

EIGENVALUE OPTIMIZATION AND ITS APPLICATIONS IN BUCKLING AND VIBRATION

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This dissertation is dedicated
to my parents
Shri. Gopal Krishna Subramaniam
and
Smt. Gomathy Gopal Krishna
for their love, endless support and encouragement

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Abstract

Eigenvalue problems are of immense interest and play a pivotal role not only in many fields of engineering but also in pure and applied mathematics. An in-depth understanding of this class of problems is a pre-requisite for vibration and buckling analyses of structures. Design optimization of structures to prevent failure due to instability (buckling) and vibration introduces the problem of determining optimal physical parameters such that load carrying capacity or the fundamental natural frequency is maximized. A classical example of such problems is the Lagrange problem of determining the shape of the strongest column against buckling. The primary objective of this research is to develop discrete models for column buckling and to solve the problem of finding the strongest column by applying fundamental principles of optimization.

The physical parameters of optimal discrete link-spring models, which maximize the buckling loads, are reconstructed. It is shown that the optimal system can be determined recursively by using a one parameter iterative loop. The mathematical problem of determining parameters of an affine sum such that the system has extremum eigenvalues was derived and solved. Numerical techniques were developed to aid in the optimization process and were utilized to optimize mass-spring systems. The more complicated problem of finding the shape of the strongest column was also defined as an affine sum by applying finite difference schemes to both the second and fourth order governing differential equations. Optimization techniques and numerical methods were developed to arrive at the shape of the strongest clamped-free and pinned-pinned column. Unimodal solutions of the Lagrange problem were also obtained for the special case where minimum area constraints were given. A mathematical model for columns on elastic foundation also was derived and transformed to an affine sum problem. Unimodal solution of shape of the strongest pinned-pinned column on an elastic foundation was obtained. In addition the application in vibration and buckling, it is believed that the optimization principles and numerical methods developed in this research will be applicable in other fields such as optimal control.

Chapter 1

Introduction

The mathematical models or differential equations that govern a number of engineering problems ranging from structural stability to vibration and control are classified as eigenvalue problems. The element of interest in eigenvalue problems is the existence of the trivial or zero solution, which does not have any importance in practical applications. Differential equations that govern the boundary value problems associated with vibration and buckling may be represented as Sturm-Liouville differential equations that have the following general form:

$$(r(x)y')' + (q(x) + \lambda p(x))y = 0 \quad (1.1)$$

In Equation (1.1) the primes illustrate derivatives with respect to x . The obvious solution of the differential equation in (1.1) is the trivial solution of $y = 0$, but this is of no practical use. Hence, a non-trivial solution of (1.1) that satisfies the problem's boundary conditions is to be determined. Such a solution of (1.1), if it exists, is called an eigenfunction and the scalar λ for which an eigenfunction exists is called an eigenvalue. There may be an infinite number of such eigenvalues and eigenfunctions solutions of (1.1) that satisfy the corresponding problem's boundary conditions.

1.1 Eigenvalue Problems in Buckling and Vibration

The differential equations that govern both the static problem of buckling and the dynamic problem of vibration are special cases of the Sturm-Liouville differential equations shown in (1.1). To explain this in detail, let us consider both these problems separately one after the other.

1.1.1 Buckling

Buckling is a mode of failure due to elastic instability. The fourth order differential equation that governs the loss of stability in columns, also known as column buckling is

given by

$$\frac{d}{dx^2} \left(E(x)I(x) \frac{d^2y}{dx^2} \right) + P \frac{d^2y}{dx^2} = 0, \quad (1.2)$$

where P is the compressive load applied, x is the axial coordinate, $I(x)$ is the moment of inertia, $E(x)$ is the modulus of elasticity and $y(x)$ is the non-trivial equilibrium mode of the buckled column. The obvious solution of (1.2) is $y(x) = 0$ with a corresponding buckling load $P = 0$ and this is not of any practical application. The non-trivial solution i.e. $y(x) \neq 0$ is obtained by solving this eigenvalue problem (1.2). The eigenvalues are the loads P that would render the column unstable and the corresponding $y(x)$ is the mode shape associated with that load. The smallest load P that can maintain the column in its non-trivial equilibrium is called the buckling load.

1.1.2 Vibration

Vibration is the mechanical oscillation of structures about an equilibrium position. Vibration in continuous structures is governed by partial differential equations that are functions of time and the spatial coordinates. For example, the simple case of an axially vibrating rod is governed by the partial differential equation

$$\frac{\partial}{\partial x} \left(E(x)A(x) \frac{\partial u(x, t)}{\partial x} \right) = \rho A(x) \frac{\partial^2 u(x, t)}{\partial t^2}. \quad (1.3)$$

The partial differential equation in (1.3) is solved by assuming a solution $u(x, t) = v(x) \sin(\omega t)$, which yields the differential equation

$$\frac{d}{dx} \left(E(x)A(x) \frac{dv(x)}{dx} \right) + \omega^2 v(x) = 0. \quad (1.4)$$

The non-trivial solution $v(x) \neq 0$ is obtained by solving eigenvalue problem (1.4), where the eigenvalues ω are the natural frequencies and eigenvectors $v(x)$ are the corresponding vibrating mode shapes.

1.2 Direct and Inverse Eigenvalue Problems

The eigenvalue problems associated with vibration and buckling analysis of structures, using continuous models, was shown in the previous section. However, for the cases where the shape of the structure is not uniform, in both buckling and vibration analyses, it may be more convenient to use discrete models that approximate the continuous system. In particular, finite difference or finite element models are the most widely used discrete model approximations. These discrete models usually consist of linear equations that can be represented as algebraic eigenvalue problems and are of the form

$$(\mathbf{A} - \lambda\mathbf{B}) \mathbf{x} = \mathbf{0}. \quad (1.5)$$

In algebraic eigenvalue problems, the non-trivial solution $\mathbf{x} \neq \mathbf{0}$ that satisfies Equation (1.5) is called an eigenvector and λ its corresponding eigenvalue. Eigenvalue problems that are based on either continuous system or its discrete model approximation can be used to solve two broad classifications of problems:

1. Direct eigenvalue problems, and
2. Inverse eigenvalue problems

In direct eigenvalue problems the physical parameters (area, length, modulus of elasticity, density, etc.) of the system are known and these are used to determine the unknown spectral data (eigenvalues and eigenvectors) of the system. In contrast, in inverse eigenvalue problems some or all of the spectral data are known and these are used to determine the unknown physical parameters. A graphical explanation of direct and inverse eigenvalue problems can be obtained from the block diagrams in Figure 1.1.

Introduction of optimization principles in mathematical models that are used for buckling and vibration analysis creates a new genre of problems. These problems are a mixture of both direct and inverse eigenvalue problems. The aim here is to find the optimal physical parameters that yield extremum eigenvalues, subject to certain constraints on these phys-

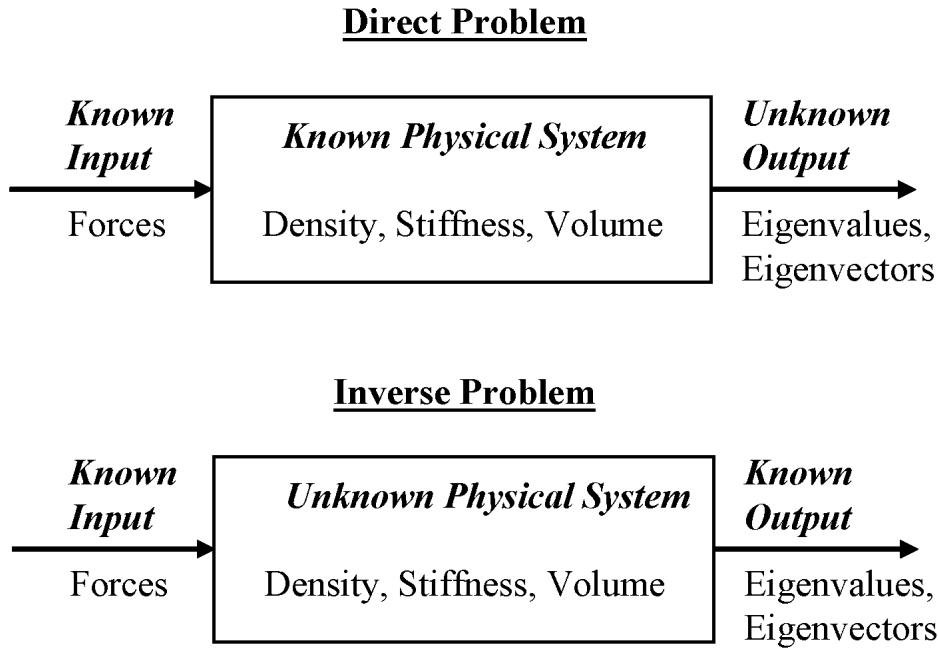


Figure 1.1: Direct and inverse problems

ical parameters. For example, the problem of finding the shape of the strongest column against buckling whose length and volume are specified belongs to this genre. The block diagram representation of such a problem is shown in Figure 1.2

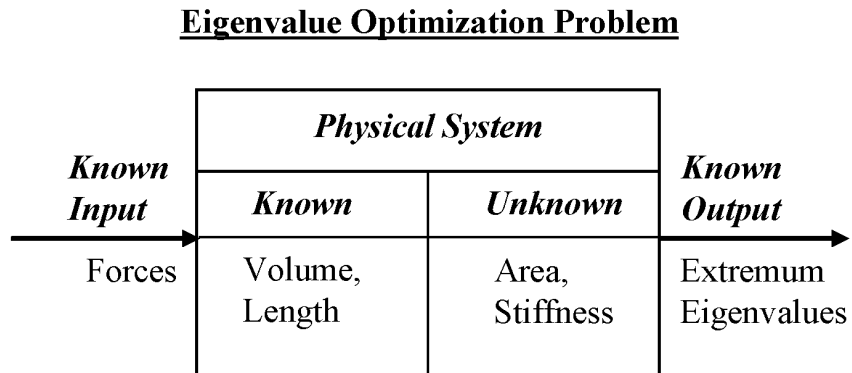


Figure 1.2: Eigenvalue optimization problem

1.3 Eigenvalue Optimization Problems

A more comprehensive understanding of eigenvalue optimization problems may be gained by means of the following example. Consider a two degree of freedom spring mass system as

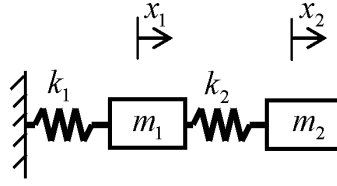


Figure 1.3: Two degree of freedom mass-spring system

shown in Figure 1.3, with spring stiffnesses $k_1 = k_2 = 1$. The objective of the optimization problem is to find the masses m_1 and m_2 that makes the natural frequencies of the system extremum, subject to a total mass constraint given by

$$m_1 + m_2 = 1. \quad (1.6)$$

The dynamic response of this system is governed by the second order differential equation

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}, \quad (1.7)$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix}, \quad (1.8)$$

and

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad (1.9)$$

and

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T, \quad (1.10)$$

where the dots illustrates differentiation with respect to time t . The general solution of the

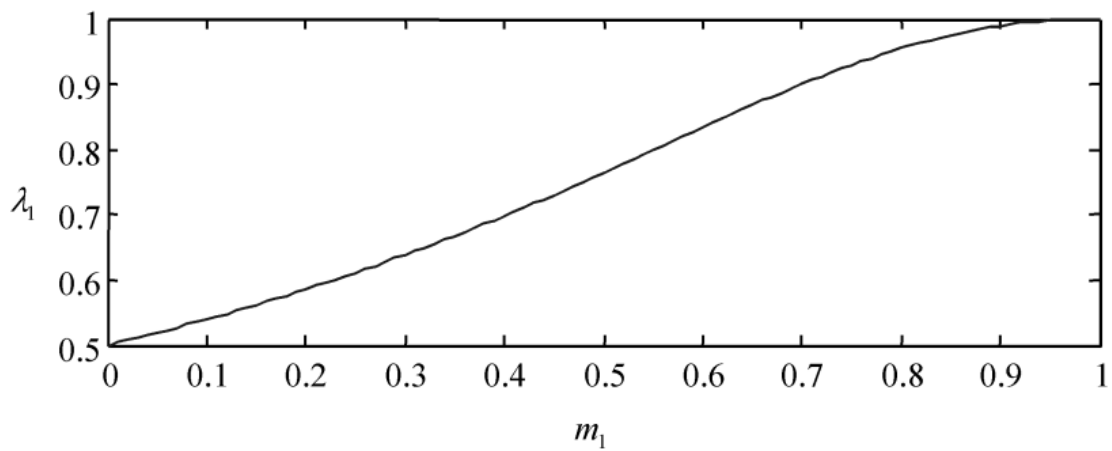
differential equation in (1.7) is

$$\mathbf{x} = \mathbf{A}_1 \sin(\sqrt{\lambda}t) + \mathbf{A}_2 \cos(\sqrt{\lambda}t) \quad (1.11)$$

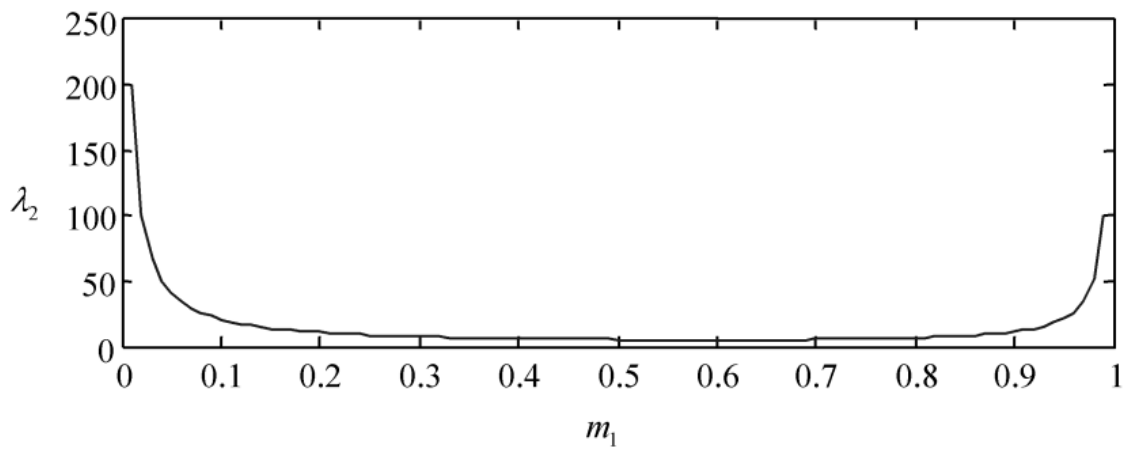
where

$$\lambda = \text{eigenvalue}(\mathbf{K}, \mathbf{M}) \quad (1.12)$$

and \mathbf{A}_1 and \mathbf{A}_2 are constants that depend on the initial conditions. The eigenvalues of the



(a)



(b)

Figure 1.4: (a) Variation of λ_1 with m_1 (b) Variation of λ_2 with m_1

system defined in (1.12) is obtained by solving the equation

$$|\mathbf{K} - \lambda\mathbf{M}| = \begin{vmatrix} k_1 + k_2 - \lambda m_1 & -k_2 \\ -k_2 & k_2 - \lambda m_2 \end{vmatrix} = 0 \quad (1.13)$$

which when evaluated gives the characteristic equation of the form

$$m_1 m_2 \lambda^2 - \lambda (m_1 k_2 + m_2 (k_1 + k_2)) + k_1 k_2 = 0. \quad (1.14)$$

It is evident from equation (1.14) that the eigenvalue λ is dependent on the unknown masses m_1 and m_2 . To graphically determine the maximum and minimum eigenvalue λ , all possible values of one mass, say m_1 is considered, and the corresponding mass m_2 is found such that it satisfies the constraint equation in (1.6). The two roots of equation (1.14) are the first and second eigenvalue of the vibrating mass-spring system. The variation of λ_1 and λ_2 with respect to m_1 is shown in Figure 1.3(a) and Figure 1.3(b), respectively.

The procedure described above for finding the extremum eigenvalues graphically often becomes very tedious or sometimes even impossible as the order of the system increases. Hence, there is a necessity for developing a simpler, equally defensible procedure to find extremum eigenvalues. This dissertation proposes enhanced methods to do the same.

1.4 Literature Review and Overview

Lagrange coined the problem of finding the shape of the strongest column in the 18th century. Even though the Lagrange problem can be stated easily, it has been one of the most difficult ones to solve. Many authors have made substantial contributions in solving this problem, but many have also made some serious errors, including Lagrange himself.

Lagrange approached the problem by considering all possible solids of revolution (solids with circular cross-sections) and attempted to find the shape that would give maximum relative strength, i.e. strength to weight ratio. Lagrange concluded that the right circular cylinder is one but not the only solid of revolution that would have maximum relative

strength. His work was found erroneous by Young (1855).

Fiegen (1952), showed that the optimal wall thickness of a round tubular column is only dependent on the load and is independent of the diameter. Following this work of Fiegen, Keller (1960) took up the problem of finding the shape of the strongest pinned-pinned column with convex cross-section. He concluded that the strongest column has an equilateral triangle as its cross-section, is tapered along its length, and is thickest in the middle and thinnest at the ends. Tadjbakhsh and Keller (1962) determined the best shape of columns with circular cross-sections for the clamped-free, clamped-pinned and clamped-pinned boundary conditions. The shapes of the optimal clamped-clamped and clamped-pinned columns had two and one, vanishing internal areas, respectively. Keller and Niordson (1966) extended the work of Tadjbakhsh and Keller (1962) by considering the weight of the column, and found the height of the tallest column. The only limitation on height in this case was lateral buckling of the column.

The solution of the fixed-fixed column given by Tadjbakhsh and Keller (1962) was claimed to be incorrect by Olhoff and Rasmussen (1977). They argued the assumption that the optimum solution in the former work was simple, i.e. has a multiplicity of one, and was incorrect. Olhoff and Rasmussen obtained a solution for the strongest clamped-clamped column by applying a minimum area constraint. However, the details of the numerical approximation procedure and the proof of validity of the solution were missing. Moreover, this optimal column has a double least eigenvalue.

Thus, the controversy associated with the Lagrange problem was started. Some authors supported the Tadjbakhsh and Keller (1962) solutions and others particularly, Masur (1984) and Seyranian (1983,1984) supported the Olhoff and Rasmussen solution. Seyranian (1983), derived the bimodal optimality condition for a fixed-fixed column in the form of elliptic integrals. Further, Cox and Overton (1992) also proved that the Olhoff and Rasmussen's (1977) clamped-clamped column satisfies the first order necessary condition of Clarke (1990). The theoretical and numerical techniques in Cox and Overton (1992) are

based on convex analysis and its generalizations given by Rockafellar (1970) and Clarke (1990). However, Cox and Overton (1992) also introduced an error by claiming that the clamped-pinned solution of Tadjbakhsh and Keller (1962) was incorrect. Further details on the history of the shape of the strongest column may be found in the works of Cox (1992), Lewis and Overton (1996), Seyranian and Privalova (2003) and others.

The applications of optimal eigenvalue extends far beyond the best shape problem. There are many instances in inverse vibration problems and optimal control where extreme eigenvalues are of interest. We discuss some of these applications in the subsequent chapters.

1.5 Research Objectives

The following issues are addressed in this work:

- Develop discrete models to shed light on the controversy associated with the strongest column.
- Use the discrete models to eliminate the controversy associated with the strongest columns.
- Give solutions for more general boundary conditions such as spring supported boundaries.
- Develop generalized mathematical and numerical techniques that can be used in problems not only in the field of buckling as well as in many other mechanical engineering fields such as vibration where extremum eigenvalues are of interest.
- Use the developed generalized optimization techniques to find optimal vibratory systems with various boundary conditions.
- Develop solutions of optimal columns with minimum area constraint.
- Extend the optimization techniques to determine the shape of strongest columns on elastic foundation.

Chapter 2

Motivation of Research

2.1 Introduction to Instability in Columns

Columns are structural members that carry loads in the axial direction and transmit loads in compression. While the use of columns in architectural construction probably dates back to the very first man-made construction, the concept of instability in columns was not understood until Euler in the 18th century derived the classical expression for the critical buckling load of columns. Throughout this entire dissertation the term column would mean Euler-Bernoulli columns, i.e. a long and slender structure whose length is much higher than its radius at any section along its span.

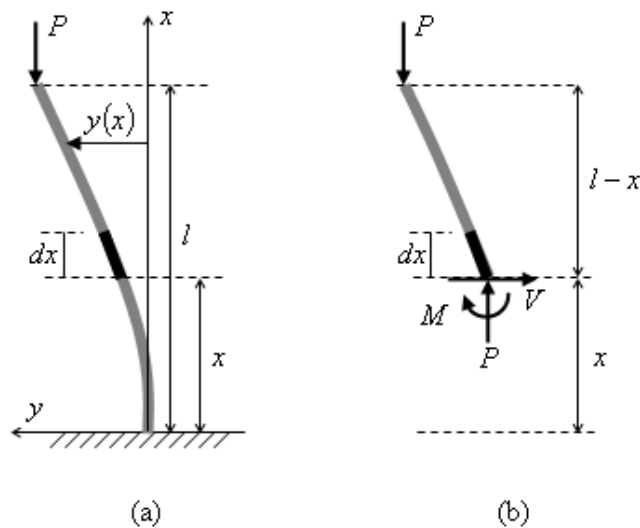


Figure 2.1: (a) Column deformed by the applied compressive load (b) Free-body diagram of one section of the column

Consider a column of length l , variable cross-sectional area $a = a(x)$, which is deformed under the applied load P , as shown in Figure 2.1(a). The minimum static load P that the column can carry to maintain the statically bent shape as shown in Figure 2.1(a) is known as the critical buckling load of the column. A free-body diagram of the upper side of the column is shown in Figure 2.1(b). Since the column is in static equilibrium, using the

free-body diagram of the column, it can also be inferred that the shear force at any section of the column vanishes

$$V = 0 \tag{2.1}$$

In order to derive the eigenvalue problem associated with column buckling we draw, in view of (2.1), a free-body diagram for a small element of length dx , as shown in Figure 2.2, and upon summing the moments obtain

$$-M + \left(M + \frac{dM}{dx} dx \right) + \left(P + \frac{dP}{dx} dx \right) y' dx = 0, \tag{2.2}$$

which simplifies to

$$\frac{dM}{dx} dx + P y' dx + y' \frac{dP}{dx} (dx)^2 = 0 \tag{2.3}$$

Dividing (2.3) by dx and setting $dx \rightarrow 0$ we obtain

$$\frac{dM}{dx} + P y' = 0. \tag{2.4}$$

For the sign convention being used the moment at any section of the beam column is given by

$$M = EI \frac{d^2 y}{dx^2}. \tag{2.5}$$

Applying (2.5) in (2.4) we obtain

$$\frac{d}{dx} \left(EI \frac{d^2 y}{dx^2} \right) + P \frac{dy}{dx} = 0. \tag{2.6}$$

Differentiating (2.6) with respect to x we obtain the following fourth-order differential equation associated with column buckling

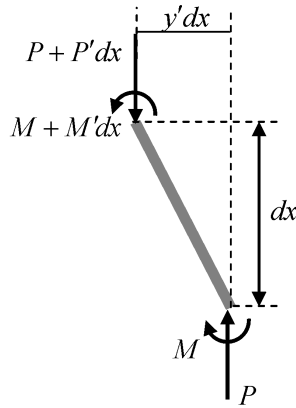


Figure 2.2: Free body diagram of a small section of the column

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 y}{dx^2} \right) + P \frac{d^2 y}{dx^2} = 0 \quad (2.7)$$

Since the governing differential equation of column buckling was derived by considering the free-body diagram of a small section of the column and not the entire column, it is independent of boundary conditions. This fourth-order differential equation in (2.7) along with various boundary conditions may be used to determine the buckling load and mode shapes. The various classical boundary conditions are:

a) Clamped-Free Boundary Conditions

$$y(0) = 0 \quad (2.8a)$$

$$\frac{dy(0)}{dx} = 0 \quad (2.8b)$$

$$EI \frac{d^2 y(l)}{dx^2} = 0 \quad (2.8c)$$

$$P \frac{dy(l)}{dx} + \frac{d}{dx} \left(EI \frac{d^2 y(l)}{dx^2} \right) = 0 \quad (2.8d)$$

b) Clamped-Pinned Boundary Conditions

$$y(0) = 0 \quad (2.9a)$$

$$\frac{dy(0)}{dx} = 0 \quad (2.9b)$$

$$y(l) = 0 \quad (2.9c)$$

$$EI \frac{d^2y(l)}{dx^2} = 0 \quad (2.9d)$$

c) Clamped-Clamped Boundary Conditions

$$y(0) = 0 \quad (2.10a)$$

$$\frac{dy(0)}{dx} = 0 \quad (2.10b)$$

$$y(l) = 0 \quad (2.10c)$$

$$\frac{dy(l)}{dx} = 0 \quad (2.10d)$$

d) Pinned-Pinned Boundary Conditions

$$y(0) = 0 \quad (2.11a)$$

$$EI \frac{d^2y(0)}{dx^2} = 0 \quad (2.11b)$$

$$y(l) = 0 \quad (2.11c)$$

$$EI \frac{d^2y(l)}{dx^2} = 0 \quad (2.11d)$$

2.2 Non-Dimensionalization of the Governing Differential Equation

Equation (2.7) along with the classical boundary conditions is a fourth-order self-adjoint differential equation, whose first eigenvalue is the critical buckling load of the column. If we consider columns with circular cross-sections, then the moment of inertia can be expressed

in terms of the cross-sectional area a as

$$I = \alpha a^2, \quad (2.12)$$

where the constant α is

$$\alpha = \frac{1}{4\pi} \quad (2.13)$$

The differential equation of buckling and the boundary conditions are all functions of physical dimensions. Since, it is more appropriate and perhaps easier to work with non-dimensional parameters we begin the process of non-dimensionalizing by multiplying (2.7) with l^2 and obtain

$$\frac{d^2}{dx^2} \left(EI l^2 \frac{d^2 y}{dx^2} \right) + P l^2 \frac{d^2 y}{dx^2} = 0. \quad (2.14)$$

Using (2.13) in (2.14) and simplifying we have

$$\frac{d^2}{dx^2} \left(a^2 l^2 \frac{d^2 y}{dx^2} \right) + \frac{P l^2}{\alpha E} \frac{d^2 y}{dx^2} = 0. \quad (2.15)$$

We introduce the non-dimensional spatial parameter ξ such that

$$\xi = \frac{x}{l} \quad (2.16)$$

and denote

$$u(\xi) = a^2 l^2 \frac{d^2 y}{dx^2}. \quad (2.17)$$

Using this transformation, the fourth-order differential equation in (2.7) is transformed into

$$\frac{d^2 u}{d\xi^2} + \lambda l^2 \frac{d^2 y}{dx^2} = 0; \quad (2.18)$$

since

$$\frac{d^2u}{dx^2} = \frac{1}{l^2} \frac{d^2u}{d\xi^2}, \quad (2.19)$$

where the eigenvalue λ is defined as

$$\lambda = \frac{Pl^2}{\alpha E}. \quad (2.20)$$

Multiplying both sides of equation (2.18) with a^2 and using (2.17) we obtain the second order differential equation in terms of u and ξ as

$$a^2 \frac{d^2u}{d\xi^2} + \lambda u = 0. \quad (2.21)$$

Equation (2.21) is the equivalent second-order, non-dimensional differential equation that governs buckling of column. It is interesting to note that (2.21) is in terms of moment whereas (2.7) is in terms of lateral displacement at any point along the length of the column. The non-dimensionalization of the boundaries conditions is shown in the next section.

2.3 Non-Dimensionalization of Boundary Conditions

To express the boundary conditions in terms of the non-dimensional parameter ξ we first substitute for u in (2.21) using (2.17) and obtain

$$a^2 \frac{d^2u}{d\xi^2} + \lambda a^2 l^2 \frac{d^2y}{dx^2} = 0. \quad (2.22)$$

Integrating equation (2.22) with respect to ξ from $\xi = 0$ to ξ , we have

$$\frac{du(\xi)}{d\xi} - \frac{du(0)}{d\xi} + \lambda l \left(\frac{dy(x)}{dx} - \frac{dy(0)}{dx} \right). \quad (2.23)$$

The various non-dimensional boundary conditions can be obtained using (2.23) and their respective boundaries that are in terms of displacements.

a) Clamped-Free Boundary Conditions

Using equation (2.8b) in (2.23) we obtain

$$\frac{dy(x)}{dx} = \frac{1}{\lambda l} \left(\frac{du(0)}{d\xi} - \frac{du(\xi)}{d\xi} \right). \quad (2.24)$$

Multiplying equation (2.8c) with l^2 and using (2.12) and (2.17) we get

$$\alpha E u(1) = 0, \quad (2.25)$$

and since α and E are non-zero constants we obtain the non-dimensional boundary condition

$$u(1) = 0 \quad (2.26)$$

Upon multiplying equation (2.8d) with l^2 and using (2.12) and (2.17) at $x = l$ or $\xi = 1$, and simplifying we obtain

$$\frac{Pl^2}{\alpha E} \frac{dy(l)}{dx} + \frac{d}{dx} (u(1)) = 0. \quad (2.27)$$

Introducing λ using (2.20) and using (2.24) at $x = l$ or $\xi = 1$, it simplifies to

$$\frac{1}{l} \left(\frac{du(0)}{d\xi} - \frac{du(1)}{d\xi} \right) - \frac{1}{l} \frac{du(1)}{d\xi} = 0. \quad (2.28)$$

or

$$\frac{du(0)}{d\xi} = 0 \quad (2.29)$$

since $\frac{du}{dx} = \frac{1}{l} \frac{du}{d\xi}$. Hence the boundary conditions for the fixed-free column in terms of the non-dimensional parameters are given by equations (2.26) and (2.29).

b) Clamped-Pinned Boundary Conditions

Since the boundary conditions at the base of the clamped-pinned column are same as the

boundary conditions of the clamped-free column, equation (2.26) is a valid non-dimensional boundary for the former. To obtain the other boundary condition we integrate equation (2.24) with respect to ξ from $\xi = 0$ to $\xi = 1$ and obtain

$$\frac{y(x) - y(0)}{l} = \frac{1}{\lambda L} \left(\xi \frac{du(0)}{d\xi} - u(\xi) + u(0) \right). \quad (2.30)$$

Upon applying the boundary conditions represented by equation (2.9a) in (2.30) we get

$$y(x) = \frac{1}{\lambda} \left(\xi \frac{du(0)}{d\xi} - u(\xi) + u(0) \right). \quad (2.31)$$

The boundary condition in (2.9c) is obtained from (2.31) at $x = l$ or $\xi = 1$ which simplifies to

$$y(l) = \frac{1}{\lambda} \left(\frac{du(0)}{d\xi} - u(1) + u(0) \right) = 0. \quad (2.32)$$

Using equation (2.26) in (2.32) we obtain the other boundary to be

$$\frac{du(0)}{d\xi} + u(0) = 0 \quad (2.33)$$

Hence the boundary condition for the clamped-pinned column in terms of the non-dimensional parameters is given by equations (2.26) and (2.33).

c) Clamped-Clamped Boundary Conditions

Applying the boundary condition (2.10b) in equation (2.23) we obtain equation (2.24). Now applying the boundary condition (2.10d) in (2.24) for $x = l$ or $\xi = 1$ we obtain the first non-dimensional boundary condition to be

$$\frac{du(0)}{d\xi} - \frac{du(1)}{d\xi} = 0. \quad (2.34)$$

Now integrating (2.24) with respect to ξ from $\xi = 0$ to ξ and applying the boundary condition (2.10a) we obtain equation (2.31). When $x = l$ and $\xi = 1$ equation (2.31)

simplifies to yield the second boundary condition

$$\frac{du(0)}{d\xi} - u(1) + u(0) = 0. \quad (2.35)$$

Hence the boundary conditions in terms of non-dimensional parameters for the fixed-fixed column are given by equations (2.34) and (2.35).

d) Pinned-Pinned Boundary Conditions

The boundary conditions for the pinned-pinned column can be obtained very easily by considering the boundaries (2.11b) and (2.11d) and the definition of $u(\xi)$ given by equation (2.17). The boundary conditions in terms of non-dimensional parameters for the hinged-hinged column are

$$u(0) = 0, \quad (2.36)$$

$$u(1) = 0. \quad (2.37)$$

2.4 The Essential Rule of Optimization

The Lagrange problem that was introduced in Chapter 1 deals with determining the shape of the column that can carry the largest load, subject to a specified volume constraint. Keller and Tadjbakhsh and Keller have used the essential rule of optimization,

$$u^2 = a^3 \quad (2.38)$$

without giving detailed or clear proof of its validity. This essential rule is based on the tactical assumption that larger cross-sectional area is required at those points on the length of the column where the bending moment is large. Using elementary principles of calculus of variations the essential rule of optimization in (2.38) other wise called the meta-theorem by Cox (1992) can be derived.

Let us consider a column of specified length l and volume V . The total volume of the

column is related to the non-dimensional cross-sectional area $a(\xi)$ via the simple integral relationship

$$V = \int_0^1 a(\xi) d\xi. \quad (2.39)$$

The governing self-adjoint second order differential equation for buckling is given by (2.21) along with corresponding boundary conditions. The smallest eigenvalue of the self-adjoint equation is given by

$$\lambda = \min_u \frac{\int_0^1 u'^2 d\xi}{\int_0^1 a^{-2} u^2 d\xi}, \quad (2.40)$$

or this is equivalent to minimizing λ where

$$\lambda = \min_u \int_0^1 u'^2 d\xi \quad (2.41)$$

subject to the constraint

$$\int_0^1 a^{-2} u^2 d\xi = C, \quad (2.42)$$

where C is an arbitrary non-zero constant. Hence, the problem may be recast as, to find the shape of the column that maximizes λ subject to the constraint (2.42) and the volume constraints in (2.39). The solution is then a stationary value of the Lagrange function given by

$$L = \int_0^1 u'^2 d\xi - \lambda \int_0^1 a^{-2} u^2 d\xi + \eta \int_0^1 a d\xi, \quad (2.43)$$

which simplifies to

$$L = \int_0^1 \left(u'^2 - \lambda a^{-2} u^2 + \eta a \right) d\xi, \quad (2.44)$$

where the admissible functions must satisfy the boundary conditions, and λ and η are Lagrange multipliers. Hence, by defining functional as

$$F = (u, u', a) = u'^2 - \lambda a^{-2} u^2 + \eta a \quad (2.45)$$

the Euler-Lagrange equations with respect to u gives

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} = -2\lambda a^{-2} u - 2u'' = 0 \quad (2.46)$$

which is the same as the differential equation (2.21). The Euler-Lagrange equations with respect to a gives

$$\frac{\partial F}{\partial a} = 2\lambda a^{-3} u^2 + \eta = 0, \quad (2.47)$$

which can be simplified to

$$a^{-3} u^2 = \frac{-\eta}{2\lambda}. \quad (2.48)$$

Since η and λ are constants it implies

$$a^{-3} u^2 = \text{const} \quad (2.49)$$

Further, as u is an eigenvector it can be normalized arbitrarily, thus with appropriate scaling we may rewrite (2.49) as

$$u^2 = a^3. \quad (2.50)$$

This is the essential rule of optimization or the meta-theorem that enables a column of fixed volume to carry maximum axial load without buckling. Hence, at all sections along the length of the column the square of the moment must be equal to the cube of the

cross-sectional area. This relation also concurs with the physical intuition, that a larger cross-sectional is required at places where the bending moment or the bending stress is higher. But this mathematical relation in (2.50) does not take the axial compressive stress due to the load into consideration.

2.5 The Optimal Columns

The shape of the optimal or the strongest columns may be obtained by means of the meta-theorem given in (2.50). Substituting (2.50) in (2.21) we obtain

$$\frac{d^2u}{d\xi^2} + \lambda u^{-1/3} = 0. \quad (2.51)$$

Keller and Tadjbakhsh (1962) and Keller (1960) have given a parametric solution for the strongest columns that involve solving non-linear equations. The solution of the strongest column for various boundary conditions is then obtained by simultaneously solving the non-linear equations for each case as shown below.

a) Clamped-Free Column

$$a(\xi) = \frac{4V}{3l} \sin^2 \theta(\xi), \quad \frac{-\pi}{2} \leq \theta \leq 0 \quad (2.52)$$

and

$$\theta - \frac{1}{2} \sin 2\theta + \frac{\pi}{2} = \frac{\pi}{2} \xi, \quad 0 \leq \xi \leq 1 \quad (2.53)$$

The non-linear equations in (2.52) and (2.53) can be solved using well known numerical methods such as the ‘bisection method’, and the optimal shape is shown in Figure 2.3(a). At $\xi = 0$ the non-dimensional area $\frac{al}{V} = 1.33$ and at $\xi = 1$ is $\frac{al}{V} = 0$. The eigenvalue corresponding to this column is $\lambda = \frac{1}{3}\pi^2 \left(\frac{V}{l}\right)^2$. The variation of the bending stress along the column is shown in Figure 2.3(b).

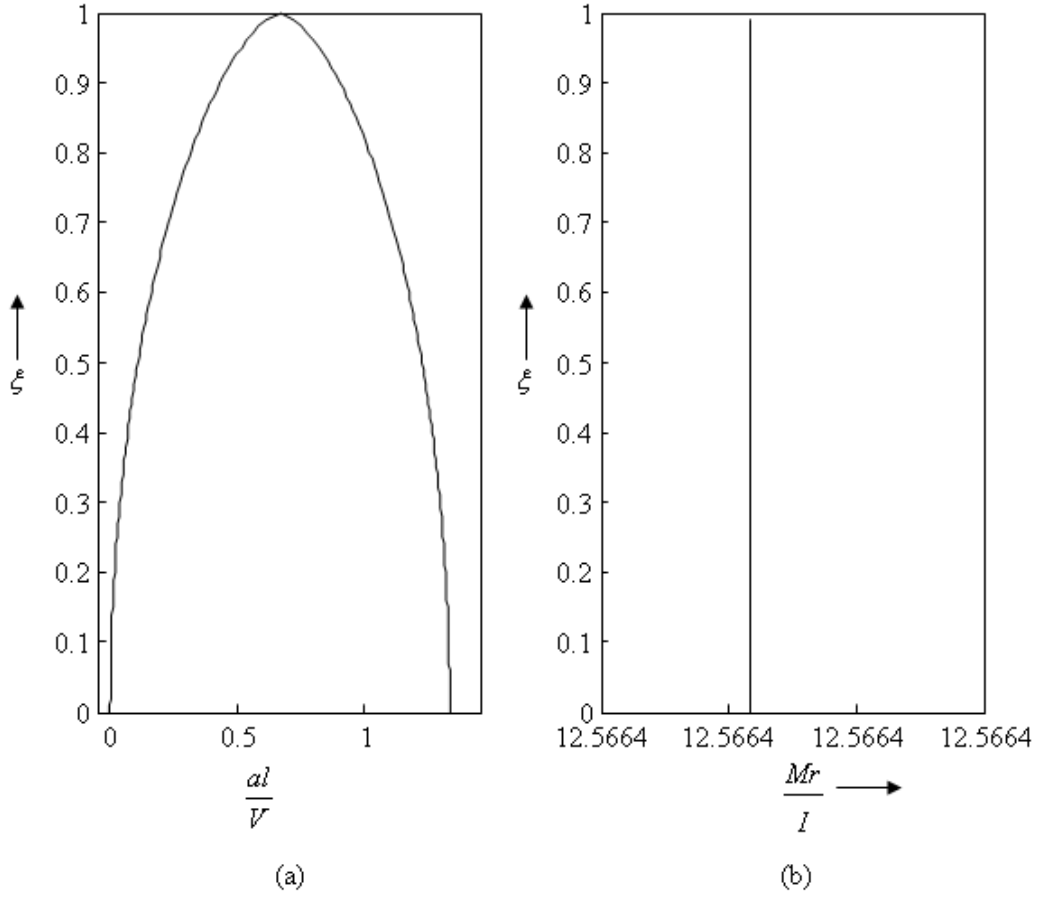


Figure 2.3: (a) Shape of the strongest clamped-free column (b) Bending stress along the length

b) Clamped-Pinned Column

$$a(\xi) = \frac{4V \sin^2 \theta(\xi)}{3l \sin^2 \theta(0)}, \quad \theta(0) \leq \theta \leq \pi \quad (2.54)$$

$$\theta - \frac{1}{2} \sin 2\theta + a = 2 \left(\frac{\lambda}{3} \right)^{1/2} a_0^{-1} \xi, \quad 0 \leq \xi \leq 1 \quad (2.55)$$

$$\pi + \alpha = 2 \left(\frac{\lambda}{3} \right)^{1/2} a_0^{-1} \quad (2.56)$$

$$a_0 = \frac{4}{3 \sin^2 \theta(0)} \frac{V}{l} \quad (2.57)$$

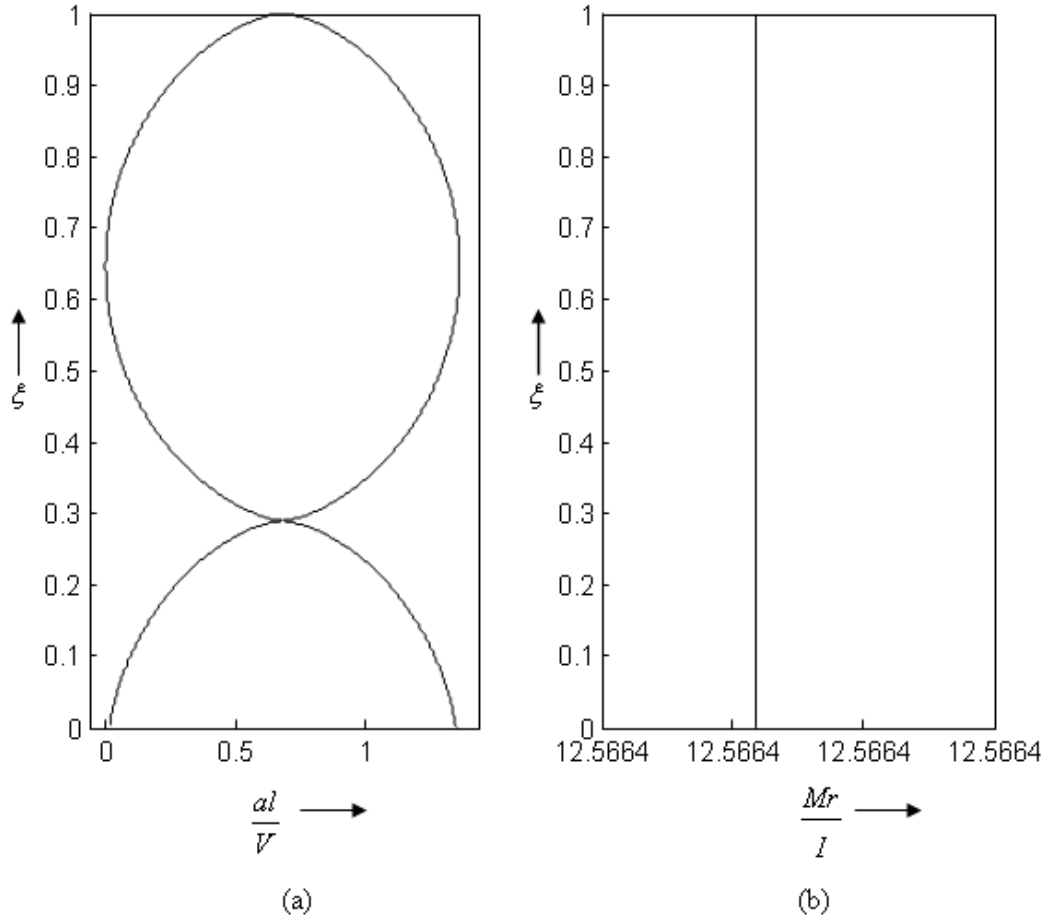


Figure 2.4: (a) Shape of the strongest clamped-pinned column (b) Bending stress along the length

where

$$\theta(0) = -1.4243 \quad (2.58)$$

and

$$\lambda = \frac{16}{27} \tan^2 \theta(0) \left(\frac{V}{l} \right)^2 \approx 27.22 \left(\frac{V}{l} \right)^2. \quad (2.59)$$

Solving the set of equation in (2.54)-(2.59) we obtain the shape of the column shown in Figure 1.4. At $\xi = 0$ the non-dimensional area is $\frac{al}{V} = 1.33$ and the area vanishes at $\xi = 1$ and $\xi = 0.2895$.

c) Clamped-Clamped Column

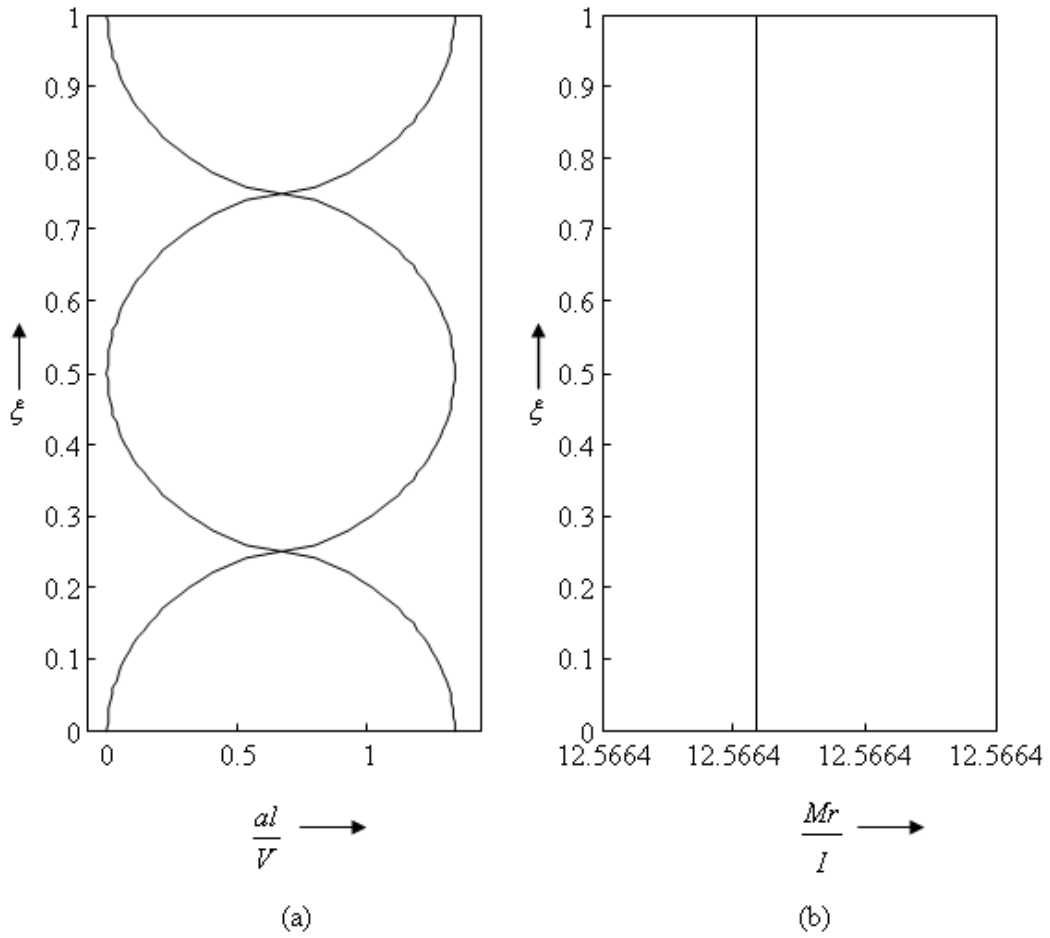


Figure 2.5: (a) Shape of the strongest clamped-clamped column (b) Bending stress along the length

$$a(\xi) = \frac{4V}{3l} \sin^2 \theta(\xi), \quad -\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \quad (2.60)$$

and

$$\theta - \frac{1}{2} \sin 2\theta + \frac{\pi}{2} = 2\pi\xi, \quad 0 \leq \xi \leq 1 \quad (2.61)$$

Solving equations (2.60) and (2.61) simultaneously we obtain the shape of the columns shown in Figure 1.5. At $\xi = 0$ and $\xi = 1$ the non-dimensional area is $\frac{al}{V} = 1.33$ and at the area vanishes at $\xi = 0.25$ and $\xi = 0.75$. The eigenvalue corresponding to this column is

$$\lambda = \frac{16\pi^2}{3} \left(\frac{V}{l} \right)^2.$$

d) Pinned-Pinned Column

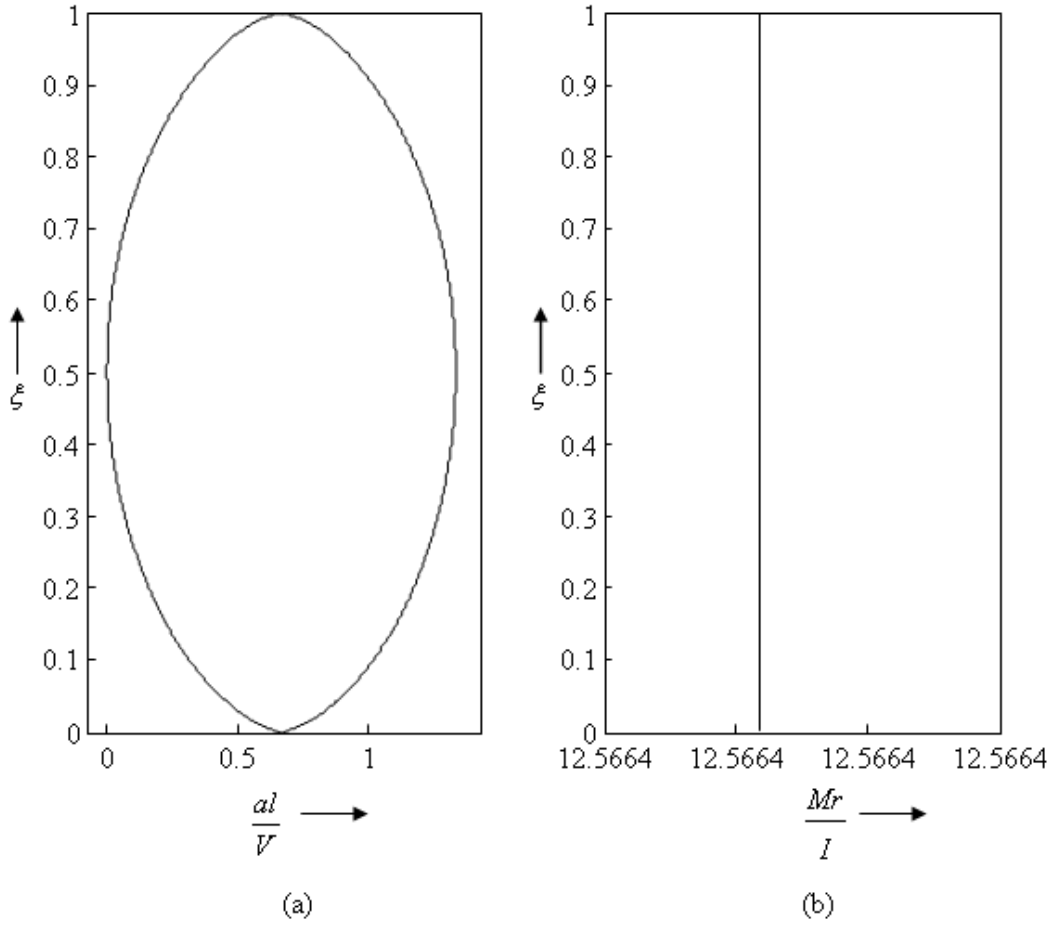


Figure 2.6: (a) Shape of the strongest pinned-pinned column (b) Bending stress along the length

$$a(\xi) = \frac{4V}{3l} \sin^2 \theta(\xi), \quad 0 \leq \theta \leq \pi \quad (2.62)$$

and

$$\theta - \frac{1}{2} \sin 2\theta - \frac{\xi}{\pi} = 0, \quad 0 \leq \xi \leq 1 \quad (2.63)$$

Solving equations (2.62) and (2.63) simultaneously we obtain the shape of the columns shown in Figure 1.5. The maximum area $\frac{al}{V} = 1.33$ is at $\xi = 0.5$ and the area vanishes at

$\xi = 0$ and $\xi = 1$. The eigenvalue corresponding to this column is $\lambda = \frac{4\pi^2}{3} \left(\frac{V}{l}\right)^2$.

The Lagrange problem of finding the shape of the strongest column has been plagued by erroneous solutions right from the beginning. As explained in Chapter 1, many researchers both in the field of engineering and mathematics have attempted to solve this problem and have made some critical errors in their attempts. Further, even the correct solutions (clamped-free and pinned-pinned) are not very clear and require complicated numerical algorithms to obtain the shape. A method that would circumvent the tedious procedure of solving the non-linear differential equation in (2.51) is needed. This dissertation address the problems of problem of finding the shape of the strongest column using discrete models. As a subset to this problem, an affine sum model that can be used to determine mass-spring systems with extremum natural frequencies, will also be developed.

Chapter 3

Discrete Model Analysis of Optimal Columns

The continuous model developed in Chapter 2, may be represented by an equivalent discrete link-spring models having appropriate spring stiffness and lengths of links. The various classical boundaries and other non-classical ones may be obtained by using appropriate support springs at the ends of the link-spring system. The details of the procedure used to obtain the optimal columns using these discrete models are given in this chapter.

3.1 Discrete Model Analysis of the Strongest Clamped-Free Column

The clamped-free column may be represented using a link-spring model with $n + 1$ inflexible or rigid links each of length $h = l/n$. Two adjacent links are connected using a moment or torsion spring of stiffness τ_i , as shown in Figure 3.1. We denote the cross-sectional area of the i -th spring by a_i , and in order to maintain an analogy with the continuous column, the stiffness of the torsion spring must be

$$\tau_i = \frac{EI_i}{h} = \frac{Ea_i^2}{4\pi h}, \quad i = 1, 2, \dots, n \quad (3.1)$$

The problem of finding the shape of the strongest column using the discrete link-spring model may equivalently stated as, to find the cross-sectional areas of the links, subject to the constraint

$$V = h \sum_{i=1}^n a_i, \quad (3.2)$$

such that the critical load P , the one which is able to maintain the system in non-trivial equilibrium, is largest. Note that the assumption that the links are inflexible implies that, the links maintain constant distance between adjacent nodes. In particular, even if the area of some links vanishes the distance between the two nodes that are on either side of

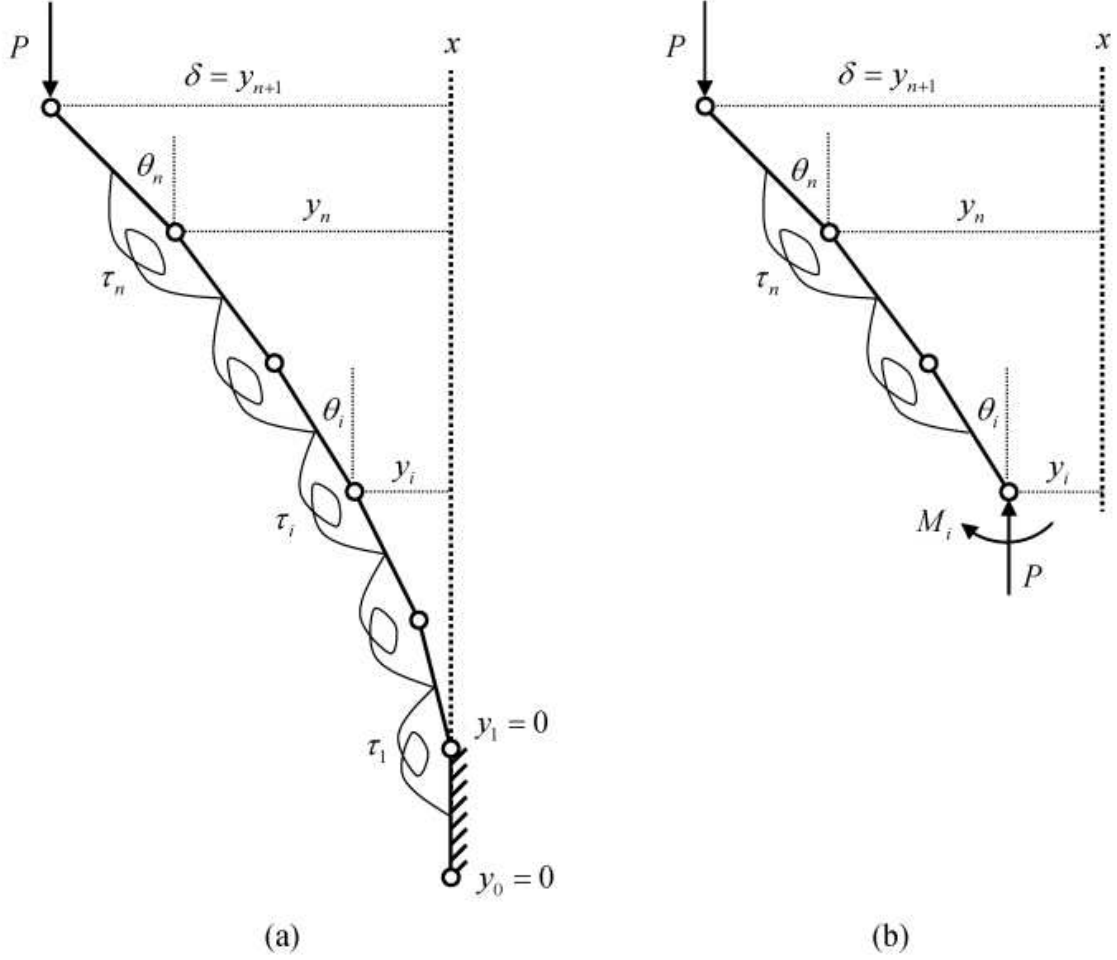


Figure 3.1: The link-spring system (a) System configuration (b) Free-body diagram of a typical segment of the system

this link still remains constant.

From the free-body diagram for a segment of the link-spring system, shown in Figure 3.1(b), the bending moment M_i at the i -th node is

$$M_i = P(\delta - y_i) = \tau_i(\theta_i - \theta_{i-1}), \quad i = 1, 2, \dots, n \quad (3.3)$$

where θ_i is the slope or the angle of the i -th link, measured from the vertical. The slopes can be expressed in terms of displacement of the nodes as,

$$\theta_i = \frac{y_{i+1} - y_i}{h}, \quad i = 1, 2, \dots, n. \quad (3.4)$$

Replacing τ_i and θ_i in (3.3) using (3.1) and (3.4) respectively we obtain,

$$M_i = P(\delta - y_i) = \frac{Ea_i^2}{4\pi h} \left(\frac{y_{i+1} - y_i}{h} - \frac{y_i - y_{i-1}}{h} \right), \quad (3.5)$$

which upon simplification yields,

$$P(\delta - y_i) = \frac{Ea_i^2}{4\pi h^2} (y_{i-1} - 2y_i + y_{i+1}) \quad i = 1, 2, \dots, n. \quad (3.6)$$

In order to eliminate δ from equation (3.6) we introduce the transformation,

$$u_i = y_i - \delta \quad i = 1, 2, \dots, n, \quad (3.7)$$

in (3.7) and obtain

$$\frac{Ea_i^2}{4\pi h^2} (-u_{i-1} + 2u_i - u_{i+1}) - Pu_i \quad i = 1, 2, \dots, n, \quad (3.8)$$

where $u_0 = u_1 = -\delta$ and $u_n + 1 = 0$.

The eigenvalue problem that equation (3.8) defines, is of the form

$$(\mathbf{K} - \mu\mathbf{M}) \mathbf{u} = \mathbf{0}, \quad (3.9)$$

where the matrix \mathbf{K} is tridiagonal of the form

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix} \quad (3.10)$$

and \mathbf{M} is diagonal of the the form

$$\mathbf{M} = \begin{bmatrix} a_1^{-2} & & & \\ & a_2^{-2} & & \\ & & \ddots & \\ & & & a_n^{-2} \end{bmatrix} \quad (3.11)$$

with eigenvalue

$$\mu = \frac{\lambda}{n^2} = \frac{4P\pi h^2}{E}, \quad (3.12)$$

and eigenvector $\mathbf{u} = (u_1 \ u_2 \ \dots \ u_n)^T$. The Courant-Fischer characterization of eigenvalues for the symmetric eigenvalue problem gives

$$\mu = \min_{\mathbf{u}} \frac{\mathbf{u}^T \mathbf{K} \mathbf{u}}{\mathbf{u}^T \mathbf{M} \mathbf{u}} \quad (3.13)$$

indicates that

$$\mu_{max} = \max_{\mathbf{a}} \min_{\mathbf{u}} \frac{\mathbf{u}^T \mathbf{K} \mathbf{u}}{\mathbf{u}^T \mathbf{M} \mathbf{u}} \quad (3.14)$$

Hence, we may determine the maximum eigenvalue μ_{max} by solving

$$\mu_{max} = \max_{\mathbf{a}} \min_{\mathbf{u}} \mathbf{u}^T \mathbf{K} \mathbf{u} \quad (3.15)$$

subject to the constraint

$$\mathbf{u}^T \mathbf{M} \mathbf{u} = C \quad (3.16)$$

and the volume constraint given in (3.2), where $C \neq 0$ is an arbitrary constant. A solution of this optimization problem, may be obtained by defining Lagrange function as

$$L = \mathbf{u}^T \mathbf{K} \mathbf{u} - \mu \mathbf{u}^T \mathbf{M} \mathbf{u} + \eta (a_1 + a_2 + \dots + a_n), \quad (3.17)$$

where μ and η are Lagrange multipliers. This Lagrange function must be stationary or invariant with respect to both u_i and a_i . By virtue of (3.11), equation (3.17) can be written in equivalent form,

$$L = \mathbf{u}^T \mathbf{K} \mathbf{u} - \mu (a_1^{-2} u_1^2 + a_2^{-2} u_2^2 + \cdots + a_n^{-2} u_n^2) + \eta (a_1 + a_2 + \cdots + a_n), \quad (3.18)$$

Variation of (3.18) with respect to \mathbf{u} leads to the eigenvalue problem (3.9)-(3.12), and variation with respect to a_i gives

$$\frac{\partial L}{\partial a_i} = 2\mu a_i^{-3} u_i^2 + \eta = 0, \quad i = 1, 2, \dots, n. \quad (3.19)$$

The Lagrange multiplier η in (3.19) does not have any significance, and to eliminate it we subtract (3.19) for $i = 2, 3, \dots, n$ from (3.19) for $i = 1$ and obtain

$$\frac{\partial L}{\partial a_i} - \frac{\partial L}{\partial a_1} = 2\mu a_i^{-3} u_i^2 - 2\mu a_1^{-3} u_1^2 = 0, \quad i = 2, 3, \dots, n \quad (3.20)$$

or

$$a_i^{-3} u_i^2 = a_1^{-3} u_1^2, \quad i = 1, 2, \dots, n. \quad (3.21)$$

Equation (3.21) is the discrete model analogy of the meta-theorem of the continuous model in equation (2.38).

If the volume constrain is changed from V to $\hat{V} = \alpha V$, where $\alpha \neq 0$ is some constant, then from (3.2) the cross sections of all links comprising the model are scaled by α . Further, it follows from (3.9) that such a scaling of volume would scale the eigenvalue μ by α^2 . Hence instead of finding the cross-sectional areas of the links in a model with volume V , we may find the cross-sections \hat{a} of an equivalent link-spring model of volume \hat{V} , and then scale this model to satisfy the original volume constraint (3.2). Obviously, such a scaling of the volume would yield $\hat{a} = 1$ and $\hat{\mu} = \mu/\alpha^2$. Moreover, since \mathbf{u} is an eigenvector of (3.9) it can be normalized such that $\hat{u}_1 = 1$ and obtain from (3.21)

$$\hat{u}_i = \hat{a}_i^{3/2}, \quad i = 1, 2, \dots, n. \quad (3.22)$$

Substituting for \hat{u} using (3.22) in the eigenvalue problem defined in (3.9)-(3.12) the following recursion is obtained:

$$\hat{a}_2 = (1 - \hat{\mu})^{2/3}, \quad (3.23)$$

$$\hat{a}_i = \left(2\hat{a}_{i-1}^{3/2} - \hat{a}_{i-2}^{3/2} - \hat{\mu}\hat{a}_{i-1}^{-1/2} \right)^{2/3}, \quad i = 3, 4, \dots, n. \quad (3.24)$$

The last equation in (3.9) gives

$$\hat{\mu} = 2\hat{a}_n^2 - \hat{a}_{n-1}^{3/2} \sqrt{\hat{a}_n}. \quad (3.25)$$

Thus the problem may be solved by a one parameter iterative process as follows. We may start with an initial guess for $\hat{\mu}$ and obtain via (3.23) and (3.24) approximations to \hat{a}_i , $i = 2, 3, \dots, n$. Then equation (3.25) can be used to improve the initial guess for $\hat{\mu}$. By repeating this process iteratively $\hat{\mu}$ and \hat{a}_i may be determined to a required accuracy. In particular, the bisection method could be employed with $\hat{\mu}$ located within the initial interval

$$0 < \hat{\mu} < \frac{\pi^2 h^2}{4}. \quad (3.26)$$

The upper bound in (3.26) corresponds to the smallest eigenvalue of a uniform cylinder of length l and cross-sectional area $a = 1$. The solution to the problem is then obtained by scaling,

$$a_i = \frac{V \hat{a}_i}{n}, \quad (3.27)$$

$$h \sum_{j=1} a_j$$

$$\mu = a_1^2 \hat{\mu}. \quad (3.28)$$

Solving the recursive solution using bisection method we obtain the optimal solution

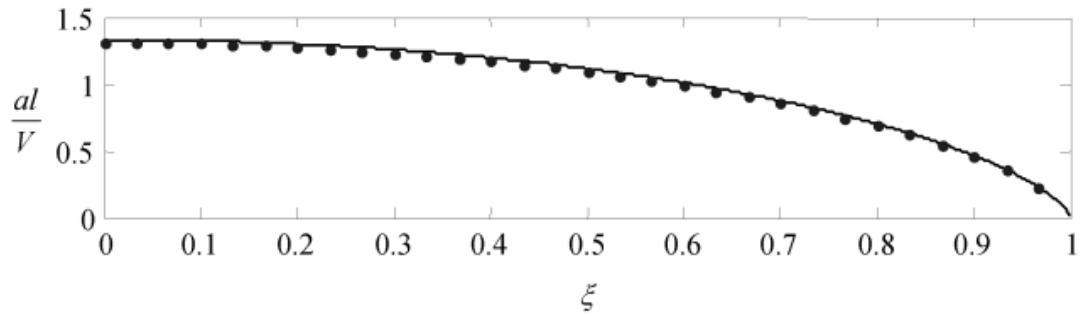


Figure 3.2: The strongest column based on the analytical solution (-), and the discrete link-spring model of order $n = 30$ (.)

of the strongest link-spring model. Figure 3.2 shows the strongest column obtained by the analytical solution of the continuous system given by Tadjbakhsh and Keller (1962), and its approximation by using a discrete link-spring system of order $n = 30$. It demonstrates that in the discrete model the area of the upper link does not vanish, although it approaches zero as n increases.

3.2 Discrete Model Analysis of the Strongest Column under the Pinned-Spring-Supported Boundaries

Consider the case where one end of the system is pinned to the ground, and a spring of stiffness k is attached to the other, as shown in Figure 3.3. We wish to determine the cross-sectional areas a_i , $i = 1, 2, \dots, n$ which maximize the critical load P , subject to the volume constraint (3.2). Instead of using the free-body-diagram we now use the energy method to determine the governing equation for buckling. The elastic energy stored in the springs is given by

$$U_1 = \frac{1}{2} \sum_{j=2}^n \tau_j (\theta_j - \theta_{j-1})^2 + \frac{h^2 k}{2} (\theta_1 + \theta_2 + \dots + \theta_n)^2, \quad (3.29)$$

and the virtual work done by the applied load P is

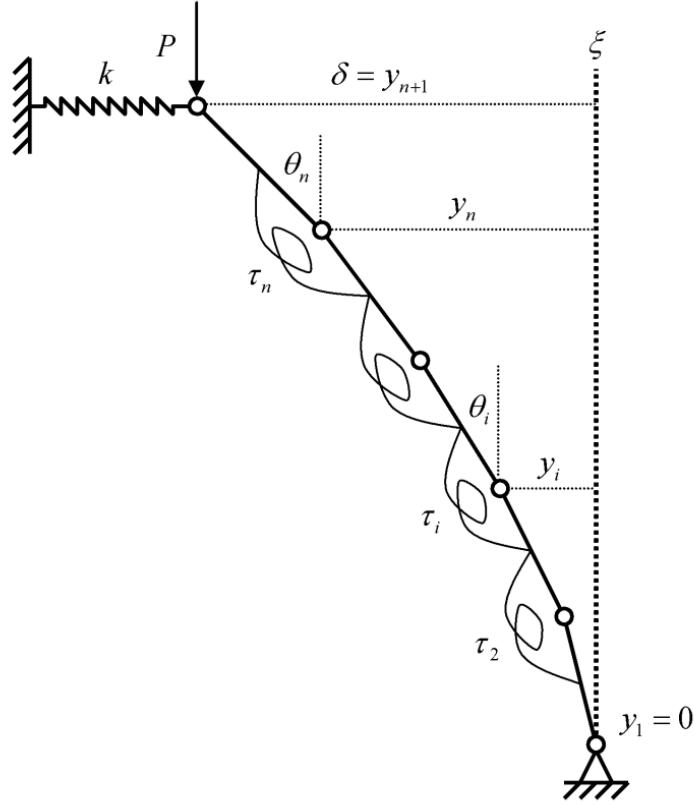


Figure 3.3: Link-spring system with a spring at its top and pin-ended at the bottom

$$W = Ph \sum_{j=1}^n (1 - \cos \theta_j), \quad (3.30)$$

which describes the product of the load P with the x -component of the deflection at the point of application. In order to linearize the equations Taylor series expansion,

$$\cos \theta = 1 - \frac{\theta^2}{2} + O(\theta^4), \quad (3.31)$$

is applied to (3.30) and when the terms of order θ_i^4 and higher are ignored gives

$$W = \frac{Ph}{2} \sum_{j=1}^n \theta_j^2. \quad (3.32)$$

Note that the terms of order θ^2 are not ignored as the elastic energy in U_1 also includes

the same order terms. The total potential energy stored in the system is then obtained by, subtracting the work done from the elastic energy, and is given by

$$U = U_1 - W = \frac{1}{2} \sum_{j=2}^n \tau_j (\theta_j - \theta_{j-1})^2 + \frac{h^2 k}{2} \left(\sum_{j=1}^n \theta_j \right)^2 - \frac{Ph}{2} \sum_{j=1}^n \theta_j^2. \quad (3.33)$$

Substituting for τ and μ using (3.1) and (3.12) in (3.33) gives

$$U = \frac{E}{8\pi h} \left(\sum_{j=2}^n a_j^2 (\theta_j - \theta_{j-1})^2 + \phi \left(\sum_{j=1}^n \theta_j \right)^2 - \mu \sum_{j=1}^n \theta_j^2 \right), \quad (3.34)$$

where

$$\phi = \frac{4\pi h^3 k}{E}. \quad (3.35)$$

Hence, the solution of the strongest link-spring model is a stationary value of

$$L = \frac{E}{8\pi h} \left(\sum_{j=2}^n a_j^2 (\theta_j - \theta_{j-1})^2 + \phi \left(\sum_{j=1}^n \theta_j \right)^2 - \mu \sum_{j=1}^n \theta_j^2 + \eta \sum_{j=2}^n a_j \right), \quad (3.36)$$

with respect to variations in both θ_i and a_i . Variation of (3.36) with respect to θ_i gives

$$\frac{\partial L}{\partial \theta_i} = \frac{E}{8\pi h} \left(2a_i^2 (\theta_i - \theta_{i-1}) - 2a_{i+1}^2 (\theta_{i+1} - \theta_i) + 2\phi \sum_{j=1}^n \theta_j - 2\mu\theta_i \right), \quad i = 1, 2, \dots, n \quad (3.37)$$

which can be simplified to

$$a_i^2 (\theta_i - \theta_{i-1}) - a_{i+1}^2 (\theta_{i+1} - \theta_i) + \phi \sum_{j=1}^n \theta_j - \mu\theta_i = 0, \quad i = 1, 2, \dots, n \quad (3.38)$$

where $a_1 = a_{n+1} = 0$.

The of n equations in (3.38) are linear in terms of $\theta, i = 1, 2, \dots, n$ and may be written in matrix form to obtain the following eigenvalue problem,

$$\begin{pmatrix} a_2^2 + \phi & -a_2^2 + \phi & \phi & \cdots & \phi & \phi \\ -a_2^2 + \phi & a_2^2 + a_3^2 + \phi & -a_3^2 + \phi & \cdots & \phi & \phi \\ \phi & -a_3^2 + \phi & a_3^2 + a_4^2 + \phi & \cdots & \phi & \phi \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi & \phi & \phi & \cdots & a_{n-1}^2 + a_n^2 + \phi & -a_n^2 + \phi \\ \phi & \phi & \phi & \cdots & -a_n^2 + \phi & a_n^2 + \phi \end{pmatrix} - \mu \mathbf{I} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_{n-1} \\ \theta_n \end{pmatrix} = \mathbf{0} \quad (3.39)$$

where \mathbf{I} is the identity matrix of appropriate dimensions. Adding the equations in (3.39) gives

$$(n\phi - \mu) \sum_{j=1}^n \theta_j = 0. \quad (3.40)$$

One solution of (3.40) that can be obtained by inspection is

$$\mu = n\phi, \quad \theta_i = C, \quad (3.41)$$

where $C \neq 0$ is an arbitrary constant. Clearly the solution in (3.41) represents a case where the buckling load is independent of the cross-sectional area of the links. Hence, these cross-sectional areas may be assigned randomly as long as the volume constraint in (3.2) is satisfied. This issue will be further demonstrated and explained in the next section. The other solutions of equation (3.39) satisfy

$$\sum_{i=1}^n \theta_i = 0. \quad (3.42)$$

However, each solution of (3.40) that satisfies the constraint (3.42) is independent of ϕ .

Further, equation (3.42) implies that the displacement at the top of the system vanishes. *We, therefore, deduce that such solutions are eigenvalues and eigenvectors of strongest pinned-pinned link-spring system.* It thus follows that critical load for the link-spring system under the pinned-spring-supported configuration is either

$$P = lk \quad (3.43)$$

by virtue of (3.12), (3.35) and (3.41), or the critical load of the strongest pinned-pinned link-spring system for the given volume and length, whichever is smaller.

The case where (3.42) holds is the configuration of the strongest pinned-pinned link-spring system. To determine the corresponding critical load for this system, we subtract the i^{th} equation in (3.38) from its succeeding equation for $i = 1, 2, \dots, n$, and obtain

$$2a_{i+1}^2 (\theta_{i+1} - \theta_i) - a_{i+2}^2 (\theta_{i+2} - \theta_{i+1}) - a_i^2 (\theta_i - \theta_{i-1}) - \mu (\theta_{i+1} - \theta_i) = 0, \quad (3.44)$$

and this is independent of ϕ . We then denote

$$v_i = \theta_i - \theta_{i-1}, \quad (3.45)$$

and obtain via (3.44)

$$-a_{i-1}^2 v_{i-1} + 2a_i^2 v_i - a_{i+1}^2 v_{i+1} - \mu v_i = 0, \quad i = 2, 3, \dots, n. \quad (3.46)$$

Variation of the Lagrange function, defined in (3.36), with respect to a_i yields

$$\frac{\partial L}{\partial a_i} = \frac{E}{4\pi h} a_i v_i^2 + \eta = 0, \quad i = 2, 3, \dots, n. \quad (3.47)$$

Subtracting the i^{th} equation of (3.47) from the first equation for $i = 3, 4, \dots, n$, to eliminate the Lagrange multiplier η , we obtain

$$a_i v_i^2 = a_2 v_2^2, \quad i = 3, 4, \dots, n. \quad (3.48)$$

Since equations (3.47) and (3.48) define an eigenvalue problem, of which $\mathbf{v} = (v_2 \ v_3 \ \dots \ v_n)^T$ is one of its eigenvectors, we may scale \mathbf{v} such that $a_2 v_2^2 = 1$, so that (3.48) gives

$$a_i = v_i^{-2}, \quad i = 3, 4, \dots, n. \quad (3.49)$$

The system of equations in (3.46) and (3.49) together with $a_1 = a_{n+1} = 0$ and $a_2 v_2^2 = 1$ yields the following $n - 1$ equations for the n unknowns, v_2, v_3, \dots, v_n and μ ,

$$2v_2^{-3} - v_3^{-3} - \mu v_2 = 0, \quad (3.50)$$

$$-v_{i-1}^{-3} + 2v_i^{-3} - v_{i+1}^{-3} - \mu v_i = 0, \quad i = 3, 4, \dots, n - 1 \quad (3.51)$$

$$-v_{n-1}^{-3} + 2v_n^{-3} - \mu v_n = 0. \quad (3.52)$$

The set of equation in (3.50)-(3.52) may be written in matrix form as

$$\begin{bmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \begin{pmatrix} v_2^{-3} \\ v_3^{-3} \\ \vdots \\ v_{n-1}^{-3} \\ v_n^{-3} \end{pmatrix} = \mu \mathbf{I} \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} \quad (3.53)$$

where \mathbf{I} is the identity matrix of appropriate dimension. The set of equations (3.50)-(3.52) or its equivalent form in (3.53) may be solved using a strategy similar to that employed in Section 3.1, for the discrete model of the strongest clamped-free column. Rather than solving (3.49)-(3.53) for the link-spring system of volume V , we may first reconstruct the system of volume $\hat{V} = V/a_2$. The cross-sectional area of the links of this system are denoted

by \hat{a}_i with $\hat{a}_1 = 0$ and $\hat{a}_2 = 1$. The other parameters associated with this system are $\hat{\mu}$ and \hat{v}_i , $i = 2, 3, \dots, n$ with $\hat{v}_2 = 1$. The recursive equations defining the solution are,

$$\hat{v}_3 = (2 - \hat{\mu})^{-1/3}, \quad (3.54)$$

$$\hat{v}_{i+1} = (-\hat{v}_{i-1}^{-3} + 2\hat{v}_i^{-3} - \hat{\mu}\hat{v}_i)^{-1/3}, \quad (3.55)$$

and

$$-\hat{v}_{n-1}^{-3} + 2\hat{v}_n^{-3} - \hat{\mu}\hat{v}_n = 0. \quad (3.56)$$

Hence, we may use for example the bi-section method with the initial guess

$$0 > \hat{\mu} > \pi^2/n, \quad (3.57)$$

where $\hat{\mu}$ is an iterative parameter. Then $\hat{v}_3, \hat{v}_4, \dots, \hat{v}_n$ can be determined recursively from (3.54) and (3.55). Equation (3.56) may be used to bisect the starting interval (3.57), where $\hat{\mu}$ is located. This process can be repeated until $\hat{\mu}$ and \hat{v}_i are determined to a desired accuracy. Once \hat{v}_i are found, \hat{a}_i are determined by the equivalent of (3.49)

$$\hat{a}_i = \hat{v}_i^{-2}, \quad i = 2, 3, \dots, n. \quad (3.58)$$

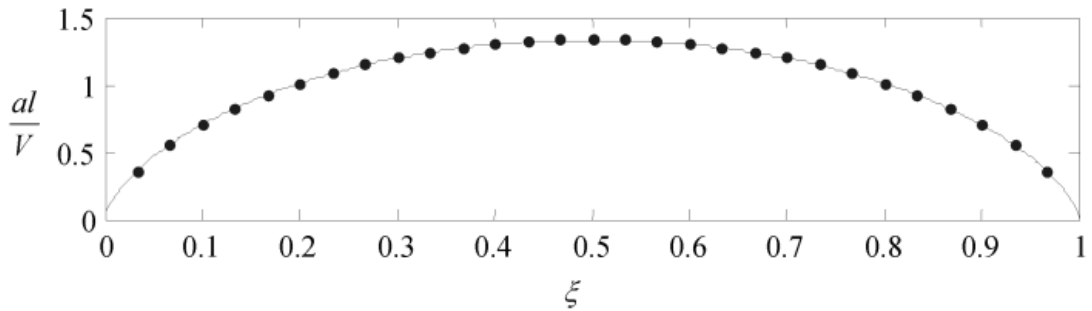


Figure 3.4: The strongest pinned-pinned column based on the analytical solution (-), and the discrete link-spring model of order $n = 30$ (.)

Finally the solution to the problem is found by using the scaling in (3.27) and (3.28). The strongest pinned-pinned column obtained by the analytical solution of the continuous system given by Keller, and its approximation by using a discrete link-spring system of order $n = 30$ is shown in Figure 3.4, which was obtained using the method described in this section.

The critical load of the discrete link-spring system depends on the length of the links in the model. The critical load of the pinned-pinned link-spring system, as a function of the model order n , is shown in Figure 3.5. The critical load of the discrete model converges to that of the continuous pinned-pinned strongest column given by Keller, $P = \pi V^2 E / 3l^5$, as n increases.

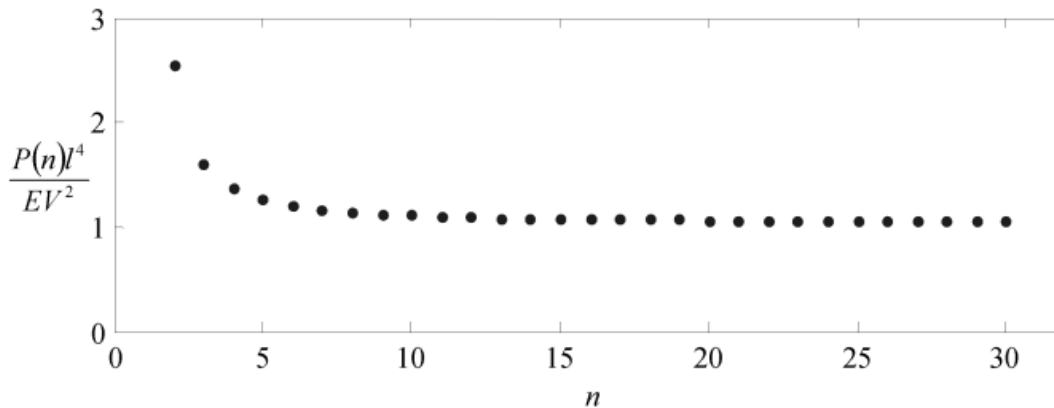


Figure 3.5: The critical load of the pinned-pinned link-spring system as function of the model order n

3.3 The Pinned-Spring-Supported System with Three Degrees of Freedom

The eigenvalue problem associated with the pinned-spring-supported system of n links is defined by equation (3.39). For the case when $n = 3$ and $h = l/3$, there are two links with non-vanishing cross-sectional area, a_2 and a_3 as shown in Figure 3.6.

The critical load for this system under the pinned-pinned configuration can be obtained from Figure 3.5. The precise value for the critical load is $P(3) = 81EV^2/16\pi l^4$. We therefore expect, as concluded in Section 3.2, that if $kl < P(3)$ there are multiple solutions

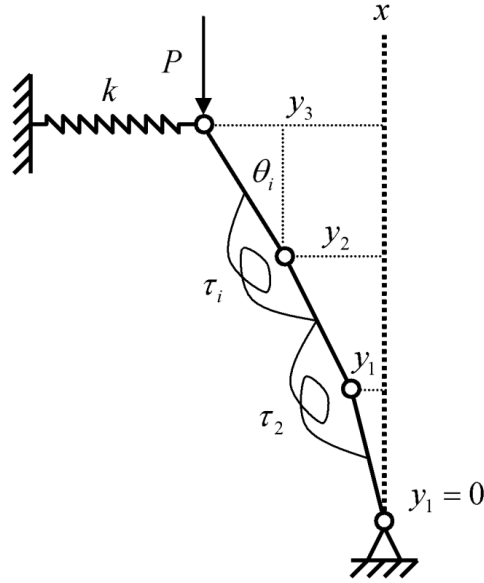


Figure 3.6: Pinned-spring-supported system with three degrees of freedom

for the strongest pinned-spring-supported system which can carry the same largest critical load, and when $kl > P(3)$ there is a unique solution, $a_2 = a_3 = V/2h$, that corresponds to the strongest pinned-pinned column with three degrees of freedom. To confirm this observation the direct problem of finding the critical load is solved, using $0 < a_2 < V/h$ as an independent variable for the cases where (a) $kl = P(3)/2$, and (b) $kl \geq P(3)/2$. The results that confirm this hypothesis are shown in Figure 3.7. It follows that in case (a) each link-spring system comprising of three links: $a_1 = 0$, $0.296 < a_2h/V < 0.704$, $a_3 = V/h - a_2$, which is pinned at the bottom and supported at its top by a spring of constant $k = 81EV^2/32\pi l^5$, will buckle by the same maximal load of $P(3)/2$. Figure 3.7(b) illustrates that for case (b), where $kl \geq P(3)$, there is a unique solution, $a_2 = a_3 = V/2h$ and this solution corresponds to the solution of the strongest pinned-pinned column. The mechanism that creates the flat portion of the graph in Figure 3.7(a) can be explained by Figure 3.8, where the variation of the two critical loads associated with the first and second buckling modes as functions of a_2 is shown. The initial part of the first buckling mode corresponds to the solution obtained in (3.42) while the second buckling mode corresponds

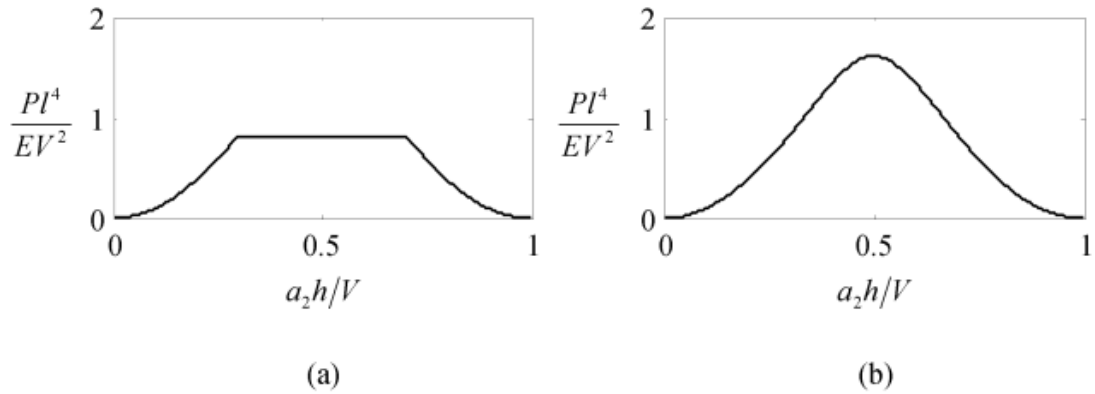


Figure 3.7: Critical load for the case $n = 3$ as a function of a_2 when: (a) $k = 81EV^2/32\pi l^5$ and (b) $k \geq 81EV^2/16\pi l^5$

to the solution in (3.41). At approximately $a_2 = 0.296V/h$, the two critical loads coincide, and for the interval $0.296V/h < a_2 < 0.704V/h$ they switch roles. The first critical load is now corresponding to the solution (3.41) and the second critical load is that of (3.42), since by definition the first critical load cannot be larger than the second. At approximately $a_2 = 0.704V/h$, they again coincide and change roles, leading to the graph shown in Figure 3.8.

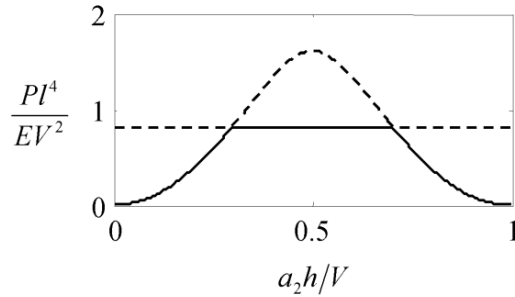


Figure 3.8: Critical load for the case $n = 3$ and $k = 81EV^2/32\pi l^5$ as a function of a_2 , - critical load for the first buckling mode, and — critical load for second buckling mode

3.4 The Continuous Pinned-Spring Supported Column

The results of the direct problem of the previous section, where a three degrees of freedom link-spring model was considered, naturally extends to a continuous model of the

column which is pinned at one end and spring supported at the other end. To clarify this first consider a non-uniform column pinned at both end and subject to an axial load P . The eigenvalue problem associated with this column is of finding the bucking load P and the non-trivial solution $u \neq 0$, such that

$$\begin{cases} (EIu'')'' + Pu'' = 0 \\ u(0) = u''(0) = u(l) = u''(l) = 0 \end{cases} \quad (3.59)$$

The spring-supported boundary condition may be obtained by replacing the pin at $x = l$ with a spring of stiffness k , then the problem transforms to

$$\begin{cases} (EIv'')'' + Qv'' = 0 \\ v(0) = v''(0) = v''(l) = (EIv'')'|_{x=l} + Qv'(l) - kv(l) = 0 \end{cases} \quad (3.60)$$

Assume that each and every non-trivial solution of (3.59) is also a solution of (3.60), i.e.

$$Q = P, v(x) = u(x). \quad (3.61)$$

The differential equation and the fist three boundary condition in (3.60) are satisfied if (3.61) holds. Further, the second equation in (3.61) transforms the last boundary condition in (3.60) to

$$(EIv'')'|_{x=l} + Qv'(l) = 0, \quad (3.62)$$

since $v(l) = 0$. Obviously the second order continuity in moment,

$$EIu'' + Pu = 0, \quad (3.63)$$

also hold for the continuous column governed by (3.59). Differentiating (3.63) once with respect to x gives the modified boundary condition in (3.62). Hence, each and every non-trivial solution of (3.59) is also a solution of (3.60), i.e. (3.61) is true. The problem (3.60)

has the additional solution

$$Q = kl, v = cx \quad (3.64)$$

where c is an arbitrary constant. The collapse of the support spring rather than the column itself is the physical representation of this case.

It thus follows that if $kl > P(\infty) = \pi V^2 E / 3l^4$ the strongest pinned-pinned column is also the strongest pinned-spring supported column, i.e. they both can carry the same maximal load. If $kl < P(\infty)$ then each and every column that satisfies the volume constraint and which can carry a load of $P = kl$ without buckling, is the strongest column for such a case.

The explanation of the process involved in determining the strongest column associated with the solution family of (3.60), the case where the solution is not unique and $k < P(\infty)$, is given below. As a first step the strongest pinned-pinned column is reconstructed for the given volume and length. It can be shown from the previous sections that if one column is obtained by scaling the cross-section of another column, while keeping the length constant, the ratio of their volumes is proportional to the square root of the ratio of their buckling loads,

$$\frac{V}{\hat{V}} \propto \sqrt{\frac{P}{\hat{P}}}. \quad (3.65)$$

So the strongest pinned-pinned column can be scaled such that the buckling load is $P = kl$. This scaled column will have a lesser volume \hat{V} . Then the excess material $V - \hat{V}$ can be added arbitrarily along the length of the column. The family of solutions obtained by this arbitrary distribution of material, satisfies the volume constraint and can carry loads that are greater than kl but less than $P(\infty) = \pi V^2 R / 3l^4$. Each of these column is a strongest column corresponding to the case when $kl < P(\infty)$. A process that is analogous to this may be applied to the discrete model.

3.5 The Clamped-Spring-Supported Link-Spring System

We now focus our attention on the problem of determining the cross-sectional area a_i , $i = 1, 2, \dots, n$ which maximize the critical load P , subject to the volume constraint (3.2), in the case where the discrete link-spring system is clamped at one end and spring-supported at the other end, as shown in Figure 3.9.

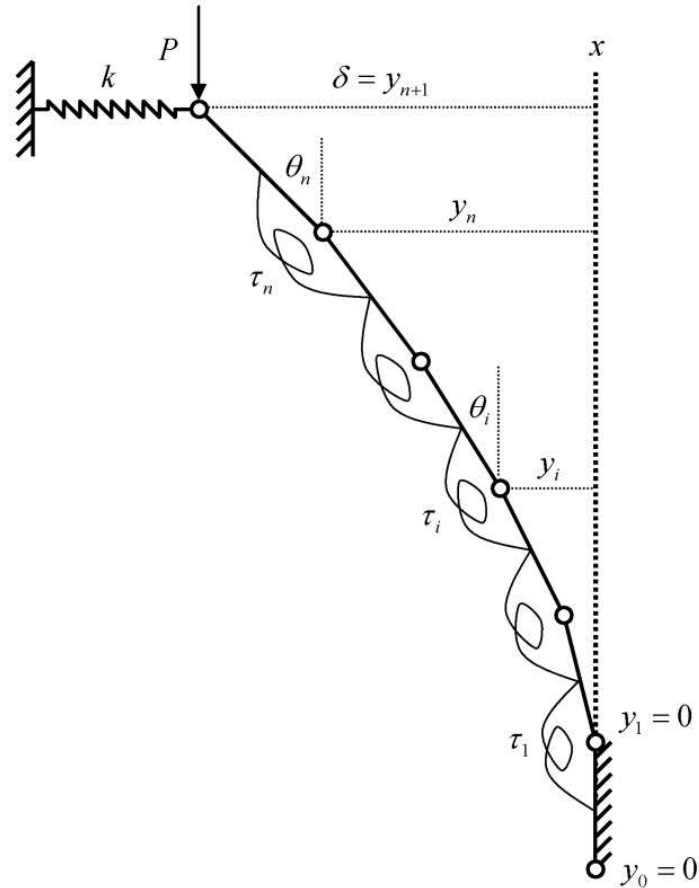


Figure 3.9: Link-spring system with a spring at its top

Using the work energy principle, the total energy stored in the system may be computed by, adding the potential energy stored in the spring and subtracting the work done by the load P . Assuming small deflections the energy stored in the system is

$$U = \frac{1}{2} \sum_{j=1}^n \tau_j (\theta_j - \theta_{j-1})^2 + \frac{h^2 k}{2} \left(\sum_{j=1}^n \theta_j \right)^2 - \frac{Ph}{2} \sum_{j=1}^n \theta_j^2 \quad (3.66)$$

where $\theta_0 = 0$.

Substituting (3.1) in (3.66) gives

$$U = \frac{E}{8\pi h} \left(\sum_{i=1}^n a_i^2 (\theta_i - \theta_{i-1})^2 + \phi \left(\sum_{i=1}^n \theta_i \right)^2 - \mu \sum_{i=1}^n \theta_i^2 \right), \quad (3.67)$$

Hence, the solution to the problem is determined by a stationary value of the Lagrangian

$$L = \frac{E}{8\pi h} \left(\sum_{i=1}^n a_i^2 (\theta_i - \theta_{i-1})^2 + \phi \left(\sum_{i=1}^n \theta_i \right)^2 - \mu \sum_{i=1}^n \theta_i^2 + \eta \sum_{i=1}^n a_i \right), \quad (3.68)$$

with respect to variations in both θ_i and a_i . Variation with respect to θ_i gives

$$a_i^2 (\theta_i - \theta_{i-1}) - a_{i+1}^2 (\theta_{i+1} - \theta_i) + \phi \sum_{j=1}^n \theta_j - \mu \theta_i = 0, \quad i = 1, 2, \dots, n \quad (3.69)$$

where $a_{n+1} = 0$. The set of equations represented by (3.69) can be written in matrix form as

$$\left(\begin{bmatrix} a_1^2 + a_2^2 + \phi & -a_2^2 + \phi & \phi & \cdots & \phi \\ -a_2^2 + \phi & a_2^2 + a_3^2 + \phi & -a_3^2 + \phi & \cdots & \phi \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \phi & \cdots & -a_{n-1}^2 + \phi & a_{n-1}^2 + a_n^2 + \phi & \phi \\ \phi & \cdots & \phi & -a_n^2 + \phi & a_n^2 + \phi \end{bmatrix} - \mu \mathbf{I} \right) \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{n-1} \\ \theta_n \end{pmatrix} = \mathbf{0}. \quad (3.70)$$

Adding the equations in (3.70) gives

$$(\mu - n\phi) \sum_{i=1}^2 \theta_i = a_1^2 \theta_1. \quad (3.71)$$

It follows from (3.70) that if $\mu = n\theta$, or equivalently, $P = lk$, then $a_1 = 0$. The critical load as a function of n , for this case is given in Figure 3.5, and the optimal configuration is that of the pinned-spring supported system. Note that if $\theta_1 = 0$ then the spring τ_1 does not apply moment, which implies that a_1 has to vanish so that the system is rendered optimal. The other conclusion that we may draw from (3.70) is that when $k \rightarrow \infty$, we have $\sum_{i=1}^2 \theta_i = 0$ which implies that $y_{n+1} = 0$. The buckling load for this case is thus that of the clamped-pinned system.

In order to determine a recursive algorithm for reconstructing the strongest link-spring system, similar to the cases shown in the previous sections, we subtract the i^{th} equation in (3.69) from its succeeding equation for $i = 1, 2, \dots, n-1$, and obtain

$$2a_{i+1}^2(\theta_{i+1} - \theta_i) - a_{i+2}^2(\theta_{i+2} - \theta_{i+1}) - a_i^2(\theta_i - \theta_{i-1}) - \mu(\theta_{i+1} - \theta_i) = 0, \quad i = 1, 2, \dots, n-1 \quad (3.72)$$

or this can be rewritten as

$$-a_{i-1}^2 v_{i-1} + 2a_i^2 v_i - a_{i+1}^2 v_{i+1} - \mu v_i = 0, \quad i = 2, 3, \dots, n \quad (3.73)$$

where v_i are as defined in (3.45).

Variation of the Lagrange function with respect to a_i gives

$$\frac{\partial L}{\partial a_i} = \frac{E}{4\pi h} a_i v_i^2 + \eta = 0, \quad i = 1, 2, \dots, n. \quad (3.74)$$

Subtracting the i^{th} equation of (3.74) from the last equation for $i = 1, 2, \dots, n-1$, gives

$$a_i v_i^2 = a_n v_n^2 \quad i = 1, 2, \dots, n-1. \quad (3.75)$$

To determine the relation between ϕ and a_i , a link-spring system of volume $\hat{V} = V/a_n$ with $\hat{a}_n = 1$, $\hat{v}_n = 1$, \hat{v}_i , $i = 1, 2, \dots, n-1$ and $\hat{\mu}$. Starting with an initial guess of $\hat{\mu}$, \hat{v}_i , $i = 1, 2, \dots, n-1$ is determined by using the recursion

$$\hat{v}_{n-1} = (2 - \hat{\mu}\hat{v}_n)^3, \quad (3.76)$$

$$\hat{v}_{i-1}^{-3} = 2\hat{v}_i^{-3} - \hat{v}_{i+1}^{-3} - \hat{\mu}\hat{v}_i, \quad i = n-1, n-2, \dots, 2 \quad (3.77)$$

Following this, the cross-sectional area is then obtained via

$$\hat{a}_i^2 = \hat{v}_i^{-4}, \quad i = 1, 2, \dots, n-1. \quad (3.78)$$

To modify $\hat{\mu}$ until convergence in \hat{v} and \hat{a} occurs, equation (3.72) for $i = 1$ which gives

$$(\hat{a}_1^2 - \hat{\mu})\hat{v}_1 - \hat{a}_2^2\hat{v}_2 + \hat{\phi} \sum_{i=1}^n \theta_i = 0 \quad (3.79)$$

where $\hat{\phi} = a_n^{-2}\phi$, can then be used as a single parameter iterative correction. Finally the solution to the problem is found by the scaling (3.27) and

$$\mu = a_n^2 \hat{\mu} \quad (3.80)$$

Figure 3.10 shows the cross-sectional area of the strongest link-spring system, with $n = 30$ degrees of freedom, for several spring stiffnesses in the range $0 < kl^5/EV^2 < 1.057$. Note that the case where $kl^5/EV^2 = 1.057$ ($\mu = n\phi$) corresponds to the buckling load of the pinned-pinned link-spring system, shown in Figure 3.5, where a_1 vanishes and $P = kl$. This is the transition from the clamped-free configuration when $k = 0$ to the pinned-pinned configuration when $k = P(30)/l$, where $P(30)$ is taken from Figure 3.5. For a model with n degrees of freedom, if $0 < kl < P(n)$ then the optimal column is regular in the sense that its cross-sectional area is positive including at the end points.

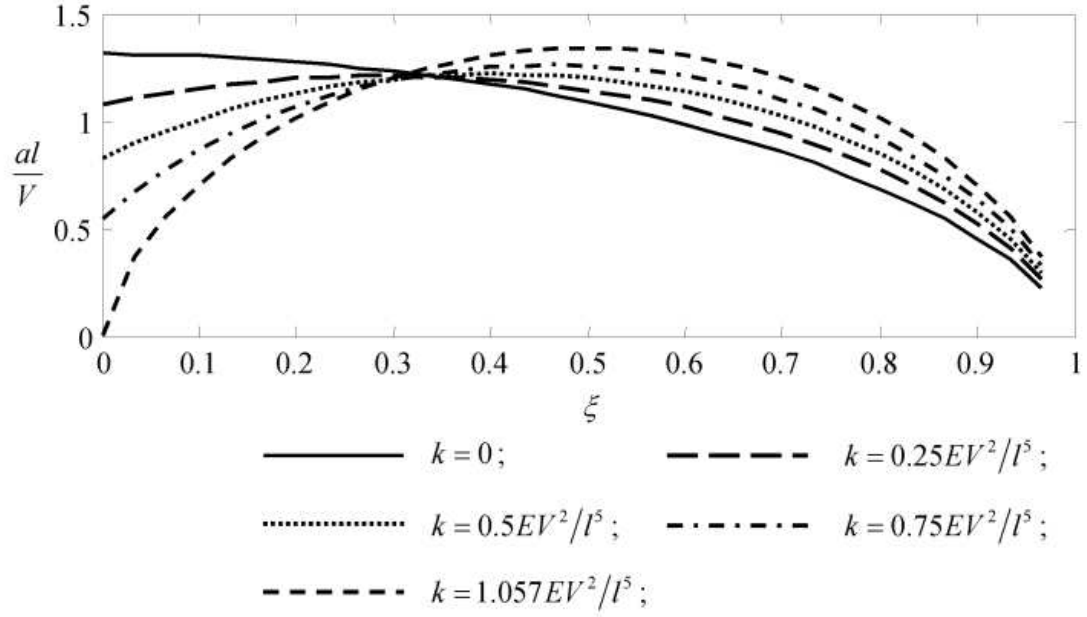


Figure 3.10: Cross-sectional area of the strongest clamped-spring-supported link-spring system, $n = 30$

3.6 The Clamped-Spring-Supported System with Two Degrees of Freedom

The two-degree of freedom link-spring system, shown in Figure 3.11, gives an insight to various aspects of the general problem studied in the previous section. For such a system, (3.70) reduces to the following equations:

$$a_1^2\theta_1 - a_2^2(\theta_2 - \theta_1) + \phi(\theta_1 + \theta_2) - \mu\theta_1 = 0, \quad (3.81)$$

and

$$a_2^2(\theta_2 - \theta_1) + \phi(\theta_1 + \theta_2) - \mu\theta_2 = 0. \quad (3.82)$$

The optimality condition in (3.75) for the two degrees of freedom gives the solution

$$a_1\theta_1^2 = a_2(\theta_2 - \theta_1)^2, \quad (3.83)$$

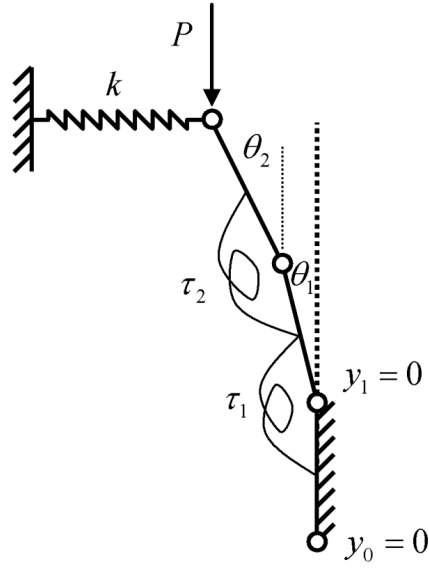


Figure 3.11: Two degree-of-freedom link-spring system

and the volume constraint reduces to

$$a_1 + a_2 = \frac{V}{h}. \quad (3.84)$$

When the constants h and ϕ are known, the known parameters in a_1 , a_2 , θ_2 and μ can be evaluated from equations (3.81)-(3.84). The value of θ_1 can be chosen arbitrarily for this evaluation. If θ_1 is chosen as zeros, $\theta_1 = 0$, then equations (3.81) - (3.84) yield $\theta_2 = 0$, $a_2 = V/h - a_1$, where μ and a_1 have arbitrary values. This is a trivial solution, since both θ_1 and θ_2 vanish, implying that the system has not buckled, which is of no interest. If we choose $\theta_1 \neq 0$, in order to discard this solution, then an auxiliary non-trivial solution of (3.81)-(3.84) can be obtained by inspection

$$a_1 = 0, \theta_1 = \theta_2, \mu = 2\phi, \text{ and } a_2 = \frac{V}{h} \quad (3.85)$$

whose importance is discussed here. In order to determine other solutions, we first eliminate μ from equations (3.81) and (3.82) and obtain after simplification

$$(a_2^2 - \phi) (\theta_2^2 - \theta_1^2) = a_1^2 \theta_2 \theta_1. \quad (3.86)$$

Then the simultaneous equations (3.83) and (3.84) are solved for a_1 and a_2 and obtain

$$a_1 = \frac{V (\theta_1^2 - \theta_1 \theta_2 + \theta_2^2)}{h (2\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2)} \quad (3.87)$$

and

$$a_2 = \frac{V \theta_1^2}{h (2\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2)} \quad (3.88)$$

Substituting for a_1 and a_2 using (3.87) and (3.88) in (3.86) gives

$$\left(V^2 \theta_1^4 - \phi h^2 (2\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2)^2 \right) (\theta_2^2 - \theta_1^2) = V^2 (\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2)^2 \theta_1 \theta_2 \quad (3.89)$$

Dividing (3.89) by $(\theta_2 - \theta_1)$ we obtain

$$\left(V^2 \theta_1^4 - \phi h^2 (2\theta_1^2 - 2\theta_1 \theta_2 + \theta_2^2)^2 \right) (\theta_1 + \theta_2) = V^2 (\theta_2 - \theta_1)^3 \theta_1 \theta_2 \quad (3.90)$$

The equation given in (3.90) is valid only if $\theta_1 \neq \theta_2$. Hence, the solutions obtained via (3.90) prevents the other auxiliary solution given in (3.85), where $\theta_1 = \theta_2$. Without loss of generality, $\theta_1 = 1$ may be substituted in (3.90) and the following polynomial form is obtained

$$\sum_{j=0}^5 c_j \theta_2^j = 0 \quad (3.91)$$

where $c_5 = \phi h^2$, $c_4 = -3\phi h^2 + V^2$, $c_3 = 4\phi h^2 - 3V^2$, $c_2 = 3V^2$, $c_1 = -4\phi h^2 - 2V^2$ and $c_0 = 4\phi h^2 - V^2$. The five different values of θ_2 may be obtained by evaluating the roots of the polynomial in (3.91). Then by (3.87), (3.84) and (3.78)

$$a_1 = \frac{V(1 - 2\theta_2 + \theta_2^2)}{h(2 - 2\theta_2 + \theta_2^2)}, \quad (3.92)$$

$$a_2 = \frac{V}{h} - a_1, \quad (3.93)$$

and

$$\mu = a_2^2 + \phi - \frac{a_2^2 - \phi}{\theta_2}. \quad (3.94)$$

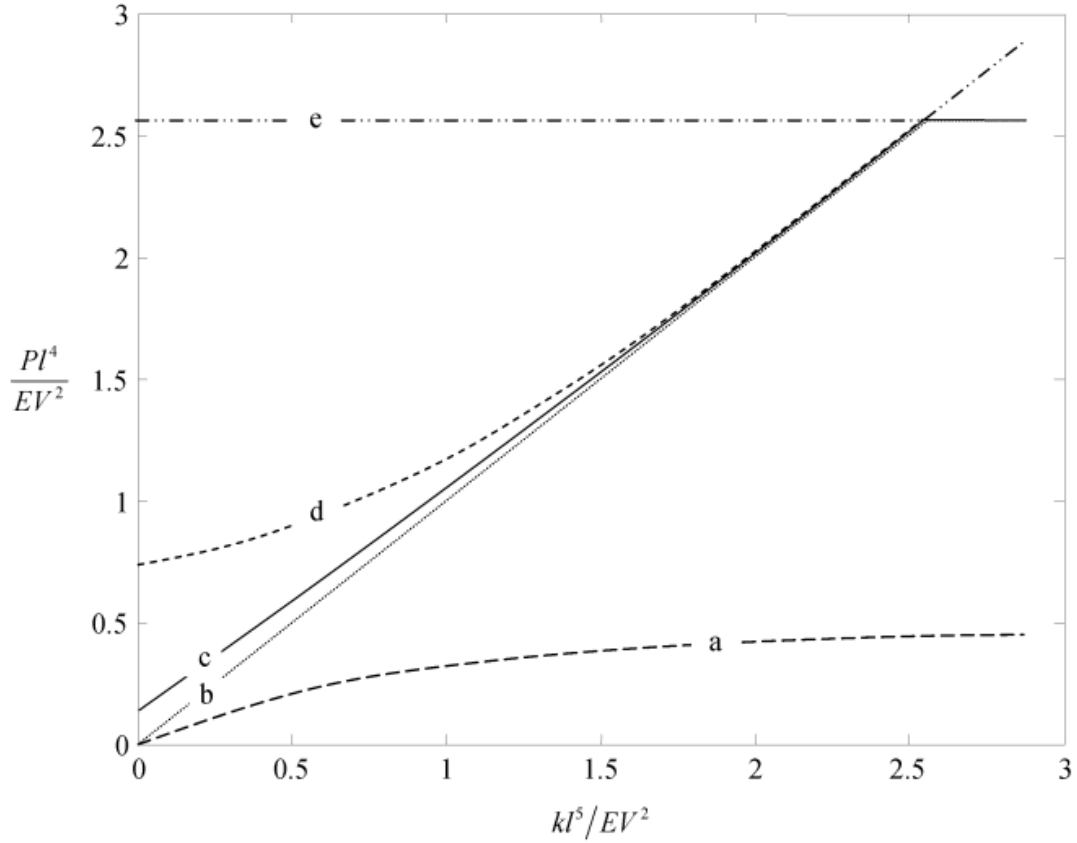


Figure 3.12: Various solution of P obtained by solving equations (3.81) through (3.84) as a function of k

The eigenvalue problem associated with the auxiliary system with vanishing a_1 is of the form

$$\begin{bmatrix} (V/h)^2 + \phi & -(V/h)^2 + \phi \\ -(V/h)^2 + \phi & (V/h)^2 + \phi \end{bmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} - \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.95)$$

The eigenvalues of the problem given in (3.95) are $\mu = 2\phi$ and $\mu = 2V^2/h^2$. Hence the critical load for this case is

$$P = \begin{cases} 2kh & k < EV^2/4\pi h^5 \\ EV^2/2\pi h^4 & k > EV^2/4\pi h^5 \end{cases} \quad (3.96)$$

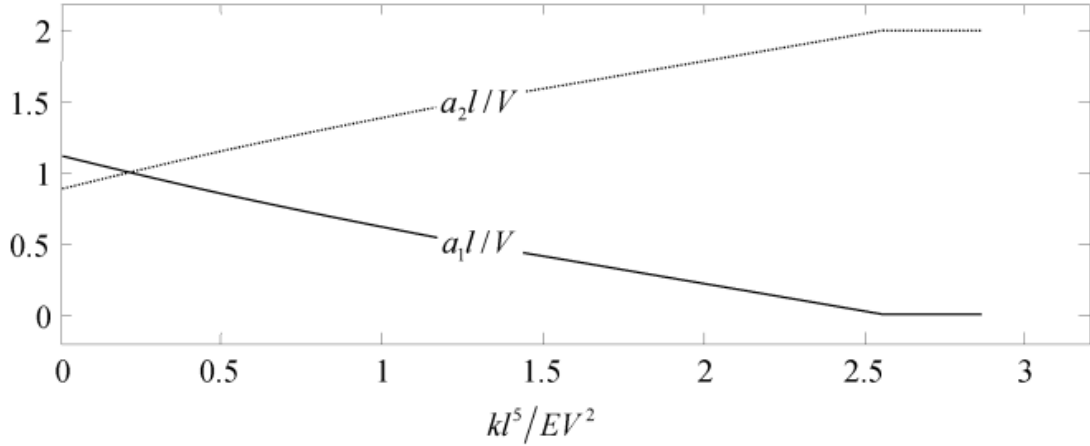


Figure 3.13: Areas of the links of the strongest two degree-of-freedom system as a function of non-dimensional k

For $0 < k < P(2)/l$, where $P(2) = 8EV^2/\pi l^4$ is shown in Figure 3.5, the polynomial (3.91) has three real roots. The corresponding values of μ together with the one associated with the auxiliary solutions (3.96), as a function of k , are shown in Figure 3.12. The graph denoted by (a) in this figure corresponds to the system that minimizes the buckling load. Graph (b) corresponds to the buckling load (3.96), associated with the case where $a_1 = 0$. Graph (c) corresponds to the critical load P of the strongest system. Graph (d) corresponds to the system which minimizes the buckling load of the second mode, and graph (e) corresponds to the system that maximizes the buckling load associated with this mode

of failure. For $k > P(2)/l$ there is one real solution to the polynomial (3.91), corresponding to graph (a) and the auxiliary solution $P = 8EV^2/\pi l^4$ associated with (3.96). Figure 3.13 shows the areas of the two links that compose the strongest system as a function of k .

The discrete link-spring model representing the strongest column subject to various boundary conditions, including that of spring support at the top, may be determined by a recursive process involving a one parameter iterative loop. Since the cases where one side of the column was either pinned or clamped and the other side was spring supported, were analyzed, it was possible to reconstruct optimal columns as a transition from the unstable pinned-free to the pinned-pinned; and from the clamped-free to the pinned-pinned configurations.

Chapter 4

Optimal Eigenvalues of Constrained Affine Sum

The problem of finding the optimal eigenvalues of a constrained affine sum was introduced in Chapter 1 with an example of a vibrating mass-spring system with two degrees of freedom. The optimization procedure that is developed in this chapter can be used in any mathematical problem that can be framed as an affine sum, where it is necessary to determine optimal eigenvalues. To illustrate the problem let us consider a simple example of a 2×2 constrained affine sum.

Example 1: We wish to find the coefficient α_1 and α_2 of the affine sum that would maximize the smallest eigenvalue of

$$\mathbf{A} = \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2, \quad (4.1)$$

subject to the constraint

$$\mathbf{c}^T \boldsymbol{\alpha} = 1, \quad (4.2)$$

where

$$\mathbf{c} = (c_1 \ c_2)^T, \quad (4.3)$$

and

$$\boldsymbol{\alpha} = (\alpha_1 \ \alpha_2)^T. \quad (4.4)$$

$$\text{Given: } \mathbf{A}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 5 & -2 \\ -2 & 4 \end{bmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For this case equations (4.1) and (4.2) are transformed into

$$\mathbf{A} = \begin{bmatrix} 2\alpha_1 + 5\alpha_2 & \alpha_1 - 2\alpha_2 \\ \alpha_1 - 2\alpha_2 & 3\alpha_1 + 4\alpha_2 \end{bmatrix}, \quad (4.5)$$

and

$$\alpha_1 + \alpha_2 = 1 \quad (4.6)$$

respectively. Eliminating α_2 from (4.5) using (4.6) we obtain

$$\mathbf{A} = \begin{bmatrix} -3\alpha_1 + 5 & 3\alpha_1 - 2 \\ 3\alpha_1 - 2 & -\alpha_1 + 4 \end{bmatrix}. \quad (4.7)$$

The eigenvalues of \mathbf{A} in (4.7) can be obtained by solving the characteristic polynomial equation, which is defined by

$$|\mathbf{A} - \lambda\mathbf{I}| = 0, \quad (4.8)$$

which for the present case is

$$f(\lambda, \alpha_1) = \begin{vmatrix} -3\alpha_1 + 5 - \lambda & 3\alpha_1 - 2 \\ 3\alpha_1 - 2 & -\alpha_1 + 4 - \lambda \end{vmatrix} = 0. \quad (4.9)$$

The characteristic equation obtained by symbolically evaluating the determinant is

$$f(\lambda, \alpha_1) = (-3\alpha_1 + 5 - \lambda)(-\alpha_1 + 4 - \lambda) - (3\alpha_1 - 2)^2 = 0. \quad (4.10)$$

The extreme or optimal eigenvalue λ , which is are the roots of (4.10), must be stationary or invariant with respect to α_1 .

Variation of (4.10) with respect to α_1 gives

$$\begin{aligned}
df(\lambda, \alpha_1) &= [(-3 - \lambda')(-\alpha_1 + 4 - \lambda) \\
&\quad + (-3\alpha_1 + 5 - \lambda)(-1 - \lambda') \\
&\quad - 6(3\alpha_1 - 2)^2] d\alpha_1 = 0.
\end{aligned} \tag{4.11}$$

where $\lambda' = \frac{d\lambda}{d\alpha_1}$. Since λ is extremum it is invariant with respect to α_1 , which implies

$$\lambda' = 0, \tag{4.12}$$

and hence (4.11) reduces to

$$df(\lambda, \alpha_1) = [4\lambda - 5 - 12\alpha_1]d\alpha_1 = 0, \tag{4.13}$$

which yields

$$\alpha_1 = \frac{4\lambda - 5}{12}. \tag{4.14}$$

Substituting for α_1 from (4.14) in (4.10) gives

$$\frac{5}{3}\lambda^2 - \frac{32}{3}\lambda + \frac{409}{24} = 0. \tag{4.15}$$

Solving equation (4.15) for lambda, we obtain the extremum λ s to be

$$\lambda_1 = \frac{16}{5} + \frac{\sqrt{6}}{20} \tag{4.16}$$

and

$$\lambda_2 = \frac{16}{5} - \frac{\sqrt{6}}{20} \tag{4.17}$$

Following the evaluation of the extreme eigenvalues, the corresponding eigenvector $\boldsymbol{\alpha}$ that renders these eigenvalues extremum can be evaluated by substituting (4.16) and (4.17) in equations (4.14) and (4.6). The parameters α_1 and α_2 that render the eigenvalues of \mathbf{A} to

be extreme are

$$\alpha_1 = 0.6908 \text{ and } \alpha_2 = 0.3092 \quad (4.18)$$

and

$$\alpha_1 = 0.6092 \text{ and } \alpha_2 = 0.3908 \quad (4.19)$$

To verify that the eigenvalues obtained are indeed extremum we determine λ for all possible combinations of α_1 and α_2 and obtain the graph shown in Figure 4.1.

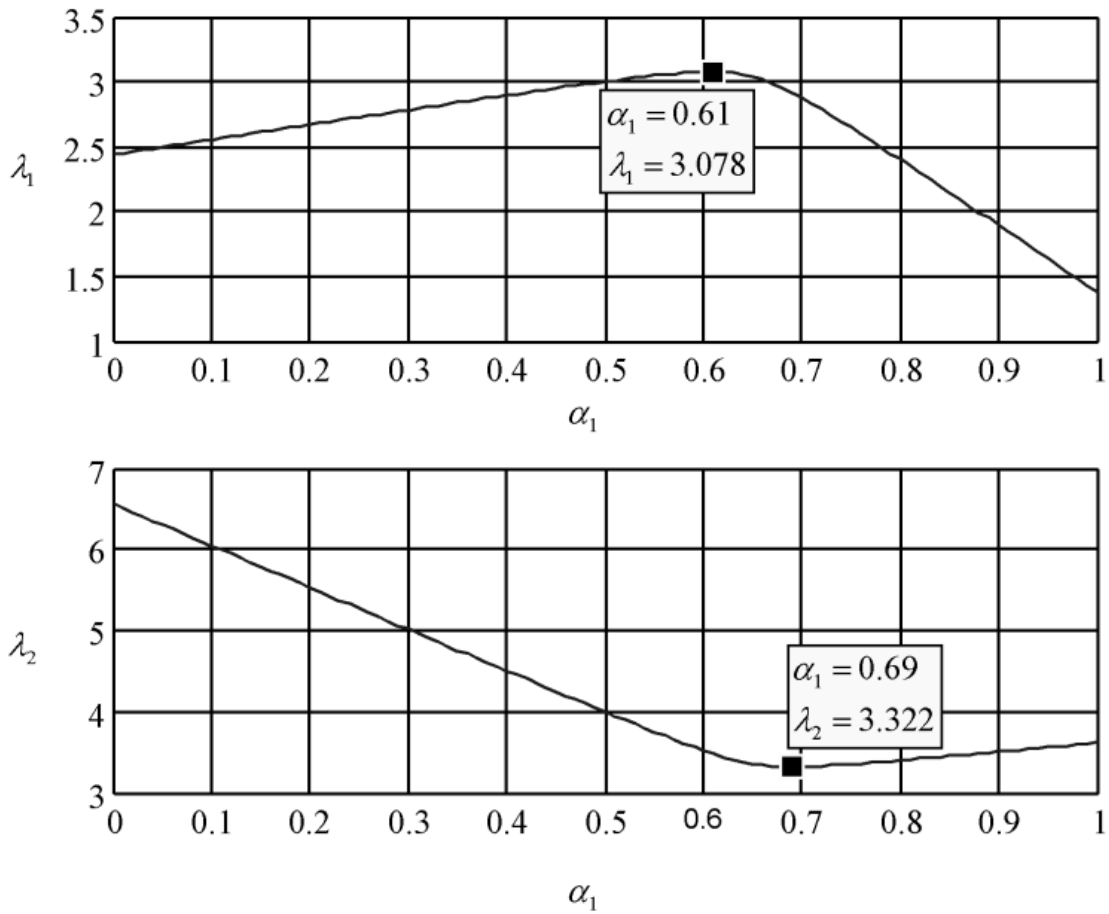


Figure 4.1: Variation of eigenvalues λ_1 and λ_2 as a function of α_1

Hence, we may infer that for any eigenvalue problem that is of the type $(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$, where \mathbf{A} is a symmetric affine sum, either the smallest eigenvalue may be maximized or the largest one minimized. Obviously, as the size of the matrix \mathbf{A} increases, the number of unknown parameters also increases, making the method described above impractical. The

development of a method that can be used for such multi-variable optimizations problems is developed in this chapter.

4.1 Problem Definition and Solution of the Affine Sum

Suppose that n symmetric matrices $\mathbf{A}_k \in \mathfrak{R}^{n \times n}$ and a vector $\mathbf{c} \in \mathfrak{R}^n$ are given. We consider the problem of finding the n coefficients α_i of the vector $\boldsymbol{\alpha} = (\alpha_1 \ \alpha_2 \ \dots \ \alpha_n)^T$ which render stationary or extremum eigenvalues of,

$$(\mathbf{A} - \lambda \mathbf{I}) \mathbf{x} = \mathbf{0}, \mathbf{x} \neq \mathbf{0}, \quad (4.20)$$

where \mathbf{A} is an affine sum defined by,

$$\mathbf{A} = \sum_{k=1}^n \alpha_k \mathbf{A}_k. \quad (4.21)$$

The constraint on $\boldsymbol{\alpha}$ for the optimization problem is

$$\mathbf{c}^T \boldsymbol{\alpha} = 1 \quad (4.22)$$

We recall from the Courant-Fischer minimax theorem that the smallest eigenvalue of a symmetric matrix \mathbf{A} , is the solution of the minimization problem

$$\lambda_1 = \min_{\mathbf{x}} \mathbf{x}^T \mathbf{A} \mathbf{x} \quad (4.23)$$

subject to the constraint

$$\mathbf{x}^T \mathbf{x} = 1. \quad (4.24)$$

Hence the vector $\boldsymbol{\alpha}$ that provides a maximum smallest eigenvalue is the solution, $\boldsymbol{\alpha}$ and \mathbf{x} of

$$\max_{\boldsymbol{\alpha}} \min_{\mathbf{x}} \mathbf{x}^T \mathbf{A} \mathbf{x}, \quad (4.25)$$

subject to the constraints (4.22) and (4.24). Similarly, the dual problem of finding the vector $\boldsymbol{\alpha}$ that minimizes the largest eigenvalue λ_n , is obtained by solving

$$\min_{\boldsymbol{\alpha}} \max_{\mathbf{x}} \mathbf{x}^T \mathbf{A} \mathbf{x}, \quad (4.26)$$

subject to (4.22) and (4.24).

This optimization problem may be addressed using fundamental principles of optimization, by defining the Lagrange function as

$$L = \mathbf{x}^T \mathbf{A} \mathbf{x} - \mu c^T \boldsymbol{\alpha} - \xi \mathbf{x}^T \mathbf{x} \quad (4.27)$$

where μ and ξ are Lagrange multipliers. For the solution to be extreme the Lagrange function must be stationary or invariant with respect to \mathbf{x} and $\boldsymbol{\alpha}$. Variation with respect to \mathbf{x} gives

$$\frac{\partial L}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x} - 2\xi\mathbf{x} = \mathbf{0}, \quad (4.28)$$

where,

$$\frac{\partial L}{\partial \mathbf{x}} = \left(\frac{\partial L}{\partial x_1} \quad \frac{\partial L}{\partial x_2} \quad \cdots \quad \frac{\partial L}{\partial x_n} \right)^T. \quad (4.29)$$

Variation of (4.27) with respect to $\boldsymbol{\alpha}$ gives

$$\frac{\partial L}{\partial \alpha_k} = \mathbf{x}^T \mathbf{A}_k \mathbf{x} - \mu c_k = 0, \quad k = 1, 2, \dots, n \quad (4.30)$$

The system of equations in (4.22), (4.24), (4.28) and (4.30) makeup a total of $2n + 2$ equations, which have to be solved simultaneously to obtain, n unknowns elements of vector \mathbf{x} , n unknowns elements of vector $\boldsymbol{\alpha}$, and the two Lagrange multipliers μ and ξ . To decouple \mathbf{x} from the equations (4.30) may be rewritten as

$$\frac{\mathbf{x}^T \mathbf{A}_k \mathbf{x}}{c_k} = \mu, \quad k = 1, 2, \dots, n \quad (4.31)$$

and defining

$$\mathbf{B} = \frac{\mathbf{A}_{k+1}}{c_{k+1}} - \frac{\mathbf{A}_k}{c_k}, \quad k = 1, 2, \dots, n-1 \quad (4.32)$$

and

$$\mathbf{B}_n = \mathbf{I}_n. \quad (4.33)$$

The Lagrange multiplier μ in (4.31) is not of any interest and can be eliminated by, subtracting equation (4.31) for k from equation (4.31) for $k+1$. This yields

$$\frac{\mathbf{x}^T \mathbf{A}_{k+1} \mathbf{x}}{c_{k+1}} - \frac{\mathbf{x}^T \mathbf{A}_k \mathbf{x}}{c_k}, \quad k = 1, 2, \dots, n-1 \quad (4.34)$$

which simplifies to

$$\mathbf{x}^T \left(\frac{\mathbf{A}_{k+1}}{c_{k+1}} - \frac{\mathbf{A}_k}{c_k} \right) \mathbf{x}, \quad k = 1, 2, \dots, n-1 \quad (4.35)$$

Using (4.32) and (4.33), in equations (4.35) and (4.24) respectively, we obtain

$$\mathbf{x}^T \mathbf{B}_k \mathbf{x} = 0, \quad k = 1, 2, \dots, n-1 \quad (4.36)$$

and

$$\mathbf{x}^T \mathbf{B}_n \mathbf{x} = 1. \quad (4.37)$$

It is evident that equation (4.36) and (4.37) are independent of $\boldsymbol{\alpha}$ and ξ and thus can be used to solve for the n unknown elements of the vector \mathbf{x} . Following the evaluation of \mathbf{x} , the unknown in ξ and vector $\boldsymbol{\alpha}$ can be determined via the set of linear equations

$$\begin{bmatrix} \mathbf{A}_1 \mathbf{x} & \mathbf{A}_2 \mathbf{x} & \cdots & \mathbf{A}_n \mathbf{x} & -\mathbf{x} \\ c_1 & c_2 & \cdots & c_n & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \xi \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \quad (4.38)$$

4.2 The Set of Quadratic Equations

Equations (4.36) and (4.37) represent a set of equations that are quadratic with respect to the n unknown elements of the vector \mathbf{x} . If n is small, then this set of quadratic equations can be solved analytically. An insight of the problem can be obtained by considering an expository example involving the case when $n = 2$. Let \mathbf{B}_1 be an arbitrary symmetric matrix of dimension 2×2 and of the form

$$\mathbf{B}_1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \quad (4.39)$$

and \mathbf{B}_2 is the identity matrix of the same dimension, i.e

$$\mathbf{B}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (4.40)$$

We wish to find the vector \mathbf{x} that satisfies the set of quadratic equations in (4.36) and (4.37), where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (4.41)$$

For this case equations (4.36) and (4.37) reduces to

$$P(\mathbf{x}) = b_{11}x_1^2 + 2b_{12}x_1x_2 + b_{22}x_2^2 = 0 \quad (4.42)$$

and

$$Q(\mathbf{x}) = x_1^2 + x_2^2 - 1 = 0 \quad (4.43)$$

The problem now is of determining all possible x_1 and x_2 that satisfies (4.42) and (4.43). To solve this problem analytically, i.e. in closed form, the eliminant can be used to eliminate either x_1 or x_2 from these equations. Without the loss of generality, x_2 can be eliminated. The process of using the eliminant starts by multiplying (4.42) and (4.43) with x_2 and x_2^2 .

Then the four equations that are obtained as a result of this multiplication can be written in matrix form as

$$\begin{bmatrix} b_{22} & 2b_{12}x_1 & b_{11}x_1^2 & 0 \\ 0 & b_{22} & 2b_{12}x_1 & b_{11}x_1^2 \\ 0 & 1 & 0 & x_1^2 - 1 \\ 1 & 0 & x_1^2 - 1 & 0 \end{bmatrix} \begin{pmatrix} x_2^4 \\ x_2^3 \\ x_2^2 \\ x_2 \end{pmatrix} = \mathbf{0}. \quad (4.44)$$

The necessary and sufficient condition ensuring that (4.42) and (4.43) have common roots is that

$$\begin{vmatrix} b_{22} & 2b_{12}x_1 & b_{11}x_1^2 & 0 \\ 0 & b_{22} & 2b_{12}x_1 & b_{11}x_1^2 \\ 0 & 1 & 0 & x_1^2 - 1 \\ 1 & 0 & x_1^2 - 1 & 0 \end{vmatrix} = 0 \quad (4.45)$$

or equivalently

$$-((b_{22} - b_{11})^2 + 4b_{12}^2)x_1^4 + 2(b_{22}(b_{22} - b_{11}) + 2b_{12}^2)x_1^2 - b_{22}^2 = 0 \quad (4.46)$$

Equation (4.46) is a fourth order polynomial in x_1 and hence has four roots. For each root of (4.46) we then obtain the corresponding x_2 using either (4.42) or (4.43). Further explanation of the procedure of using the eliminant to solve the set of quadratic equations can be obtained from the following example.

Example 2: Let the symmetric matrix \mathbf{B}_1 be

$$\mathbf{B}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad (4.47)$$

Equations (4.42) and (4.43) for this case are

$$P(\mathbf{x}) = 2x_1^2 + 2x_1x_2 = 0 \quad (4.48)$$

and

$$Q(\mathbf{x}) = x_1^2 + x_2^2 - 1 = 0 \quad (4.49)$$

respectively. The process of using the eliminant to eliminate x_2 is commenced by multiplying (4.48) and (4.49) with x_2 and x_2^2 , and writing these four equations in matrix form as

$$\begin{bmatrix} 0 & 2x_1 & 2x_1^2 & 0 \\ 0 & 0 & 2x_1 & 2x_1^2 \\ 0 & 1 & 0 & x_1^2 - 1 \\ 1 & 0 & x_1^2 - 1 & 0 \end{bmatrix} \begin{pmatrix} x_2^4 \\ x_2^3 \\ x_2^2 \\ x_2 \end{pmatrix} = \mathbf{0}. \quad (4.50)$$

Since we wish to find x_1 and x_2 that satisfy both (4.48) and (4.49), $x_2 \neq 0$. This implies that the determinant of the matrix in (4.50) has to be zero, i.e.

$$-8x_1^4 + 4x_1^2 = 0. \quad (4.51)$$

Equation (4.51) is a fourth order polynomial in x_1 and its roots are

$$x_1 = 0 \text{ or } x_1 = \frac{\sqrt{2}}{2} \text{ or } x_1 = -\frac{\sqrt{2}}{2}. \quad (4.52)$$

Solutions of x_2 that corresponds to x_1 given in (4.52) are obtained by finding the common solutions of (4.48) and (4.49), and they are

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\pm\sqrt{2}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (4.53)$$

It is interesting to note that if x_2 that corresponds to x_1 in (4.52), is determined only using (4.48), then solutions $x_1 = x_2 = \pm\sqrt{2}/2$ is obtained. However, this solution does not satisfy equation (4.48) and hence are spurious.

It is clear that the eliminant can be used to solve the set of quadratic equations as long as the number of equations and the number of unknowns are small. However, as the size

of the matrix increases the process of evaluating the determinant becomes laborious and hence impractical. To demonstrate the complexity involved in solving the set of quadratic equation in closed form we consider an example with 3×3 matrices.

Example 3:

Given:

$$\mathbf{B}_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \mathbf{B}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \mathbf{B}_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad (4.54)$$

We wish to find the vector \mathbf{x} that satisfies the set of quadratic equations defined in (4.36) and (4.37), where the vector \mathbf{x} is of the form

$$\mathbf{x} = (x_1 \ x_2 \ x_3)^T. \quad (4.55)$$

The set of quadratic equations in (4.36) and (4.37) for the current example reduces to

$$\mathbf{x}^T \mathbf{B}_1 \mathbf{x} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 = 0 \quad (4.56)$$

$$\mathbf{x}^T \mathbf{B}_2 \mathbf{x} = 2x_1^2 - 2x_2x_3 + 2x_3^2 = 0 \quad (4.57)$$

$$\mathbf{x}^T \mathbf{B}_3 \mathbf{x} = x_1^2 + x_2^2 + x_3^2 = 1 \quad (4.58)$$

To solve for \mathbf{x} from (4.56), (4.57) and (4.58), we begin by eliminating x_3 from (4.56) and (4.57) with the aid of the eliminant. Multiplying (4.56) and (4.57) with x_3 and x_3^2 and rearranging the equations in matrix form we obtain

$$\begin{bmatrix} 2 & -2x_2 & 2x_1^2 - 2x_1x_2 + 2x_2^2 & 0 \\ 0 & 2 & -2x_2 & 2x_1^2 - 2x_1x_2 + 2x_2^2 \\ 2 & -2x_2 & 2x_1^2 & 0 \\ 0 & 2 & -2x_2 & 2x_1^2 \end{bmatrix} \begin{pmatrix} x_3^4 \\ x_3^3 \\ x_3^2 \\ x_3 \end{pmatrix} = \mathbf{0}. \quad (4.59)$$

The non-trivial solution of x_3 is then obtained if the determinant of the matrix in (4.59) is zero, i.e.

$$\begin{vmatrix} 2 & -2x_2 & 2x_1^2 - 2x_1x_2 + 2x_2^2 & 0 \\ 0 & 2 & -2x_2 & 2x_1^2 - 2x_1x_2 + 2x_2^2 \\ 2 & -2x_2 & 2x_1^2 & 0 \\ 0 & 2 & -2x_2 & 2x_1^2 \end{vmatrix} = 0, \quad (4.60)$$

or equivalently

$$-2x_1x_2^3 + x_2^4 + x_1^2x_2^2 = 0. \quad (4.61)$$

Similarly, to eliminate x_3 from (4.56) and (4.58) with the aid of the eliminant, we multiply (4.56) and (4.58) with x_3 and x_3^2 and obtain

$$\begin{bmatrix} 2 & -2x_2 & 2x_1^2 - 2x_1x_2 + 2x_2^2 & 0 \\ 0 & 2 & -2x_2 & 2x_1^2 - 2x_1x_2 + 2x_2^2 \\ 1 & 0 & x_1^2 + x_2^2 - 1 & 0 \\ 0 & 1 & 0 & x_1^2 + x_2^2 - 1 \end{bmatrix} \begin{pmatrix} x_3^4 \\ x_3^3 \\ x_3^2 \\ x_3 \end{pmatrix} = \mathbf{0}. \quad (4.62)$$

The non-trivial solution of x_3 is then obtained if the determinant of the matrix in (4.59) is zero, i.e.

$$\begin{vmatrix} 2 & -2x_2 & 2x_1^2 - 2x_1x_2 + 2x_2^2 & 0 \\ 0 & 2 & -2x_2 & 2x_1^2 - 2x_1x_2 + 2x_2^2 \\ 1 & 0 & x_1^2 + x_2^2 - 1 & 0 \\ 0 & 1 & 0 & x_1^2 + x_2^2 - 1 \end{vmatrix} = 0, \quad (4.63)$$

which is equivalent to

$$x_2^4 + 2x_1^2x_2^2 - x_2^2 - 2x_1x_2 + 1 = 0 \quad (4.64)$$

Similarly, eliminating x_3 from (4.57) and (4.58), and neglecting the trivial solution of $x_3 = 0$ we obtain

$$2x_2^4 + x_1^2x_2^2 - 3x_2^2 + 1 = 0. \quad (4.65)$$

Equations (4.61), (4.64) and (4.65) are three independent polynomials in x_1 and x_2 . They are second order in terms of x_1 and fourth order in terms of x_2 . To find x_1 and x_2 that satisfy (4.61), (4.64) and (4.65), we begin by eliminating x_1 from these equations.

To use the eliminant in eliminating x_1 from (4.61) and (4.64), we multiply (4.61) and (4.64) with x_1 and x_1^2 and write the resultant equations in matrix form as

$$\begin{bmatrix} x_2^2 & -2x_2^3 & x_2^4 & 0 \\ 0 & x_2^2 & -2x_2^3 & x_2^4 \\ 2x_2^2 & -2x_2 & x_2^4 - x_2^2 + 1 & 0 \\ 0 & 2x_2^2 & -2x_2 & x_2^4 - x_2^2 + 1 \end{bmatrix} \begin{pmatrix} x_1^4 \\ x_1^3 \\ x_1^2 \\ x_1 \end{pmatrix} = \mathbf{0}. \quad (4.66)$$

Since we wish to find the non-trivial solution of x_1 ,

$$\begin{vmatrix} x_2^2 & -2x_2^3 & x_2^4 & 0 \\ 0 & x_2^2 & -2x_2^3 & x_2^4 \\ 2x_2^2 & -2x_2 & x_2^4 - x_2^2 + 1 & 0 \\ 0 & 2x_2^2 & -2x_2 & x_2^4 - x_2^2 + 1 \end{vmatrix} = 0, \quad (4.67)$$

or equivalently

$$9x_2^{12} - 18x_2^{10} + 15x_2^8 - 6x_2^6 + x_2^4 = 0. \quad (4.68)$$

The roots of the 12th order polynomial in x_2 are

$$x_2 = 0, \pm \frac{1}{6} \sqrt{18 \pm i6\sqrt{3}}. \quad (4.69)$$

Similarly, with the aid of the eliminant, x_1 is eliminated from (4.61) and (4.65), and the non-trivial solution is obtained by solving

$$\begin{vmatrix} x_2^2 & -2x_2^3 & x_2^4 & 0 \\ 0 & x_2^2 & -2x_2^3 & x_2^4 \\ x_2^2 & 0 & 2x_2^4 - 3x_2^2 + 1 & 0 \\ 0 & x_2^2 & 0 & 2x_2^4 - 3x_2^2 + 1 \end{vmatrix} = 0, \quad (4.70)$$

or equivalently

$$9x_2^{12} - 18x_2^{10} + 15x_2^8 - 6x_2^6 + x_2^4 = 0. \quad (4.71)$$

The roots of the 12th order polynomial in x_2 are

$$x_2 = 0, \pm \frac{1}{6} \sqrt{18 \pm i6\sqrt{3}}. \quad (4.72)$$

Finally, x_1 is eliminated from (4.64) and (4.65) with the aid of the eliminant, and the non-trivial solution is obtained by solving

$$\begin{vmatrix} 2x_2^2 & -2x_2^3 & x_2^4 - x_2^2 + 1 & 0 \\ 0 & 2x_2^2 & -2x_2^3 & x_2^4 - x_2^2 + 1 \\ x_2^2 & 0 & 2x_2^4 - 3x_2^2 + 1 & 0 \\ 0 & x_2^2 & 0 & 2x_2^4 - 3x_2^2 + 1 \end{vmatrix} = 0, \quad (4.73)$$

or equivalently

$$9x_2^{12} - 30x_2^{10} + 39x_2^8 - 22x_2^6 + 5x_2^4 = 0. \quad (4.74)$$

Solving equation (4.74) for x_2 we obtain

$$x_2 = 0, \pm \frac{1}{6} \sqrt{18 \pm i6\sqrt{3}} \pm \frac{1}{6} \sqrt{42 \pm i6\sqrt{11}}. \quad (4.75)$$

Since, solutions of x_2 must satisfy all the three equations given in (4.61), (4.64) and (4.65), $x_2 = \pm \frac{1}{6} \sqrt{42 \pm i6\sqrt{11}}$ is a set of spurious solutions. Further, if $x_2 = 0$, then (4.61) is trivially satisfied and (4.64) is not satisfied. Thus, only possible solution of x_2 that satisfies

all the three equations in (4.61), (4.64) and (4.65) is

$$x_2 = \pm \frac{1}{6} \sqrt{18 \pm i6\sqrt{3}}. \quad (4.76)$$

Once x_2 is determined, x_1 can be evaluated by finding the common solution of (4.61), (4.64) and (4.65). Then x_3 can be determined by finding the common solutions of (4.56), (4.57) and (4.58). Hence, the common solutions of x_1 , x_2 and x_3 obtained by solving the set of quadratic equations are

$$x_1 = \pm \frac{(18 + i6\sqrt{3})^{(3/2)}}{36(3 + i6\sqrt{3})}, x_2 = \pm \frac{1}{6} \sqrt{18 + i6\sqrt{3}}, x_3 = \pm \frac{1}{3} \sqrt{3} \sqrt{-i\sqrt{3}} \quad (4.77)$$

and

$$x_1 = \pm \frac{(18 - i6\sqrt{3})^{(3/2)}}{36(3 - i6\sqrt{3})}, x_2 = \pm \frac{1}{6} \sqrt{18 - i6\sqrt{3}}, x_3 = \pm \frac{1}{3} \sqrt{3} \sqrt{i\sqrt{3}}. \quad (4.78)$$

In general, the eliminant allows one to eliminate one variable from two equations involving polynomials of multiple variables. Hence, if the matrices \mathbf{B}_k are of size $n \times n$ and one needs to solve the n quadratic equations in (4.36) and (4.37), the eliminant can first be employed on every possible pair of equation to obtain C_2^n possible number of polynomials, where C_2^n is the binomial coefficient. These polynomials are functions of $n - 1$ variables x_1, x_2, \dots, x_{n-1} . In a similar manner, the eliminant can then be used to eliminate the variable x_{n-1} , and then x_{n-2} and so on and so forth till polynomials involving as single variable x_1 is obtained. These one variable polynomial can be solved, using well established numerical techniques, for their common solutions. Once x_1 is known, the set of polynomials which involves two variables, x_1 and x_2 , can be solved for their common solutions of x_2 . Continuing in this manner, equations (4.36) and (4.37) can be solved for all of its unknown variable.

Even though the eliminant aids in solving the set of quadratic equations by eliminating one variable after another, it becomes too laborious as the number of unknowns increase.

The process also involves the step of symbolically evaluating the determinant of matrices, umpteen number of times, which may even be impossible to compute if the matrix size increases. Further, finding solutions that are common to each and every pair of equations also becomes difficult with an increase in the unknown variables. Hence, albeit the eliminant promises to give closed form solutions, it is not of any practical use. A numerical method that circumvents the problems involved with the eliminant is developed in the next section of this chapter.

4.3 Numerical Solution

The development of a new numerical method to solve the problem of determining extremum eigenvalues of a constrained affine sum is shown in this section. The algorithm developed here primarily focuses on maximizing the smallest eigenvalue of the matrix \mathbf{A} subject to the constraint (4.22). The dual problem of minimizing the largest problem could also be solved by making minor modifications to the algorithm.

The solution of the constrained optimization problem associated with the affine sum can be obtained by solving the set of equation in (4.36) (4.37) and (4.38). As explained in the previous section, the greatest challenge is to determine the vector \mathbf{x} by solving the set of quadratic equations in (4.36) and (4.37). Obviously, if the vector \mathbf{x} can be determined using some numerical method, then the vector $\boldsymbol{\alpha}$ and the eigenvalue ξ can be determined by solving the linear equations in (4.38). The most popular and well suited numerical method to solve the non-linear equation in (4.36) and (4.37) is the Newton's method, also known as the Newton-Raphson method. However, the Newton's method is plagued by the problem of finding a suitable initial guess that would converge to the solution. For the problem in hand, every eigenvector \mathbf{x} of the matrix \mathbf{A} satisfies the set of quadratic equations in (4.36) and (4.37), and thus is one of the solutions. Since, we are interested only in the eigenvector associated with the smallest or the largest eigenvalue, a tactful way of finding a suitable initial guess is developed, such that the solution would converge to the desired eigenvector.

The input data for the algorithm are n symmetric matrices \mathbf{A}_k , $k = 1, 2, \dots, n$ and

the vector $\mathbf{c} \in \mathfrak{R}$. The goal is to determine the solution vector $\boldsymbol{\alpha}$, that maximizes the smallest eigenvalue of the matrix \mathbf{A} which is of the form shown in (4.21). The matrices \mathbf{A}_k , $k = 1, 2, \dots, n$ are evaluated using the input data via equations (4.32) and (4.33). To solve the problem we start with an initial guess $\boldsymbol{\alpha}^{(0)}$, the vector of unknown parameters, where the superscript in parenthesis indicates the iteration index. Then matrix $\mathbf{A}^{(0)}$ is then determined via (4.21). The smallest eigenvalues $\lambda^{(0)}$ of the matrix $\mathbf{A}^{(0)}$, and its corresponding eigenvector $\mathbf{x}^{(0)}$ is then be evaluated using pertinent numerical methods. One iteration of the Newton's method is performed on equations (4.36) and (4.37), to determine the vector $\boldsymbol{\delta}^{(0)}$ that corrects the guess of $\mathbf{x}^{(0)}$. The Jacobian matrix required for the Newton's method is given by

$$\mathbf{J}^{(0)} = \begin{bmatrix} \frac{\partial \mathbf{x}^{(0)T} \mathbf{B}_1 \mathbf{x}^{(0)}}{\partial x_1^{(0)}} & \frac{\partial \mathbf{x}^{(0)T} \mathbf{B}_1 \mathbf{x}^{(0)}}{\partial x_2^{(0)}} & \dots & \frac{\partial \mathbf{x}^{(0)T} \mathbf{B}_1 \mathbf{x}^{(0)}}{\partial x_n^{(0)}} \\ \frac{\partial \mathbf{x}^{(0)T} \mathbf{B}_2 \mathbf{x}^{(0)}}{\partial x_1^{(0)}} & \frac{\partial \mathbf{x}^{(0)T} \mathbf{B}_2 \mathbf{x}^{(0)}}{\partial x_2^{(0)}} & \dots & \frac{\partial \mathbf{x}^{(0)T} \mathbf{B}_2 \mathbf{x}^{(0)}}{\partial x_n^{(0)}} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial \mathbf{x}^{(0)T} \mathbf{B}_n \mathbf{x}^{(0)}}{\partial x_1^{(0)}} & \frac{\partial \mathbf{x}^{(0)T} \mathbf{B}_n \mathbf{x}^{(0)}}{\partial x_2^{(0)}} & \dots & \frac{\partial \mathbf{x}^{(0)T} \mathbf{B}_n \mathbf{x}^{(0)}}{\partial x_n^{(0)}} \end{bmatrix}_{n \times n} \quad (4.79)$$

Since the matrices \mathbf{B}_k , $k = 1, 2, \dots, n$ are symmetric

$$\frac{\partial \mathbf{x}^{(0)T} \mathbf{B}_1 \mathbf{x}^{(0)}}{\partial x_1^{(0)}} = 2\mathbf{B}_1 \mathbf{x}^{(0)}. \quad (4.80)$$

When (4.80) is applied to the expression of the Jacobian matrix defined in (4.79), we obtain

$$\mathbf{J}^{(0)} = 2 \begin{bmatrix} \mathbf{x}^{(0)T} \mathbf{B}_1 \\ \mathbf{x}^{(0)T} \mathbf{B}_2 \\ \vdots \\ \mathbf{x}^{(0)T} \mathbf{B}_n \end{bmatrix} \quad (4.81)$$

The error in the initial guess of the vector $\mathbf{x}^{(0)}$ is then obtained by one iteration of the

Newton's method, i.e. via

$$\boldsymbol{\delta}^{(0)} = -(\mathbf{J}^{(0)})^{-1} \mathbf{e}_n, \quad (4.82)$$

where \mathbf{e}_n is the n -th unit vector of dimension n which is defined as

$$\mathbf{e}_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T. \quad (4.83)$$

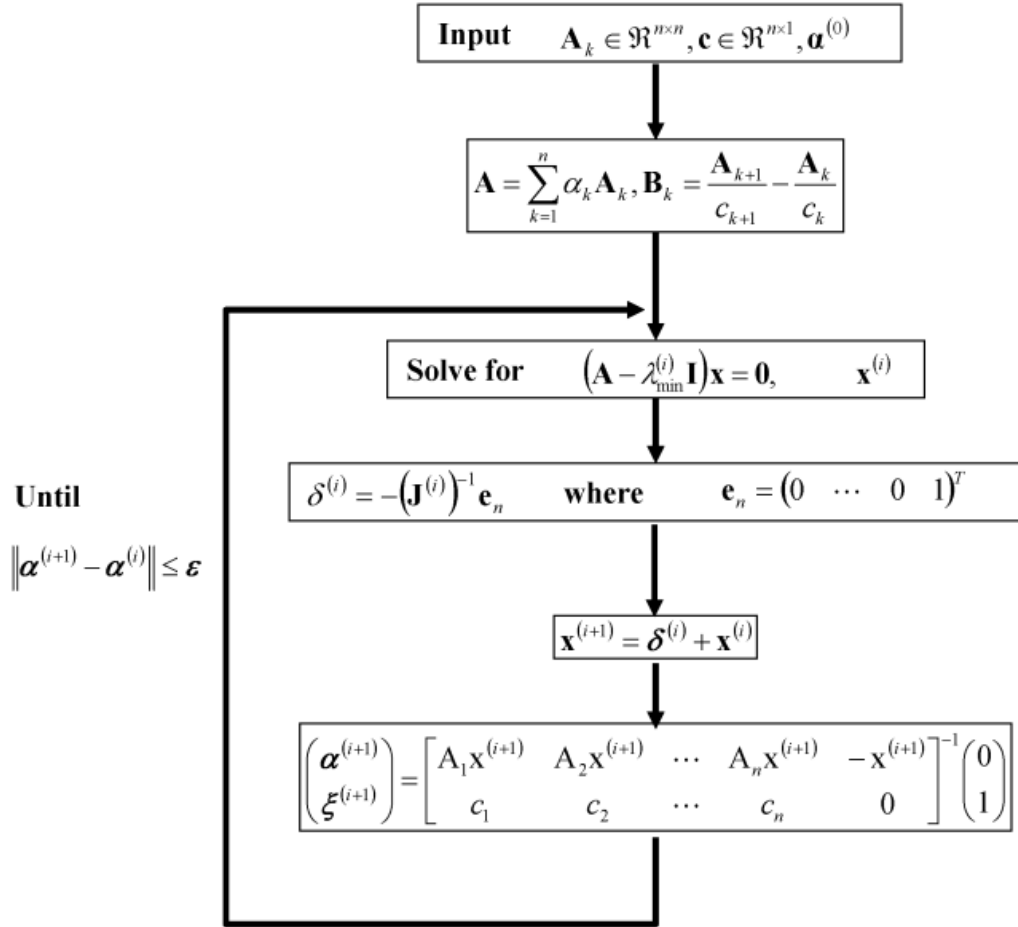


Figure 4.2: Flowchart of the modified Newton's algorithm for eigenvalue optimization

The new corrected vector \mathbf{x} is then obtained as

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \boldsymbol{\delta}^{(0)}. \quad (4.84)$$

The improved parameter vector $\boldsymbol{\alpha}$ is then determined via (4.38), i.e.,

$$\begin{bmatrix} \mathbf{A}_1 \mathbf{x}^{(1)} & \mathbf{A}_2 \mathbf{x}^{(1)} & \cdots & \mathbf{A}_n \mathbf{x}^{(1)} & -\mathbf{x}^{(1)} \\ c_1 & c_2 & \cdots & c_n & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{\alpha}^{(1)} \\ \xi^{(1)} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \quad (4.85)$$

from which the new improved matrix $\mathbf{A}^{(1)}$ can be computed. This process of the modified Newton's method is iterated $|\boldsymbol{\alpha}^{(i+1)} - \boldsymbol{\alpha}^{(i)}| < \varepsilon$, where ε is a predefined convergence tolerance. A flowchart explanation of the modified Newton's algorithm the maximizes the smallest eigenvalue is shown in Figure 4.2.

As stated before the dual problem of determining the vector $\boldsymbol{\alpha}$ that minimizes the largest eigenvalue of an affine sum matrix can be solved by making minor modifications to the proposed algorithm. Instead of choosing the eigenvector \mathbf{x} that corresponds to the smallest eigenvalue, the eigenvector \mathbf{x} that corresponds to the largest eigenvalue is chosen. Now this vector \mathbf{x} is the initial guess for one iteration of the Newton's method. The rest of the algorithm that deals with steps involved in evaluating $\boldsymbol{\alpha}$ and ξ remain unchanged. An example that utilizes the modified Newton's method is now shown.

Example 4:

Given:

The four input matrices of the affine sum are

$$\left. \begin{array}{l} \mathbf{A}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -2 & 3 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix} \\ \mathbf{A}_3 = \begin{bmatrix} 5 & -2 & 0 & 0 \\ -2 & 3 & -3 & 0 \\ 0 & -3 & 6 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -2 & 4 & -3 & 0 \\ 0 & -3 & 4 & -3 \\ 0 & 0 & -3 & 4 \end{bmatrix} \end{array} \right\} \quad (4.86)$$

and the vector for the constraint is

$$\mathbf{c} = [1 \ 1 \ 1 \ 1]^T. \quad (4.87)$$

Using (4.32) and (4.33) the matrices \mathbf{B}_k are computed to be

$$\left. \begin{aligned} \mathbf{B}_1 &= \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, & \mathbf{B}_2 &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \mathbf{B}_3 &= \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, & \mathbf{B}_4 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \right\} \quad (4.88)$$

With an initial guess of $\boldsymbol{\alpha}^{(0)}$ for the modified Newton's method as

$$\boldsymbol{\alpha}^{(0)} = [1 \ 1 \ 1 \ 1]^T, \quad (4.89)$$

the matrix $\mathbf{A}^{(0)}$ is computed using (4.21) to be

$$\mathbf{A} = \begin{bmatrix} 12 & -7 & & \\ -7 & 12 & -9 & \\ & -9 & 16 & -8 \\ & & -8 & 12 \end{bmatrix}. \quad (4.90)$$

The eigenvalues and eigenvectors of the matrix \mathbf{A} , are evaluated by solving $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ using pertinent numerical methods . The eigenvalue optimization problem when solved

using the modified Newton's method converges in seven iterations. The solution that maximizes the first eigenvalue of the affine sum matrix \mathbf{A} is

$$\boldsymbol{\alpha} = \left[\frac{7}{4} \quad \frac{-5}{4} \quad \frac{3}{4} \quad \frac{-1}{4} \right]^T, \quad (4.91)$$

and the value of the maximum smallest eigenvalue is

$$\lambda_{\max} = \frac{1}{2}. \quad (4.92)$$

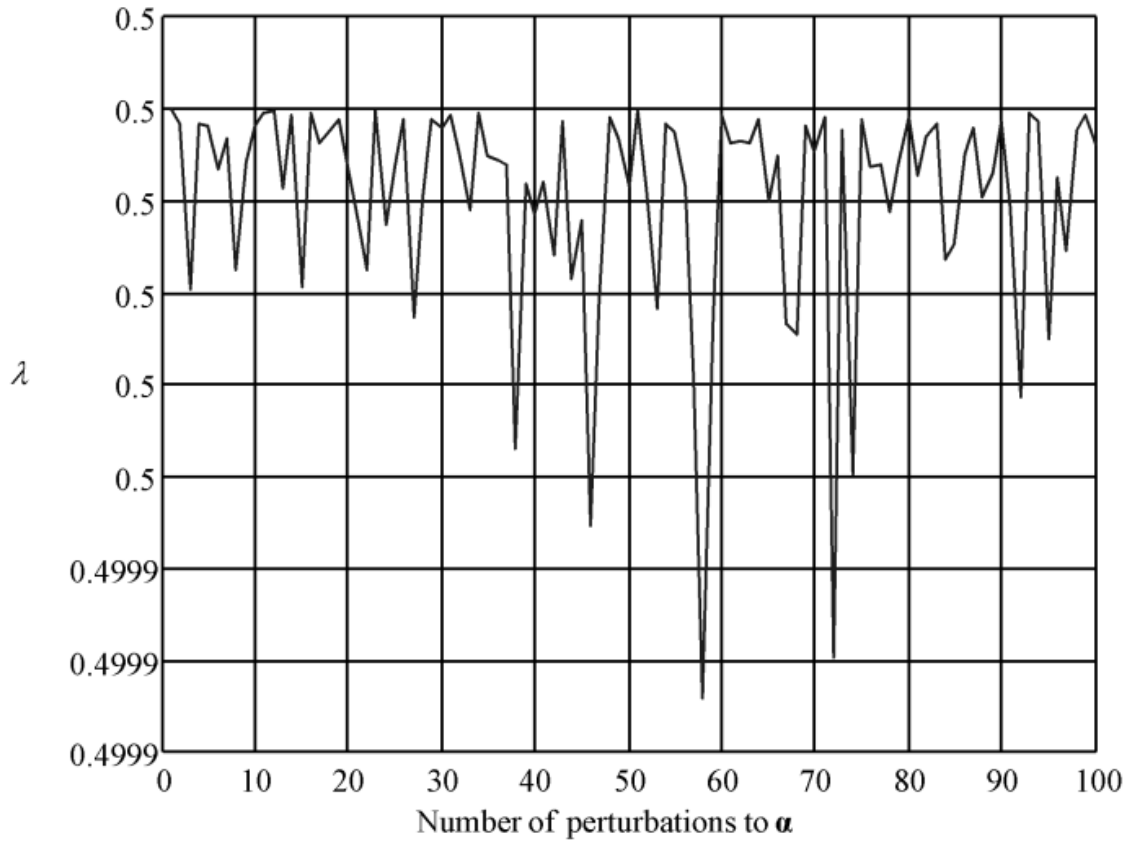


Figure 4.3: Variation of eigenvalue as a function of number of perturbations

To verify that the solution obtained using the numerical technique is indeed extremum, small perturbation are made to the optimal solution of $\boldsymbol{\alpha}$. These perturbations are made such that the constraint on $\boldsymbol{\alpha}$ given in (4.22) is satisfied and then the eigenvalue λ of the

perturbed system is computed. The variation of λ as function of the number of perturbations is shown in Figure 4.3. It is clear from Figure 4.3 that the eigenvalue is maximum for the first case, i.e. for the case for which $\boldsymbol{\alpha}$ is not perturbed, thus providing verification for the algorithm developed here.

4.4 Optimization of Vibratory Systems

The optimization procedure and the numerical method developed in the previous sections of this chapter can be applied to determine vibratory systems with extreme eigenvalues. Consider the n degrees of freedom mass-spring system shown in Figure 4.4.

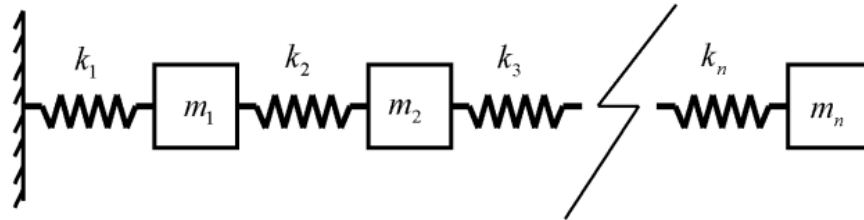


Figure 4.4: Mass-spring system with n degrees of freedom supported at one end

The objective of the optimization problem is to determine the distribution of masses, m_1, m_2, \dots, m_n , of the system, such that the mass-spring system has extremum natural frequencies (eigenvalues). A constant total mass of the system, i.e. $m_T = 1$, is the constraint for optimization. This total mass constraint can be expressed as

$$\mathbf{c}^T \mathbf{m} = 1, \quad (4.93)$$

where $\mathbf{c} = (1 \ 1 \ \dots \ 1)^T$, and $\mathbf{m} = (m_1 \ m_2 \ \dots \ m_n)^T$. The eigenvalue problem that is associated with the vibrating mass-spring system is of the form

$$(\mathbf{K} - \mu \mathbf{M}) \mathbf{z} = \mathbf{0}, \quad (4.94)$$

where the matrix \mathbf{K} is tridiagonal and for the current spring mass arrangement is

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n \end{bmatrix}_{n \times n}, \quad (4.95)$$

and the matrix \mathbf{M} is diagonal of the form

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \ddots & & \\ & & & m_{n-1} & \\ & & & & m_n \end{bmatrix}_{n \times n}. \quad (4.96)$$

The optimization procedure and the numerical technique described in Section 4.2 and Section 4.3 was for eigenvalue problems of the type in (4.20). In order to apply the optimization principles and numerical techniques for the vibrating mass-spring system, the generalized eigenvalue problem in (4.94) has to be converted to the standard form of (4.20). To begin this process, we define

$$\mathbf{z} = \mathbf{E}^{-1}\mathbf{x}, \quad (4.97)$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \vdots & \ddots & & \\ & & & -1 & 1 \end{bmatrix}_{n \times n}. \quad (4.98)$$

Pre-multiplying equation (4.94) by \mathbf{E}^{-T} we obtain

$$(\mathbf{E}^{-T}\mathbf{K} - \mu\mathbf{E}^{-T}\mathbf{M})\mathbf{z} = \mathbf{0}. \quad (4.99)$$

Replacing \mathbf{z} in (4.99) with \mathbf{x} obtained from (4.97), (4.99) takes the form

$$(\mathbf{E}^{-T}\mathbf{K} - \mu\mathbf{E}^{-T}\mathbf{M})\mathbf{E}^{-1}\mathbf{x} = \mathbf{0}, \quad (4.100)$$

equivalently

$$(\mathbf{E}^{-T}\mathbf{K}\mathbf{E}^{-1} - \mu\mathbf{E}^{-T}\mathbf{M}\mathbf{E}^{-1})\mathbf{x} = \mathbf{0}, \quad (4.101)$$

where the matrix \mathbf{E}^{-1} is lower triangular and is of the form

$$\mathbf{E}^{-1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & \\ & & & & 1 \end{bmatrix}_{n \times n}. \quad (4.102)$$

Since \mathbf{E}^{-1} is lower triangular, its transpose i.e. \mathbf{E}^{-T} is an upper triangular matrix. This knowledge reduces the first product term in (4.101) to

$$\mathbf{E}^{-T}\mathbf{K}\mathbf{E}^{-1} = \hat{\mathbf{K}} = \begin{bmatrix} k_1 & & & & \\ & k_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & k_n \end{bmatrix}_{n \times n} \quad (4.103)$$

and the second product term to

$$\mathbf{E}^{-T} \mathbf{M} \mathbf{E}^{-1} = \hat{\mathbf{M}} = \begin{bmatrix} \sum_{i=1}^n m_i & \sum_{i=2}^n m_i & \sum_{i=3}^n m_i & \cdots & m_n \\ \sum_{i=2}^n m_i & \sum_{i=2}^n m_i & \sum_{i=3}^n m_i & \cdots & m_n \\ \sum_{i=3}^n m_i & \sum_{i=3}^n m_i & \sum_{i=3}^n m_i & \cdots & m_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_n & m_n & m_n & \cdots & m_n \end{bmatrix}_{n \times n} \quad (4.104)$$

With the definitions of $\hat{\mathbf{K}}$ and $\hat{\mathbf{M}}$ equation (4.101) can be equivalently written as a generalized eigenvalue problem of the form

$$\left(\hat{\mathbf{K}} - \mu \hat{\mathbf{M}} \right) \mathbf{x} = \mathbf{0}. \quad (4.105)$$

Dividing (4.105) with μ and pre-multiplying with $\hat{\mathbf{K}}^{-1/2}$ we obtain

$$\left(\lambda \Gamma \hat{\mathbf{K}} - \Gamma \hat{\mathbf{M}} \right) \mathbf{x} = \mathbf{0}. \quad (4.106)$$

where

$$\lambda = \frac{1}{\mu} \quad (4.107)$$

and

$$\Gamma = \hat{\mathbf{K}}^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{k_1}} & & & & \\ & \frac{1}{\sqrt{k_2}} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{1}{\sqrt{k_n}} \end{bmatrix}_{n \times n} \quad (4.108)$$

Introducing $\hat{\mathbf{x}}$, such that

$$\mathbf{x} = \Gamma \hat{\mathbf{x}} \quad (4.109)$$

in equation (4.106) we obtain the eigenvalue problem

$$\left(\Gamma \hat{\mathbf{M}} \Gamma - \lambda \mathbf{I} \right) \hat{\mathbf{x}} = \mathbf{0} \quad (4.110)$$

since

$$\Gamma \hat{\mathbf{K}} \Gamma = \hat{\mathbf{K}}^{-1/2} \hat{\mathbf{K}} \hat{\mathbf{K}}^{-1/2} = \mathbf{I}. \quad (4.111)$$

Equation (4.110) is an eigenvalue problem of the form (4.20) and the matrix \mathbf{A} is an affine sum similar to that given in (4.21), where

$$\mathbf{A}_i = \begin{bmatrix} \frac{1}{\sqrt{k_1}} & & & \\ & \frac{1}{\sqrt{k_2}} & & \\ & & \dots & \\ & & & \frac{1}{\sqrt{k_n}} \end{bmatrix} \begin{matrix} i \text{ columns} \\ \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \dots & \vdots \\ 1 & \dots & 1 \end{bmatrix} \\ i \text{ rows} \end{matrix} \begin{bmatrix} \frac{1}{\sqrt{k_1}} & & & \\ & \frac{1}{\sqrt{k_2}} & & \\ & & \dots & \\ & & & \frac{1}{\sqrt{k_n}} \end{bmatrix} \quad (4.112)$$

and

$$\alpha_i = m_i. \quad (4.113)$$

The matrix with ones, i.e. the one that is independent of the stiffness k_i , has ones in the first i rows and i columns. The rest of the matrix has zeros in all other elements. The matrix \mathbf{B}_i is computed using the definition in (4.32) and for the present case it is

$$\mathbf{B}_i = \begin{bmatrix} \frac{1}{\sqrt{k_1}} & & & \\ & \frac{1}{\sqrt{k_2}} & & \\ & & \ddots & \\ & & & \frac{1}{\sqrt{k_n}} \end{bmatrix} \begin{matrix} i+1 \text{ columns} \\ \begin{bmatrix} & & & 1 \\ & & & 1 \\ & & & 1 \\ 1 & \cdots & 1 & \end{bmatrix} \\ i+1 \text{ rows} \end{matrix} \begin{bmatrix} \frac{1}{\sqrt{k_1}} & & & \\ & \frac{1}{\sqrt{k_2}} & & \\ & & \ddots & \\ & & & \frac{1}{\sqrt{k_n}} \end{bmatrix}. \quad (4.114)$$

The matrix independent of the stiffness parameter k is a zeros matrix, with ones on the $i+1^{th}$ row and column. The matrix \mathbf{B}_n is the identity matrix of dimension $n \times n$, same as the definition in (4.33). Now that the eigenvalue problem associated with the vibrating mass-spring system has been transformed to the form given in (4.20), the optimization procedure developed in the previous two sections could be used.

It is very important to know that if the smallest eigenvalue λ of (4.110) is maximized, the largest eigenvalue μ of (4.94) is minimized. This is due to the fact that λ is the reciprocal of μ according to (4.107). Hence, if we wish to find the mass configuration that would maximize first natural frequency of the spring mass system, we have to minimize the largest natural frequency of the equivalent system in (4.110). The dual problem of minimizing the largest natural frequency of the spring mass system deals with maximizing the smallest eigenvalue of (4.110).

The numerical procedure described in Section 4.3 can be used to determine the mass distribution that would maximize the smallest natural frequency or minimize the largest natural frequency of the system shown in Figure 4.4. To illustrate the results of the numerical procedure let us consider a n degree of freedom mass spring system, where the stiffness of each spring is

$$k_i = \frac{1}{n}, \quad i = 1, 2, \dots, n \quad (4.115)$$

and the total mass of the system is

$$m_{tot} = \sum_{i=1}^n m_i = 1. \quad (4.116)$$

The solution of the problem is obtained by solving the set of equations in (4.36), (4.37) and (4.38). Following the numerical procedure described in Section 4.3 and the flowchart given in Figure 4.2 the optimization problem could be solved. The problem is solved for the cases where the number of degrees of freedom are $n = 4$ and $n = 50$ and the results for these cases are shown in Figure 4.5 (a) and (b) respectively.

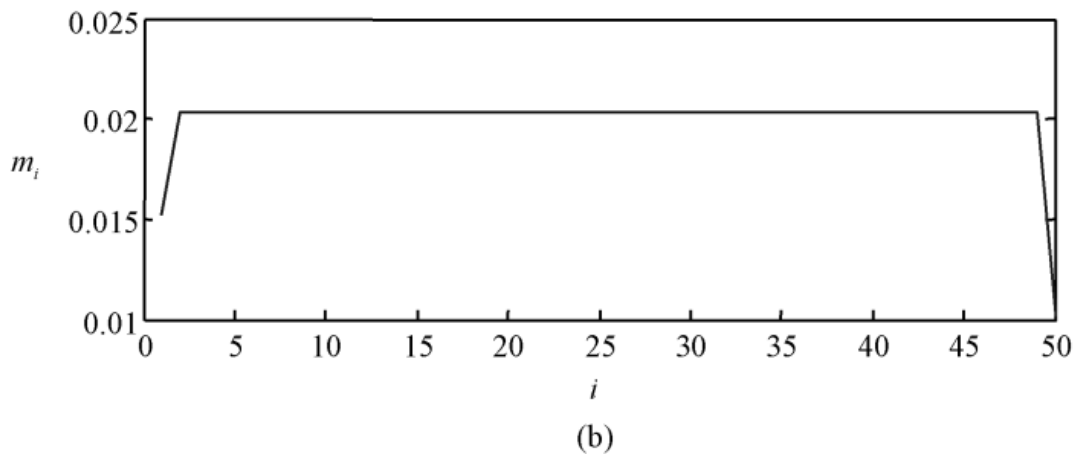
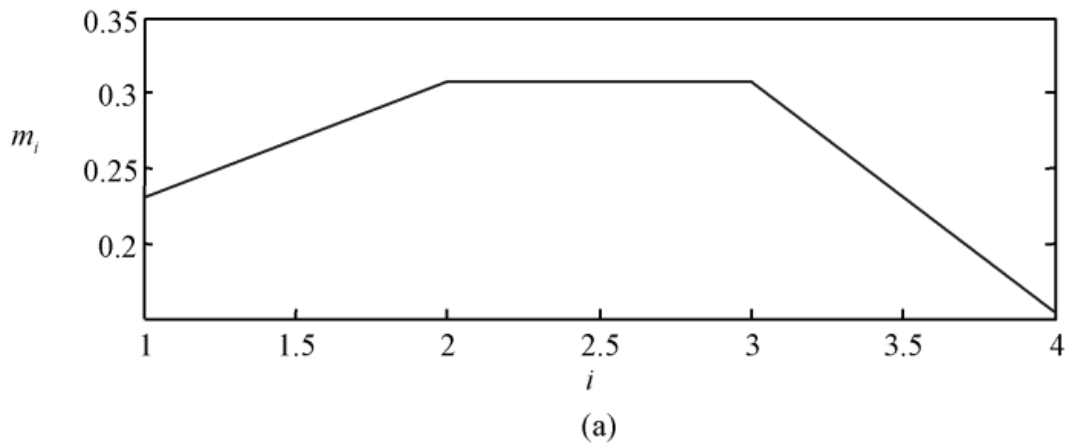


Figure 4.5: Mass distribution of mass spring system fixed at one end that minimizes the largest natural frequency (a) for $n = 4$ (b) for $n = 50$

The mass distribution as a progression for the spring mass system shown in Figure 4.4, with spring stiffnesses $k_i = 1/n$ and minimum largest natural frequency, can be obtained from Figure 4.5 (a) and (b). This series is given as the mass vector \mathbf{m} is

$$\mathbf{m} = \left[\frac{3m_T}{4n-3} \quad \frac{4m_T}{4n-3} \quad \cdots \quad \frac{4m_T}{4n-3} \quad \frac{2m_T}{4n-3} \right]^T. \quad (4.117)$$

The minimum largest eigenvalue of the vibrating mass-spring system shown in Figure 4.4 with n degrees of freedom is

$$\mu = \frac{4n-3}{n}, \quad (4.118)$$

and the natural frequency $\omega_{n(min)}$ corresponding to this eigenvalue is

$$\omega_{n(min)} = \sqrt{\mu} = \sqrt{\frac{4n-3}{n}}. \quad (4.119)$$

The mass distribution for the system shown in Figure 4.4 that has a maximum smallest eigenvalue is determined to be

$$\mathbf{m} = \left[m_T \quad 0 \quad \cdots \quad 0 \quad 0 \right]^T, \quad (4.120)$$

and the corresponding eigenvalue is

$$\mu = \frac{1}{n}, \quad (4.121)$$

and the natural frequency $\omega_{1(max)}$ is

$$\omega_{1(max)} = \sqrt{\mu} = \sqrt{\frac{1}{n}}. \quad (4.122)$$

The method described above is valid only for a mass-spring system shown in Figure 4.4. The transformation procedure described in this section cannot be used to transform the

governing generalized eigenvalue problem in (4.94) to the form in (4.20), for all mass spring configurations. The procedure to find the mass distribution of the mass-spring system with extreme eigenvalues for any configuration is described in the next section of this chapter.

4.5 Optimization of Vibratory System with Arbitrary Spring Orientations

The generalized eigenvalue problem in (4.94) is the one that governs the vibration of mass spring systems with any configuration. For all these systems, the stiffness matrix \mathbf{K} is symmetric and the mass matrix \mathbf{M} is diagonal. However, to aid the process of determining the mass distribution that would yield extreme eigenvalues, the generalized eigenvalue problem in (4.94) has to be transformed to the standard form in (4.20). The transformation process is started by pre-multiplying (4.94) with $\mathbf{K}^{-1/2}$ to obtain

$$(\mathbf{K}^{-1/2}\mathbf{K} - \mu\mathbf{K}^{-1/2}\mathbf{M})\mathbf{z} = \mathbf{0}. \quad (4.123)$$

Introducing the vector \mathbf{x} such that

$$\mathbf{z} = \mathbf{K}^{-1/2}\mathbf{x}, \quad (4.124)$$

in (4.123) we obtain

$$(\mathbf{I} - \mu\mathbf{K}^{-1/2}\mathbf{M}\mathbf{K}^{-1/2})\mathbf{x} = \mathbf{0}. \quad (4.125)$$

or equivalently

$$(\mathbf{K}^{-1/2}\mathbf{M}\mathbf{K}^{-1/2} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}, \quad (4.126)$$

since $\mathbf{K}^{-1/2}\mathbf{K}\mathbf{K}^{-1/2} = \mathbf{I}$, and λ is same as that defined in (4.107). The eigenvalue problem in (4.126) can be framed as an affine sum in mass m_i by defining

$$\mathbf{A}_i = \mathbf{K}^{-1/2} \mathbf{M}_i \mathbf{K}^{-1/2}, \quad (4.127)$$

and

$$\mathbf{A} = \sum_{i=1}^n m_i \mathbf{A}_i, \quad (4.128)$$

where \mathbf{M}_i is a $n \times n$ matrix with zeros everywhere with the exception of $\mathbf{M}_i(i, i) = 1$. The optimization explained in Sections 4.2 and 4.3 may be used to determine the mass distribution that would yield extremum eigenvalues. The matrices \mathbf{B}_i , $i = 1, 2, \dots, n$ can be evaluated from their definitions in (4.32) and (4.33) and the solution of the problem is obtained by solving the set of equations in (4.36), (4.37) and (4.38). The numerical procedure described in Section 4.3 may be used to obtain the optimal solution.

Further explanation of the procedure for obtaining extremum eigenvalues can be obtained with the help of the following example. Consider a n degree of freedom mass spring system as shown in Figure 4.6. In this case the mass spring system has $n + 1$ springs, n masses and the first and the last spring are supported by walls.

For the sake of simplicity let us assume that the stiffness of each spring is

$$k_i = \frac{1}{n + 1}, \quad i = 1, 2, \dots, n + 1 \quad (4.129)$$

and the total mass of the system is

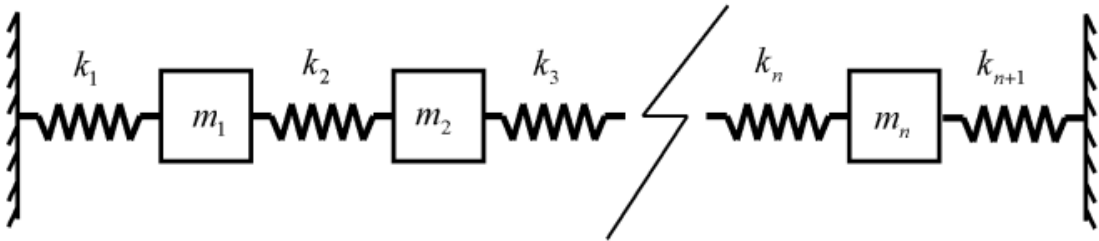


Figure 4.6: n degrees of freedom mass spring system supported at both ends

$$m_T = \sum_{i=1}^n = \mathbf{c}^T \mathbf{m} = 1 \quad (4.130)$$

where $\mathbf{c} = (1 \ 1 \ \dots \ 1)^T$ and $\mathbf{m} = (m_1 \ m_2 \ \dots \ m_n)^T$. The stiffness matrix \mathbf{K} for the mass spring system shown in Figure 4.6 is tridiagonal and of the form

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & & & & \\ -k_2 & k_2 + k_3 & -k_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n & \\ & & & -k_n & k_n + k_{n+1} & \end{bmatrix}_{n \times n}, \quad (4.131)$$

and the mass matrix \mathbf{M} is

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \ddots & & \\ & & & m_n & \end{bmatrix}_{n \times n}. \quad (4.132)$$

Using the numerical procedure described in Section 4.3 along with the flowchart given in Figure 4.2, the problem of determining the mass distribution that would yield extreme eigenvalue is solved. The solution for the case when then number of degrees of freedom $n = 4$ and $n = 50$, is shown in Figures # (a) and (b) respectively. The mass vector \mathbf{m} that would render the largest eigenvalue to be minimum is

$$\mathbf{m} = \left[\frac{3m_T}{2(2n-1)} \quad \frac{2m_T}{2n-1} \quad \dots \quad \frac{2m_T}{2n-1} \quad \frac{3m_T}{2(2n-1)} \right]. \quad (4.133)$$

The minimum largest eigenvalue is

$$\mu = \frac{4n-2}{n+1}, \quad (4.134)$$

and the natural frequency ω corresponding to this eigenvalue is

$$\omega_{n(\min)} = \sqrt{\frac{4n-2}{n+1}}. \quad (4.135)$$

The mass vector \mathbf{m} for the mass spring system that has maximum smallest eigenvalue is

$$\mathbf{m} = \left[\begin{array}{cccc} \frac{m_T}{2} & 0 & \cdots & 0 & \frac{m_T}{2} \end{array} \right]. \quad (4.136)$$

The maximum smallest eigenvalue is

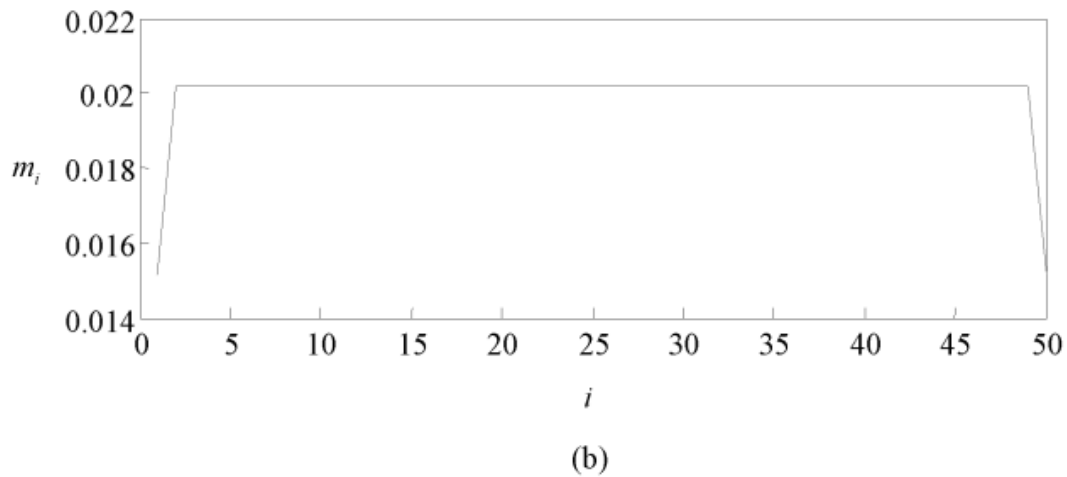
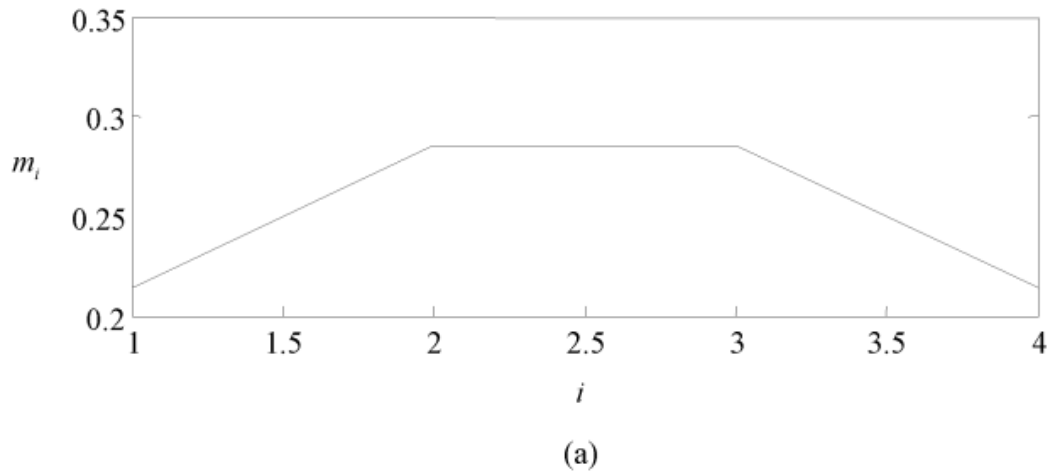


Figure 4.7: Mass distribution of mass spring system fixed at both ends that minimizes the greatest eigenvalue (a) for $n = 4$ (b) for $n = 50$

$$\mu = \frac{2}{n+1}, \quad (4.137)$$

and the natural frequency ω corresponding to this eigenvalue is

$$\omega_{1(max)} = \sqrt{\frac{2}{n+1}}. \quad (4.138)$$

The optimization procedure to determine extremum eigenvalues of a linear affine sum with a linear constraint was developed. A numerical technique based on Nuewton's method, that not only aids in solving the set of quadratic equations but also makes sure that the solution corresponds with the smallest or largest eigenvector was developed. As an application to the mathematical problem described, optimal vibrating spring mass-systems were obtained.

Chapter 5

Optimal Columns as an Affine Sum

The procedure of obtaining extremum eigenvalues of a symmetric matrix that can be expressed as a linear affine sum of matrices has been described in Chapter 4. The vibratory system considered there was expressed as a linear sum of matrices and the constraint of total mass was also a linear functions of the mass m_i . The problem of determining the shape of the strongest column has a constraint that is linear with respect to the cross-sectional area. However, the coefficients of the affine i.e. moment of inertia, varies as a square of the cross-sectional area. Hence, the procedure of finding the extremum eigenvalues of a linear affine sum cannot directly be used for the case of determining optimal columns. An extension to the affine sum problem that could be used to determine the shape of the strongest column is given in this chapter.

5.1 Affine Sum for Column Buckling Using Finite Differences on the Second-Order Differential Equation

The second order non-dimensional differential equation that governs buckling of columns with circular cross-section was derived in Section 2.2 and is given in (2.21). This second order differential equation can be converted into a set of algebraic equations by applying appropriate finite difference schemes. If the non-dimensional length of the column is divided into n equal sections, then the length of each one of these sections is

$$h = \frac{1}{n}, \quad (5.1)$$

and the cross-sectional area is a_i .

The central finite difference expansion of the second derivative is

$$\frac{d^2 u_i}{d\xi^2} = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}, \quad i = 0, 1, \dots, n \quad (5.2)$$

Replacing the second derivative in (2.21) using (5.2) we obtain

$$a_i^2 \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \quad i = 0, 1, \dots, n \quad (5.3)$$

or equivalently

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + a_i^{-2} \lambda u_i = 0, \quad i = 0, 1, \dots, n \quad (5.4)$$

The terms u_0 and u_{n+1} in (5.4) can be obtained using appropriate boundary conditions.

The n equations in (5.4) can be written in matrix form as

$$\left(\underbrace{\frac{1}{h^2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}}_{\mathbf{K}} - \lambda \underbrace{\begin{bmatrix} a_0^{-2} & & & & \\ & a_1^{-2} & & & \\ & & \ddots & & \\ & & & a_{n-1}^{-2} & \\ & & & & a_n^{-2} \end{bmatrix}}_{\mathbf{M}} \right) \underbrace{\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix}}_{\mathbf{u}} = \mathbf{0}. \quad (5.5)$$

The problem of determining the optimal shape of the column involves determining the unknown vector of areas $\mathbf{a} = [a_0 \ a_1 \ \dots \ a_n]^T$, and the eigenvector $\mathbf{u} = [u_0 \ u_1 \ \dots \ u_n]^T$ that maximizes smallest eigenvalue λ , subject to the volume constraint (3.2) which is equivalent to

$$V = \mathbf{c}^T \mathbf{a} = 1 \quad (5.6)$$

where $\mathbf{c} = [h \ h \ \dots \ h]^T$. It is evident from equation (5.5) that the unknown parameters a_i have an index of -2 , whereas the volume constraint in (5.6) is a linear function of a_i . To transform the generalized eigenvalue problem in (5.5) into the standard eigenvalue

problem of form $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ we pre-multiply (5.5) with $\mathbf{K}^{-1/2}$ and obtain

$$(\mathbf{K}^{-1/2}\mathbf{K} - \lambda\mathbf{K}^{-1/2}\mathbf{M})\mathbf{u} = \mathbf{0}. \quad (5.7)$$

Introducing the vector \mathbf{x} such that

$$\mathbf{u} = \mathbf{K}^{-1/2}\mathbf{x}, \quad (5.8)$$

in (5.7) we obtain

$$\left(\underbrace{\mathbf{K}^{-1/2}\mathbf{M}\mathbf{K}^{-1/2}}_{\mathbf{A}} - \mu\mathbf{I} \right) = \mathbf{0}, \quad (5.9)$$

where λ and μ hold the same relationship as in (4.107) and since $\mathbf{K}^{-1/2}\mathbf{K}\mathbf{K}^{-1/2} = \mathbf{I}$. The matrix \mathbf{A} is an affine sum of the form

$$\mathbf{A} = \sum_{i=0}^n a_i^{-2} \mathbf{A}_i, \quad (5.10)$$

where

$$\mathbf{A}_i = \mathbf{K}^{-1/2}\mathbf{M}_i\mathbf{K}^{-1/2}, \quad i = 0, 1, \dots, n \quad (5.11)$$

All the elements of the matrix \mathbf{M}_i is zero with the exception of $\mathbf{M}_i(i, i) = 1$. To determine the maximum smallest eigenvalue we use the Courant-Fischer characterization of eigenvalues for a symmetric eigenvalue problem and obtain

$$\mu = \min_{\mathbf{x}} \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \quad (5.12)$$

Since the objective is to maximize the smallest eigenvalue,

$$\mu_{max} = \max_{\mathbf{a}} \min_{\mathbf{x}} \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}. \quad (5.13)$$

The minimax problem in (5.13) may be recast as

$$\mu_{max} = \max_{\mathbf{a}} \min_{\mathbf{x}} \mathbf{x}^T \mathbf{A} \mathbf{x} \quad (5.14)$$

subject to the constraint

$$\mathbf{x}^T \mathbf{x} = C \quad (5.15)$$

and the constraint on volume given in (5.6), where $C \neq 0$ is an arbitrary constant. To solve this optimization problem the Lagrange function that is made up of the objective function and constraints, is defined as

$$L = \mathbf{x}^T \mathbf{A} \mathbf{x} - \mu \mathbf{x}^T \mathbf{x} + \eta \mathbf{c}^T \mathbf{a}. \quad (5.16)$$

where μ and η are Lagrange multipliers. Substituting for \mathbf{A} in (5.16) using (5.11) we obtain

$$L = \mathbf{x}^T \left(\sum_{i=0}^n a_i^{-2} \mathbf{K}^{-1/2} \mathbf{M}_i \mathbf{K}^{-1/2} \right) \mathbf{x} - \mu \mathbf{x}^T \mathbf{x} + \eta h(a_0 + a_1 + \dots + a_n). \quad (5.17)$$

Since we wish to find extreme eigenvalues, the Lagrangian defined in (5.17) has to be stationary or invariant with respect to both x_i and a_i . Variation of (5.17) with respect to \mathbf{x} reduces to the eigenvalue problem defined in (5.9) and variation with respect to a_i gives

$$\frac{\partial L}{\partial a_i} = -2a_i^{-3} \mathbf{x}^T \mathbf{A}_i \mathbf{x} + \eta = 0, \quad i = 0, 1, \dots, n \quad (5.18)$$

The Lagrange multiplier η in (5.18) is not of any interest and this η can be eliminated by subtracting (5.18) for $i = 0$ from (5.18) for $i = 1, 2, \dots, n$ to obtain

$$\frac{\partial L}{\partial a_i} - \frac{\partial L}{\partial a_0} = a_0^{-3} \mathbf{x}^T \mathbf{A}_0 \mathbf{x} - a_i^{-3} \mathbf{x}^T \mathbf{A}_i \mathbf{x}, \quad i = 1, 2, \dots, n \quad (5.19)$$

In order to make the indices of the areas positive we rewrite (5.19) as

$$a_0^3 \mathbf{x}^T \mathbf{A}_i \mathbf{x} - a_i^3 \mathbf{x}^T \mathbf{A}_1 \mathbf{x}, \quad i = 1, 2, \dots, n \quad (5.20)$$

The solution of the strongest column is then obtained by simultaneously solving the set of equations in (5.9), (5.15), (5.20) and (5.6) for the $2n + 3$ unknowns in \mathbf{x} , \mathbf{a} and μ . Since the equations in (5.9), (5.15), (5.20) and (5.6) are non-linear it is not feasible to solve these equations in closed form. A numerical technique that not only aids in solving equations but also converges to the smallest eigenvalue is developed in the next section of this chapter.

5.2 Numerical Method to Solve the Affine Sum Problem Associated with the Strongest Column

The buckling load of a column is defined as the smallest load that would keep the column in a non-trivial equilibrium and this load is a function of the smallest eigenvalue λ of (5.5). If we wish to find the maximum buckling of the column, we have to maximize the smallest eigenvalue λ of the system in (5.5). Since, $\lambda = \mu^{-1}$ we will have to minimize the largest eigenvalue μ of the eigenvalue problem defined in (5.9). We will now develop a strategy for numerically solving the set of equation in (5.9), (5.15), (5.20) and (5.6) that would minimize the largest eigenvalue of the symmetric matrix \mathbf{A} in (5.9).

The input data for the numerical algorithm are the $n + 1$ symmetric matrices \mathbf{A}_i , $i = 0, 1, \dots, n$ that can be obtained from (5.11). To solve the problem we start with an initial guess of the vector $\mathbf{a}^{(0)}$, the vector of unknown parameters, where the superscript in parenthesis denotes the iteration index. The matrix \mathbf{A} is then computed using (5.10). The eigenvalue problem in (5.9) is solved using pertinent numerical techniques and the largest eigenvalue $\mu^{(0)}$ and its corresponding eigenvector $\mathbf{x}^{(0)}$ are computed. Once $\mathbf{x}^{(0)}$ is known, equations (5.6) and (5.20) can be used to find the new improved vector $\mathbf{a}^{(i)}$. Since (5.20) consists of a set of non-linear equations, we use the Newton-Raphson method to solve the problem. The vector \mathbf{f} that arises from the set of equations in (5.6) and (5.20) is given by

convergence tolerance. A flowchart explains the numerical technique described above is shown in Figure 5.1.

The numerical technique described here can be used to determine the shape of the strongest columns with clamped-free and pinned-pinned boundary conditions. The finite difference scheme was applied to the second order differential equation of columns, which was in terms of moments. Due to this, only the \mathbf{K} matrix in the equation (5.9) will change along with changes in the boundary conditions. The solution of the optimal column using the numerical method for the clamped-free and pinned-pinned case is given in the next section.

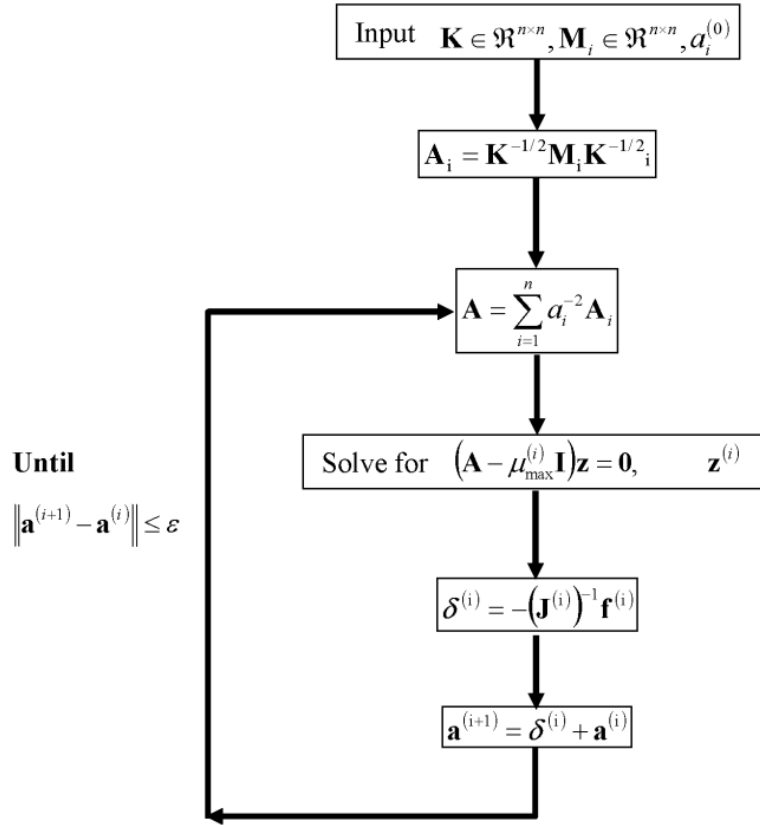


Figure 5.1: Flowchart of the numerical technique to determine the shape of the strongest column

5.3 Solution of the Strongest Columns

The process of converting the generalized eigenvalue problem, obtained using finite difference expansion of the second order governing differential equation, to an affine sum in terms of the unknown areas is shown in Section 5.1. The optimization procedure and the numerical technique to determine the shape of the strongest column is described in the previous two sections. A detailed solution of the shape of the strongest column for the clamped-free and pinned-pinned cases is shown in this section.

5.3.1 Strongest Clamped-Free Column

The finite difference expansion of the second order, non-dimensional, governing differential equation for column buckling is given in (5.4). The terms u_0 and u_{n+1} in (5.4) have to be replaced using the appropriate boundary conditions given in (2.26) and (2.29). From (2.26) we obtain

$$u_n = 0, \tag{5.26}$$

which implies that the last equation in (5.5) disappears. The other boundary condition in (2.29) when expanded using a central finite difference scheme gives

$$\frac{u_1 - u_{-1}}{2h} = 0, \tag{5.27}$$

from which we obtain

$$u_{-1} = u_1. \tag{5.28}$$

Applying (5.28) to the first equation in (5.4) we obtain

$$\frac{-2u_0 + 2u_1}{h^2} + a_0^{-2}\lambda u_0 = 0. \tag{5.29}$$

Equations (5.26) and (5.28) transforms the $n + 1$ equations in (5.5) in to the following n equations

$$\left(\underbrace{\frac{1}{h^2} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}}_{\mathbf{K}} - \lambda \underbrace{\begin{bmatrix} a_0^{-2}/2 & & & & \\ & a_1^{-2} & & & \\ & & \ddots & & \\ & & & a_{n-2}^{-2} & \\ & & & & a_{n-1}^{-2} \end{bmatrix}}_{\mathbf{M}} \right) \underbrace{\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix}}_{\mathbf{u}} = \mathbf{0}. \quad (5.30)$$

The numerical technique described in Section 5.3 can be used to determine the shape of the strongest clamped-free column. Using the flowchart given in Figure 5.1, the shape of the strongest column shown in Figure 5.2 is obtained for $n = 50$ degrees of freedom.

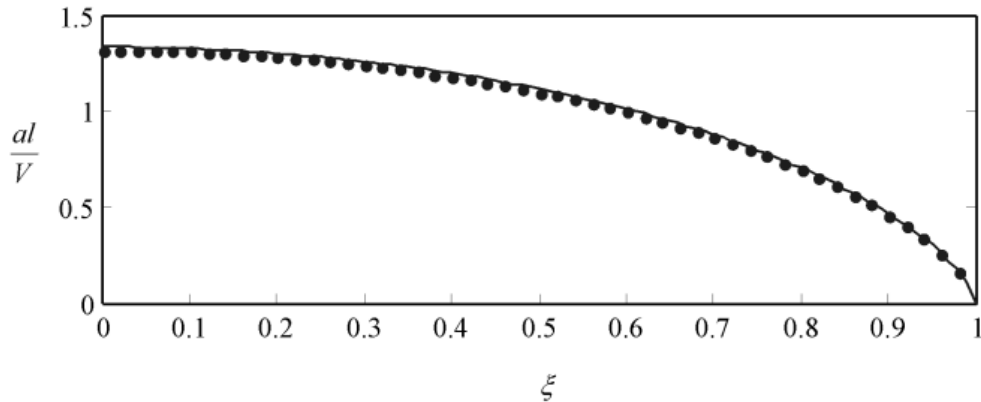


Figure 5.2: The strongest clamped-free column based on the analytical solution (-), and the finite difference approximation of order $n = 50$ (.)

Using a model order of $n = 50$ for the finite difference approximation of the second order differential equation, the eigenvalue obtained is

$$\lambda = 3.166 \left(\frac{V}{l} \right)^2 \quad (5.31)$$

and the exact value is $\lambda = \frac{1}{3}\pi^2 \left(\frac{V}{l}\right)^2 \approx 3.2898 \left(\frac{V}{l}\right)^2$. The buckling load corresponding to this eigenvalue in (5.31) is

$$P = 0.2519 \frac{EV^2}{l^4} \quad (5.32)$$

whereas the exact value of the buckling load is $P = 0.2618 \frac{EV^2}{l^4}$.

5.3.2 Strongest Pinned-Pinned Column

The boundary conditions required for the second order differential equation in (2.21) for the case of the pinned-pinned column are given in (2.36) and (2.37). The equivalent finite difference expansions of the boundary conditions in (2.36) and (2.37) are

$$u_0 = 0, \quad (5.33)$$

and

$$u_n = 0, \quad (5.34)$$

respectively. When equations (5.33) and (5.34) are used to eliminate u_0 and u_n in (5.5), the first and the last equation in (5.5) disappears. The set of equations that represent (5.5) for the case of the pinned-pinned column is

$$\left(\underbrace{\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}}_{\mathbf{K}} - \lambda \underbrace{\begin{bmatrix} a_1^{-2}/2 & & & & \\ & a_2^{-2} & & & \\ & & \ddots & & \\ & & & a_{n-2}^{-2} & \\ & & & & a_{n-1}^{-2} \end{bmatrix}}_{\mathbf{M}} \right) \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix}}_{\mathbf{u}} = \mathbf{0}. \quad (5.35)$$

The solution of the strongest column is obtained by determining the $2n - 1$ unknowns in \mathbf{a}, \mathbf{u} and λ . The flowchart given in Figure 5.1 is used to evaluate the shape of the strongest column and the solution of which is shown in Figure 5.3.

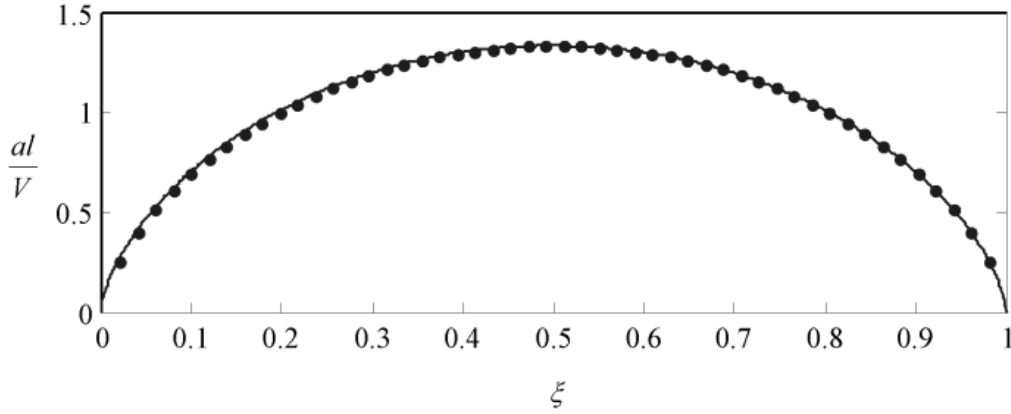


Figure 5.3: The strongest pinned-pinned column based on the analytical solution (-), and the finite difference approximation of order $n = 50$ (.)

Using the model order of $n = 50$ for the finite difference approximation of the second order differential equation, the eigenvalue obtained is

$$\lambda = 13.217 \left(\frac{V}{l} \right)^2 \quad (5.36)$$

and the exact value is $\lambda = \frac{4}{3} \pi^2 \left(\frac{V}{l} \right)^2 \approx 13.1595 \left(\frac{V}{l} \right)^2$. The buckling load corresponding to this eigenvalue in (5.36) is

$$P = 1.0518 \frac{EV^2}{l^4} \quad (5.37)$$

whereas the exact value of the buckling load is $P = 1.04719 \frac{EV^2}{l^4}$.

The process of using the finite difference expansion to convert the governing second order non-dimensional differential equation of column buckling to determine the strongest column has been explained with examples. The error in the solution of the discrete model may be attributed to the inaccuracy of the finite difference approximation. Using a higher order of discretization, or a better finite difference scheme may yield results that are more accurate. However, it should be noted that regardless of the discretization size or finite difference scheme, the solution may approach but will never exactly match with that of the analytical solution.

5.4 Strongest Column with Minimum Area Constraint

Strongest column for the clamped-free and pinned-pinned boundary conditions have been constructed using the method of affine sum in Section 5.3 and the discrete link-spring model in Sections 3.1 and 3.2. It is obvious from the shapes of the optimal columns that the cross-sectional area approaches zero, at the free end for the clamped-free column and at both the ends for the pinned-pinned column, as the model order is increased. Questions regarding physical feasibility arises, when applying loads or placing supports at the ends of these optimal columns with zero cross-sectional areas. Hence, it is necessary to construct optimal columns that have a constraint on their minimum cross-sectional area.

The problem of determining the shape of the strongest column with a minimum area constraint is an inequality constrained optimization problem. The inequality constraint on the cross-sectional area is

$$a_i \geq \beta, \quad i = 1, 2, \dots, n \quad (5.38)$$

where β is the least area that the column could possibly have anywhere along its length.

The Lagrange function associated with this optimization problem with both equality and inequality constraints is

$$L = \sum_{i=1}^n \mathbf{x}^T \mathbf{A}_i \mathbf{x} - \mu \mathbf{x}^T \mathbf{x} + \eta h (a_1 + a_2 + \cdots + a_n) + \sum_i^n \psi_i a_i \quad (5.39)$$

where \mathbf{A}_i is defined in (5.11) and μ , η and ψ_i are Lagrange multipliers. The Lagrange function in (5.39) must be stationary with respect to \mathbf{a} , \mathbf{x} , ψ_i and the feasible solutions must satisfy the volume constraint in (5.6) and the minimum area constraint in (5.38). Variation of the Lagrange function with respect to \mathbf{x} gives the eigenvalue problem in (5.9) and variation with respect to a_i yields

$$\frac{\partial L}{\partial a_i} = \frac{-2}{a_i^{-3}} \mathbf{x}^T \mathbf{A}_i \mathbf{x} + \eta h + \psi_i = 0 \quad (5.40)$$

which can be rewritten as

$$\frac{-2}{h a_i^{-3}} \mathbf{x}^T \mathbf{A}_i \mathbf{x} + \frac{\psi_i}{h} + \eta = 0. \quad (5.41)$$

The Lagrange multiplier η in (5.41) is not of any significance or interest and can be eliminated by subtracting (5.41) for $i = 1$ from (5.41) for $i = 2, 3, \dots, n$ to obtain

$$\frac{2}{a_i^3} \mathbf{x}^T \mathbf{A}_i \mathbf{x} - \frac{2}{a_1^3} \mathbf{x}^T \mathbf{A}_1 \mathbf{x} - \psi_i + \psi_1 = 0, \quad i = 2, 3, \dots, n \quad (5.42)$$

The inequality constraint in (5.38) can be written as

$$\psi_i (a_i - \beta) = 0 \quad i = 1, 2, \dots, n. \quad (5.43)$$

This constrained optimization problem has a total of $3n + 1$ unknown parameters in \mathbf{x} , \mathbf{a} , $\psi_i, i = 1, 2, \dots, n$ and μ . The $n + 1$ unknowns in \mathbf{x} and μ can be evaluated by solving the eigenvalue problem in (5.9) along with the constraint (5.15), using pertinent numerical methods. The remaining $2n$ unknowns in \mathbf{a} and $\psi_i, i = 1, 2, \dots, n$ can be determined by

solving the $2n - 1$ equations in (5.42) and (5.43) along with the volume constraint in (5.6).

The numerical technique described in Section 5.2 can be modified to solve this inequality constrained optimization problem. The matrices \mathbf{A}_i , $i = 1, 2, \dots, n$ that are defined in (5.11) are inputs to the numerical method. To begin with, a starting guess of the vector of areas $\mathbf{a}^{(0)}$ is made and the matrix \mathbf{A} is computed using (5.10). Following this, the largest eigenvalue $\mu^{(0)}$ and its eigenvector $\mathbf{x}^{(0)}$ are evaluated by solving (5.9) using pertinent numerical techniques. Following the determination of $\mathbf{x}^{(0)}$, equations (5.42), (5.43) and (5.6) are used to find the improved vector $\mathbf{a}^{(i)}$. The correction to improve the initial guess of $\mathbf{a}^{(0)}$, is obtained from one iteration of the Newton-Raphson method. Even though a majority of the algorithm is similar to the one described in Section 5.2, the number of unknown parameters this case are higher. Thus the vector \mathbf{f} and the Jacobian matrix \mathbf{J} are different from its counterparts in Section 5.2. The vector \mathbf{f} that is made up of the equation in (5.42), (5.43) and (5.6) is given by

$$\mathbf{f} = \begin{bmatrix} \frac{2\mathbf{x}^T \mathbf{A}_2 \mathbf{x}}{a_2^3} - \frac{2\mathbf{x}^T \mathbf{A}_1 \mathbf{x}}{a_1^3} - \psi_2 + \psi_1 \\ \frac{2\mathbf{x}^T \mathbf{A}_3 \mathbf{x}}{a_3^3} - \frac{2\mathbf{x}^T \mathbf{A}_1 \mathbf{x}}{a_1^3} - \psi_3 + \psi_1 \\ \vdots \\ \frac{2\mathbf{x}^T \mathbf{A}_n \mathbf{x}}{a_n^3} - \frac{2\mathbf{x}^T \mathbf{A}_1 \mathbf{x}}{a_1^3} - \psi_n + \psi_1 \\ h(a_1 + a_2 + \dots + a_n) - 1 \\ \psi_1(a_1 - \beta) \\ \vdots \\ \psi_n(a_n - \beta) \end{bmatrix}_{2n \times 1} \quad (5.44)$$

and the Jacobian matrix is of the form

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{f}_1}{\partial a_1} & \cdots & \frac{\partial \mathbf{f}_1}{\partial a_n} & \frac{\partial \mathbf{f}_1}{\partial \psi_1} & \cdots & \frac{\partial \mathbf{f}_1}{\partial \psi_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial \mathbf{f}_{2n}}{\partial a_1} & \cdots & \frac{\partial \mathbf{f}_{2n}}{\partial a_n} & \frac{\partial \mathbf{f}_{2n}}{\partial \psi_1} & \cdots & \frac{\partial \mathbf{f}_{2n}}{\partial \psi_n} \end{bmatrix}_{2n \times 2n}. \quad (5.45)$$

The Jacobian matrix obtained by evaluating the derivatives in (5.45) is

$$\mathbf{J} = \begin{bmatrix} \frac{6\mathbf{x}^T \mathbf{A}_1 \mathbf{x}}{a_1^4} & \frac{-6\mathbf{x}^T \mathbf{A}_2 \mathbf{x}}{a_2^4} & 0 & 0 & 1 & -1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots & 0 & \ddots & 0 \\ \frac{6\mathbf{x}^T \mathbf{A}_1 \mathbf{x}}{a_1^4} & 0 & 0 & \frac{-6\mathbf{x}^T \mathbf{A}_n \mathbf{x}}{a_n^4} & 1 & 0 & 0 & -1 \\ h & h & \cdots & h & 0 & \cdots & \cdots & 0 \\ \psi_1 & & & & a_1 - \beta & & & \\ & \psi_2 & & & & a_2 - \beta & & \\ & & \ddots & & & & \ddots & \\ & & & \psi_n & & & & a_n - \beta \end{bmatrix}_{2n \times 2n} \quad (5.46)$$

The vector $\boldsymbol{\delta}^{(0)}$ that gives the error in the initial guess of $\mathbf{a}^{(0)}$ and $\boldsymbol{\psi}$ is then obtained by (5.24) and the improved \mathbf{a} is then obtained via

$$a_i^{(1)} = a_i^{(0)} + \delta_i^{(0)} \quad i = 1, 2, \dots, n \quad (5.47)$$

and the improved $\boldsymbol{\psi}$ is obtained via

$$\psi_i^{(1)} = \psi_i^{(0)} + \delta_{n+i}^{(0)} \quad i = 1, 2, \dots, n \quad (5.48)$$

A flowchart that represents the algorithm of the numerical technique explained above is shown in Figure 5.4. Using this numerical algorithm, the shape of the strongest columns with minimum area constraint are determined, for the clamped-free and pinned-pinned

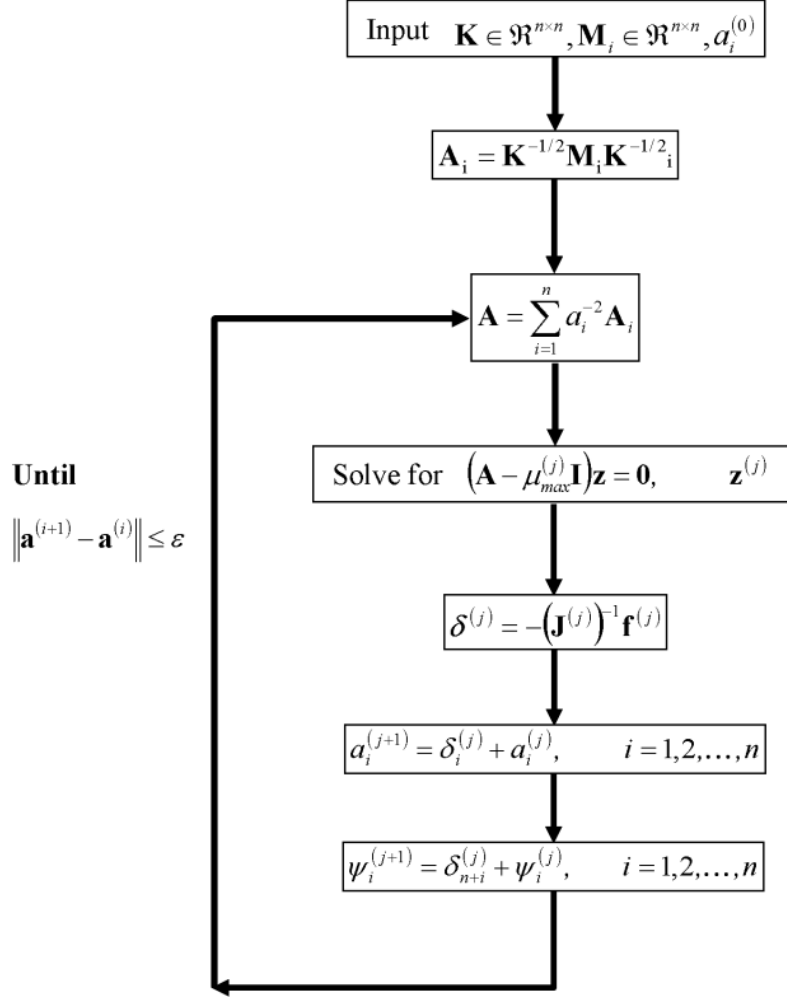


Figure 5.4: Flowchart of the numerical technique to determine the shape of the strongest column with minimum area constraint

boundary conditions. For the case of the clamped-free column, the \mathbf{K} matrix required as an input for the numerical technique is obtained from (5.30). The problem is solved for three cases where the minimum non-dimensional areas are (a) $\frac{al}{V} \geq 0.4$, (b) $\frac{al}{V} \geq 0.6$ and (c) $\frac{al}{V} \geq 0.8$. The results for these three cases is shown along with the strongest column without any minimum area constraint in Figure 5.5.

It can be inferred from Figure 5.5 that as the value of the minimum non-dimensional area $\frac{al}{V}$ increases, a larger length of the column tends to have cross-sectional area equal to this minimum value. Further, points along the length of the column that have cross-sectional area greater than the required minimum area, have areas that are smaller than

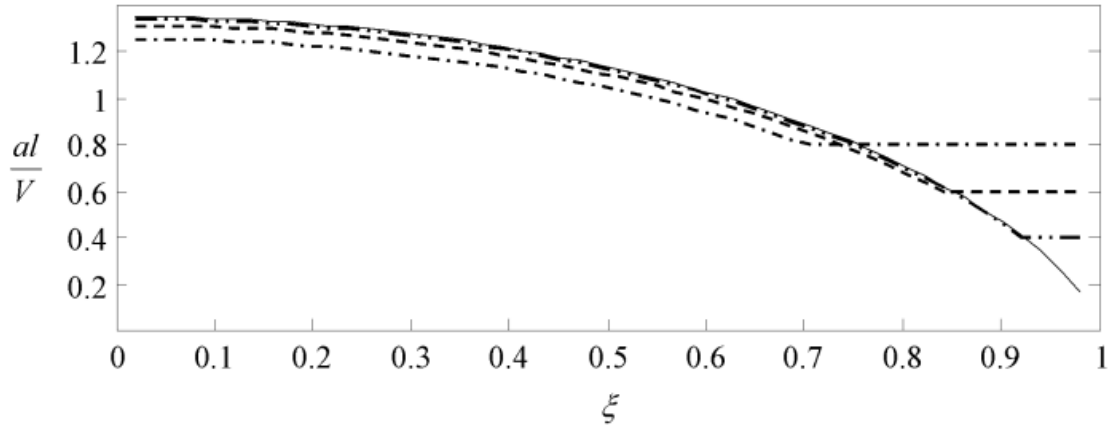


Figure 5.5: Strongest clamped-free column with minimum area constraint obtained using $n = 50$, $-\cdot-\cdot-$ $al/V \geq 0.4$, $--$ $al/V \geq 0.6$ and $- \cdot -$ $al/V \geq 0.8$

that of the optimal column without any minimum area constraint. This phenomenon seems logical because material is added to parts of the column where the area is less than that of the minimum required, but in order to satisfy the volume constraint this added material is removed from rest of the column. To verify that the shape of the column obtained is indeed the strongest, the column is given small random perturbations and the buckling load the perturbed column is determined. These perturbations are made such that the perturbed column does not violate both the minimum area and the volume constraint. The results of the perturbation analysis is shown in Figure 5.6. Clearly it is evident from Figure 5.6 that the unperturbed column i.e case one, has the maximum lowest eigenvalue.

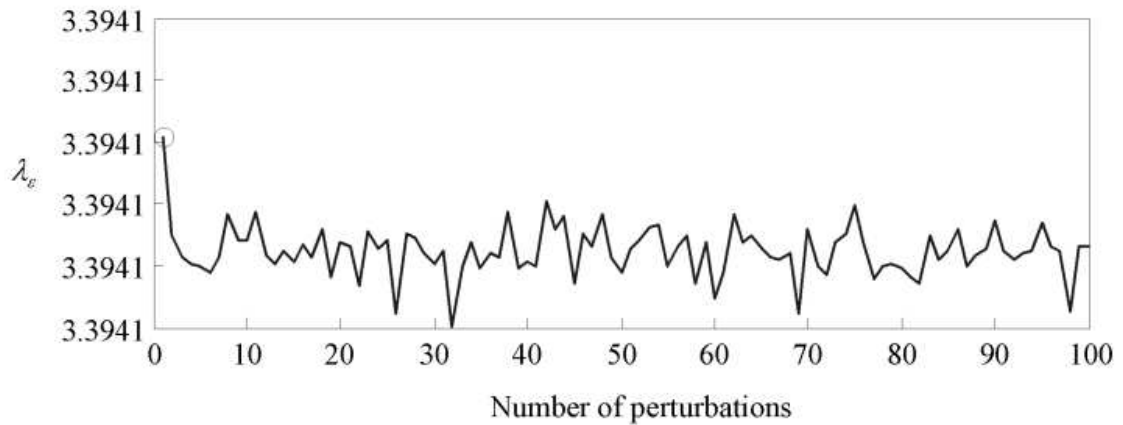


Figure 5.6: Variation of the smallest eigenvalue of the strongest column with the number of perturbations given to its optimal shape

A similar analysis can be performed on the pinned-pinned column as well. The shape of the strongest column with minimum area as an inequality constraint can be determined using the algorithm given in Figure 5.4. The problem is solved for three cases where the constraint on the non-dimensional areas are (a) $\frac{al}{V} \geq 0.4$, (b) $\frac{al}{V} \geq 0.6$ and (c) $\frac{al}{V} \geq 0.8$ and the shapes of the strongest columns obtained are shown in Figure 5.7.

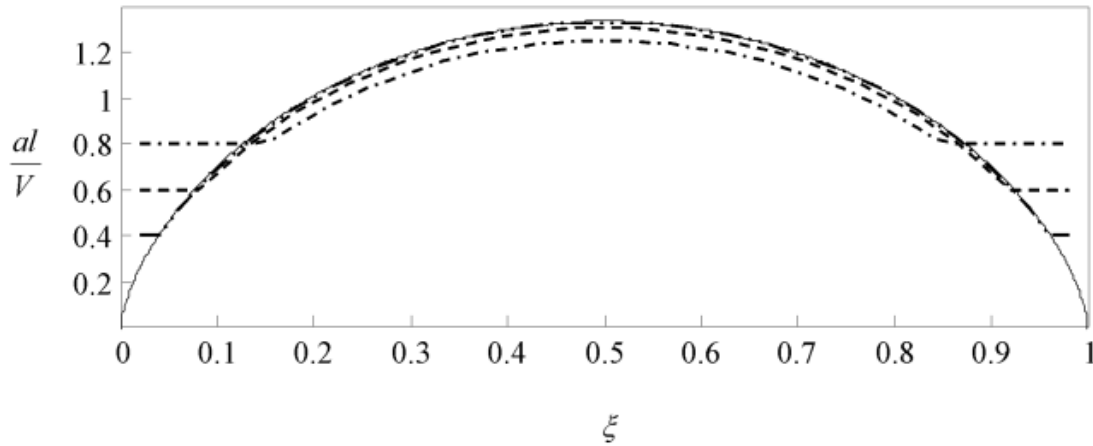


Figure 5.7: Strongest pinned-pinned column with minimum area constraint obtained using $n = 50$, $-\cdot-\cdot-$ $al/V \geq 0.4$, $- -$ $al/V \geq 0.6$ and $- \cdot -$ $al/V \geq 0.8$

Perturbation analysis similar to the one performed on the clamped-free optimal column is also performed on the strongest pinned-pinned column, to verify that the column is indeed the strongest. Results of this perturbation analysis is shown in Figure 5.8. As in the case of the clamped-free column, it is evident that even for the pinned-pinned column that column without any perturbations i.e case one, has the maximum smallest eigenvalue (see Figure 5.8).

5.5 Affine Sum for Column Buckling Using Finite Differences Expansion of the Fourth-Order Differential Equation

Solutions of the strongest column as an affine sum using finite difference expansion of the second-order non-dimensional differential equation has been shown in Sections 5.1, 5.2, and 5.3. This second order differential equations is in terms of the moments along the length of the column. It is not possible to express all classical boundary conditions involving

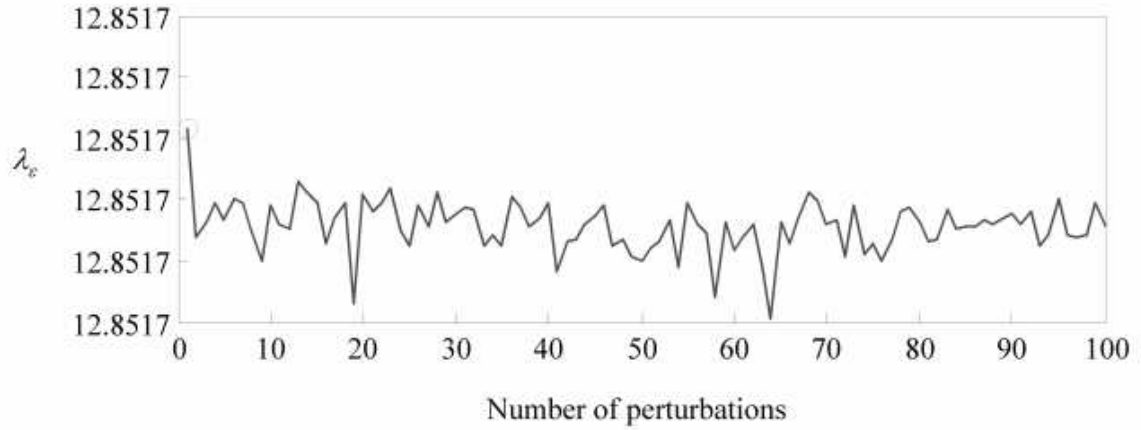


Figure 5.8: Variation of the smallest eigenvalue of the strongest column with the number of perturbations given to its optimal shape

column buckling as a self-adjoint system using this second-order differential equation. The fourth order differential equation associated with column buckling is given in (2.7) and the boundary conditions for various classical boundaries is given in Section 2.1. This fourth-order differential equation is in terms of displacements and thus enables us to express all classical boundaries as a self-adjoint system. The procedure of determining optimal columns using the fourth-order differential equation in (2.7) is given in this section. The fourth-order differential equation in (2.7) is in terms of the moment of inertia I and can be written in terms of the cross-sectional area a , for columns with circular cross-sections, using (2.12) as

$$\frac{d^2}{dx^2} \left(a^2 \frac{d^2 y}{dx^2} \right) + \lambda \frac{d^2 y}{dx^2} = 0 \quad (5.49)$$

where

$$\lambda = \frac{P}{\alpha E} \quad (5.50)$$

The finite difference expansion of the first term in equation (5.49) is

$$(a_i^2 y_i'')'' = \frac{1}{h^4} \begin{pmatrix} a_{i-1}^2 y_{i-2} - 2(a_{i-1}^2 + a_i^2) y_{i-1} + (a_{i-1}^2 + 4a_i^2 + a_{i+1}^2) y_i \\ -2(a_i^2 + a_{i+1}^2) y_{i+1} + a_{i+1}^2 y_{i+2} \end{pmatrix} \quad (5.51)$$

and the expansion of the second term is same as in equation (5.2). The terms y_0 , y_{-1} , y_n and y_{n+1} can be obtained from the boundary conditions. The boundary conditions for a pinned-pinned column are given in (2.11). From (2.11a) and (2.11c) we have $y_0 = 0$ and $y_n = 0$ respectively, and the finite difference expansion of equation (2.11b) gives

$$\frac{y_{-1} - 2y_0 + y_1}{h^2} = 0 \quad (5.52)$$

from which we obtain $y_{-1} = -y_1$. Similarly, the finite difference expansion of (2.11d) gives

$$\frac{y_{n-1} - 2y_n + y_{n+1}}{h^2} = 0 \quad (5.53)$$

from which we obtain $y_{n+1} = -y_{n-1}$. The set of equations in (5.51) can be written in the form of an affine sum of symmetric matrices as

$$\mathbf{K} = \sum_{i=1}^{n-1} a_i^2 \mathbf{K}_i \quad (5.54)$$

where

$$\mathbf{K}_1 = \begin{bmatrix} 4 & -2 & & \\ -2 & 1 & & \\ & & & \\ & & & \end{bmatrix}, \quad (5.55)$$

pre-multiply equation (5.59) with $\mathbf{M}^{-1/2}$ and obtain

$$(\mathbf{M}^{-1/2}\mathbf{K} - \lambda\mathbf{M}^{-1/2}\mathbf{M})\mathbf{y} = \mathbf{0}. \quad (5.60)$$

Then introducing the vector

$$\mathbf{x} = \mathbf{M}^{1/2}\mathbf{y} \quad (5.61)$$

in (5.60) we obtain

$$\left(\underbrace{\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}}_{\mathbf{A}} - \lambda\mathbf{I} \right) \mathbf{x} = \mathbf{0}. \quad (5.62)$$

where the matrix \mathbf{A} can be written in the form of an affine sum as

$$\mathbf{A} = \sum_{i=1}^{n-1} a_i^2 \underbrace{\mathbf{M}^{-1/2}\mathbf{K}_i\mathbf{M}^{-1/2}}_{\mathbf{A}_i}. \quad (5.63)$$

The Lagrange function associated with the optimization problem is then defined as

$$L = \mathbf{x}^T \mathbf{A} \mathbf{x} - \lambda \mathbf{x}^T \mathbf{x} + \eta \mathbf{c}^T \mathbf{a} \quad (5.64)$$

where λ and μ are Lagrange multipliers. At the optimal solution, the Lagrange function is stationary or invariant with respect to both \mathbf{x} and \mathbf{a} . Variation of the Lagrange function with respect to \mathbf{x} gives the eigenvalue problem in equation (5.62) and variation with respect to a_i gives

$$\frac{\partial L}{\partial a_i} = 2a_i \mathbf{x}^T \mathbf{A}_i \mathbf{x} + \eta = 0 \quad i = 1, 2, \dots, n-1 \quad (5.65)$$

The Lagrange multiplier η is not of any interest and can be eliminated by subtracting the first equation of (5.65) i.e. for $i = 1$, from the rest of the equation i.e. for $i = 2, 3, \dots, n-1$ and obtain

$$a_i \mathbf{x}^T \mathbf{A}_i \mathbf{x} - a_1 \mathbf{x}^T \mathbf{A}_1 \mathbf{x} = 0 \quad i = 2, 3, \dots, n-1 \quad (5.66)$$

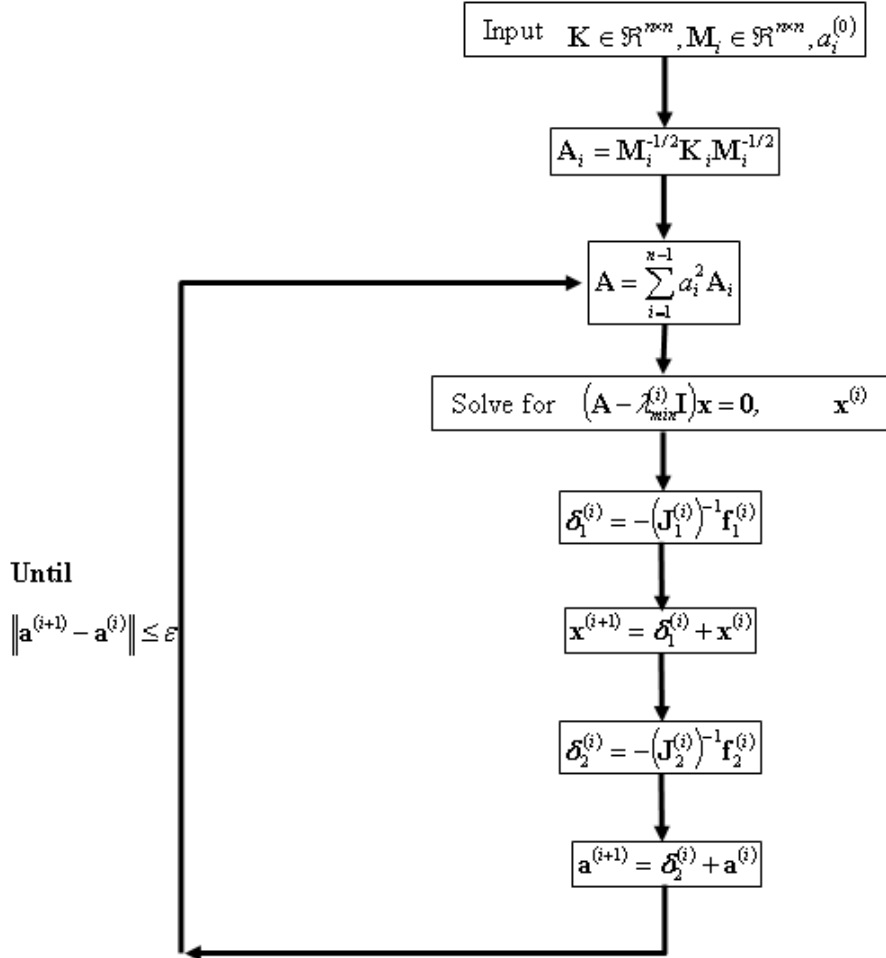


Figure 5.9: Flowchart of the numerical technique to determine the shape of the strongest column as an affine sum using the fourth order differential equation

The $n - 2$ equations in (5.66) and the constraint on \mathbf{x} , i.e. $\mathbf{x}^T \mathbf{x} = 1$ can be used to evaluate the $n - 1$ unknowns in the eigenvector \mathbf{x} . Since the set of equations in (5.66) are nonlinear, Newton's method may be used to evaluate \mathbf{x} and the initial guess required for the iterative process may be obtained by solving the eigenvalue problem in (5.62). The unknown areas a_i and the eigenvalue λ can be evaluated by solving the set of equations in (5.62) and the volume constraint in (3.2). To make the numerical technique of the Newton's

method stable, \mathbf{x} and \mathbf{a} are evaluated using two separate Newton's algorithms as shown in the flowchart given in Figure 5.9. The Jacobian matrix that is required to evaluate \mathbf{x} is given by

$$\mathbf{J}_1 = \begin{bmatrix} 2a_2\mathbf{x}^T \mathbf{A}_2 - 2a_1\mathbf{x}^T \mathbf{A}_1 \\ 2a_3\mathbf{x}^T \mathbf{A}_3 - 2a_1\mathbf{x}^T \mathbf{A}_1 \\ \vdots \\ 2a_{n-1}\mathbf{x}^T \mathbf{A}_{n-1} - 2a_1\mathbf{x}^T \mathbf{A}_1 \\ 2\mathbf{x}^T \end{bmatrix}_{n-1 \times n-1} \quad (5.67)$$

and the Jacobian matrix required for the evaluation of \mathbf{a} is

$$\mathbf{J}_2 = \begin{bmatrix} 2a_1\mathbf{A}_1\mathbf{x} & 2a_2\mathbf{A}_2\mathbf{x} & \cdots & 2a_{n-1}\mathbf{A}_{n-1}\mathbf{x} & -\mathbf{x} \\ h & h & \cdots & h & 0 \end{bmatrix}_{n \times n} \quad (5.68)$$

The solution of the strongest column obtained using the affine sum of the fourth-order differential equation, along with that of the continuous model solution is given in Figure 5.10.

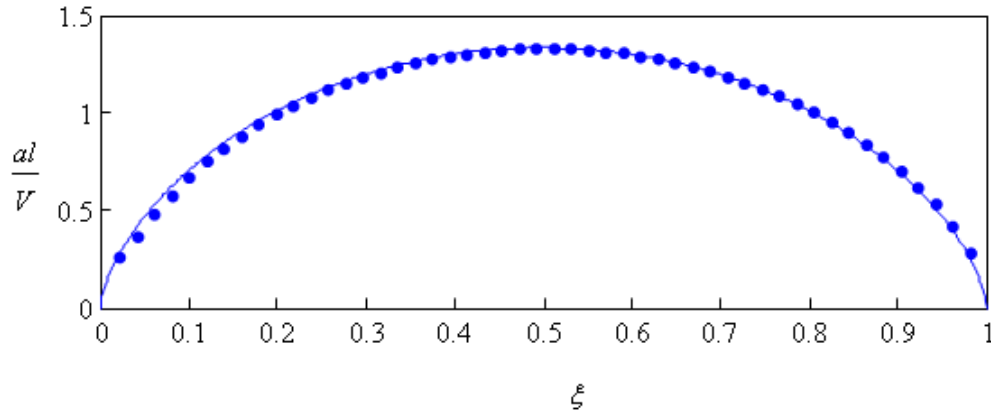


Figure 5.10: Optimal column obtained using affine sum of the fourth-order differential equation with $n = 50$ (.) and the continuous model (-)

5.6 Optimal Pinned-Pinned Column on Elastic Foundation

The solution of the strongest column using the affine sum expansion of the fourth order differential equation may be extended to determine the shape of the optimal pinned-pinned column on an elastic foundation.

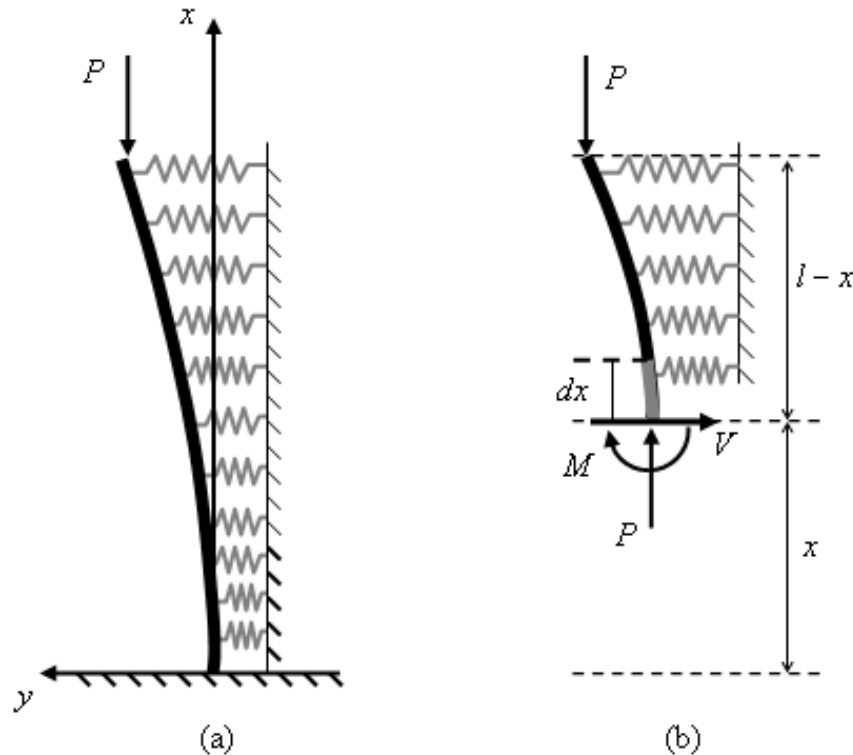


Figure 5.11: (a) Column on elastic foundation in its deformed configuration (b) Free body diagram of one section of the column

Consider a column of length l resting on an elastic foundation of stiffness $k = k(x)$, with a variable cross-sectional area $a = a(x)$, which is deformed under the applied load P , as shown in Figure 5.11(a). The minimum static load P that the column can carry to maintain the statically bent shape as shown in Figure 5.11(a) is known as the critical buckling load of the column. A free-body diagram of the upper side of the column is shown in Figure 5.11(b). Since the column is in static equilibrium, from the free-body diagram of the column it can also be inferred that the shear force at any section of the column is

$$V = - \int_x^l ky(x)dx \quad (5.69)$$

Differentiating equation (5.69) once with respect to x we obtain

$$\frac{dV}{dx} = ky. \quad (5.70)$$

In order to derive the eigenvalue problem associated with column buckling we draw, in view of (5.69), a free-body diagram for the small element of length dx , as shown in Figure 5.12, and upon summing the moments obtain

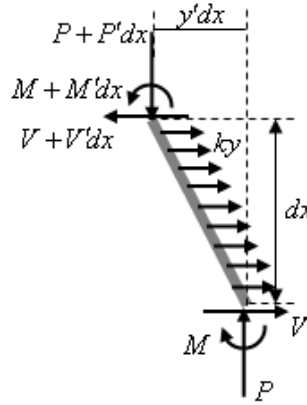


Figure 5.12: Free-body diagram of a small section of the column on elastic foundation

$$-M + \left(M + \frac{dM}{dx}dx \right) + \left(P + \frac{dP}{dx}dx \right) \frac{dy}{dx}dx + \left(V + \frac{dV}{dx}dx \right) dx = 0, \quad (5.71)$$

which simplifies to

$$\frac{dM}{dx}dx + P \frac{dy}{dx}dx + \frac{dy}{dx} \frac{dP}{dx} (dx)^2 + V dx + \frac{dV}{dx} dx^2 = 0 \quad (5.72)$$

Dividing (5.72) by dx and setting $dx \rightarrow 0$ we obtain

$$\frac{dM}{dx} + P \frac{dy}{dx} + V = 0. \quad (5.73)$$

Differentiating (5.73) once with respect to x and substituting for M using (2.5) and for $\frac{dV}{dx}$ using (5.70) we obtain the following fourth-order governing differential equation

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 y}{dx^2} \right) + P \frac{d^2 y}{dx^2} + ky = 0 \quad (5.74)$$

Since the governing differential equation was derived by summing the moments for a small section of the column, the differential equation is independent of boundary conditions. In other words the differential equation in (5.74) can be used to determine the critical buckling load of columns on elastic foundation with any boundary conditions not just the clamped-free configuration shown in Figure 5.11. For the special case of a column with circular cross-section, the differential equation in (5.74) can be written in terms of area as

$$\frac{d^2}{dx^2} \left(a^2 \frac{d^2 y}{dx^2} \right) + \lambda \frac{d^2 y}{dx^2} + \frac{k}{\beta} y = 0 \quad (5.75)$$

where the definition of λ is the same as in (5.50) and

$$\beta = \alpha E = \frac{E}{4\pi} \quad (5.76)$$

Finite differences is used to transform the differential eigenvalue problem in (5.75) to an algebraic eigenvalue problem. The finite difference expansion of the fourth derivative given in (5.51) and that for the second derivative given in (5.2) is applied to the differential equation given in (5.75). The third term in (5.75) would yield a diagonal matrix which is of the form

$$\mathbf{D} = \frac{1}{\beta} \begin{bmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_{n-1} \end{bmatrix} \quad (5.77)$$

The algebraic eigenvalue problem for the case of buckling of columns on elastic foundation

is of the form

$$(\mathbf{K} + \mathbf{D} - \lambda\mathbf{M})\mathbf{y} = \mathbf{0}. \quad (5.78)$$

To convert this generalized eigenvalue problem to the standard form of $(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$, we pre-multiply equation (5.59) with $\mathbf{M}^{-1/2}$ and obtain

$$(\mathbf{M}^{-1/2}\mathbf{K} + \mathbf{M}^{-1/2}\mathbf{D} - \lambda\mathbf{M}^{-1/2}\mathbf{M})\mathbf{y} = \mathbf{0}. \quad (5.79)$$

Following this, using the definition of \mathbf{x} given in (5.61) we obtain

$$\left(\underbrace{\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}}_{\mathbf{A}} + \underbrace{\mathbf{M}^{-1/2}\mathbf{D}\mathbf{M}^{-1/2}}_{\mathbf{B}} - \lambda\mathbf{I} \right) \mathbf{x} = \mathbf{0}, \quad (5.80)$$

where the matrix \mathbf{A} can be written in the form of an affine sum that is identical to (5.63). To determine the shape of the strongest column on an elastic foundation with pinned-pinned boundary conditions, the optimization procedure given in Section 5.5 may be used. Since, the matrix \mathbf{B} in (5.80) is independent of the cross-sectional area, variation of the Lagrange function with respect to a_i is same as in (5.65). Variation of the Lagrange function with respect to \mathbf{x} yields the eigenvalue problem given in (5.80). The numerical method that is to be employed to obtain the strongest column on elastic foundation is similar to the one described in Section 5.5, but with a few minor modifications. The flowchart that encompasses these minor modifications is given in Figure 5.13. The Jacobian matrices required for the modified Newton's method are given in (5.67) and (5.68).

The solution of the strongest column on an elastic foundation using the affine sum of the fourth-order differential equation for a foundation stiffness $\frac{k}{E} = \pi^3$ and $\frac{k}{E} = 0$ is shown in Figure 5.14. The value of the stiffness $\frac{k}{E} = \pi^3$ is the limiting value of the stiffness for which the solution of is unimodal. This limiting value for the elastic foundation stiffness $\frac{k}{E} = \pi^3$ is identical to the case of a uniform circular column of same length and volume on

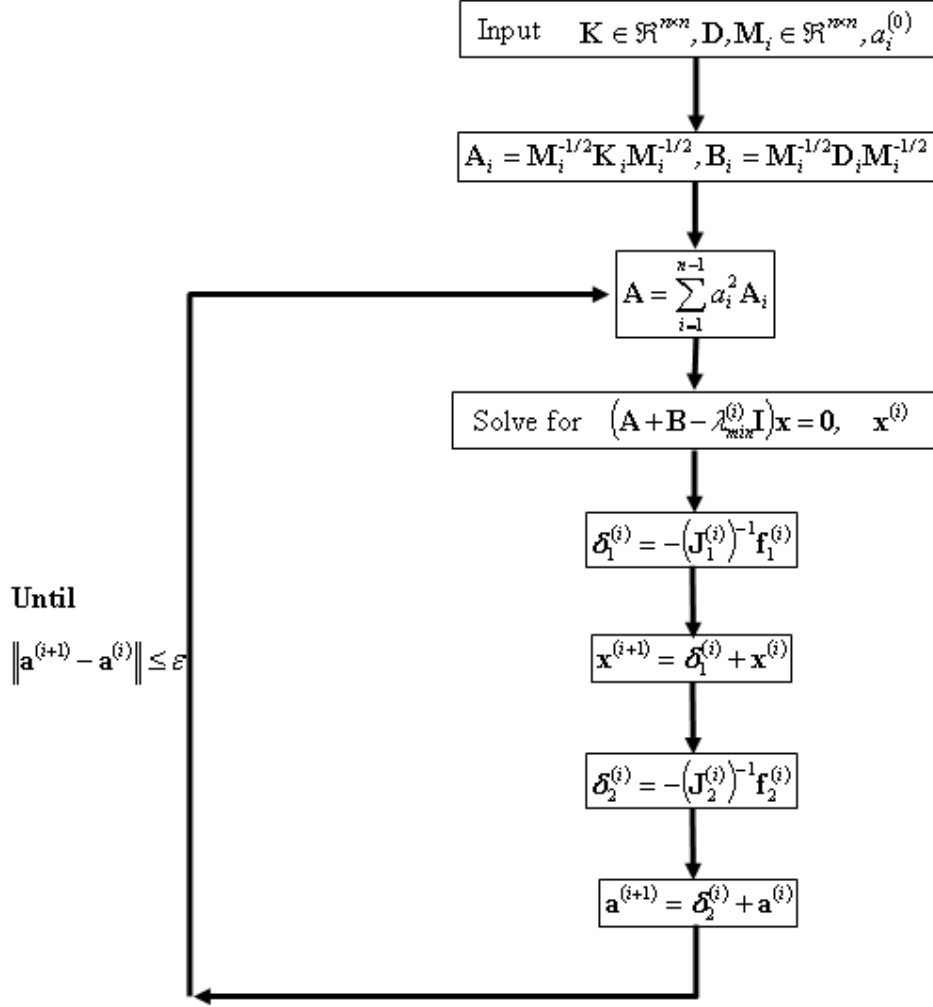


Figure 5.13: Flowchart of the numerical technique to determine the shape of the strongest column on an elastic foundation as an affine sum using the fourth order differential equation a uniform elastic foundation. For elastic foundation stiffness $\frac{k}{E} \geq \pi^3$ a bimodal analysis, which is out of the scope of this research, is to be performed to obtain the shape of the optimal column. The variation of the non-dimensional buckling load as function of the non-dimensional stiffness is shown in Figure 5.15.

The procedure for obtaining optimal eigenvalues of an affine sum was extended to solve the more complicated problem of determining the shape of the strongest columns. The

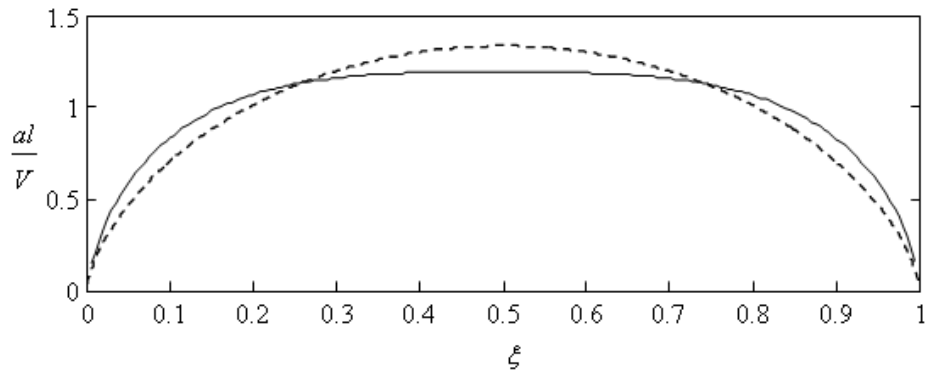


Figure 5.14: Optimal pinned-pinned column on elastic foundation of stiffness $\frac{k}{E} = \pi^3$ (-) and $\frac{k}{E} = 0$ (- -)

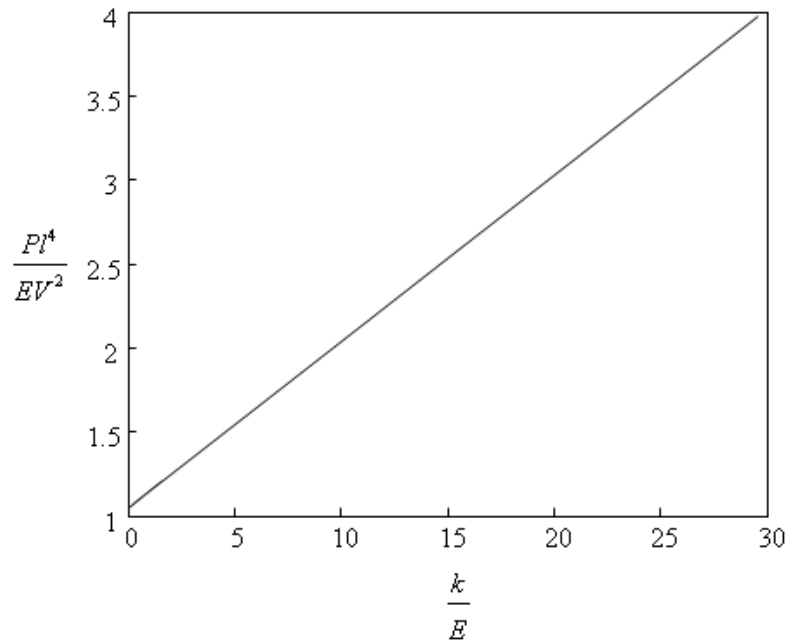


Figure 5.15: The buckling load of a pinned-pinned column on an elastic foundation as a function of the stiffness

shape of the strongest column with a minimum area constraint, was obtained by applying finite difference expansion to the second-order governing differential equation. The shape of the strongest pinned-pinned column on an elastic foundation, with uniform stiffness, was obtained by applying finite differences to the fourth-order governing differential equation. It was observed that the solution becomes bimodal beyond a certain foundation stiffness.

Chapter 6

Concluding Remarks and Future Work

Vibration and buckling analysis of real structures are represented by mathematical models that are classified as differential eigenvalue problems, as natural frequencies and buckling loads are essentially the eigenvalues of the system. Eigenvalue problems are of immense interest and play a pivotal role not only in many fields of engineering but also in pure and applied mathematics. Eigenvalue problems have numerous applications in the mechanical engineering field ranging from the fundamental design application of calculating principle stresses to more complicated applications such as stability analysis and vibration and control. Due to the dependence of structural stability on the buckling load, it is important for engineers to design structures such that buckling load is higher. Several structures such as rods and beams are used to mount or support machines that apply periodic loads. If the frequency of the applied periodic load is close to the natural frequency of the structure, resonance is induced in the structure which eventually leads to failure. This warrants the need for engineers to design support structures whose fundamental natural frequency is greater than the frequency of excitation. Optimization of these structures becomes important either when the amount of material that could be used to build these structures is limited or when the weight of the structure is a limiting factor.

Optimization of structural properties to obtain extremum eigenvalues using the differential eigenvalue problem is tedious because it involves the application of complicated principles of calculus of variations. The problem in the continuous domain can be brought to a discrete domain by using discrete models such as finite element and finite differences that approximate the real structures. This circumvents the complicated variational analysis of the continuous system and involves optimization of algebraic equations in the discrete domain. This dissertation focused on determining optimal structural properties that yield

extreme eigenvalues based on unimodal optimization in discrete domain

The initial portion of the dissertation (chapter 2) provided an extensive literature review on buckling analysis of optimal columns in the continuous domain. Following the work of Keller (1960) and Tadjbakhsh and Keller (1962), an in depth variational analysis was performed for the non-dimensional second order differential equation and unimodal solutions of the strongest column were obtained for various classical boundary conditions. To obtain solutions for non-classical boundaries such as spring supports a discrete link-spring model was developed in Chapter 3 and the conclusions are

- The algebraic equations required for the optimization were derived using both Newton's method and the concept of minimization of potential energy.
- A recursive solution for the strongest clamped-free column was obtained by appropriate scaling of the volume constraint.
- The solution of the strongest clamped-free and pinned-pinned columns agrees with that of the continuous model solution obtained by Tadjbakhsh and Keller (1962).
- It was shown that the pinned-spring supported column has infinite optimal solutions if the support spring at the top of the column fails.
- For the case where the support spring does not fail, a recursive solution of the optimal pinned-pinned column that agrees with that of the continuous model solution of Keller (1960), was obtained.
- The method to obtain the shape of the strongest pinned-spring supported column for the case where the support spring fails was arrived at considering the continuous model of the column.
- The clamped-spring supported column was analyzed to obtain the transition of the shape of the optimal column from clamped-free to the pinned-pinned configurations by increasing the support spring stiffness.

- Analysis of the two degrees of freedom clamped-spring supported column yields the shape of the strongest column as solutions of a polynomial expression. The true and various auxiliary solutions were analyzed and variation of the eigenvalues with respect to the support spring stiffness was obtained.

Optimization procedures to determine extremum eigenvalues of generic algebraic eigenvalue problem that can be expressed as a linear affine, sum were developed in Chapter 4. Beginning with simple numerical examples, graphical method to determine optimal parameters that produce extreme eigenvalues is described. The conclusions of the analysis of higher order systems are:

- The optimization procedure involved the setting up the Lagrange function using the Courant-Fischer minimax theorem for symmetric matrices. The optimal solution is then arrived at by solving a set of quadratic equations.
- Numerical and parametric examples that yield the set of quadratic equations were solving with the aid of the eliminant. It was inferred from the examples that obtaining closed solutions using the eliminant becomes very tedious or sometimes even impossible as the size of the matrices increases.
- A modified Newton-Raphson's method that circumvents the eliminant was developed. This new numerical algorithm not only aids in solving the set of quadratic equations but also provides a means to converge either to the smallest or largest eigenvalue.
- Numerical examples were solved to obtain extremum eigenvalues, using the modified Newton's method. Perturbation analysis was performed on the solution to validate the same.
- The optimization technique and the numerical algorithm was then extended to determine the optimal mass distribution in vibrating spring-mass system with arbitrary spring arrangements.

- Solutions for optimal mass distribution in vibrating spring-mass system was obtained for various spring arrangements.

The Lagrange problem of determining the shape of the strongest column against buckling can also be setup as an affine sum in squares of the unknown areas. However, since the volume constraint which is a function of the linear sum of the unknown areas, this problem is more complicated than that of the vibrating spring-mass system. The details of the optimization procedure using finite difference schemes on the second order and fourth order governing differential equations can be summarized as follows:

- The generalized algebraic eigenvalue problem associated with column buckling was setup using finite difference expansion of both the second-order non-dimensional differential equation which is in terms of moments and the fourth-order differential equation in terms of displacements.
- Numerical techniques based on modifying the Newton-Raphson's method was developed to solve the set of nonlinear optimality equations.
- Shapes of the strongest pinned-pinned and clamped-free columns were obtained and these shapes agree with the continuous model solutions given by Keller (1960) and Tadjbakhsh and Keller (1962).
- As an extension of the affine sum model based the second-order differential equation, the optimization problem was solved with minimum area constraint. The shapes of the optimal pinned-pinned and clamped-free column with various minimum area constraint was solved.
- The governing differential associated with buckling of column on an elastic foundation was derived. For a given elastic foundation stiffness the optimization problem was formulated as an affine sum using finite difference expansion of the fourth order differential equations.

- The strongest pinned-pinned column on elastic foundation was obtained, and the variation of buckling load as a function of foundation stiffness was analyzed until the point beyond which the solution does not remain unimodal.

The fundamental problem of determining physical parameters that yield extremum eigenvalue has been studied in detail in this dissertation. The Lagrange problem of finding the shape of the strongest column was solved using the link-spring model, the affine sum expansion of the second and fourth order differential equations. The affine sum method was extended to find the shape of the strongest column on elastic foundation. As an offshoot of the affine sum optimization, a method to obtain optimal spring-mass systems was developed.

As future research challenges, fundamental problems such as determining the shape of rods and beams that have maximum fundamental natural frequency can be addressed. Since the optimization procedure of affine sums is based on fundamental principles of calculus, the procedure can be extended to other discrete models such as the finite element model in the future. The domain of the finite element model enables one to easily extend the optimization principles to higher dimensional structures such as plates and frames. The discrete model optimization principles developed may also have applications in optimal control.

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