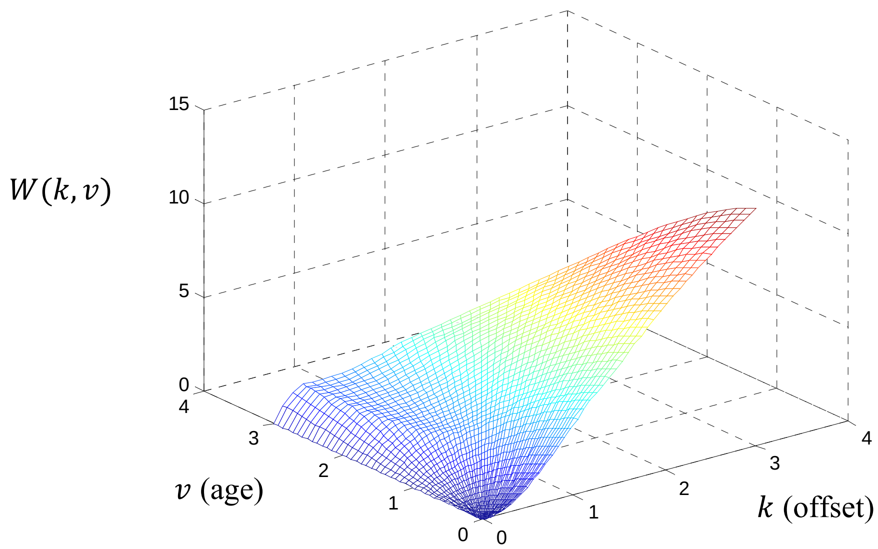

Figure 4.5: Total Cost Function $W(k,v)$

Figure 4.5 shows the optimal cost function $W(k, v)$ at possible offset k and virtual age v combinations $\{(k, v)|k = x + v \leq W,\ k, x, v \geq 0\}$ for a finite maintenance horizon $W = 3$. Figure 4.5 confirms the monotony of $W(k, v)$ in both age v and offset k. The apparent non-smoothness of the surface is caused by the discontinuity in the optimal preventive replacement policy, which is more explicitly presented in Figure 4.6, where the optimal age limit for preventive replacement against the offset k is displayed. The three curves made up by small circles represent the preventive replacement age limit, where the discontinuity is obvious.

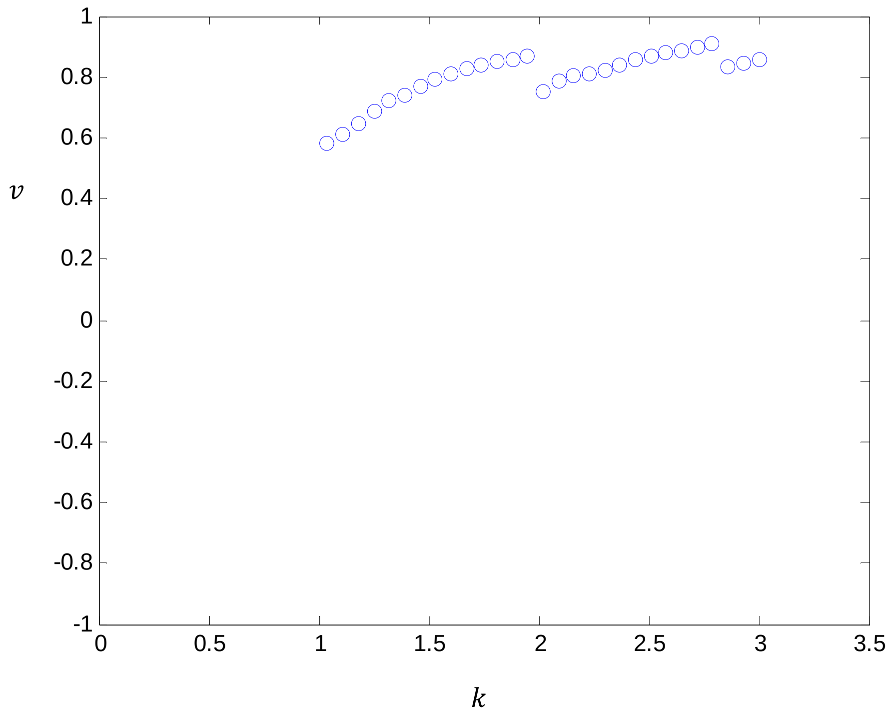

Figure 4.6: Preventive Replacement Age-Limit $APL(n, x)$

Numerical experiments have been conducted that all indicate that the general pattern of optimal the cost function as well as APL policy are rather stable. See Figure 4.7 for a sample of the experiment.

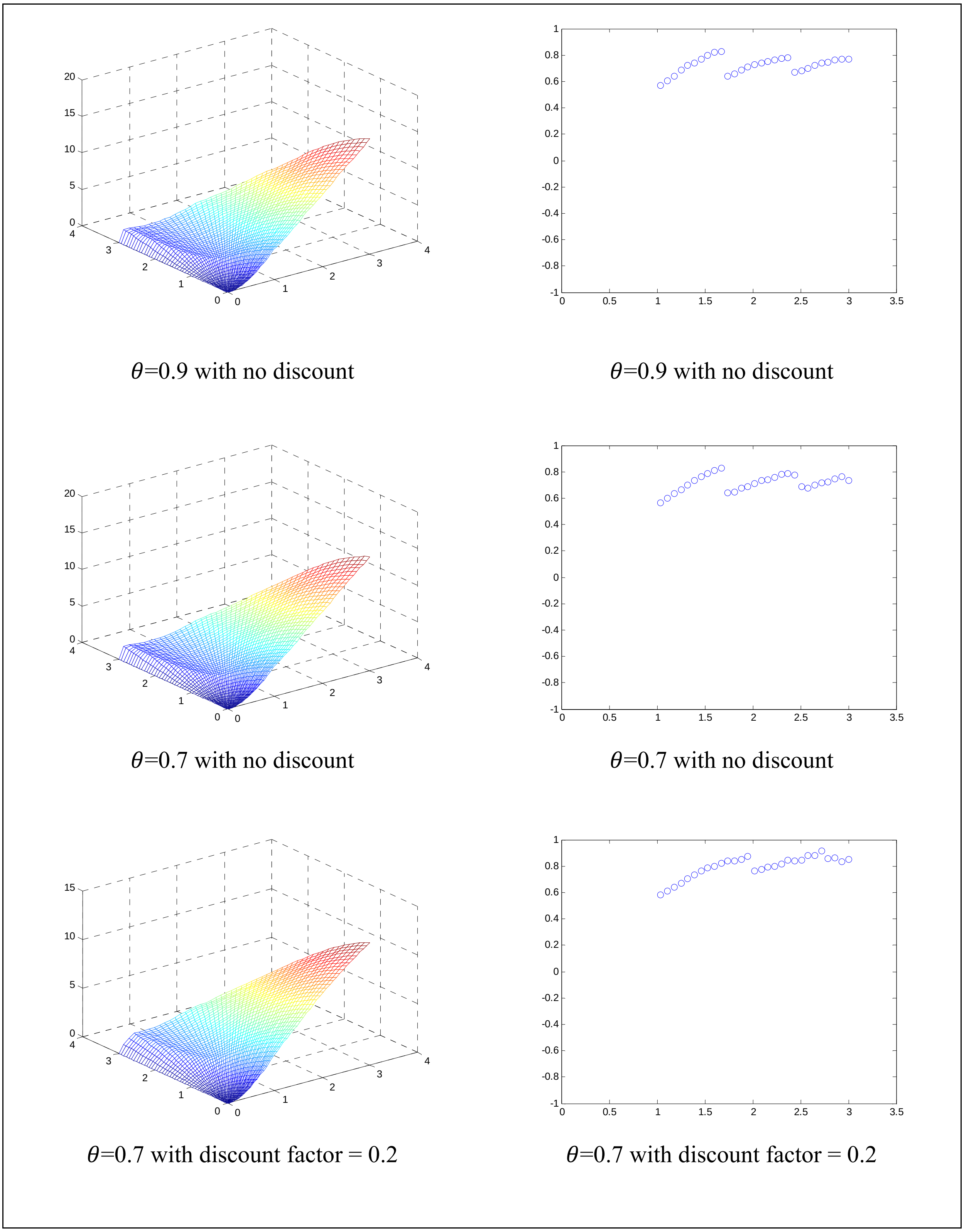

Figure 4.7: Sensitivity Analysis

The age-limit preventive replacement policy $APL(x)$ is illustrated in Figure 4.8. As shown in Figure 4.8, the service period is divided into two parts: Area 1 and Area 2. In Area 1, there will be no preventive replacement to be performed. Intuitively, for any new item having $x < 1$ unit of time left in the service period, for instance, at Point A, which represents an item with 0.6 units of time left, no preventive replacement should be considered. In Area 2, a new item starts from point B at age 0 and moves upwards D. It moves upwards because the offset $k = v + x$ is a constant before any repair or replacement. If no failure occurs before it reaches the age limit, a preventive replacement is to be performed at the age limit D. After the replacement, the age of the new item will be reset to zero, and the state of the item will be moved to point E in this figure. However, if a failure does occur at point C, and a repair is performed to rectify the failure, the system condition moves to point F which corresponds to a younger age and a smaller offset. If no failure occurs before the system reaches G, then a preventive replacement is to be carried out at G.

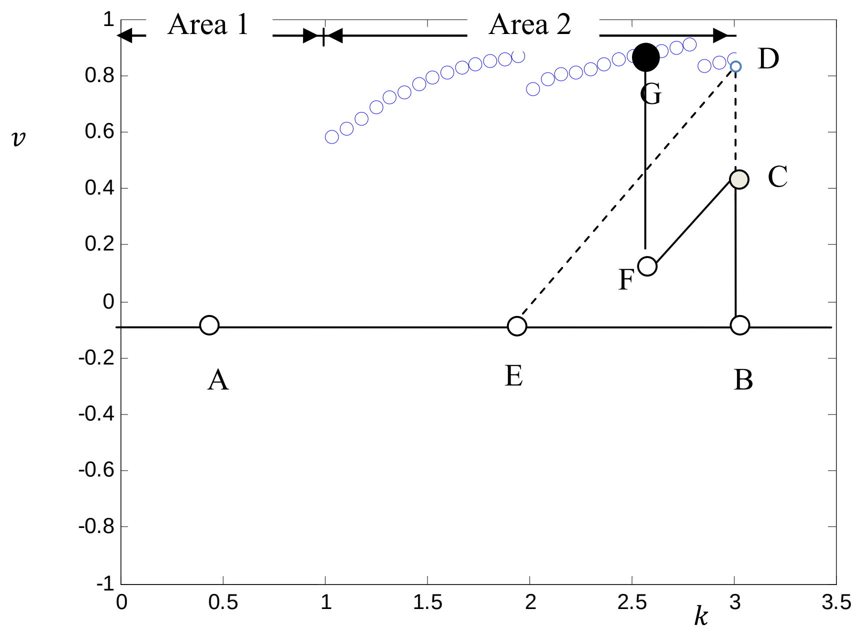

Figure 4.8: Age-Limit Preventive Replacement Policy

Figure 4.9 reveals an important characteristic that we call near-periodicity. Plot four straight lines in the figure. Three of them pass through the original with slopes of 1/4, 1/3, and 1/2, respectively, and the horizontal line depicts the average of the preventive replacement age limits. It is remarkable to notice that the preventive replacement age limits are very much determined by these straight lines. The explanation is as follows: for the age limits on the slope of 1/4, it is expected that three more preventive replacements will be performed. Therefore, the remaining service period will be covered relatively evenly by four items, each of which will have its share of 1/4 of the total service time. A longer service period corresponds to more preventive replacement actions, which stabilize the preventive replacement age limit to be around its average, which is about 0.82 in this case.

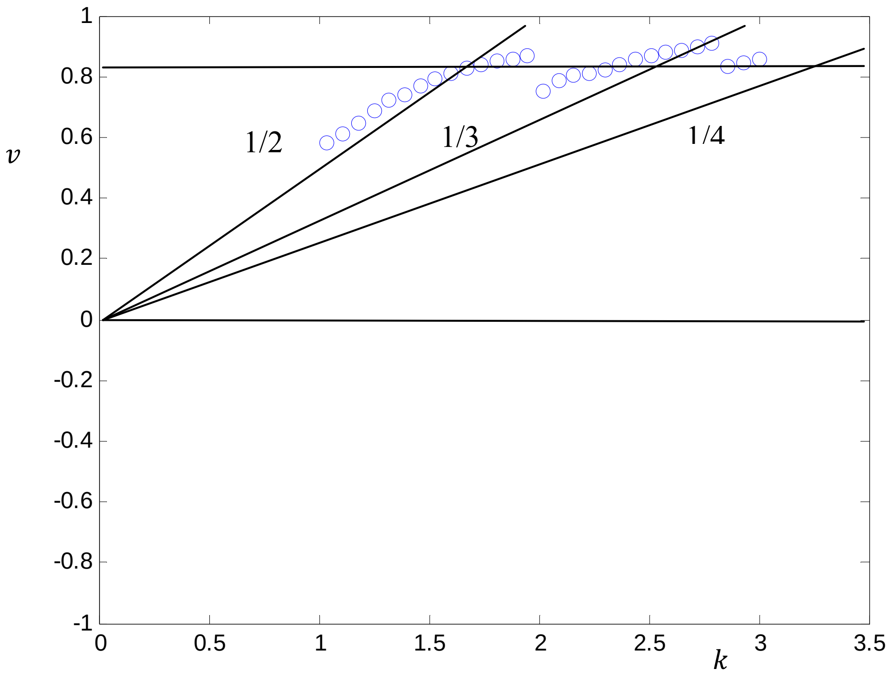

Figure 4.9: Near-Periodicity of Age Limit

An important special case of the repair-cost-limit policy is the age-limit replacement policy, which is optimal for the system with deterministic repair costs. Figure 4.10 illustrates such an important age limit: if an item fails at an age below the failure replacement limit, it will be repaired. Similarly to the case in Figure 4.8, we have three areas here. In area 1, there will be no preventive replacement or failure replacement. If any item fails, only repair will be taken. We can take Point A as an example. In Area 2, there will be no preventive replacement, but failure replacement will be considered. Look at point B in the figure; it represents that an item fails at age 0.3 and has 0.8 left during the service period. Since B is above the failure replacement limit, we will replace this item with a new one. In Area 3, if an item fails at point C, repair will be chosen; if an item fails at point D, the item will be replaced. If an item reaches the preventive replacement limit, we will conduct the preventive replacement at point E.

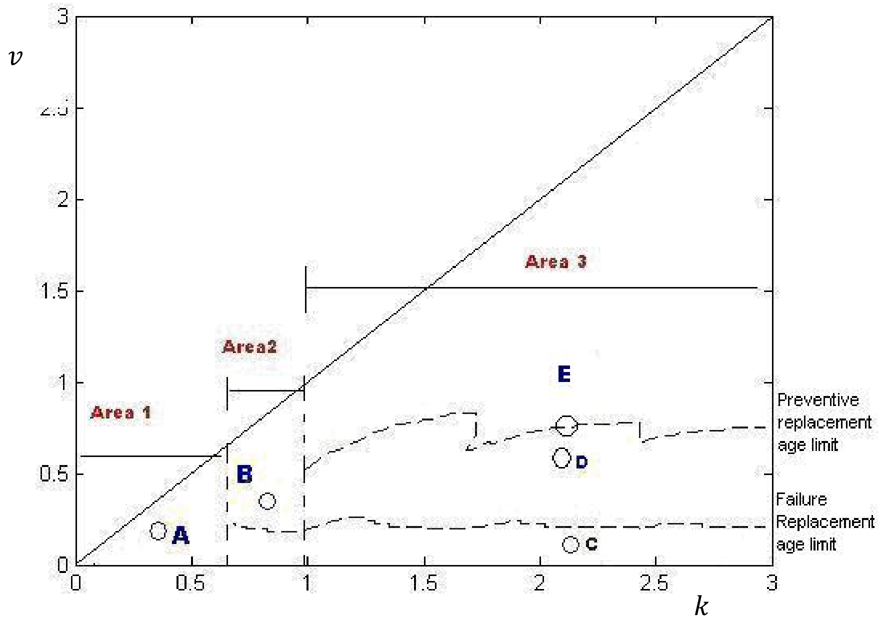

Figure 4.10: Age-Limit Failure and Preventive Replacement Policy

4.5 Summary

In summary, we proposed a general maintenance model in this chapter based on the modeling framework developed in the previous chapter. The $(RCL(n,v,x), APL(n,x))$ policy is proved to be optimal. A computational algorithm is developed, based on which numerical examples are developed which further offer additional insights to the optimal policy.

CHAPTER 5
POLICY OPTIMALITY AND COMPARISON

In this chapter, the focus is extended from a single maintenance model to the 258 models constructed in Section 3.1. By the standard optimization procedure discussed in Section 3.2 and further illustrated in Section 3.3 and Chapter 4, 40 maintenance policies are identified and 38 of them are mapped to those 258 models as the corresponding optimal policies. It turns out that each of the 36 optimal policies has the repair cost limit policy form or its derived policy form for failure rectification, and an age-based policy form for preventive replacement.

Moreover, the unified treatment of diverse models arising from a common modeling framework further enables policy comparison through model comparison, based on which structural properties on the policy space can be deduced from those of the underlying model space.

5.1 Policy Optimality

As discussed in Chapter 2, the existing maintenance policies can be classified into one of the following three categories: heuristic method, Markov method, and optimal stopping method. By the heuristic method, various forms of maintenance policies are proposed, policy performance is analytically expressed, and policy parameter(s) are then optimized accordingly. By the Markov method, optimal policies are represented as solutions to the underlying Markov decision processes. The optimal stopping method derives the optimal policy without presetting specific policy forms or restricting within Markov policy class, and guarantees the optimality of the derived optimal stopping policy in the widest sense, which further enables appropriate comparisons among policies in accordance with the comparisons among their associated maintenance models.

The common modeling framework and the unified optimization methodology based on optimal stopping allow one to solve a large number of models through a standard procedure. The two concrete models solved in Section 3.3 and Chapter 4 illustrates the procedure. Familiarity and intuitions gained from that experience make it now a rather straightforward routine to solve many models arising from the common modeling framework.

In this chapter, we will focus on those 258 models that have been explicitly constructed in Chapter 3. Application of the optimization procedure generates 36 maintenance policies, which are displayed in Table 5.1 as Policies $\boldsymbol{P_1}$ to $\boldsymbol{P_{36}}$. Four more well-known policies, $\boldsymbol{P_{37}}$ to $\boldsymbol{P_{40}}$, are also included in the table to serve as baseline policies in policy comparison. Detailed descriptions of these policies are listed in Appendix B.

As can be noticed from the table, each of these 40 policies is indexed by a sequence number (for example,$\boldsymbol{P_1}$), and supplied with a codified descriptor capturing the essence of the policy form. Formation of the policy descriptors is based on the following notational convention:

RCL: Repair-Cost-Limit (RCL) policy

ARL: Age-Limit Failure Replacement policy

APL: Age-Based Preventive Replacement policy

NRL: Number-Limit Failure Replacement policy

As an example, $\boldsymbol{P_1}$ is identified as $[RCL(n,v,x), APL(n,v,x)]$, whose detailed description in Appendix B reads as follows.

$\boldsymbol{P_1}$: $[RCL(n,v,x), APL(n,v,x)]$ Policy: A failure from condition (n,v,x) is rectified by $A_g$if $RCL(n,v_+,x) \ > C(n,v_-,x)$; otherwise, it is rectified by A_r. For the system resumes its operation from condition (n,v,x), A_p is to be scheduled for the system starting from condition (n,v,x) at virtual age $APL(n,v,x)$. Here v_+ denotes the virtual age after repair, and v_- the virtual age just prior to the failure. Condition(n, v, x) describes the number of failures (n), the virtual age, and the remaining service time (x), respectively.

Table 5.1: Forty Maintenance Policies

#	Policy Descriptor	Parameterization with $\boldsymbol{S}$	
$\boldsymbol{P_1}$:	$[RCL(n,v,x), APL(n,v,x)]$ Policy	$[RCL^S(v), APL^S(v)]$	$S=(n,x)$
$\boldsymbol{P_2}$:	$[RCL(n,v,x), APL(n,x)]$ Policy	$[RCL^S(v), APL^S]$	$S=(n,x)$
$\boldsymbol{P_3}$:	$[RCL(n,v), APL(n,v)]$ Policy	$[RCL^S(v), APL^S(v)]$	$S=(n)$
$\boldsymbol{P_4}$:	$[RCL(n,v), APL(n)]$ Policy	$[RCL^S(v), APL^S]$	$S=(n)$
$\boldsymbol{P_5}$:	$[RCL(v,x), APL(v,x)]$ Policy	$[RCL^S(v), APL^S(v)]$	$S=(x)$
$\boldsymbol{P_6}$:	$[RCL(v,x), APL(x)]$ Policy	$[RCL^S(v), APL^S]$	$S=(x)$
$\boldsymbol{P_7}$:	$[ARL(n,v,x), APL(n,v,x)]$ Policy	$[ARL^S(v), APL^S(v)]$	$S=(n,x)$
$\boldsymbol{P_8}$:	$[ARL(n,x), APL(n,x)]$ Policy	$[ARL^S, APL^S]$	$S=(n,x)$
$\boldsymbol{P_9}$:	$[RCL(v), APL(v)]$ Policy	$[RCL^S(v), APL^S(v)]$	$S=\phi$
$\boldsymbol{P_{10}}$:	$[RCL(v), APL]$ Policy	$[RCL^S(v), APL^S]$	$S=\phi$
$\boldsymbol{P_{11}}$:	$[ARL(n,v), APL(n,v)]$ Policy	$[ARL^S(v), APL^S(v)]$	$S=(n)$
$\boldsymbol{P_{12}}$:	$[ARL(n), APL(n)]$ Policy	$[ARL^S, APL^S]$	$S=(n)$
$\boldsymbol{P_{13}}$:	$[ARL(v,x), APL(v,x)]$ Policy	$[ARL^S(v), APL^S(v)]$	$S=(x)$
$\boldsymbol{P_{14}}$:	$[ARL(x), APL(x)]$ Policy	$[ARL^S, APL^S]$	$S=(x)$
$\boldsymbol{P_{15}}$:	$[ARL(v), APL(v)]$ Policy	$[ARL^S(v), APL^S(v)]$	$S=\phi$
$\boldsymbol{P_{16}}$:	$[ARL, APL]$ Policy	$[ARL^S, APL^S]$	$S=\phi$
$\boldsymbol{P_{17}}$:	$RCL(n,v,x)$ Policy	$RCL^S(v)$	$S=(n,x)$
$\boldsymbol{P_{18}}$:	$RCL(n,v)$ Policy	$RCL^S(v)$	$S=(n)$
$\boldsymbol{P_{19}}$:	$RCL(v,x)$ Policy	$RCL^S(v)$	$S=(x)$
$\boldsymbol{P_{20}}$:	$ARL(n,v,x)$ Policy	$ARL^S(v)$	$S=(n,x)$
$\boldsymbol{P_{21}}$:	$ARL(n,x)$ Policy	ARL^S	$S=(n,x)$
$\boldsymbol{P_{22}}$:	$RCL(v)$ Policy	$RCL^S(v)$	$S=\phi$
$\boldsymbol{P_{23}}$:	$ARL(n,v)$ Policy	$ARL^S(v)$	$S=(n)$
$\boldsymbol{P_{24}}$:	$ARL(n)$ Policy	ARL^S	$S=(n)$
$\boldsymbol{P_{25}}$:	$ARL(v,x)$ Policy	$ARL^S(v)$	$S=(x)$
$\boldsymbol{P_{26}}$:	$ARL(x)$ Policy	ARL^S	$S=(x)$
$\boldsymbol{P_{27}}$:	$ARL(v)$ Policy	$ARL^S(v)$	$S=\phi$
$\boldsymbol{P_{28}}$:	ARL Policy	ARL^S	$S=\phi$
$\boldsymbol{P_{29}}$:	NRL Policy	Discrete-Time ARL^S	$S=\phi$
$\boldsymbol{P_{30}}$:	$RCL(n)$ Policy	Discrete-Time RCL^S	$S=\phi$
$\boldsymbol{P_{31}}$:	$[A_m$ or $A_g,\ APL(n,x)]$ Policy	$[A_m$ or $A_g, APL^S]$	$S=(n,x)$
$\boldsymbol{P_{32}}$:	$[A_m$ or $A_g,\ APL(n)]$ Policy	$[A_m$ or $A_g, APL^S]$	$S=(n)$
$\boldsymbol{P_{33}}$:	$[A_m$ or $A_g, APL(x)]$ Policy	$[A_m$ or $A_g, APL^S]$	$S=(x)$
$\boldsymbol{P_{34}}$:	$[A_m$ or $A_g, APL]$ Policy	$[A_m$ or $A_g, APL^S]$	$S=\phi$
$\boldsymbol{P_{35}}$:	$[A_r, APL(x)]$ Policy	$[A_r, APL^S]$	$S=(x)$
$\boldsymbol{P_{36}}$:	$[A_r, APL]$ Policy	$[A_r, APL^S]$	$S=\phi$

(table 5.1 continued)

$\boldsymbol{P_{37}}$:	(p,q) Policy	Sub-optimal
$\boldsymbol{P_{38}}$:	A_m *or* A_g Policy	Sub-optimal
$\boldsymbol{P_{39}}$:	A_r Policy	Sub-optimal
$\boldsymbol{P_{40}}$:	Periodic/Bloc Replacement Policy	Sub-optimal

Solving the 258 models in essence corresponds to mapping each model to its optimal policy. The results are presented in Appendix A. In particular, Chapter 4 is dedicated to the optimality of $\boldsymbol{P_2}$ with respect to Model M_{222}, Section 3.3 discusses the optimality of $\boldsymbol{P_4}$ with respect to M_{126} derived as an illustrative example. Remark 4.4 can be simply translated as an affirmation of the optimality of $\boldsymbol{P_1}$ with respect to M_{221} . Optimality of $\boldsymbol{P_3}$ with respect to M_{137} is identified in Jiang *et al.* (2001). Policies $\boldsymbol{P_1}$ to $\boldsymbol{P_4}$ represent the four most comprehensive maintenance policies among the policies listed in Table 5.1, and all of the remaining policies are their degenerated cases or derived forms.

The optimal policies listed in Table 5.1 exhibits appealing commonality that can be explained from a more abstract perspective.

Let S describe the parameter(s) needed to characterize a policy. It takes one of the following four possible values, $(n, x), (n), (x)$ and ϕ, where by naming convention, n refers to the number of failures and x the length of remaining service time, and ϕ denotes the empty set, indicating no parameter is needed. The third column of Table 5.1 shows that many policies have degenerated or induced forms of the following generic form:

$$[RCL^S(v), APL^S(v)] \tag{5.1}$$

In other words, these policies can be understood as parameterized Repair-Cost-Limit policies (for failure replacement) and Age-limit policies (for preventive replacement) with respect to parameter *S*. The above statement is supported by the following observations.

Observation 1: ARL as a special case of RCL policy

A seeming discrepancy between RCL and ARL policies needs to be resolved first. In fact, there are two possible ways to derive an ARL policy from an RCL policy. The first way is through specialization, and the second is based on the notion of information projection, whose discussion is to be deferred until Section 5.3.

The first way can be explained in a straightforward manner through Figure 5.1 below. Let's restrict the repair cost $C(n, v)$ to be deterministic and keep its monotonicity in n and v. Due to the monotonicity of the repair cost limit, RCL^S, C intersects RCL^S from below. The interval of age to the left of the crossing point, ARL^S, thus defines the repair zone, and to the right defines the replacement zone, which consequently translates the RCL into an ARL policy and makes the crossing point ARL^S, the Age Replacement Limit.

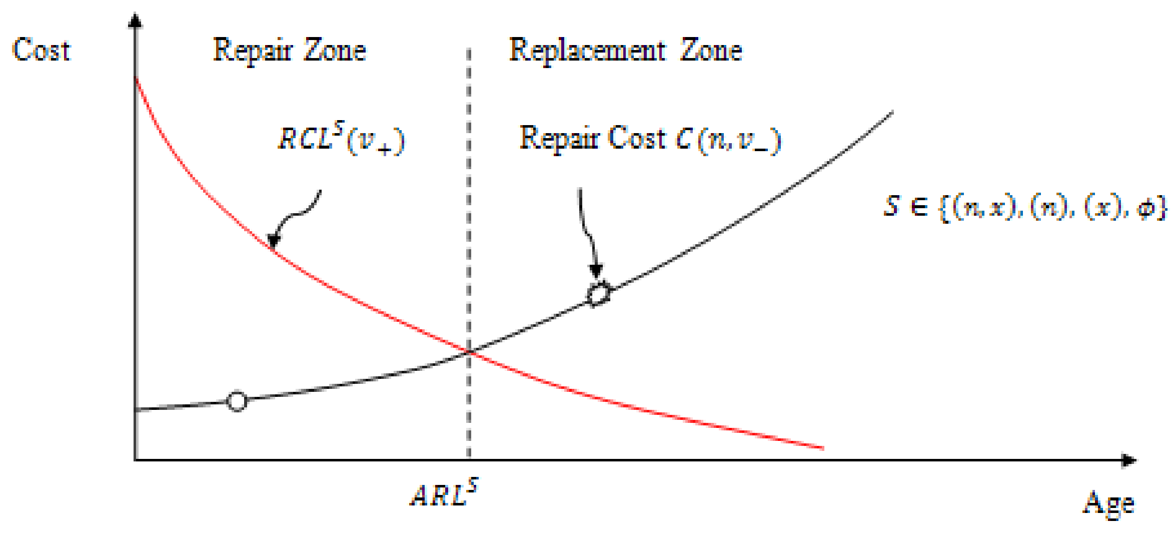

Figure 5.1: ARL Policy Viewed as a Special Case of RCL through Specialization

Whether ARL^S, APL^S depends on v or not is determined by the fact whether the virtual age after repair, v_+, can be determined by v_-, the virtual age just prior to the current failure. In the cases of Type I general repair, we have

$$v_+ = v + \theta(v_- - v),$$

where v is the virtual age of the system when it resumed its operation after the rectification of the *previous* failure, thus $(v_- - v)$ is the run length since the previous failure. Therefore, the determination of v_+ through the unique intersection between $RCL^S(v_+)$ and $C(n, v_-)$ would require the knowledge on v, which gives rise to $v_+^* = ARL^S(v)$, a quantity depending on v.

On the other hand, in the cases of Type II general repair (which includes minimal repair as a special case), $v_+ = \theta v_-$, thus the unique intersection between $RCL^S(v_+)$ and $C^S(v_+/\theta)$ identifies a unique v_+, which gives rise to a constant ARL^S which is independent on v.

Observation 2: Policies without Preventive Replacement as a special case of APL policy

In the sense of formality, maintenance policies that make no use of preventive replacement can be thought of as having a form of APL policy with age limit $APL \equiv \infty$. Moreover, for $C_f \equiv 0$, the APL policy can be rigorously proved to be $APL \equiv \infty$. Intuitively, preventive replacement does not do anything good other than cut the system's useful life time shorter.

Observation 3: A_r, A_m and A_g policies as a special case of RCL policy

The A_r policy involved in $\boldsymbol{P_{35}}$ and $\boldsymbol{P_{36}}$ can be derived from the RCL policy by assuming in the modeling stage that

$$C(n, v) \equiv C_P.$$

Similarly, the A_m or A_g policy in $\boldsymbol{P_{31}}$ to $\boldsymbol{P_{34}}$ can be reinterpreted as RCL by slightly modifying the cost structure during the model construction stage in order to guarantee that under the optimal policy, the failure replacement, A_r, would never be chosen.

Observation 4: Policies P_{29} and P_{30} as a special case of RCL policy

A convenient way to interpret Policies P_{29} and P_{30} in terms of repair-cost-limit policy is to paraphrase the original model on a continuous-time horizon as a model of discrete-time, where the notion of time represents the failure count, for which Policies P_{29} and P_{30} become the counterpart of ARL and RCL policies (defined on continuous-time), respectively.

In summary, all policies except P_{38} and P_{40} can be reinterpreted as a special form of repair-cost-limit policy for decision at failure epochs and age-limit policy for preventive maintenance. Starting from the next section, we will show that these policies will not only share this appealing commonality, they also carry unique characteristics and possess their unique positions in relation with other policies.

5.2 Policy Comparison: Overview

The rules that govern policy comparison are based on a simple idea that can be roughly described as follows: the optimal policy of a general model is no worse than any policies for any sub-model of the former one, where sub-model refers to a specialization of the general model. In this way, comparison between policies is converted to the comparison between models. We identify here the following four intuitive rules:

Rule 1: For systems on a finite horizon, optimal policy utilizes the knowledge on the remaining service period. For systems on an infinite horizon, the optimal policy can be found among stationary policies.

For a stationary policy, we refer to the policy such that all life cycles (defined as time intervals between two consecutive replacements) are stochastically identical and independent. Within each of the life cycles, the policy is defined by an optimal stopping time. Rule 1 basically states that when the system operates on an infinite horizon, the optimal policy is simply repetitive application of a single optimal stopping policy for all life cycles. However, since the system operates on a finite horizon, each of the life cycles is confined by the remaining service period; thus the optimal policy must utilize such information and becomes non-stationary.

Rule 2: The larger the maintenance action set, the better its corresponding optimal policy.

Rule 2 is trivial. It simply means it is always better to have more options for maintenance.

Rule 3: The higher the information level, the better its corresponding optimal policy.

Rule 4: Randomization of deterministic policies does not improve the performance.

Rules 3 and 4 are associated with the fact that a higher information level corresponds to a larger admissible policy set, thus a better optimum, yet incorporation of irrelevant information does not enhance the performance. A more in-depth comprehension of Rules 3 and 4 can be gained through the notion of projection of smooth semi-martingale (SSM) that transforms SSM from its original filtration to sub-filtrations, where projection can be intuitively understood as "conditioning". In this way, the model with a lower information level can be derived from its "super" model through the operation of projection. Consequently, the optimal policy of the "super" model is better than any of the policies in the projected models.

The application of these four rules of policy comparison endows a partially ordered structure among maintenance policy forms. In particular, Figure 5.2 depicts an organization of the 40 policy forms listed in Table 5.1 through a three dimensional geometrical representation. The order relation between a pair of policy forms, say P_a and P_b, is represented by $P_a \rightarrow P_b$, standing for "P_a is better than P_b," or more precisely "P_a is no worse than P_b". To be more rigorous, we say the optimal policy in the form of P_a is no worse than the optimal policy in the form of P_b.

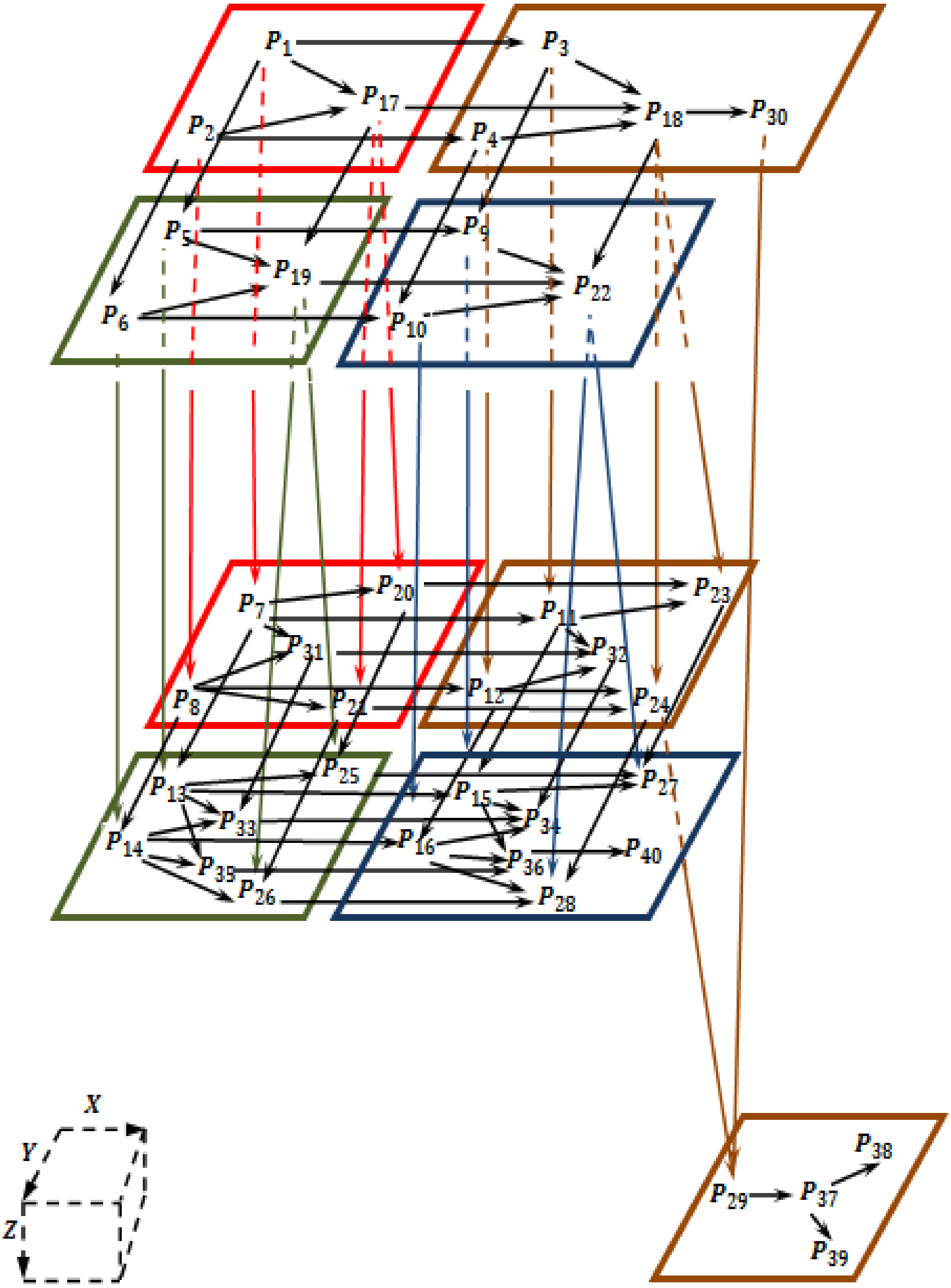

Figure 5.2: Partial Order Structure Induced by Policy Comparison

Remark 5.1: It is straightforward to verify that the relation defined by "→" enjoys the three fundamental properties of a partial order relation: reflexivity, anti-symmetry, and transitivity, i.e., for all P_a, P_b and P_c in the policy set, we have that:

i) $P_a \rightarrow P_a$ (reflexivity)

ii) if $P_a \rightarrow P_b$ and $P_b \rightarrow P_a$ then $P_a = P_b$ (antisymmetry)

iii) if $P_a \rightarrow P_b$ and $P_b \rightarrow P_c$ then $P_a \rightarrow P_c$ (transitivity)

Remark 5.2: The focus of policy comparison is placed on different forms of maintenance policies instead of individual policies, i.e., the policies are in the same form but are parameterized differently. For this reason, we have not included the discussion on some other interesting comparison issues. For example, we have the following two additional rules of policy comparison:

Rule 5: The system with the smaller the repair degree θ performs better than the one with a larger θ.

Rule 6: The system with Type II general repair performs better than the one with Type I general repair.

Rule 5 and rule 6 are associated with the effect of repair actions. As described in the previous chapter, the repair degree θ is in [0, 1] and a smaller value means a better repair result. For two systems fail at the same virtual age, it is also obvious that a type II general repair would always result in a larger age reduction than a type I general repair, given their corresponding repair degrees, denoted as θ_I and θ_{II}, are identical. As a consequence, the system with Type II general repair performs better than the system with Type I general repair.

Remark 5.3: While only 40 maintenance policy forms are identified in Table 5.1. They are the most representative ones in the sense all optimal policies of the 258 models under investigation are included. Obviously, various combinations of two or more of these policy forms could significantly increase the number of policies, which has contributed to a substantial number of existing publications. With the ordered structure identified in Figure 5.2, it is clear that those combinations are sub-optimal at the best.

We conclude this section by presenting a top view of the 3-D representation of the partially ordered structure induced by policy comparison as depicted in Figure 5.2. It is clear that each policy resides on one of the three distinct (horizontal) layers that correspond to the three distinct information levels with which each of the policies is associated. The three layers of policy space are shown in Figures 5.3-5. It is notable that there are four sub-domains on each of the top two layers, and the order relations between these sub-domains exhibit appealing uniformity. The detailed order structure within a layer will be analyzed in Section 5.3 through the notion of specialization, and the structure between the layers will be discussed in Section 5.4 through projection of filtration; intuitively, it refers to information reduction.

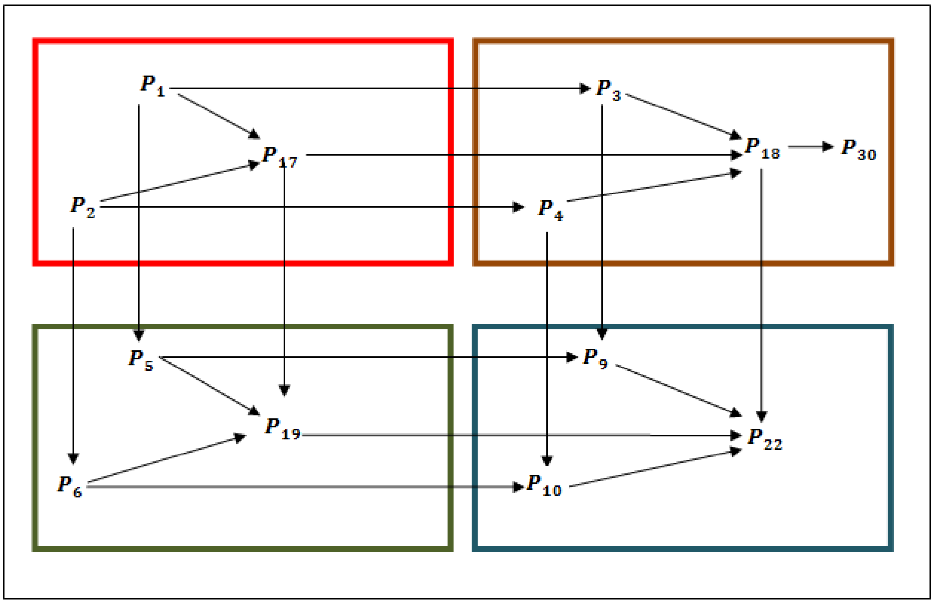

Figure 5.3: Partial Order Structure on Policies that Rely on Information about Random Repair Cost, Virtual Age, and Failure Counts

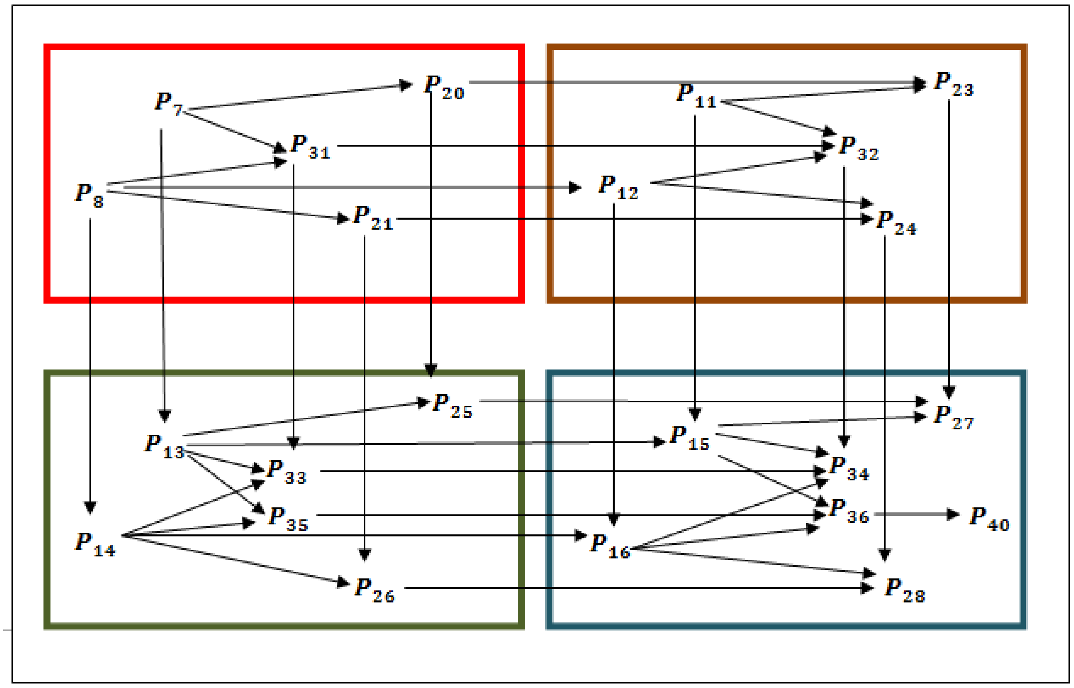

Figure 5.4: Partial Order Structure on Policies that Rely on Information about Virtual Age and Failure Count

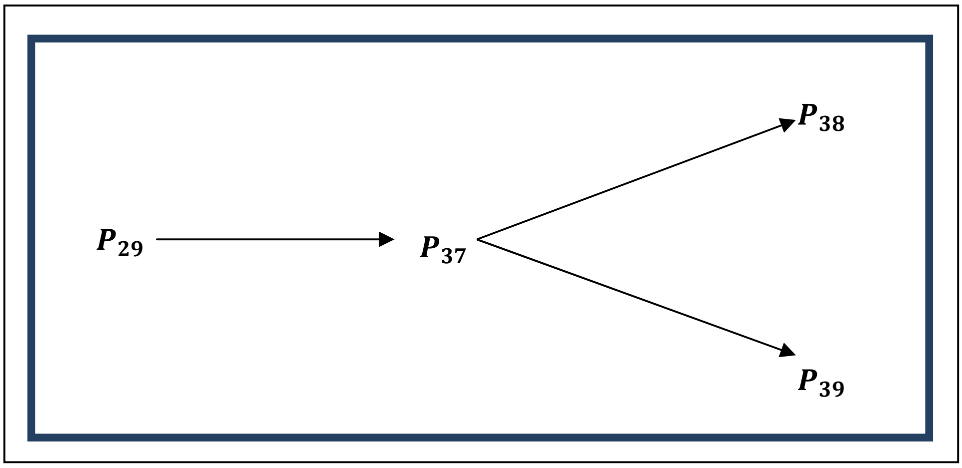

Figure 5.5: Partial Order Structure on Policies that Rely on Information about Failure Count

5.3 Policy Comparison through Specialization

The policy comparison within each of the three horizontal layers shown in Figures 5.3-5.5 is based on the notion of specialization. By "specialization", we refer to an almost trivial assertion that the optimal policy of a general model cannot be worse than any policy defined for any subset of models that can be generated from the general model by instantiating the modeling parameters.

In particular, we consider three categories of specialization: specialization on maintenance horizon, on the dependence on failure count, and on the maintenance action set.

5.3.1 Specialization on Maintenance Horizon: from $W < \infty$ to $W = \infty$

The first category of specializations is to restrict the maintenance horizon from an arbitrary horizon (denoted as $W < \infty$) to an infinite horizon, $W = \infty$. In Figure 5.2, this category is shown as the arrows along the X-axis pointing from the domains on the west to those on the east. Using $P_1 \rightarrow P_3$ in Figure 5.6 as an example, P_1 is the optimal policy of model M_{221} on a finite horizon $W < \infty$, whereas P_3 is the optimal policy of model M_{125} which differ respectively from M_{221} and M_{125} only by its horizon $W = \infty$. Recall that P_1 has the form of $[RCL(n, v, x), APL(n, v, x)]$ and P_3 has the form of $[RCL(n, v), APL(n, v)]$. It is clear that when P_3 is applied to M_{221} where $W < \infty$, it ignores the information on the remaining service time x, thus could not adaptively make maintenance decision according to x, which clearly implies the non-optimality of P_3 and confirms the claim of $P_1 \rightarrow P_3$.

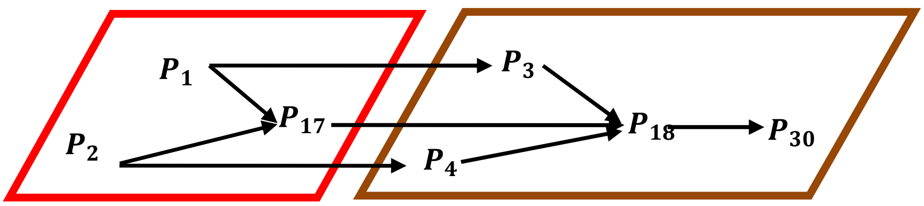

Figure 5.6: Comparison through Specialization on Maintenance Horizon: from $W < \infty$ to $W = \infty$

The following 17 of the order relations belong to Category 1 specialization: $P_1 \to P_3$; $P_{17} \to P_{18}$; $P_2 \to P_4$; $P_5 \to P_9$; $P_{19} \to P_{22}$; $P_6 \to P_{10}$; $P_{20} \to P_{23}$; $P_{31} \to P_{32}$; $P_8 \to P_{12}$; $P_{21} \to P_{24}$; $P_{25} \to P_{27}$; $P_{13} \to P_{15}$; $P_{33} \to P_{34}$; $P_{35} \to P_{36}$; $P_{14} \to P_{16}$; $P_{25} \to P_{28}$; $P_7 \to P_{11}$.

5.3.2 Specialization on the Dependence on Failure Count: From *n*-dependent Cases to *n*-independent Cases

The second category of specializations is to disallow the model dynamics to depend on failure count. In Figure 5.2, this category is shown as the arrows along the Y-axis pointing from north to south. Using $P_1 \to P_5$ in Figure 5.7 as an example, P_1 is the optimal policy of model M_{221} in which failure rate $h(n, v)$ and repair costs $C(n, v)$ are dependent on n, and P_5 is the optimal policy of model M_{245}, which differ respectively from M_{221} and M_{245} only by restricting $h(n, v) \equiv h(v)$ and $C(n, v) \equiv C(v)$. Recall that P_1 has the form of $[RCL(n, v, x), APL(n, v, x)]$ and P_5 has the form of $[RCL(v, x), APL(v, x)]$. It is clear that when P_5 is applied to M_{221}, it ignores the information on the current failure count, n, thus could not adaptively make maintenance decision according to n, which clearly implies the non-optimality of P_5 and confirms the claim of $P_1 \to P_5$.

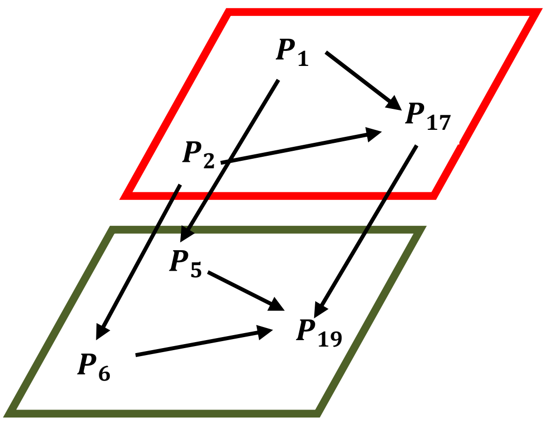

Figure 5.7: Comparison through Specialization on the Dependence with Respect to Failure Count: from n-dependent cases to *n*-independent

The following 16 of the order relations belong to Category 2 specialization: $P_2 \rightarrow P_6$; $P_1 \rightarrow P_5$; $P_{17} \rightarrow P_{19}$; $P_4 \rightarrow P_{10}$; $P_3 \rightarrow P_9$; $P_{18} \rightarrow P_{22}$; $P_8 \rightarrow P_{14}$; $P_7 \rightarrow P_{13}$; $P_{31} \rightarrow P_{33}$; $P_{21} \rightarrow P_{26}$; $P_{20} \rightarrow P_{25}$; $P_{12} \rightarrow P_{16}$; $P_{11} \rightarrow P_{15}$; $P_{32} \rightarrow P_{34}$; $P_{24} \rightarrow P_{28}$; $P_{23} \rightarrow P_{27}$.

5.3.3 Specialization on the Maintenance Action Set

The third category of specializations is to narrow the choice of maintenance actions from a full range of maintenance actions including A_p, A_r and A_m or A_g. In Figure 5.2, this category is shown as the arrows within each of the sub-domains. Apparently, the optimal policy using a smaller maintenance action set is inferior to the optimal policy that utilizes a larger maintenance action set, which is essentially what Rule 2 in Section 5.1 is about. Using $P_1 \rightarrow P_{17}$ in Figure 5.8 as an example, P_{17} has the form of $RCL(n, v, x)$ and P_1 has the form of $[RCL(n, v, x), APL(n, v, x)]$. Essentially, P_{17} corresponds to the best strategy without involving preventive replacement. Restrictions on the options at failure epochs, such as allowing only repairs $A_m(A_g)$ or only replacement A_r consequentially result in inferior policies. Examples falling in these scenarios include $P_{16} \rightarrow P_{34}$ and $P_{16} \rightarrow P_{36}$.

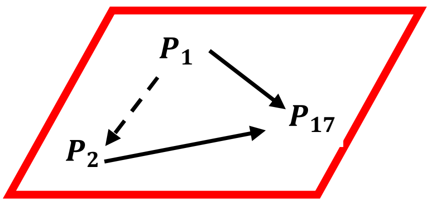

Figure 5.8: Comparison through Specialization on Maintenance Action Set

It is worthy to note that in Figure 5.8 we have inserted a dashed arrow from P_1 to P_2 which have not been displayed anywhere else. From a sense of pure formality, the form of P_1,$[RCL(n, v, x), APL(n, v, x)]$ contains the form of P_2, $[RCL(n, v, x), APL(n, x)]$ as a special case; thus the claim of $P_1 \rightarrow P_2$ is valid. However, we have chosen not to display this order relation due to the fact that such a relation is not derived from the grand scheme of model comparison and does not carry useful information on policy comparison for individual models.

To be more specific, recall that P_1 and P_2 correspond to the optimal policies of models with Type I and Type II A_g, respectively. These two types of general repairs are incompatible with each other excepting the special case when the repair degrees θ_I and θ_{II} are both equal to 1, i.e., they both degenerate to the case of minimal repair A_m. Yet in this case, P_2 is optimal, whereas P_1 demands redundant information for preventive replacement. In this way, the claim of $P_1 \rightarrow P_2$ may misleadingly create an impression that information on v, which is the virtual age of the system when it was recovered from the previous failure, plays a role in the scheduling of preventive replacement.

Based on the above discussion, we identify the following 30 Category 3 specializations: $P_1 \rightarrow P_{17}$; $P_2 \rightarrow P_{17}$; $P_3 \rightarrow P_{18}$; $P_4 \rightarrow P_{18}$; $P_5 \rightarrow P_{19}$; $P_6 \rightarrow P_{19}$; $P_9 \rightarrow P_{22}$; $P_{10} \rightarrow P_{22}$; $P_7 \rightarrow P_{20}$; $P_7 \rightarrow P_{31}$; $P_8 \rightarrow P_{31}$; $P_8 \rightarrow P_{21}$; $P_{11} \rightarrow P_{23}$; $P_{11} \rightarrow P_{32}$; $P_{12} \rightarrow P_{32}$; $P_{12} \rightarrow P_{24}$; $P_{13} \rightarrow P_{25}$; $P_{13} \rightarrow P_{33}$; $P_{13} \rightarrow P_{35}$; $P_{14} \rightarrow P_{33}$; $P_{14} \rightarrow P_{35}$; $P_{14} \rightarrow P_{26}$; $P_{15} \rightarrow P_{27}$; $P_{15} \rightarrow P_{34}$; $P_{15} \rightarrow P_{36}$; $P_{16} \rightarrow P_{34}$; $P_{16} \rightarrow P_{36}$; $P_{16} \rightarrow P_{28}$; $P_{37} \rightarrow P_{38}$; $P_{37} \rightarrow P_{39}$.

Putting all the comparison results obtained through Categories 1-3 specializations, we have obtained all the policy comparison results on all of the three horizontal layers in Figure 5.2 excepting $P_{18} \rightarrow P_{30}$, $P_{29} \rightarrow P_{37}$ and $P_{36} \rightarrow P_{40}$, which are to be discussed in the next section.

5.4 Policy Comparison through Information Reduction

In this section, we analyze order relations between information layers, which are shown as vertical arrows in Figure 5.2, and a portion of them are shown in Figure 5.9. While the majority of the relations can be understood again through specification by restricting the random repair cost to a deterministic one, which causes the repair-cost-limit policy become to an age-limit policy, a more profound interpretation is available by employing the notion of projection on filtration, which rigorously validates the policy comparisons results governed by Rules 3 and 4 presented in Section 5.1. In this section, we will present additional mathematical tools in 5.4.1. The policy comparison results and their analysis are presented in 5.4.2, which concludes the whole discussion of policy comparison in this dissertation.

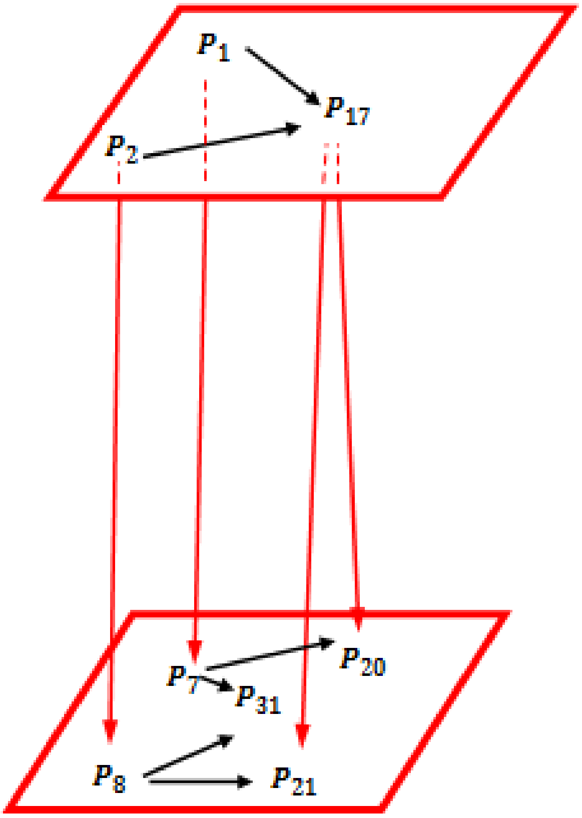

Figure 5.9: Policy Comparison through Information Reduction

5.4.1. Randomization, Sufficiency and Projection Theorem

In this subsection, two additional concepts, Randomization and Sufficiency, which are closely related to the description of information level, are introduced. For conciseness, we precede the discussion on a discrete-time setting following Shiryayev (1978). The notion of projection follows Jensen (1989). See also Section 3.2 and 3.3 for additional information on background and notation.

Definition 5.1 if (F_n) is the subfiltration of (g_n), i.e., $\forall\, n \in N_0$, $F_n \subset g_n$, and $\forall\, A \subset B(\bigcup_1^\infty F_k)$, $P(A|g_n) = P(A|F_n)$, then we call $\{g_n\}$ a randomization of $\{F_n\}$.

Theorem 5.2 Randomization has no effect on the value of $\{Y_n; F_n\}_1^\infty$.

Let (U_t) be the subfiltration of (F_n). Denote

$C^F = \{\tau|(F_t) - stopping\ time, EY_\tau^- < \infty\}$,

$C^U = \{\tau|(U_t) - stopping\ time, \tau \in C^F\}$.

Definition 5.3 If (U_t) is subfiltration of (F_t), and $\sup\{EY_\tau|\tau \in C^U\} = \sup\{EY_\tau|\tau \in C^F\}$, then (U_t) sufficient in (F_t).

Theorem 5.4 If (U_n) is the subfiltration of (F_n), (Y_n) is (U_n)-adapted, and for all F_{n+1}-measurable random variable Y, $E(Y|U_n) = E(Y|F_n)$, then (U_n) is sufficient in (F_n).

Theorems 5.2 and 5.4 are the mathematical expressions of Rule 3 and 4 presented in Section 5.1. Intuitively, a subfiltration means a lower information level, and a lower information level then represents a smaller stopping time class, which consequently corresponds to a sub-model with an inferior optimization policy.

While Theorem 5.4 also lays out the condition under which partial information does not actually cause performance degradation, i.e., reduction of the optimal value of the original problem, the following theorem tells how the original optimal stopping problem is transformed to another when it is subjected to information reduction.

For the continuous case, let (U_t) be the subfiltration of (F_t), let Y be a (F_t)-SSM. Then under the partial information level (U_t), the optimal stopping problem is to find a (U_t)-stopping time σ which satisfies $EY_\sigma = \sup\{EY_\tau | \tau \in C^U\}$.

Theorem 5.5 (Projection Theorem) If $Y_t = (f, M)$ is a (F_t)-SSM, then $\overline{Y}_t = E(Y_t|U_t)$ is a (U_t)-SSM of form $\overline{Y}_t = (E(f|U), \overline{M})$, i.e.,

$$\overline{Y}_t = \overline{Y}_0 + \int_0^t E(f_x|U_x)dx + \overline{M}_t. \tag{5.2}$$

For the discrete case,

$$\overline{Y}_n = \overline{Y}_0 + \sum_1^n E(f_i|U_i) + \overline{M}_n. \tag{5.3}$$

Intuitively, what Theorem 5.5 tells is, in the case of information reduction from (F_t) to (U_t), the new optimization problem is derived from the original one by computing the conditional expectation $E(f_x|U_x)$.

5.4.2. Policy Comparison Results and Discussion

The layered structure of policies depicted in Figure 5.2 corresponds to three information levels of the underlying models that are associated with these policies. The policies with information on random repair cost, (virtual) age, and failure count are on the top layer; the policies with age and count information are on the middle level; and the policies with failure count information only are shown on the bottom layer. The only exception is P_{30}. Policy P_{30} corresponds to the information level that keeps track of random repair costs and the failure count, but not the age.

Two types of information reduction are relevant: with/without the information on random repair cost, and with/without information on virtual age. Full combination of these two types results in four different projections. Denote $\{C, v, n\}$ as the information level that contains information on random repair cost C, virtual age v and failure count n. The subsets of $\{C, v, n\}$ are interpreted accordingly. The four projections can be specified as follows:

A. Projection from $\{C, v, n\}$ to $\{v, n\}$

B. Projection from $\{C, v, n\}$ to $\{C, n\}$

C. Projection from $\{C, n\}$ to $\{n\}$

D. Projection from $\{v, n\}$ to $\{n\}$

To facilitate the discussion, we denote several alternative filtrations that account for different information levels. Let S_n, C_n be the n-failure time and its associated repair cost, define

$$H_n^1 = B\big((C_i, S_i), i \leq n\big)$$

$$H_n^2 = B(C_i, i \leq n)$$

$$H_n^3 = B(S_i, i \leq n)$$

$$H_n^4 = B(i, i \leq n)$$

$$H_n^5 \equiv \{\emptyset, \Omega\}$$

It is clear that filtrations (H_n^1), $(H_n^2), (H_n^3,)(H_n^4)$ are the formal characterization of information levels $\{C, v, n\}$, $\{C, n\}, \{v, n\}$ and $\{n\}$, respectively. In addition, (H_n^5) corresponds to a trivial case which basically contains null information on system condition.

Group A. Projection from $\{C, v, n\}$ to $\{v, n\}$

This group of order relations corresponds to all of the 16 vertical arrows between the top two layers in Figure 5.2, which include: $P_6 \rightarrow P_{14}$; $P_5 \rightarrow P_{13}$; $P_2 \rightarrow P_8$; $P_1 \rightarrow P_7$; $P_{19} \rightarrow P_{26}$; $P_{19} \rightarrow P_{25}$; $P_{17} \rightarrow P_{21}$; $P_{17} \rightarrow P_{20}$; $P_{10} \rightarrow P_{16}$; $P_9 \rightarrow P_{15}$; $P_4 \rightarrow P_{12}$; $P_3 \rightarrow P_{11}$; $P_{22} \rightarrow P_{28}$; $P_{22} \rightarrow P_{27}$; $P_{18} \rightarrow P_{24}$; $P_{18} \rightarrow P_{23}$.

Derivation of these order relations can be illustrated by $P_{18} \rightarrow P_{23}, P_{24}$. Recall that the basic idea supporting policy comparison reads as follows: if P_a and P_b are policies of a common maintenance model and P_a is the optimal one, then $P_a \to P_b$. Consider model M_{118}, M_{119} and M_{120} which have an information level of (H_n^1) and Policy P_{18} is optimal. For an illustrative purpose, the remaining discussion in this section is limited to the average cost criterion.

When information is reduced to the level of (H_n^3), the original objective function Y_n, as an output from Step 3 of the standard optimization procedure where λ −maximization is applied, is projected from (H_n^1) to (H_n^3), which leads to a new optimal stopping problem with respect to an objective function $\bar{Y}_n$, such that

$$\bar{Y}_n = \lambda S_n - \left(\sum_1^{n-1}\left(E\left(C(i, v_i)\middle|H_i^3\right)\right) + C_r\right) + \bar{M}_n$$

$$= \lambda S_n - \left(\sum_1^{n-1}\left(EC(i, v_i)\right) + C_r\right) + \bar{M}_n.$$

The second equality holds when independency between repair costs, $C(n, v_n)$ for all n is assumed.

This objective function corresponds to a specialization on maintenance cost where $C(n, v) \triangleq EC(n, v)$, i.e., $C(n, S_n)$ is a deterministic function. Depending on whether it has a Type I or a Type II repair, the optimal policy will degenerate to P_{23} or P_{24}, as has been carefully examined in Section 4.4. Ultimately, these two degenerated forms imply the relations $P_{18} \to P_{23}$, and $P_{18} \to P_{24}$. In accordance with this example, all other cases in Group A can be similarly treated by substituting the expected value of the random repair cost in the objective function and preserving optimality at the reduced information level.

Group B. Projection from $\{C, v, n\}$ to $\{C, n\}$

There is only one relation in this group: $P_{18} \to P_{30}$, which corresponds to the projection from (H_n^1) to (H_n^2). Assume independence among all repair costs, and further restrict them to take the form of $C(i)$, i.e., they do not depend on (virtual) age, thus could not offer information beyond the count i to estimate v_i. Then the objective function under projection becomes

$$\bar{\bar{Y}}_n = \sum_{i=1}^{n} \left(\lambda E(X_i|H_i^2) - C(i-1)\right) - C_f + \bar{\bar{M}}_n$$

$$= \sum_{i=1}^{n} \left(\lambda E(X_i) - C(i-1)\right) - C_f + \bar{\bar{M}}_n$$

where $C(0) = C_p$, and X_i is the i-th run length, i.e., the duration of time between $(i-1)$- and i-the failures.

Solving to this discrete-time optimal stopping problem yields Policy P_{30}, which is nothing but the counterpart of the RCL policy in a discrete-time setting, where "time" refers to the failure count.

Group C. Projection from $\{C, n\}$ to $\{n\}$

There is only one relation in this group, $P_{30} \rightarrow P_{29}$, which corresponds to the projection from (H_n^2) to (H_n^4). Again, following the same idea worked for Group A, the repair costs under projection become $E(C(i)|H_n^4) = E\left(C(i)\right)$, where the equality is supported by independence among random repair costs. The resulting objective function, denoted by $\tilde{Y}_n$, becomes

$$\tilde{Y}_n = E\left(\bar{\bar{Y}}_n \middle| H_n^4\right)$$

$$= \lambda E(S_n) - \left(\sum_{i=1}^{n-1} (EC(i)) + C_r\right) + \widetilde{M}_n$$

$$= \sum_{i=1}^{n} \left(\lambda EX_i - EC(i)\right) - C_f + \widetilde{M}_n$$

The optimal stopping time becomes

$$\sigma_\lambda = sup\{n|\lambda EX_{n-1} > EC(S_{n-1})\}.$$

Combined with $EY_{\sigma_\lambda} = 0$ as a standard boundary condition demanded by λ −maximization,

$$\lambda = (C_r + \textstyle\sum_1^{n-1}(EC(i)))/E(S_n)$$

and

$$n^* = sup\{n|(C_r + \textstyle\sum_1^{n-1}\left(EC(i)\right))/E(S_n) > EC(i)/EX_{n-1}\}$$

which gives rise to P_{29}. Policy is the so-called N-policy in literature, which is basically the discrete-time version of ARL policy, which is also known as the T-policy.

Group D. Projection from $\{\boldsymbol{v}, \boldsymbol{n}\}$ to $\{\boldsymbol{n}\}$

There is only one relation in this group, $P_{24} \rightarrow P_{29}$, which corresponds to the projection from (H_n^3) to (H_n^4). The resulting objective function, is still the same $\tilde{Y}_n$ in Group C, however it is transformed from $\tilde{Y}_n = E(\bar{Y}_n | H_n^4)$.

5.4.3 Policy Comparison Involving Non-Optimal Policies

To close our discussion on policy comparison, we now consider the remaining four order relations that involve non-optimal policies: $P_{29} \rightarrow P_{37}$; $P_{37} \rightarrow P_{38}$; $P_{37} \rightarrow P_{39}$; $P_{36} \rightarrow P_{40}$. Whereas the claims of $P_{37} \rightarrow P_{38}$; P_{39} are trivial, we discuss the remaining two below.

$\boldsymbol{P_{29}} \rightarrow \boldsymbol{P_{37}}$: N-Policy Is Better Than (p,q)-Policy

Randomize $H_n^3 = B(S_1, \dots, S_n)$ as following: Let $C_n \triangleq B(S_1, \dots, S_n; b_1, \dots, b_n)$, where b_i is i.i.d. Bernoulli trials, in which Prob$\{b = 1\}$= p, and Prob$\{b = 0\}$= 1-p. Clearly, filtration (C_n) is a randomization of (H_n^3), and it is easy to verify that (H_n^3) is sufficient in (C_n). Consequently, the N- policy P_{29} remains optimal with respect to filtration (C_n).

Interestingly enough, the (p, q)- policy of imperfect repair can be constructed with respect to filtration (C_n) as follows: At the *i*-th failure epoch, take A_r if $b_i = 1$; otherwise, repair the system. In other words, both P_{29} and P_{37} are stopping times adapted to filtration (C_n). The claim of $P_{29} \rightarrow P_{37}$ follows immediately from the optimality of P_{29} among all (C_n) −stopping time policies.

$\boldsymbol{P_{36}} \rightarrow \boldsymbol{P_{40}}$: Age Preventive Replacement Is Better Than Block Replacement

The essence of the optimal stopping approach is to focus on a single lifecycle of the system (i.e., the time interval between two consecutive replacements). Given the system is on an infinite horizon, the optimal maintenance policy is obtained by applying the same optimal stopping rule in all lifecycles without the constraints that would have been imposed by a finite horizon $W < \infty$. A rigorous discussion can be found in Bergman (1978) and Jiang (1995), which are essentially based on an extended renewal theory.

Recall policy P_{36} and P_{40} are the so-called Age Preventive Replacement Policy and Periodic or Block Preventive Replacement Policy, respectively. The former is an optimal stopping based policy for M_{16} and the latter can be understood as a composition of non-optimal strategies in each lifecycle, which automatically leads to the conclusion of $P_{36} \rightarrow P_{40}$. Intuitively, the calendar time based policy P_{40} disallows the optimal preventive replacement according to the age, thus the system is preventively replaced at improper times when the system could be too young in some cases and too old in others.

Finally, P_{40} can be understood as a policy for a system on a finite horizon, where the horizon is defined by the period between two preventive replacements. In this way, our previous discussions of models on an infinite horizon and on a finite horizon merge in an interesting way, and, as a consequence, we could rigorously show that even a finite horizon model is often more complicated. It in general has an inferior performance compared with its counterpart on the infinite horizon.

CHAPTER 6
CONCLUSION AND FUTURE WORKS

In this study, we have proposed a unified maintenance modeling and optimization methodology for single-unit repairable systems, with which policy comparison and optimality verification are carried out in a systematic manner and a natural structure among many models is established.

6.1 Conclusion

The whole procedure of analyzing a maintenance policy problem can be described as follows:

(i) Specify the six factors under the modeling framework

(ii) Find the optimal policy in admitted stopping time class for each model

(iii) Evaluate policies based on the related model comparisons

In Chapter 3, a general maintenance modeling framework with six classifying factors was developed for formulation and analysis of a wide range of maintenance systems, under which many existing models in literature could be nicely incorporated and reformulated as optimal stopping models; a systematic optimization methodology was developed based on optimal stopping, semi-martingale, and λ-maximization techniques; a concrete model was presented and solved as an example to illustrate the proposed methodology, where the numerical analysis leads to some additional insights. The unified modeling framework and optimization procedures are summarized below.

The Unified Maintenance Modeling Framework: The unified maintenance modeling framework proposed in this study includes six factors:

(i) Maintenance Horizon

(ii) System Deterioration Dynamics

(iii) Maintenance Actions

(iv) Cost Structures

(v) Information Level

(vi) Objective Criterion

Optimization Procedure:

Step 1: λ-maximization technique. This technique is introduced for the following two purposes:

(i) It transforms the original objective function to an additive function, where the optimal stopping theory can be readily applied.

(ii) It absorbs a substantial amount of formulating and computational complexity into parameter λ. More insights on intrinsic properties such as monotonicity and convexity can be gained, and less burden of numerical computation can be achieved.

Step 2: Characterization of stopping time for jump process. Noting that the system failure dynamics form a jump process, it becomes obvious that the explicit characterization of stopping time for jump process in continuous time developed in Markis *et al.* (2000) leads to a critical simplification: each optimal stopping time can be decomposed into two simpler ones, one for failure replacement and the other for preventive replacement. With the decomposition, the two simpler stopping times can be obtained in sequence without loss of optimality.

Step 3: Smooth semi-martingale decomposition. This technique is essentially to separate the deterioration trend (the progressive part), with the noninformative randomness (the martingale part), and allow one to consider only the trend part without loss of the optimality (guaranteed by the Optional Stopping Theorem). Critical properties such as monotonicity and Markov can be identified in this step.

Step 4: Dynamical programming. With the Markov property being identified, dynamic programming can be applied to yield the dynamical equation of the value function. Further analysis of the value function reveals the nice-structure of the optimal policy.

In chapter 4, a maintenance model for a system with a finite maintenance service period and full information level is studied. This model is generated under the unified modeling framework in Chapter 3. It is shown that a generalized repair-cost-limit and age-limit ($RCL(n, v), APL(x, n)$) policy is optimal.

The optimality of maintenance policies and comparisons between them are studied in Chapter 5 based on the unified methodology developed in Chapter 3. The general model developed in Chapter 4 contains many other models as its transformations or degenerated forms and its maintenance policy leads to many other policies through specialization. The policy comparison discussed in Chapter 5 was based on the following simple idea. The optimal policy of a general model is no worse than any policies in any sub-model of the original. In this way, the comparison between policies is lifted to the comparison between models, and the following four intuitive rules can be summarized.

Rule 1: For systems on a finite horizon, optimal policy utilizes the knowledge on the remaining service period. For systems on an infinite horizon, the optimal policy can be found among stationary policies.

Rule 2: The larger the maintenance action set, the better its corresponding optimal policy.

Rule 3: The higher the information level, the better its corresponding optimal policy.

Rule 4: Randomization of deterministic policies does not improve the performance.

6.2 Future Research Directions

It is of great importance to put theoretical investigation into practice. The proposed modeling framework and optimization procedure in fact are very convenient for the design and implementation of a computer-based maintenance optimization system. From a user's perspective, the model construction process makes it very easy to select a model that is most appropriate for one's own application need. From a designer's point of view, the common optimization procedure enables efficient implementation of the system. Availability of such a system further closes the gap between theoretical results and practical needs.

In the theoretical front, the proposed methodology can be relatively easy to extend to some other maintenance models beyond Kijima types of repairs. For example, the proportional (hazards) intensity reduction models (see Chan and Shaw (1993)) can be readily solved without additional technical difficulty.

The most promising and challenging topic in the field of maintenance is the emerging Condition-Based Maintenance (CBM), also known as Sensor-Based Maintenance (SBM), which is both technology-intense and information-intense. The optimal stopping approach represents a very appropriate tool due to its strength in representing, processing, and utilizing information. While some works have applied optimal stopping to CBM, a systematic investigation on this topic is still lacking. It is our strong belief that serious studies in CBM with Optimal Stopping methodology could contribute significantly to the pursuit of maintenance excellence.

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APPENDIX A:
FULL PRESENTATION OF MODELS CORRESPONDING OPTIMAL POLICIES WITHIN THE UNIFIED MODELING FRAMEWORK

Model #	System	Objective	Maintenance Horizon	Failure Count Dependence	Information Level	Maintenance Actions	Optimal Policy
M_1	NR	Criterion 2	$W = \infty$	$h(n, v)$	(n)	A_r	P_{39}
M_2	NR	Criterion 2	$W = \infty$	$h(v)$	(n)	A_r	P_{39}
M_3	NR	Criterion 1	$W < \infty$	$h(v)$	(n, v)	A_r	P_{39}
M_4	NR	Criterion 1	$W = \infty$	$h(v)$	(n, v)	A_r	P_{39}
M_5	NR	Criterion 1	$W < \infty$	$h(n, v)$	(n, v)	A_r	P_{39}
M_6	NR	Criterion 1	$W = \infty$	$h(n, v)$	(n, v)	A_r	P_{39}
M_7	NR	Criterion 2	$W < \infty$	$h(v)$	(n, v)	A_r	P_{39}
M_8	NR	Criterion 2	$W = \infty$	$h(v)$	(n, v)	A_r	P_{39}
M_9	NR	Criterion 2	$W < \infty$	$h(n, v)$	(n, v)	A_r	P_{39}
M_{10}	NR	Criterion 2	$W = \infty$	$h(n, v)$	(n, v)	A_r	P_{39}
M_{11}	NR	Criterion 1	$W < \infty$	$h(v)$	(n, v)	(A_r, A_p)	P_{35}
M_{12}	NR	Criterion 1	$W = \infty$	$h(v)$	(n, v)	(A_r, A_p)	P_{36}
M_{13}	NR	Criterion 1	$W < \infty$	$h(n, v)$	(n, v)	(A_r, A_p)	P_{35}
M_{14}	NR	Criterion 1	$W = \infty$	$h(n, v)$	(n, v)	(A_r, A_p)	P_{36}
M_{15}	NR	Criterion 2	$W < \infty$	$h(v)$	(n, v)	(A_r, A_p)	P_{35}
M_{16}	NR	Criterion 2	$W = \infty$	$h(v)$	(n, v)	(A_r, A_p)	P_{36}
M_{17}	NR	Criterion 2	$W < \infty$	$h(n, v)$	(n, v)	(A_r, A_p)	P_{35}
M_{18}	NR	Criterion 2	$W = \infty$	$h(n, v)$	(n, v)	(A_r, A_p)	P_{36}
M_{19}	R	Criterion 1	$W = \infty$	$h(n, v)$, $C(n)$	(n)	A_m	P_{38}
M_{20}	R	Criterion 1	$W = \infty$	$h(n, v)$, $C(n)$	(n)	A_g I	P_{38}
M_{21}	R	Criterion 1	$W = \infty$	$h(n, v)$, $C(n)$	(n)	A_g II	P_{38}
M_{22}	R	Criterion 1	$W = \infty$	$h(n, v)$, $C(n)$	(n)	(A_r, A_m)	P_{29}
M_{23}	R	Criterion 1	$W = \infty$	$h(n, v)$, $C(n)$	(n)	(A_r, A_g I)	P_{29}
M_{24}	R	Criterion 1	$W = \infty$	$h(n, v)$, $C(n)$	(n)	(A_r, A_g II)	P_{29}
M_{25}	R	Criterion 1	$W = \infty$	$h(v)$, C	(n)	A_m	P_{38}
M_{26}	R	Criterion 1	$W = \infty$	$h(v)$, C	(n)	A_g I	P_{38}
M_{27}	R	Criterion 1	$W = \infty$	$h(v)$, C	(n)	A_g II	P_{38}
M_{28}	R	Criterion 1	$W = \infty$	$h(v)$, C	(n)	(A_r, A_m)	P_{29}
M_{29}	R	Criterion 1	$W = \infty$	$h(v)$, C	(n)	(A_r, A_g I)	P_{29}
M_{30}	R	Criterion 1	$W = \infty$	$h(v)$, C	(n)	(A_r, A_g II)	P_{29}
M_{31}	R	Criterion 2	$W = \infty$	$h(n, v)$, $C(n)$	(n)	A_m	P_{38}
M_{32}	R	Criterion 2	$W = \infty$	$h(n, v)$, $C(n)$	(n)	A_g I	P_{38}
M_{33}	R	Criterion 2	$W = \infty$	$h(n, v)$, $C(n)$	(n)	A_g II	P_{38}
M_{34}	R	Criterion 2	$W = \infty$	$h(n, v)$, $C(n)$	(n)	(A_r, A_m)	P_{29}

M_{35}	R	Criterion 2	$W = \infty$	$h(n,v), C(n)$	(n)	$(A_r, A_g$ I)	P_{29}
M_{36}	R	Criterion 2	$W = \infty$	$h(n,v), C(n)$	(n)	$(A_r, A_g$ II)	P_{29}
M_{37}	R	Criterion 2	$W = \infty$	$h(v), C$	(n)	A_m	P_{38}
M_{38}	R	Criterion 2	$W = \infty$	$h(v), C$	(n)	A_g I	P_{38}
M_{39}	R	Criterion 2	$W = \infty$	$h(v), C$	(n)	A_g II	P_{38}
M_{40}	R	Criterion 2	$W = \infty$	$h(v), C$	(n)	(A_r, A_m)	P_{29}
M_{41}	R	Criterion 2	$W = \infty$	$h(v), C$	(n)	$(A_r, A_g$ I)	P_{29}
M_{42}	R	Criterion 2	$W = \infty$	$h(v), C$	(n)	$(A_r, A_g$ II)	P_{29}
M_{43}	R	Criterion 1	$W = \infty$	$h(n,v), C(n)$	(n,c)	A_m	P_{38}
M_{44}	R	Criterion 1	$W = \infty$	$h(n,v), C(n)$	(n,c)	A_g I	P_{38}
M_{45}	R	Criterion 1	$W = \infty$	$h(n,v), C(n)$	(n,c)	A_g II	P_{38}
M_{46}	R	Criterion 1	$W = \infty$	$h(n,v), C(n)$	(n,c)	(A_r, A_m)	P_{30}
M_{47}	R	Criterion 1	$W = \infty$	$h(n,v), C(n)$	(n,c)	$(A_r, A_g$ I)	P_{30}
M_{48}	R	Criterion 1	$W = \infty$	$h(n,v), C(n)$	(n,c)	$(A_r, A_g$ II)	P_{30}
M_{49}	R	Criterion 1	$W = \infty$	$h(v), C$	(n,c)	A_m	P_{38}
M_{50}	R	Criterion 1	$W = \infty$	$h(v), C$	(n,c)	A_g I	P_{38}
M_{51}	R	Criterion 1	$W = \infty$	$h(v), C$	(n,c)	A_g II	P_{38}
M_{52}	R	Criterion 1	$W = \infty$	$h(v), C$	(n,c)	(A_r, A_m)	P_{30}
M_{53}	R	Criterion 1	$W = \infty$	$h(v), C$	(n,c)	$(A_r, A_g$ I)	P_{30}
M_{54}	R	Criterion 1	$W = \infty$	$h(v), C$	(n,c)	$(A_r, A_g$ II)	P_{30}
M_{55}	R	Criterion 2	$W = \infty$	$h(n,v), C(n)$	(n,c)	A_m	P_{38}
M_{56}	R	Criterion 2	$W = \infty$	$h(n,v), C(n)$	(n,c)	A_g I	P_{38}
M_{57}	R	Criterion 2	$W = \infty$	$h(n,v), C(n)$	(n,c)	A_g II	P_{38}
M_{58}	R	Criterion 2	$W = \infty$	$h(n,v), C(n)$	(n,c)	(A_r, A_m)	P_{30}
M_{59}	R	Criterion 2	$W = \infty$	$h(n,v), C(n)$	(n,c)	$(A_r, A_g$ I)	P_{30}
M_{60}	R	Criterion 2	$W = \infty$	$h(n,v), C(n)$	(n,c)	$(A_r, A_g$ II)	P_{30}
M_{61}	R	Criterion 2	$W = \infty$	$h(v), C$	(n,c)	A_m	P_{38}
M_{62}	R	Criterion 2	$W = \infty$	$h(v), C$	(n,c)	A_g I	P_{38}
M_{63}	R	Criterion 2	$W = \infty$	$h(v), C$	(n,c)	A_g II	P_{38}
M_{64}	R	Criterion 2	$W = \infty$	$h(v), C$	(n,c)	(A_r, A_m)	P_{30}
M_{65}	R	Criterion 2	$W = \infty$	$h(v), C$	(n,c)	$(A_r, A_g$ I)	P_{30}
M_{66}	R	Criterion 2	$W = \infty$	$h(v), C$	(n,c)	$(A_r, A_g$ II)	P_{30}
M_{67}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	A_m	P_{38}
M_{68}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	A_g I	P_{38}
M_{69}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	A_g II	P_{38}
M_{70}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	(A_r, A_m)	P_{24}
M_{71}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_r, A_g$ I)	P_{23}
M_{72}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_r, A_g$ II)	P_{24}
M_{73}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	(A_m, A_p)	P_{32}
M_{74}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_g$ I$, A_p)$	P_{32}

M_{75}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_g \text{ II}, A_p)$	P_{32}
M_{76}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	(A_m, A_p, A_r)	P_{12}
M_{77}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_g\text{I}, A_p, A_r)$	P_{11}
M_{78}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_g\text{II}, A_p, A_r)$	P_{12}
M_{79}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	A_m	P_{38}
M_{80}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	A_g I	P_{38}
M_{81}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	A_g II	P_{38}
M_{82}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	(A_r, A_m)	P_{24}
M_{83}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_r, A_g \text{ I})$	P_{23}
M_{84}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_r, A_g \text{ II})$	P_{24}
M_{85}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	(A_m, A_p)	P_{32}
M_{86}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_g \text{ I}, A_p)$	P_{32}
M_{87}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_g \text{ II}, A_p)$	P_{32}
M_{88}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	(A_m, A_p, A_r)	P_{12}
M_{89}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_g\text{I}, A_p, A_r)$	P_{11}
M_{90}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v)	$(A_g\text{II}, A_p, A_r)$	P_{12}
M_{91}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	A_m	P_{38}
M_{92}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	A_g I	P_{38}
M_{93}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	A_g II	P_{38}
M_{94}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	(A_r, A_m)	P_{28}
M_{95}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_r, A_g \text{ I})$	P_{27}
M_{96}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_r, A_g \text{ II})$	P_{28}
M_{97}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	(A_m, A_p)	P_{34}
M_{98}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_g \text{ I}, A_p)$	P_{34}
M_{99}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_g \text{ II}, A_p)$	P_{34}
M_{100}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	(A_m, A_p, A_r)	P_{16}
M_{101}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_g\text{I}, A_p, A_r)$	P_{15}
M_{102}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_g\text{II}, A_p, A_r)$	P_{16}
M_{103}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	A_m	P_{38}
M_{104}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	A_g I	P_{38}
M_{105}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	A_g II	P_{38}
M_{106}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	(A_r, A_m)	P_{28}
M_{107}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_r, A_g \text{ I})$	P_{27}
M_{108}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_r, A_g \text{ II})$	P_{28}
M_{109}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	(A_m, A_p)	P_{34}
M_{110}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_g \text{ I}, A_p)$	P_{34}
M_{111}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_g \text{ II}, A_p)$	P_{34}
M_{112}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	(A_m, A_p, A_r)	P_{12}
M_{113}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_g\text{I}, A_p, A_r)$	P_{11}
M_{114}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v)	$(A_g\text{II}, A_p, A_r)$	P_{12}

M_{115}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	A_m	P_{38}
M_{116}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	A_g I	P_{38}
M_{117}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	A_g II	P_{38}
M_{118}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_r, A_m)	P_{18}
M_{119}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_r, A_g I)	P_{18}
M_{120}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_r, A_g II)	P_{18}
M_{121}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_m, A_p)	P_{32}
M_{122}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_g I, A_p)	P_{32}
M_{123}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_g II, A_p)	P_{32}
M_{124}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_m, A_p, A_r)	P_4
M_{125}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_gI, A_p, A_r)	P_3
M_{126}	R	Criterion 1	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_gII, A_p, A_r)	P_4
M_{127}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	A_m	P_{38}
M_{128}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	A_g I	P_{38}
M_{129}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	A_g II	P_{38}
M_{130}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_r, A_m)	P_{18}
M_{131}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_r, A_g I)	P_{18}
M_{132}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_r, A_g II)	P_{18}
M_{133}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_m, A_p)	P_{32}
M_{134}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_g I, A_p)	P_{32}
M_{135}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_g II, A_p)	P_{32}
M_{136}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_m, A_p, A_r)	P_4
M_{137}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_gI, A_p, A_r)	P_3
M_{138}	R	Criterion 2	$W = \infty$	$h(n,v), C(n,v)$	(n,v,c)	(A_gII, A_p, A_r)	P_4
M_{139}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	A_m	P_{38}
M_{140}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	A_g I	P_{38}
M_{141}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	A_g II	P_{38}
M_{142}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	(A_r, A_m)	P_{22}
M_{143}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	(A_r, A_g I)	P_{22}
M_{144}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	(A_r, A_g II)	P_{22}
M_{145}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	(A_m, A_p)	P_{34}
M_{146}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	(A_g I, A_p)	P_{34}
M_{147}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	(A_g II, A_p)	P_{34}
M_{148}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	(A_m, A_p, A_r)	P_{10}
M_{149}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	(A_gI, A_p, A_r)	P_9
M_{150}	R	Criterion 1	$W = \infty$	$h(v), C(v)$	(n,v,c)	(A_gII, A_p, A_r)	P_{10}
M_{151}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v,c)	A_m	P_{38}
M_{152}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v,c)	A_g I	P_{38}
M_{153}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v,c)	A_g II	P_{38}
M_{154}	R	Criterion 2	$W = \infty$	$h(v), C(v)$	(n,v,c)	(A_r, A_m)	P_{22}

M_{155}	R	Criterion 2	$W=\infty$	$h(v), C(v)$	(n, v, c)	$(A_r, A_g \text{ I})$	P_{22}
M_{156}	R	Criterion 2	$W=\infty$	$h(v), C(v)$	(n, v, c)	$(A_r, A_g \text{ II})$	P_{22}
M_{157}	R	Criterion 2	$W=\infty$	$h(v), C(v)$	(n, v, c)	(A_m, A_p)	P_{34}
M_{158}	R	Criterion 2	$W=\infty$	$h(v), C(v)$	(n, v, c)	$(A_g \text{ I}, A_p)$	P_{34}
M_{159}	R	Criterion 2	$W=\infty$	$h(v), C(v)$	(n, v, c)	$(A_g \text{ II}, A_p)$	P_{34}
M_{160}	R	Criterion 2	$W=\infty$	$h(v), C(v)$	(n, v, c)	(A_m, A_p, A_r)	P_{10}
M_{161}	R	Criterion 2	$W=\infty$	$h(v), C(v)$	(n, v, c)	$(A_g\text{I}, A_p, A_r)$	P_9
M_{162}	R	Criterion 2	$W=\infty$	$h(v), C(v)$	(n, v, c)	$(A_g\text{II}, A_p, A_r)$	P_{10}
M_{163}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	A_m	P_{38}
M_{164}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	A_g I	P_{38}
M_{165}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	A_g II	P_{38}
M_{166}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	(A_r, A_m)	P_{21}
M_{167}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_r, A_g \text{ I})$	P_{20}
M_{168}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_r, A_g \text{ II})$	P_{21}
M_{169}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	(A_m, A_p)	P_{31}
M_{170}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_g \text{ I}, A_p)$	P_{31}
M_{171}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_g \text{ II}, A_p)$	P_{31}
M_{172}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	(A_m, A_p, A_r)	P_8
M_{173}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_g\text{I}, A_p, A_r)$	P_7
M_{174}	R	Criterion 1	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_g\text{II}, A_p, A_r)$	P_8
M_{175}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	A_m	P_{38}
M_{176}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	A_g I	P_{38}
M_{177}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	A_g II	P_{38}
M_{178}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	(A_r, A_m)	P_{21}
$M1_{79}$	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_r, A_g \text{ I})$	P_{20}
M_{180}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_r, A_g \text{ II})$	P_{21}
M_{181}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	(A_m, A_p)	P_{31}
M_{182}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_g \text{ I}, A_p)$	P_{31}
M_{183}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_g \text{ II}, A_p)$	P_{31}
M_{184}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	(A_m, A_p, A_r)	P_8
M_{185}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_g\text{I}, A_p, A_r)$	P_7
M_{186}	R	Criterion 2	$W<\infty$	$h(n,v), C(n,v)$	(n, v)	$(A_g\text{II}, A_p, A_r)$	P_8
M_{187}	R	Criterion 1	$W<\infty$	$h(v), C(v)$	(n, v)	A_m	P_{38}
M_{188}	R	Criterion 1	$W<\infty$	$h(v), C(v)$	(n, v)	A_g I	P_{38}
M_{189}	R	Criterion 1	$W<\infty$	$h(v), C(v)$	(n, v)	A_g II	P_{38}
M_{190}	R	Criterion 1	$W<\infty$	$h(v), C(v)$	(n, v)	(A_r, A_m)	P_{26}
M_{191}	R	Criterion 1	$W<\infty$	$h(v), C(v)$	(n, v)	$(A_r, A_g \text{ I})$	P_{25}
M_{192}	R	Criterion 1	$W<\infty$	$h(v), C(v)$	(n, v)	$(A_r, A_g \text{ II})$	P_{26}
M_{193}	R	Criterion 1	$W<\infty$	$h(v), C(v)$	(n, v)	(A_m, A_p)	P_{33}
M_{194}	R	Criterion 1	$W<\infty$	$h(v), C(v)$	(n, v)	$(A_g \text{ I}, A_p)$	P_{33}

M_{195}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v)	$(A_g \text{ II}, A_p)$	P_{33}
M_{196}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v)	(A_m, A_p, A_r)	P_{14}
M_{197}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v)	$(A_g\text{I}, A_p, A_r)$	P_{13}
M_{198}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v)	$(A_g\text{II}, A_p, A_r)$	P_{14}
M_{199}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	A_m	P_{38}
M_{200}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	A_g I	P_{38}
M_{201}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	A_g II	P_{38}
M_{202}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	(A_r, A_m)	P_{26}
M_{203}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	$(A_r, A_g \text{ I})$	P_{25}
M_{204}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	$(A_r, A_g \text{ II})$	P_{26}
M_{205}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	(A_m, A_p)	P_{33}
M_{206}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	$(A_g \text{ I}, A_p)$	P_{33}
M_{207}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	$(A_g \text{ II}, A_p)$	P_{33}
M_{208}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	(A_m, A_p, A_r)	P_{14}
M_{209}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	$(A_g\text{I}, A_p, A_r)$	P_{13}
M_{210}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v)	$(A_g\text{II}, A_p, A_r)$	P_{14}
M_{211}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	A_m	P_{38}
M_{212}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	A_g I	P_{38}
M_{213}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	A_g II	P_{38}
M_{214}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	(A_r, A_m)	P_{17}
M_{215}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_r, A_g \text{ I})$	P_{17}
M_{216}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_r, A_g \text{ II})$	P_{17}
M_{217}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	(A_m, A_p)	P_{31}
M_{218}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_g \text{ I}, A_p)$	P_{31}
M_{219}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_g \text{ II}, A_p)$	P_{31}
M_{220}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	(A_m, A_p, A_r)	P_2
M_{221}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_g\text{I}, A_p, A_r)$	P_1
M_{222}	R	Criterion 1	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_g\text{II}, A_p, A_r)$	P_2
M_{223}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	A_m	P_{38}
M_{224}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	A_g I	P_{38}
M_{225}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	A_g II	P_{38}
M_{226}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	(A_r, A_m)	P_{17}
M_{227}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_r, A_g \text{ I})$	P_{17}
M_{228}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_r, A_g \text{ II})$	P_{17}
M_{229}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	(A_m, A_p)	P_{31}
M_{230}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_g \text{ I}, A_p)$	P_{31}
M_{231}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_g \text{ II}, A_p)$	P_{31}
M_{232}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	(A_m, A_p, A_r)	P_2
M_{233}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_g\text{I}, A_p, A_r)$	P_1
M_{234}	R	Criterion 2	$W < \infty$	$h(n, v), C(n, v)$	(n, v, c)	$(A_g\text{II}, A_p, A_r)$	P_2

M_{235}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	A_m	P_{38}
M_{236}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	A_g I	P_{38}
M_{237}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	A_g II	P_{38}
M_{238}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_r, A_m)	P_{19}
M_{239}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_r, A_g I)	P_{19}
M_{240}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_r, A_g II)	P_{19}
M_{241}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_m, A_p)	P_{33}
M_{242}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_g I, A_p)	P_{33}
M_{243}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_g II, A_p)	P_{33}
M_{244}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_m, A_p, A_r)	P_6
M_{245}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_gI, A_p, A_r)	P_5
M_{246}	R	Criterion 1	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_gII, A_p, A_r)	P_6
M_{247}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	A_m	P_{38}
M_{248}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	A_g I	P_{38}
M_{249}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	A_g II	P_{38}
M_{250}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_r, A_m)	P_{19}
M_{251}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_r, A_g I)	P_{19}
M_{252}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_r, A_g II)	P_{19}
M_{253}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_m, A_p)	P_{33}
M_{254}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_g I, A_p)	P_{33}
M_{255}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_g II, A_p)	P_{33}
M_{256}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_m, A_p, A_r)	P_6
M_{257}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_gI, A_p, A_r)	P_5
M_{258}	R	Criterion 2	$W < \infty$	$h(v), C(v)$	(n, v, c)	(A_gII, A_p, A_r)	P_6

APPENDIX B:
COMPLETE DESCRIPTIONS OF MAINTENANCE POLICIES

$\boldsymbol{P_1}$: $[RCL(n, v, x), APL(n, v, x)]$ Policy: A failure from condition (n, v, x) is rectified by A_m or A_g if $RCL(n, v_+, x) > C(n, v_-, x)$; otherwise, it is rectified by A_r. For the system resumes its operation from condition(n, v, x), A_p is to be scheduled at virtual age $APL(n, v, x)$. Here v_+ denotes the virtual age after repair, and v_- the virtual age just prior to the failure. Condition (n, v, x) describes the number of failures (n), the virtual age, and the remaining service time (x), respectively.

$\boldsymbol{P_2}$: $[RCL(n, v, x), APL(n, x)]$ Policy: A failure from condition (n, v, x) is rectified by A_m or $A_g$if $RCL(n, v_+, x) > C(n, v_-, x)$, otherwise, by A_r. For the system resumes its operation from condition (n, v, x), A_p is to be scheduled at age $APL(n, x)$.

$\boldsymbol{P_3}$: $[RCL(n, v), APL(n, v)]$ Policy: A failure from condition (n, v) is rectified by A_m or $A_g$if $RCL(n, v_+) > C(n, v_-)$; otherwise, it is rectified by A_r. For the system resumes its operation from condition(n, v, x), A_p is to be scheduled at virtual age $APL(n, v)$.

$\boldsymbol{P_4}$: $[RCL(n, v), APL(n)]$ Policy: A failure from condition (n, v) is rectified by A_m or $A_g$if $RCL(n, v_+) > C(n, v_-)$, otherwise, by A_r. For the system resumes its operation from condition (n, v), A_p is to be scheduled at age $APL(n)$.

$\boldsymbol{P_5}$: $[RCL(v, x), APL(v, x)]$ Policy: A failure from condition (v, x) is rectified by A_m or $A_g$if $RCL(v_+, x) > C(v_-, x)$; otherwise, it is rectified by A_r. For the system resumes its operation from condition(v, x), A_p is to be scheduled at virtual age $APL(v, x)$.

$\boldsymbol{P_6}$: $[RCL(v, x), APL(x)]$ Policy: A failure from condition (v, x) is rectified by A_m or $A_g$if $RCL(v_+, x) > C(v_-, x)$, otherwise, by A_r. For the system resumes its operation from condition (v, x), A_p is to be scheduled at age $APL(x)$.

$\boldsymbol{P_7}$: $[ARL(n, v, x), APL(n, v, x)]$ Policy: A failure is rectified by A_m or A_g if $v_- < ARL(n, v, x)$ otherwise, it is rectified by A_r . A_p is scheduled at virtual age $APL(n, v, x)$. Here v_-, v_+ and v refer to the virtual age immediately prior to, after the current failure, and immediately after the previous failure.

$\boldsymbol{P_8}$: $[ARL(n, x), APL(n, x)]$ Policy: A failure from condition (n, x) is rectified by A_m or $A_g$if $ARL(n, x) > v$, otherwise, by A_r. For the system resumes its operation from condition (n, x), A_p is to be scheduled at age $APL(n, x)$.

$\boldsymbol{P_9}$: $[RCL(v), APL(v)]$ Policy: A failure from condition (n, v) is rectified by A_m or $A_g$if $RCL(n, v_+) > C(n, v_-)$; otherwise, it is rectified by A_r. For the system resumes its operation from condition(n, v, x), A_p is to be scheduled at virtual age $APL(n, v)$.

$\boldsymbol{P_{10}}$: $[RCL(v), APL]$ Policy: A failure from condition (n, v) is rectified by A_m or A_g if $RCL(v_+) > C(v_-)$, otherwise, by A_r. For the system resumes its operation from condition (n, v), A_p is to be scheduled at age APL.

$\boldsymbol{P_{11}}$: $[ARL(n, v), APL(n, v)]$ Policy: A failure from condition (n, v) is rectified by A_m or A_g if $ARL(n, v) > v$; otherwise, it is rectified by A_r. For the system resumes its operation from condition(n, v), A_p is to be scheduled at virtual age $APL(n, v)$.

$\boldsymbol{P_{12}}$: $[ARL(n), APL(n)]$ Policy: A failure from condition (n, v) is rectified by A_m or A_g if $ARL(n) > v$, otherwise, by A_r. For the system resumes its operation from condition (n, v), A_p is to be scheduled at age $APL(n)$.

$\boldsymbol{P_{13}}$: $[ARL(v, x), APL(v, x)]$ Policy: A failure from condition (v, x) is rectified by A_m or A_g if $ARL(v, x) > v$; otherwise, it is rectified by A_r. For the system resumes its operation from condition(v, x), A_p is to be scheduled at virtual age $APL(v, x)$.

$\boldsymbol{P_{14}}$: $[ARL(x), APL(x)]$ Policy: A failure from condition (v, x) is rectified by A_m or A_g if $ARL(v, x) > v$, otherwise, by A_r. For the system resumes its operation from condition (v, x), A_p is to be scheduled at age $APL(x)$.

$\boldsymbol{P_{15}}$: $[ARL(v), APL(v)]$ Policy: A failure from condition (v) is rectified by A_m or A_g if $ARL(v) > v$; otherwise, it is rectified by A_r. For the system resumes its operation from condition(v), A_p is to be scheduled at virtual age $APL(v)$.

$\boldsymbol{P_{16}}$: $[ARL(.), APL(.)]$ Policy: A failure from condition (v) is rectified by A_m or A_g if $ARL > v$, otherwise, by A_r. For the system resumes its operation from condition (v), A_p is to be scheduled at age APL.

$\boldsymbol{P_{17}}$: $RCL(n, v, x)$ Policy: A failure from condition (n, v, x) is rectified by A_m or A_g if $RCL(n, v_+, x) > C(n, v_-, x)$; otherwise, it is rectified by A_r.

$\boldsymbol{P_{18}}$: $RCL(n, v)$ Policy: A failure from condition (n, v) is rectified by A_m or A_g if $RCL(n, v_+) > C(n, v_-)$.

$\boldsymbol{P_{19}}$: $RCL(v, x)$ Policy: A failure from condition (v, x) is rectified by A_m or A_g if $RCL(v_+, x) > C(v_-, x)$.

$\boldsymbol{P_{20}}$: $ARL(n, v, x)$ Policy: A failure from condition (n, v, x) is rectified by A_m or A_g if $ARL(n, v_+, x) > v$ otherwise, it is rectified by A_r.

$\boldsymbol{P_{21}}$: $ARL(n, x)$ Policy: A failure from condition (n, v, x) is rectified by A_m or A_g if $ARL(n, x) > v$, otherwise, by A_r.

$\boldsymbol{P_{22}}$: $RCL(v)$ Policy: A failure from condition (v) is rectified by A_m or A_g if $RCL(v) > C$; otherwise, it is rectified by A_r.

$\boldsymbol{P_{23}}$: $ARL(n, v)$ Policy: A failure from condition (n, v) is rectified by A_m or A_g if $ARL(n, v) > v$; otherwise, it is rectified by A_r.

$\boldsymbol{P_{24}}$: $ARL(n)$ Policy: A failure from condition (n, v) is rectified by A_m or A_g if $ARL(n) > v$, otherwise, by A_r.

$\boldsymbol{P_{25}}$: $ARL(v, x)$ Policy: A failure from condition (v, x) is rectified by A_m or A_g if $ARL(v, x) > v$; otherwise, it is rectified by A_r.

$\boldsymbol{P_{26}}$: $ARL(x)$ Policy: A failure from condition (v, x) is rectified by A_m or A_g if $ARL(x) > v$, otherwise, by A_r.

$\boldsymbol{P_{27}}$: $ARL(v)$ Policy: A failure from condition (v) is rectified by A_m or A_g if $ARL(v) > v$; otherwise, it is rectified by A_r.

$\boldsymbol{P_{28}}$: $ARL(.)$ Policy: A failure from condition (v) is rectified by A_m or A_g if $ARL > v$, otherwise, by A_r.

$\boldsymbol{P_{29}}$: NRL Policy: A failure from condition (n) is rectified by A_m or A_g if $RCL(n) > C$; otherwise, it is rectified by A_r.

$\boldsymbol{P_{30}}$: $RCL(n)$ Policy: A failure from condition (v) is rectified by A_m or A_g if $RCL(n) > C$; otherwise, it is rectified by A_r.

$\boldsymbol{P_{31}}$: $[A_m / A_g,\ APL(n, x)]$ Policy: A failure from condition (n, v, x) is rectified by A_m or A_g. For the system resumes its operation from condition(n, v, x), A_p is to be scheduled at virtual age $APL(n, x)$.

$\boldsymbol{P_{32}}$: $[A_m / A_g,\ APL(n)]$ Policy: A failure from condition (n, v) is rectified by A_m or A_g. For the system resumes its operation from condition(n, v), A_p is to be scheduled at virtual age $APL(n)$.

$\boldsymbol{P_{33}}$: $[A_m / A_g, APL(x)]$ Policy: A failure from condition (v, x) is rectified by A_m or A_g. For the system resumes its operation from condition(v, x), A_p is to be scheduled at virtual age $APL(x)$.

$\boldsymbol{P_{34}}$: $[A_m / A_g, APL]$ Policy: A failure from condition (v) is rectified by A_m or A_g. For the system resumes its operation from condition(v), A_p is to be scheduled at virtual age APL.

$\boldsymbol{P_{35}}$: $[A_r, APL(x)]$ Policy: A failure from condition (v, x) is rectified by A_r. For the system resumes its operation from condition(v, x), A_p is to be scheduled at virtual age $APL(x)$.

$\boldsymbol{P_{36}}$: $[A_r, APL]$ Policy: A failure from condition (v) is rectified by A_r. For the system resumes its operation from condition(v), A_p is to be scheduled at virtual age APL.

$\boldsymbol{P_{37}}$: (p,q) Policy: A failure is rectified by $A_m$with probability p or by A_r with probability q.

$\boldsymbol{P_{38}}$: A_m / A_g Policy: Each failure is removed by minimal repair A_m / general repairA_g.

$\boldsymbol{P_{39}}$: A_r Policy: Each failure is removed by failure replacement A_r.

$\boldsymbol{P_{40}}$: Periodic/Block Policy: A failure is rectified by $A_m$or A_r, and $A_p$is taken at a fixed calendar time.

VITA

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