

ERROR ESTIMATES FOR STABILIZED APPROXIMATION  
METHODS FOR SEMIGROUPS

A Dissertation

Submitted to the Graduate Faculty of the  
Louisiana State University and  
Agricultural and Mechanical College  
in partial fulfillment of the  
requirements for the degree of  
Doctor of Philosophy

in

The Department of Mathematics

by

Sarah Campbell McAllister

A.S., Ulster County Community College, 1999

B.A., Empire State College, 2000

M.S., Louisiana State University, 2001

August 2005

©Copyright 2005  
Sarah Campbell McAllister  
All rights reserved

# Acknowledgments

This dissertation would not be possible without several contributions. It is a pleasure to thank my advisor, Professor Frank Neubrandner for suggesting this topic and for his guidance and encouragement. It is also a pleasure to thank my graduate committee: Professors R.A. Litherland, W. Metcalf, R. Perlis, S. Shipman and P. Wolenski. It is a pleasure also to thank Soula O'Bannon, Anthony Picado and Jeff Sheldon for providing me with a pleasant working environment.

I extend the most sincere gratitude to a few distinguished teachers whose tutelage I enjoyed during my primary school and undergraduate years. A very special thanks to Laura Beyer, who was the first to show me a glimpse of mathematics beyond arithmetic. A very special thanks also to Jackie McCracken and Ed Fritz of Highland Elementary, whose classrooms were always entertaining and enlightening. Special thanks to Tom Davis of SUNY-Ulster for helping to keep the wonder alive and for the exciting discussions about string theory. It is a pleasure to thank Ed Peifer of SUNY-Ulster for being an excellent calculus teacher. Very special thanks to Dr. Salvatore Anastasio of SUNY-New Paltz and Gail Young for mentoring and preparing me for doctoral study in mathematics. It is a pleasure to thank Dr. Mick Adams of SUNY-New Paltz for making my foray into graduate study a delightful and challenging one. It is also a pleasure to thank my undergraduate mentor, Steve Lewis of Empire State College.

For their friendship, encouragement and moral support, I would like to thank all of the following people: Dr. Robin Blankenship, Michelle Cranford, M. Y. Farooq, Valerie Jordan Fayad, Kaye B. Goodrich, Javed Iqbal, Dr. Jinko Kanno, Molly McCoy, Vicki Morgan, Tawanna Richard Ott, Hester and Tom Sofranko, Dilek

and Ibrahim Suslu, Dr. Sami Ullah and Dr. Shirley Yap. I extend sincere thanks to my parents, Florence Campbell McAllister and Michael Ford McAllister for supporting my nascent academic career. Special thanks to my sister, Blythe B. Fletcher, my nephew Joseph R. Fletcher and my niece, Kayla Jasmine Liscombe: I love you all and your friendship has been invaluable. Finally, I acknowledge the One and Only Sustainer of the universe for giving me life, the supportive people in my life, the talent, wisdom and opportunity to achieve my dreams. This dissertation is dedicated to the memory of my maternal grandparents, Rua Priscilla Houston Campbell (1907-1997) and Dr. Douglas Whitney Campbell (1906-1985). Although my time with them was brief, they taught me much and inspire me to this day.

# Table of Contents

<b>Acknowledgments</b> .....	<b>iii</b>
<b>Abstract</b> .....	<b>vi</b>
<b>Introduction</b> .....	<b>1</b>
<b>Chapter 1. Preliminaries</b> .....	<b>11</b>
1.1 Time-Discretization . . . . .	11
1.2 Stabilization Methods . . . . .	21
<b>Chapter 2. Stabilized Approximation Methods</b> .....	<b>24</b>
2.1 Error Estimates for Arbitrary Initial Data . . . . .	25
2.2 Higher Order Estimates for Smooth Initial Data . . . . .	30
2.3 Examples . . . . .	41
2.4 Abstract Stabilization Results . . . . .	50
<b>Chapter 3. Semigroup Approximation via Numerical Integration</b> ..	<b>66</b>
3.1 Quadrature Methods for Vector-Valued Functions . . . . .	67
3.2 Approximation of Semigroups Generated by Fractional Powers of a Closed Operator . . . . .	72
3.3 Error Estimates for the Approximation of Semigroups Generated by the Square Root of a Closed Operator . . . . .	76
<b>References</b> .....	<b>84</b>
<b>Appendix A. Brownian Motion: Constructing the Wiener Space</b> ..	<b>87</b>
A.1 A Short Biography of N. Wiener . . . . .	90
A.2 The Wiener Measure . . . . .	95
A.3 The Wiener Theorem . . . . .	98
<b>Appendix B. The Brownian Motion Process</b> .....	<b>108</b>
B.1 Brownian Motion with Zero Drift . . . . .	108
B.2 Brownian Motion with Positive Drift . . . . .	114
<b>Vita</b> .....	<b>115</b>

# Abstract

In this work we analyze error estimates for rational approximation methods, and their stabilizations, for strongly continuous semigroups. Chapter 1 consists of a brief survey of time discretization methods for semigroups. In Chapter 2, we demonstrate a new method for obtaining convergent approximations in the absence of stability for strongly continuous semigroups with arbitrary initial data. In Section 2.2, we state the stabilization result in more general form and show that this method can be used to improve known error estimates by a magnitude of up to one half for smooth initial data. In Section 2.3, we give concrete examples of some of these stabilizers. Section 2.4 concerns abstract stabilization results, including stabilized Trotter-Kato and Lax-Chernoff theorems. In Chapter 3, we use numerical quadrature formulas for Banach space valued functions in order to approximate semigroups that can be represented via the Hille-Phillips functional calculus. In particular, we find error estimates for our approximation method for the semigroup generated by the square root of a semigroup generator.

# Introduction

This dissertation concerns the error estimates associated with approximation methods for

- (i) strongly continuous semigroups  $T(t)$  (see Chapter 2) and
- (ii) operators of the form  $f(A) = \int_0^\infty T(t) d\alpha(t)$  (see Chapter 3),

where  $A$  generates the strongly continuous semigroup  $T(t)$  and  $f$  is an analytic function that has Laplace-Stieltjes representation  $f(\lambda) = \int_0^\infty e^{\lambda t} d\alpha(t)$  for some normalized function  $\alpha$  of bounded variation on  $[0, \infty)$ .

Strongly continuous semigroups  $T(t) = e^{tA}$  and their generators  $A$  play a pivotal role in the study of the abstract Cauchy problem

$$\begin{aligned} u'(t) &= Au(t) & (0.0.1) \\ u(0) &= x \in D(A), \end{aligned}$$

where  $t \geq 0$  and  $A$  is a closed linear operator with domain  $D(A)$  and range in a complex Banach space  $X$ .

A family  $(T(t))_{t \geq 0}$  of bounded linear operators on a Banach space  $X$  is called a *semigroup* on  $X$  if it satisfies the functional equation

$$T(t+s) = T(t)T(s) \text{ and } T(0) = I \quad (0.0.2)$$

for every  $t, s \geq 0$ . The semigroup property (0.0.2) was isolated by Cauchy in the one dimensional case; in particular, he posed the problem to characterize all functions  $T(\cdot) : [0, \infty) \rightarrow \mathbb{C}$  that satisfy the semigroup property (0.0.2). For more details on Cauchy's functional equation, see section I.1 of [EN].

Let  $\mathcal{L}(X)$  denote the space of all bounded linear operators on  $X$ . We say that a semigroup on  $X$  is *uniformly continuous* if the map  $t \mapsto T(t) \in \mathcal{L}(X)$  is continuous with respect to the uniform operator topology on  $\mathcal{L}(X)$ . It is well known that every uniformly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $X$  is of the form  $T(t) = e^{tA}$  for some bounded linear operator  $A \in \mathcal{L}(X)$ . A semigroup on  $X$  is called *strongly continuous* if the orbit maps  $u_x : [0, \infty) \rightarrow X, t \mapsto T(t)x$  are continuous for every  $x \in X$ . In other words,  $t \mapsto T(t)$  is continuous as a function  $[0, \infty) \rightarrow \mathcal{L}(X)$  when  $\mathcal{L}(X)$  is endowed with the strong operator topology. For every strongly continuous semigroup  $(T(t))_{t \geq 0}$ , there exist constants  $\omega \in \mathbb{R}$  and  $M \geq 1$  such that  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . We say that such a semigroup is *bounded* if we can take  $\omega = 0$ . We say that such a semigroup is *contractive* if we can take  $\omega = 0$  and  $M = 1$ . We define the *exponential growth bound* of the semigroup  $T$  by  $\omega = \omega(T) := \omega(\|T\|) = \inf\{\omega \in \mathbb{R} : \sup_{t \geq 0} \|e^{-\omega t} T(t)\| < \infty\}$ . By the uniform boundedness principle,  $\omega(T) = \sup\{\omega(\xi_x) : x \in X\}$ .

The *generator*  $(A, D(A))$  of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $X$  is the operator  $Ax := \lim_{h \searrow 0} \frac{1}{h}(T(h)x - x)$  defined for every  $x$  in its domain  $D(A) := \{x \in X : t \mapsto T(t)x \text{ is differentiable at } t = 0\}$ . If  $A$  generates a strongly continuous semigroup  $T(t)$  with  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ , then  $R(\lambda, A) := (\lambda I - A)^{-1}$  exists for all  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > \omega$  and is given by  $R(\lambda, A) = \hat{T}(\lambda) = \int_0^\infty e^{-\lambda t} T(t)x dt$ , for each  $x \in X$ . In particular, if  $A$  generates a strongly continuous semigroup, then the resolvent set  $\rho(A) := \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(X)\}$  contains a half-plane  $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > \omega\}$ .

In the finite dimensional setting, we have the well known approximant  $e^{tA} = \lim_{n \rightarrow \infty} (1 + \frac{t}{n}A)^n$ , known as the Forward Euler method. The Backward Euler method  $e^{tA} = \lim_{n \rightarrow \infty} (1 - \frac{t}{n}A)^{-n} = \lim_{n \rightarrow \infty} ([\frac{n}{t}(\frac{n}{t}I - A)]^{-n})$  is of great practical use in the infinite dimensional Banach space setting where, in many applications,

$A$  is an unbounded operator with bounded inverses  $(\lambda I - A)^{-1}$ . We may interpret the above approximant by making use of the resolvent operators  $\lambda \mapsto R(\lambda, A) := (\lambda - A)^{-1}$ , where

$$\lambda \in \rho(A) := \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is bijective}\}.$$

If  $e^{tA}x = \lim_{n \rightarrow \infty} [\frac{n}{t}R(\frac{n}{t}, A)]^n x$  exists for all  $x \in X$ , then we define

$$e^{tA}x := \lim_{n \rightarrow \infty} [\frac{n}{t}R(\frac{n}{t}, A)]^n x. \quad (0.0.3)$$

This formula is fundamental to the celebrated Hille-Yosida Generation Theorem, which characterizes those operators  $A$  such that the limit in (0.0.3) exists. Let  $(A, D(A))$  be a linear operator on a Banach space  $X$ . Then the following statements are equivalent:

- (a)  $(A, D(A))$  generates a strongly continuous semigroup satisfying  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ ,
- (b)  $(A, D(A))$  is closed, densely defined, and there exists  $\omega_0 \in \mathbb{R}$  such that  $(\omega_0, \infty) \subset \rho(A)$  and  $\lim_{n \rightarrow \infty} (\frac{n}{t})^n R(\frac{n}{t}, A)x$  exists for every  $x \in X$ ,
- (c)  $(A, D(A))$  is closed, densely defined, and for every  $\lambda \in \mathbb{R}$  with  $\lambda > \omega$  we have that  $\lambda \in \rho(A)$  and  $\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}$  for every  $n \in \mathbb{N}$ .

Let us reconsider the abstract Cauchy problem, or (ACP), given by equation (0.0.1). We say that a function  $u : [0, \infty) \rightarrow X$  is a *classical solution* of (ACP) if  $u$  is continuously differentiable, solves (0.0.1), and  $u(t) \in D(A)$  for all  $t \geq 0$ . If  $(A, D(A))$  generates a strongly continuous semigroup  $T(t)$  on  $X$  and if  $x \in D(A)$ , then  $T(t)x \in D(A)$  and  $\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x$  for every  $t \geq 0$ . Hence we see that  $T(t)x$  is the unique classical solution of (ACP). On the other hand, it is

always the case that

$$\int_0^t T(s)x ds \in D(A)$$

for  $t \geq 0$  and  $x \in X$ . With this in mind, we make the following definition.

A continuous function  $u : [0, \infty) \rightarrow X$  is called a *mild solution* of (ACP) if

$\int_0^t u(s) ds \in D(A)$  for all  $t \geq 0$  and

$$u(t) = A \int_0^t u(s) ds + x.$$

Thus if  $(A, D(A))$  generates a strongly continuous semigroup  $T(t)$  on  $X$  then for each  $x \in X$ , the orbit map  $u_x(t) := T(t)x$  is the unique mild solution of the associated abstract Cauchy problem (ACP). In summary, let  $(A, D(A))$  be a closed operator on a Banach space  $X$ . Then the following are equivalent.

- (a) For each  $x \in X$ , there exists a unique mild solution of (ACP)
- (b)  $(A, D(A))$  generates a strongly continuous semigroup
- (c)  $\rho(A) \neq \emptyset$  and for each  $x \in D(A)$ , there exists a unique classical solution of (ACP).

When these assertions hold, the classical and mild solutions of (ACP) are given by

$$u(t) = T(t)x = \lim_{n \rightarrow \infty} \left(\frac{n}{t} R\left(\frac{n}{t}, A\right)\right)^n x.$$

Many different methods for the approximation of  $T(t) = e^{tA}$  have been developed during the last two centuries. This is because in many applications either the limit is difficult to compute explicitly, or the rate of convergence is quite slow. Let us first consider the finite dimensional setting  $X = \mathbb{C}^n$ . If  $A \in \mathcal{L}(\mathbb{C}^n)$ , we know from standard linear ODE theory that the solution to a linear system of equations

$x'(t) = Ax(t)$  is given by the matrix exponential applied to the initial data  $x$ ; that is,

$$e^{tA}x := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x = U \left( \sum_{n=0}^{\infty} \frac{t^n}{n!} J^n x \right) U^{-1}, \quad (0.0.4)$$

where  $J$  is the Jordan Normal Form of  $A$ ,  $U$  is the matrix whose columns consist of the (generalized) eigenvectors of  $A$  and  $A = UJU^{-1}$ .

If  $A$  is an  $n \times n$  matrix and  $n$  is not too large, then  $(\lambda I - A)^{-1}$  can be computed explicitly without too much difficulty. In this case a computationally more often appealing representation of the matrix exponential is given by the Cauchy formula

$$e^{tA}x := \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} (\lambda I - A)^{-1} x d\lambda,$$

where all the eigenvalues of  $A$  lie on the inside of the simple closed curve  $\gamma$ . In this case, the main theorems of the Dunford Functional Calculus (cf. [DS]) verify that the exponential  $e^{tA}$  given by the formula above satisfies  $e^{0 \cdot A} = I$  and the semigroup property (0.0.2). In the case that  $A$  is an  $n \times n$  matrix, the resolvents are of the form  $R(\lambda, A) = (a_{i,j}(\lambda))_{1 \leq i,j \leq n}$ , where the  $a_{i,j}(\cdot)$  are rational functions given by  $a_{i,j}(\lambda) = \frac{p_{i,j}(\lambda)}{\det(\lambda I - A)}$  for some polynomial  $p_{i,j}(\lambda)$  of degree not more than  $n - 1$ . Therefore, without having to know the generalized eigenvectors as in (0.0.4), one can compute

$$e^{tA} = \left( \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} \frac{p_{i,j}(\lambda)}{\det(\lambda I - A)} d\lambda \right)_{1 \leq i,j \leq n}$$

explicitly using basic techniques from complex analysis as long as the eigenvalues of  $A$  can be computed.

If  $(\lambda I - A)^{-1}$  can be computed efficiently but not the roots of  $\det(\lambda I - A)$  (that is, the eigenvalues of  $A$ ), then numerical approximation methods can be used.

Inspired by the fact that the numerical exponential

$$e^{ta} = \lim_{n \rightarrow \infty} \left(1 + \frac{t}{n}a\right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{t}{n}a\right)^{-n} = \lim_{n \rightarrow \infty} \left(1 + \frac{t}{2n}a\right)^n \left(1 - \frac{t}{2n}a\right)^{-n}$$

may be expressed in many ways as the limit of rational expressions, we investigate rational approximation schemes  $V(t) := r(tA)$ , where  $z \mapsto r(z)$  approximates the numerical exponential function  $z \mapsto e^z$ . In applications, one is interested to know how fast approximation methods converge. Therefore, we investigate approximation methods of order  $m$ , that is we use rational schemes such that  $r(z) - e^z = O(z^{m+1})$  for  $z$  of ‘sufficiently small’ modulus, that is, the first  $m$  terms of the Taylor expansions of  $r(z)$  and  $e^z$  coincide. Examples are  $r(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$  and  $r(z) = \frac{1+\frac{z}{2}}{1-\frac{z}{2}} = 1 + z + \frac{z^2}{2} + \frac{z^3}{8} + \dots$ .

In general, if  $r$  is any continuously differentiable real valued function with  $r(0) = 1$  and  $r'(0) = 1$ , then  $r(z) > 0$  for  $|z|$  sufficiently small. Therefore,

$$\ln(r(\frac{z}{n})^n) = \frac{\ln(r(\frac{z}{n})) - \ln(r(0))}{\frac{1}{n}} \longrightarrow z$$

as  $n \rightarrow \infty$  or, equivalently,  $r(\frac{z}{n})^n \longrightarrow e^z$ . In order to obtain error estimates, we need some additional assumptions on  $r$ . If  $r(z)$  approximates the exponential of order  $m$ , then

$$r(\frac{z}{n})^n - e^z = r(\frac{z}{n})^n - (e^{\frac{z}{n}})^n = (r(\frac{z}{n}) - e^{\frac{z}{n}}) \sum_{j=0}^{n-1} r(\frac{z}{n})^j (e^{\frac{z}{n}})^{n-j}.$$

If we assume in addition that  $|r(z)| \leq 1$  for  $\text{Re}(z) \leq 0$ , then

$$\begin{aligned} |r(\frac{z}{n})^n - e^z| &\leq n|r(\frac{z}{n}) - e^{\frac{z}{n}}| \\ &\leq nM \frac{|z|^{m+1}}{n^{m+1}} \leq M|z|^{m+1} \frac{1}{n^m} \end{aligned}$$

for  $\text{Re}(z) \leq 0$ . The condition that “ $|r(z)| \leq 1$  for  $\text{Re}(z) \leq 0$ ” is called *A-stability* and will play a major role in our discussions. Widely used A-stable numerical methods include the Backward Euler method

$$e^{tA}x = \lim_{n \rightarrow \infty} (I - \frac{t}{n}A)^{-n}x \tag{0.0.5}$$

and the Crank-Nicolson method

$$e^{tA}x = \lim_{n \rightarrow \infty} \left(I + \frac{t}{2n}A\right)^n \left(I - \frac{t}{2n}A\right)^{-n}x. \quad (0.0.6)$$

Suppose that  $A$  generates a strongly continuous semigroup  $T(\cdot)$  on a Banach space  $X$  and let  $\{V(t); t \in [0, \tau]\}$  be an approximation scheme of bounded linear operators with  $V(0) = I$  that satisfies the *consistency* condition

$$\lim_{t \rightarrow 0} \frac{V(t)x - x}{t} = Ax$$

for all  $x$  in a set  $D \subset D(A)$  that is dense in  $X$ . Then it was shown by Lax and Richtmyer in 1956 [LR] (with a stronger consistency condition) and in final form by Chernoff in 1974 [Ch] that the following statements are equivalent:

- (i)  $V(0) = I$  is *stable*; that is, there exist  $\omega, M \geq 0$  such that  $\|V(t)^n\| \leq Me^{\omega nt}$  for each  $n \in \mathbb{N}_0$  and for each  $t \in [0, \tau]$ ,
- (ii)  $\lim_{n \rightarrow \infty} V(\frac{t}{n})^n x = T(t)x$  for all  $t \geq 0$  and  $x \in X$ .

Examples of consistent rational approximation schemes are the Backward Euler scheme  $V_{BE}(t) := (I - tA)^{-1}$  and the Crank-Nicolson scheme

$$V_{CN} := \left(I + \frac{t}{2}A\right)\left(I - \frac{t}{2}A\right)^{-1}.$$

Indeed, for the Backward Euler scheme, we have

$$\begin{aligned} (I - tA)^{-1}x &= x + (I - tA)^{-1}tAx = x + tAx + (I - tA)^{-1}t^2A^2x \\ &= x + tAx + f(tA)x, \end{aligned}$$

with  $\|f(tA)x\| \leq Mt^2\|A^2x\|$  since  $\|(I - tA)^{-1}\| = \|\frac{1}{t}R(\frac{1}{t}, A)\| \leq M$  for each  $t > 0$ . Therefore we obtain consistency, since  $\frac{V_{BE}(t)x - x}{t} - Ax = \frac{f(tA)x}{t} \rightarrow 0$  for all

$x \in D(A^2)$ . Furthermore, for the Crank-Nicolson scheme we may calculate

$$\begin{aligned}
& (I + \frac{t}{2}A)(I - \frac{t}{2}A)^{-1}x \\
&= (I + \frac{t}{2}A)(x + \frac{t}{2}Ax + (I - \frac{t}{2}A)^{-1}\frac{t^2}{4}A^2x) \\
&= (I + \frac{t}{2}A)(x + \frac{t}{2}Ax + \frac{t^2}{4}A^2x + (I - \frac{t}{2}A)^{-1}\frac{t^3}{8}A^3x) \\
&= x + tAx + \frac{t^2}{2}A^2x + \frac{t^3}{8}A^3x + (I + \frac{t}{2}A)(I - \frac{t}{2}A)^{-1}\frac{t^3}{8}A^3x \\
&= x + tAx + \frac{t^2}{2}A^2x + f(tA)x,
\end{aligned}$$

with  $\|f(tA)x\| \leq Mt^3\|A^3x\|$  since

$$\begin{aligned}
I + (I + \frac{t}{2}A)(I - \frac{t}{2}A)^{-1} &= [(I - \frac{t}{2}A) + (I + \frac{t}{2}A)](I - \frac{t}{2}A)^{-1} \\
&= 2I(I - \frac{t}{2}A)^{-1}.
\end{aligned}$$

Therefore, consistency follows from  $\frac{V_{CN}(t)x - x}{t} - Ax = \frac{t}{2}A^2x + \frac{f(tA)x}{t} \rightarrow 0$  for all  $x \in D(A^3)$ .

There are two main shortcomings to the Lax Equivalence Theorem. Firstly, the Lax Equivalence Theorem does not provide any information about the speed of convergence. Secondly, many consistent schemes become unstable when dealing with nonanalytic semigroups. Indeed, it was shown by T. Kato (see page 224 of [CHMM], pages 77-78 of [CLPT], or Theorem 3.1.2 of [Ko]), that the consistent Crank-Nicolson scheme  $V_{CN}(t) := (I + \frac{t}{2}A)(I - \frac{t}{2}A)^{-1}$  is unstable for the shift semigroup  $T(t)f(x) := f(x + t)$  on  $L^1(\mathbb{R})$ . In fact, there exists a constant  $C$  such that  $\|V_{CN}(\frac{t}{n})^n\| \geq C\sqrt{n}$ . By the Lax Equivalence Theorem, there exists  $x \in X$  such that  $V_{CN}(\frac{t}{n})^n x$  does not converge to  $T(t)x$ . One of the main purposes of this dissertation is to show that the Crank-Nicolson scheme, or any other A-stable rational approximation method, can be stabilized so that the stabilized schemes converge to  $T(t)x$  for all  $x \in X$ .

Suppose now that  $r(\cdot)$  is an A-stable rational approximation of the exponential; that is,  $|r(z)| \leq 1$  for  $\operatorname{Re}(z) \leq 0$ . In 1979, P. Brenner and V. Thomée established error estimates of the form

$$\|V(\frac{t}{n})^n x - T(t)x\| \leq CM \frac{1}{n^\gamma} \|A^k x\| \quad (0.0.7)$$

for some  $C, \gamma > 0$ ,  $k \in \mathbb{N}$  and all  $t \in [0, \tau]$ ,  $n \in \mathbb{N}$  and  $x \in D(A^k)$  (see Chapter 1 or [BT]). The estimate (0.0.7) is only useful if  $\|A^k x\|$  is known and not too large. For instance, if  $A$  is a differential operator then the initial data  $x$  should be sufficiently smooth and not wildly oscillatory. Therefore it is highly desirable to obtain sharp error estimates for arbitrary initial data  $x \in X$ . In the 1970's, Luskin and Rannacher [LuR] introduced a stabilization method that achieves this goal in the case that  $A$  generates an analytic semigroup with negative spectral bound. This method was brought into final form by A. Hansbo in 1999 [Ha] (see Section 1.2).

In Chapter 2, we present a method for obtaining convergence results for arbitrary initial data  $x \in X$  in the absence of stability. The method of Chapter 2 does not require any additional assumptions on the semigroup (such as analyticity or spectral restrictions) and it gives estimates for  $x \in D(A^k)$  that show that the speed of convergence given by  $\gamma$  in (0.0.7) can be improved. In Chapter 3, we develop a blueprint for approximation methods for semigroups of the form  $f_t(A)x := \int_0^\infty T(s)x d\alpha_t(s)$ , where  $A$  generates a strongly continuous semigroup  $T(s)$  and  $f_t(\lambda) = \int_0^\infty e^{\lambda s} d\alpha_t(s)$ , for all  $\operatorname{Re}(\lambda) \leq 0$  and for a function  $\alpha_t : [0, \infty) \rightarrow \mathbb{C}$  of bounded total variation on  $[0, \infty)$ . In particular, we approximate the semigroup

$$e^{-t\sqrt{-A}}x = \int_0^\infty e^{sA}x h_t(s) ds,$$

where

$$h_t(s) := \frac{t}{2\sqrt{\pi}} e^{\frac{-t^2}{4s}} s^{-\frac{3}{2}}.$$

Our approach is based on the approximation of the semigroup  $T(s) = e^{sA}$  and circumvents the need for a detailed knowledge of the operator  $f(A)$  and its resolvent.

# Chapter 1. Preliminaries

We saw in the introduction that the function  $t \mapsto T(t)x$  provides the unique mild solution to the abstract Cauchy problem (0.0.1). In the first section of this chapter, we discuss approximation methods for strongly continuous semigroups based on time discretizations of the map  $t \mapsto T(t)x$ . In section two, we discuss the literature on stabilization techniques for time-discretization methods for analytic semigroups.

## 1.1 Time-Discretization

Inspired by the fact that the numerical exponential function may be represented in various ways as the limit of readily computable rational expressions, we consider *rational approximation schemes*  $V(t) := r(tA)$  of the strongly continuous semigroup  $T(t)$ , where we take  $r(\cdot) : \mathbb{C} \rightarrow \mathbb{C}$  to be a rational function such that

$$r(z) - e^z = O(z^{m+1}) \tag{1.1.1}$$

for some  $m \in \mathbb{N}$  and for all  $z$  of ‘sufficiently small’ modulus. In the above situation, we say that  $r(\cdot) : \mathbb{C} \rightarrow \mathbb{C}$  is a *rational approximation of the exponential of approximation order  $m$* . The following lemma is due to Padé and may be found in [Pad].

**Lemma 1.1 (Padé).** *If  $r(\cdot) = \frac{p(\cdot)}{q(\cdot)}$  is a rational approximation of the exponential of approximation order  $m$ , then*

$$m \leq p + q,$$

*where  $p$  and  $q$  denote the degree of  $p(\cdot)$  and the degree of  $q(\cdot)$ , respectively.*

A rational approximation scheme  $r(\cdot)$  of maximal approximation order  $m = p + q$  is called a *Padé approximation*, or an *approximation of Padé-type*. Given  $p$  and

$q$ , there is exactly one rational Padé approximation  $r(\cdot) = \frac{p(\cdot)}{q(\cdot)}$  of the exponential with  $\deg p(\cdot) = p$ , and  $\deg q(\cdot) = q$ , and  $q(0) = 1$  (we must normalize  $q(\cdot)$  in order to achieve uniqueness).

A rational approximation  $r(\cdot)$  of the exponential with the property that  $|r(z)| \leq 1$  for  $\operatorname{Re}(z) \leq 0$  is known as an *A-stable* scheme. Unfortunately, this terminology often results in ambiguous use of the letter A; notice that the above definition of A-stability *does not refer* to the particular semigroup generator  $(A, D(A))$ .

The following result identifies which Padé approximations are A-stable. For a proof, see page 60 of [HW].

**Theorem 1.2.** *If  $r(\cdot)$  is of Padé-type, then  $r(\cdot)$  is A-stable if and only if  $q - 2 \leq p \leq q$ .*

A proof of the following statement can be found in [AS].

**Theorem 1.3.** *A Padé approximation  $r(\cdot)$  has all of its poles in the open right halfplane if and only if  $q - 4 \leq p \leq q$ .*

For example, when  $m = p + q = 1$ , we must have that  $p = 0, q = 1$  and we obtain the Padé approximation

$$r_{BE}(z) = \frac{1}{1 - z},$$

which is also known as the *Backward Euler* approximation of the exponential. When  $m = p + q = 2$ , two cases arise. Either we may have  $p = q = 1$ , or we may have  $p = 0$  while  $q = 2$ . In the case  $p = q = 1$ , we obtain the *Crank-Nicholson* approximation

$$r_{CN}(z) = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}.$$

Both the Backward Euler scheme and the Crank-Nicolson scheme are A-stable approximation schemes of Padé type. However, there are examples of A-stable

approximations which are not of Padé type, and of a Padé approximations which are not A-stable [Fl]. An example of an A-stable rational approximation scheme that is not a Padé approximation is the so-called Calahan scheme of approximation order  $m = 3$  with  $p = q = 2$  given by

$$r(z) = \frac{6 - 2\sqrt{3}z - (1 + \sqrt{3})z^2}{(2 + \sqrt{3})(z - \frac{3+\sqrt{3}}{2+\sqrt{3}})^2}.$$

On the other hand, the Padé approximation

$$r(z) = \frac{1}{1 - z + \frac{1}{2}z^2 - \frac{1}{6}z^3}$$

has order  $m = 3$  with  $p = 0, q = 3$ . This scheme is not A-stable since  $p = q - 3$ . In fact,  $|r(\frac{i}{2})| = |\frac{6}{3-5i}| = |\frac{9+15i}{17}| > 1$ .

In the case that  $(A, D(A))$  generates a bounded strongly continuous semigroup, Padé approximations or A-stable rational approximations of the exponential  $V(t) = r(tA)$  are bounded linear operators. In fact, using the partial fraction decomposition of the rational function  $r(\cdot)$ , we see that  $V(t) = r(tA)$  can be written in the form

$$V(t) = r(tA) = \sum a_i R(b_i, tA)^{n_i}$$

for appropriately chosen complex numbers  $a_i, b_i$  and  $n_i \in \mathbb{N}$ . Hence, these approximations are each bounded linear operators that commute with the resolvent  $R(\lambda, A)$ . Since  $T(t)x = \lim_{n \rightarrow \infty} (\frac{n}{t})^n R(\frac{n}{t}, A)^n x$  for all  $x \in X$  and  $t > 0$ , it follows that these approximations also commute with the semigroup  $T(t)$ .

We recall from the Introduction that an approximation method  $r(t)$  is consistent on a dense set  $D \subset D(A)$  if

$$\lim_{t \rightarrow 0} \frac{V(t)x - x}{t} - Ax = 0$$

for all  $x \in D$ . Since  $\frac{V(t)x - T(t)x}{t} = \frac{V(t)x - x}{t} - \frac{T(t)x - x}{t}$  and  $\lim_{t \rightarrow 0} \frac{T(t)x - x}{t} = Ax$  for all  $x \in D(A)$ , this is equivalent to the statement

$$\lim_{t \rightarrow 0} \left\| \frac{V(t) - T(t)}{t} x \right\| = 0$$

for each  $x \in D$ . It was shown by S. Flory in Theorem 2.3 of [Fl] that all A-stable rational approximation schemes are consistent.

We say that a rational approximation scheme is *stable* on  $[0, \tau]$  if there are nonnegative constants  $\omega$  and  $M$  for which

$$\|V(t)^n\| \leq Me^{\omega nt}$$

for each  $n \in \mathbb{N}_0$  and for each  $t \in [0, \tau]$ . Motivation for the above definitions may be found in the commutative case of the ‘only if’ implication of the following theorem.

**Theorem 1.4 (Lax-Chernoff).** *Let  $X$  be a Banach space and let  $(A, D(A))$  be an operator on  $X$  that generates a strongly continuous semigroup  $T(\cdot)$ . Suppose that  $V(\cdot)$  is a consistent rational approximation scheme with  $V(t) \in \mathcal{L}(X)$  for  $t \in [0, \tau]$ . Then  $V(\cdot)$  is stable if and only if  $V(\frac{t}{n})^n x \rightarrow T(t)x$  for all  $t \geq 0$  and  $x \in X$ .*

In the case that  $V(\cdot)$  commutes with  $T(\cdot)$  we may use the semigroup property and subsequently the binomial theorem to calculate

$$\begin{aligned} V\left(\frac{t}{n}\right)^n x - T(t)x &= V\left(\frac{t}{n}\right)^n x - T\left(\frac{t}{n}\right)^n x \\ &= \frac{t}{n} \sum_{j=0}^{n-1} V\left(\frac{t}{n}\right)^{n-1-j} T\left(\frac{t}{n}\right)^j \left( \frac{V\left(\frac{t}{n}\right) - T\left(\frac{t}{n}\right)}{\frac{t}{n}} \right) x. \end{aligned}$$

Applying norms and applying basic properties of the norm yields

$$\left\| V\left(\frac{t}{n}\right)^n x - T(t)x \right\| \leq \frac{t}{n} \sum_{j=0}^{n-1} \left\| V\left(\frac{t}{n}\right) \right\|^{n-1-j} \left\| T\left(\frac{t}{n}\right) \right\|^j \left\| \frac{V\left(\frac{t}{n}\right) - T\left(\frac{t}{n}\right)}{\frac{t}{n}} \right\|.$$

The stability condition allows us to find for all  $\tau > 0$  a constant  $M_\tau > 0$  such that

$\left\| V\left(\frac{t}{n}\right) \right\|^{n-1-j} \left\| T\left(\frac{t}{n}\right) \right\|^j \leq M$  for all  $t \in [0, \tau]$  and  $n \in \mathbb{N}$ . Thus,

$$\left\| V\left(\frac{t}{n}\right)^n x - T(t)x \right\| \leq tM_\tau \left\| \frac{V\left(\frac{t}{n}\right) - T\left(\frac{t}{n}\right)}{\frac{t}{n}} \right\|$$

and we see that indeed for each  $x$  in the consistency domain  $D$ ,  $\|V(\frac{t}{n})^n x - T(\frac{t}{n})x\|$  must converge to zero as  $n$  approaches infinity. Since  $D$  is dense in  $X$  and since the operators  $V(\frac{t}{n}) - T(\frac{t}{n})$  are uniformly bounded for  $t \in [0, \tau]$  and  $n \in \mathbb{N}$ , it follows that  $V(\frac{t}{n})x - T(\frac{t}{n})x \rightarrow 0$  for all  $x \in X$ .

As important as it is historically, practically and theoretically, the Lax-Chernoff Theorem has two major shortcomings. First of all, it does not provide any error estimates that tell us “how fast”  $V(\frac{t}{n})^n x$  converges towards  $T(t)x$ . Second of all, widely used consistent approximation methods such as the Crank-Nicolson scheme turn out to be unstable if  $A$  generates a nonanalytic semigroup. Many of the results that address this problem are due to Brenner and Thomée [BT], and were based on earlier work by Hersh and Kato [HK]. The results at the end of this section concerning analytic semigroups are due to Larsson, Thomée and Wahlbin [LTW]. The main idea of the Brenner-Thomée and Hersh-Kato papers is to use the Hille-Phillips functional calculus to make the representation  $r(\frac{t}{n}A)^n = \int_0^\infty T(s) d\mu_{t,n}(s)$  for  $r(z) = \int_0^\infty e^{sz} d\mu(s)$ , and to show that  $\mu_{t,n}(s) \rightarrow H_t(s)$  (in an appropriate sense and with precise error estimates), where

$$H_t(s) := \begin{cases} 0 & \text{if } 0 \leq s < t, \\ \frac{1}{2} & \text{if } s = t \\ 1 & \text{if } s > t, \end{cases}$$

is the normalized Heaviside function with jump at time  $t > 0$ . We note that  $r(\frac{t}{n}z)^n = \int_0^\infty e^{sz} d\mu_{t,n}(s)$ , where  $\mu_{t,n}(s)$  is the  $n$ th convolution of  $\mu(\frac{ns}{t})$  with itself. If the semigroup  $T(t)$  is bounded and if  $\mu$  is of total bounded variation, then

$r(\frac{t}{n}A)^n x = \int_0^\infty T(t)x d\mu_{t,n}(s)$  and integration by parts yields

$$\begin{aligned} r(\frac{t}{n}A)^n x - T(t)x &= \int_0^\infty T(s)x d[\mu_{n,t}(s) - H_t(s)] \\ &= \int_0^\infty T(s)x d[\mu_{n,t}(s) - H_t(s)] \\ &= \int_0^\infty [H_t(s) - \mu_{n,t}(s)]T(s)Ax ds. \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \|\mu_{n,t} - H_t\|_1 = 0$  (see [Ko], Theorem 2.3.4), we have that

$$\begin{aligned} r^n(\frac{t}{n}A)x - T(t)x &= \int_0^\infty T(s)x d[\mu_{n,t}(s) - H_t(s)] \\ &= \int_0^\infty [H_t(s) - \mu_{n,t}(s)]T(s)Ax ds \longrightarrow 0 \end{aligned}$$

as  $n \longrightarrow \infty$ .

The Brenner-Thomée Theorem 1.5 below appears in more general form as Theorem 3.2.1 of [Ko] and is a refinement of the results in the ground-breaking paper of Hersh and Kato [HK].

**Theorem 1.5 (Brenner-Thomée).** *Let  $A$  be the generator of a strongly continuous semigroup  $T(\cdot)$  with  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . Let  $r(\cdot)$  be an  $A$ -stable rational approximation of the exponential of approximation order  $m$  and define  $V(t) := r(tA)$ . Then there exist constants  $C, c > 0$  such that*

$$\|V(\frac{t}{n})^n x - T(t)x\| \leq CMte^{c\omega t}(\frac{t}{n})^m \|A^{m+1}x\| \quad (1.1.2)$$

for all  $t \geq 0, n \in \mathbb{N}$ , and  $x \in D(A^{m+1})$ . In general, for  $s = 0, 1, 2, \dots, m+1$  with  $s \neq \frac{m+1}{2}$  there are positive constants  $c$  and  $C$  (depending only on  $r(\cdot)$ ) such that

$$\|V(\frac{t}{n})^n x - T(t)x\| \leq CM e^{c\omega t} t^{s-\beta(s)} (\frac{t}{n})^{\beta(s)} \|A^s x\|$$

for every  $t \geq 0, n \in \mathbb{N}$ , and  $x \in D(A^s)$ , where

$$\beta(s) := s \frac{m}{m+1} + \min(0, \frac{s}{m+1} - \frac{1}{2}) = \begin{cases} s - \frac{1}{2} & \text{if } 0 \leq s < \frac{m+1}{2}, \\ s \frac{m}{m+1} & \text{if } \frac{m+1}{2} < s \leq m+1. \end{cases}$$

Observe that the Brenner-Thomée Theorem does not provide error estimates for  $x \in D(A^s)$  when  $s = \frac{m+1}{2}$ . In particular, Theorem 1.5 does not apply in the case that  $s = m = 1$ . The following theorem fills the gap for the Backwards Euler scheme. A proof may be found in [FNW]. The proof relies on the Riesz-Stieltjes Representation Theorem to guarantee the existence of an isometric isomorphism

$$\text{Lip}_0([0, \infty), X) \longrightarrow \mathcal{L}(L^1([0, \infty)), X)$$

$$F \mapsto T_F,$$

where

$$T_F(g) = \int_0^\infty g(s) dF(s). \quad (1.1.3)$$

The identity (1.1.3) is used to obtain a crucial estimation, which is then applied to a particular Lipschitz continuous function  $F(t) := T(t)x - x$ . This estimation together with the Hille-Yosida theorem yields the result.

**Theorem 1.6 (Flory).** *Let  $A$  be the generator of a strongly continuous semigroup  $T(\cdot)$  with  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . Then, for every  $x \in D(A)$ ,*

$$\|V_{BE}(\frac{t}{n})^n x - T(t)x\| \leq M(M+2) \frac{t}{\sqrt{n}} \|Ax\|.$$

There are two negative messages contained in the Brenner-Thomée estimate

$$\|V(\frac{t}{n})^n x - T(t)x\| \leq CM e^{\omega t} t^{s-\beta(s)} (\frac{t}{n})^{\beta(s)} \|A^s x\|,$$

where

$$\beta(s) := s \frac{m}{m+1} + \min(0, \frac{s}{m+1} - \frac{1}{2}) = \begin{cases} s - \frac{1}{2} & \text{if } 0 \leq s < \frac{m+1}{2}, \\ s \frac{m}{m+1} & \text{if } \frac{m+1}{2} < s \leq m+1. \end{cases}$$

First, for arbitrary initial data  $x \in X$  (that is, if  $s = 0$ ), Theorem 1.5 predicts nonconvergence. In fact, divergence may occur as quickly as the sequence  $n \mapsto \sqrt{n}$

diverges. Secondly, no matter what the order  $m$  is of the scheme we use, Theorem 1.5 only predicts convergence like  $\frac{1}{\sqrt{n}}$  for initial data  $x \in D(A)$ . Indeed,  $\beta(1) = \frac{m}{m+1} + \min(0, \frac{1}{m+1} - \frac{1}{2}) = \frac{m}{m+1} + \frac{1}{m+1} - \frac{1}{2} = \frac{1}{2}$ . The Brenner-Thomée Theorem 1.5 establishes rates of convergence on  $D(A^s)$  for A-stable rational approximation schemes of order  $m$  in terms of the function  $\beta(s)$ . The following table shows the values of  $\beta(s)$  for  $s = 0, 1, 2, 3, 4$  and  $m = 1, 2, 3, 4, 5, 6, 7$ . The symbol  $\star$  indicates that  $\beta(s)$  is undefined for the given value of  $m$ . The value for  $s = 1$  and  $m = 1$  is given by Theorem 1.6 above.

Table 1.1 Values of the function  $\beta(s)$

$x \in D(A^s)$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$
$s = 0$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$s = 1$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$s = 2$	1	$\frac{4}{3}$	$\star$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$s = 3$	$\star$	2	$\frac{9}{4}$	$\frac{12}{5}$	$\star$	$\frac{5}{2}$	$\frac{5}{2}$
$s = 4$	$\star$	$\star$	3	$\frac{16}{5}$	$\frac{20}{6}$	$\frac{24}{7}$	$\star$

Since  $\beta(0) = -\frac{1}{2}$  for all  $m$ , the Brenner-Thomée Theorem predicts that these methods are not stable in general, since  $\|V(\frac{t}{n})^n x - T(t)x\|$  may grow as fast as  $\sqrt{n}$  if  $x \in X$  has no smoothness properties. In fact, it was shown by Kato (see [CHMM], p. 224) that the Crank-Nicolson scheme satisfies

$$\|V_{CN}(\frac{t}{n})^n\| \geq c\sqrt{n}$$

for the shift semigroup  $T(t)f(x) := f(x + t)$  on  $L^1(\mathbb{R})$ . In particular, it follows from the uniform boundedness principle that for all  $0 < \varepsilon < \frac{1}{2}$ , there exists  $c_0 > 0$

and  $x \in X$  such that  $\|V_{CN}(\frac{t}{n})^n x\| \geq c_0 n^{\frac{1}{2}-\varepsilon}$ . Hence there exists  $\tilde{c} > 0$  such that

$$\begin{aligned} \|V_{CN}(\frac{t}{n})^n x - T(t)x\| &\geq \| \|V_{CN}(\frac{t}{n})^n x\| - \|T(t)x\| \| \\ &= c_0 n^{\frac{1}{2}-\varepsilon} - \|T(t)x\| \geq \tilde{c} n^{\frac{1}{2}-\varepsilon} \end{aligned}$$

for large  $n$ . The consistent, A-stable Crank-Nicolson scheme  $V_{CN}(t) := (I + \frac{t}{2}A)(I - \frac{t}{2}A)^{-1}$  is also unstable for the shift semigroup  $T(t)f(x) := f(x+t)$  on  $C_0(\mathbb{R})$ . In fact, there exist constants  $C, c > 0$  such that  $c\sqrt{n} \leq \|V_{CN}(\frac{t}{n})^n - T(t)\| \leq C\sqrt{n}$  (see [Ko], Theorem 3.1.2). Therefore for arbitrary semigroups, A-stable approximation methods are in general not by themselves suitable to approximate mild solutions of (ACP). The following theorem shows that in fact a divergence of order  $\sqrt{n}$  is the worst possible case.

**Theorem 1.7 (Brenner-Thomée).** *Let  $A$  be the generator of a strongly continuous semigroup  $T(\cdot)$  with  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . Let  $r(\cdot)$  be an A-stable rational approximation of the exponential of order  $m$  and define  $V(t) := r(tA)$ . Then, there are positive constants  $C$  and  $c$  such that*

$$\|V(\frac{t}{n})^n\| \leq CM e^{c\omega t} \sqrt{n}$$

for every  $t \geq 0$  and  $n \in \mathbb{N}$ . Particularly, for each  $\tau > 0$ , there is a positive constant  $C_\tau$  such that

$$\|V(\frac{t}{n})^n\| \leq C_\tau \sqrt{n} \tag{1.1.4}$$

for every  $t \in [0, \tau]$  and  $n \in \mathbb{N}$ .

The second negative message contained in the Brenner-Thomée Theorem 1.5, is that without additional assumptions on the semigroup, the rate of convergence on  $D(A)$  is maximally that of  $\frac{1}{\sqrt{n}}$ , no matter what rational approximation scheme one chooses. That is, the smoothness of the initial data sets a barrier for the rate

of convergence that cannot be overcome by any A-stable rational approximation method. To overcome this barrier, one either modifies the schemes appropriately (this is what we do in Chapter 2) or else one considers special classes of semigroups, such as analytic semigroups.

In order to achieve stability for general A-stable rational approximation schemes, and in particular for the Crank-Nicolson scheme, we must turn to the class of analytic semigroups. Analytic semigroups may be defined using the Cauchy integral formula for a sectorial operator. We write  $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \theta\} \setminus \{0\}$ , the sector of angle  $\theta$ . An analytic semigroup is defined for  $z$  in a sector  $\Sigma_\theta \cup \{0\}$ . We make the following formal definition.

A family  $(T(z))_{z \in \Sigma_\theta \cup \{0\}}$  of operators on  $X$  is called an *analytic semigroup* if

- (i)  $T(0) = I$  and  $T(z + z') = T(z)T(z')$  for all  $z, z' \in \Sigma_\theta$ ,
- (ii) the map  $z \mapsto T(z)$  is analytic in  $\Sigma_\theta$ , and
- (iii)  $\lim_{z \rightarrow 0, z \in \Sigma_{\theta'}} T(z)x = x$  for all  $x \in X$  and  $0 < \theta' < \theta$ .

If, in addition,

- (iv)  $\|T(z)\|$  is bounded in  $\Sigma_{\theta'}$  for every  $0 < \theta' < \theta$ ,

then we call  $(T(z))_{z \in \Sigma_\theta \cup \{0\}}$  a *bounded analytic semigroup*.

If  $A$  generates a bounded analytic semigroup, then the following result may be obtained. The following result shows, roughly, that the order of convergence for analytic semigroups can be improved by  $\frac{1}{2}$  on  $D(A^s)$  if  $0 \leq s < \frac{m+1}{2}$  and by  $\frac{s}{m+1}$  if  $\frac{m+1}{2} < s \leq m+1$ .

**Theorem 1.8.** *Let  $A$  be the generator of a bounded analytic semigroup  $T(t)$  and let  $r(\cdot)$  be an A-stable rational approximation of the exponential of approximation order  $m$ . Then*

- (i) the rational approximation scheme  $V(t) = r(tA)$  is stable,
- (ii)  $\|V(\frac{t}{n})^n x - T(t)x\| \leq C \frac{t^s}{n^s} \|A^s x\|$  for all  $1 \leq s \leq m, x \in D(A^s), t \in [0, \tau]$ , and
- (iii) if  $|r(\infty)| < 1$ , then  $\|V(\frac{t}{n})^n x - T(t)x\| \leq M_\tau \frac{1}{n^m} \|x\|$  for all  $x \in X, t \in [0, \tau]$ .

In particular, (i) and (ii) hold for the Crank-Nicolson scheme.

*Proof.* The first part of the statement is proven by Thomée in [Th], Theorem 8.2, under the additional assumption that  $s(A) < 0$ . The remaining two parts were proven for the case  $s(A) < 0$  by Larsson, Thomée, and Wahlbin in [LTW]. In the 1993 paper [CLPT], Crouzeix, Larsson, Piskarev and Thomée observed that the condition  $s(A) < 0$  may be weakened to include bounded analytic semigroups with  $s(A) \leq 0$ . In particular, if we define,  $\Sigma_\alpha := \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \pi - \alpha\}$ , then  $\overline{\Sigma_\alpha} \setminus \{0\} \subseteq \rho(A)$  for some  $\alpha \in (0, \frac{\pi}{2})$  and  $\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}$  for  $\lambda \in \Sigma_\alpha$  suffices. In this case, we may take  $\varepsilon > 0$  and define  $A_\varepsilon := A - \varepsilon I$  and observe that  $A_\varepsilon$  is an operator with  $0 \in \rho(A)$  and  $\|R(\lambda, A)\| \leq \frac{M}{|\lambda| \sin(\alpha)}$ . Indeed, let  $\lambda \in \Sigma_\alpha \setminus \{0\}$  with  $\operatorname{Re}(\lambda) \leq 0$  and  $\operatorname{Im}(\lambda) > 0$ . Then  $\lambda = |\lambda|e^{i\alpha'}$ , where  $\alpha < \alpha' < \frac{\pi}{2}$  and so  $|\lambda| \sin(\alpha) < |\lambda| \sin(\alpha') = \operatorname{Im}(\lambda) \leq |\lambda + \varepsilon|$ . Hence we may apply Theorem 1.8 to the new operator  $A_\varepsilon$ . On the other hand, by taking a partial fractions expansion of  $r(z)$ , we may write  $r(\frac{t}{n}A_\varepsilon)^n$  as a sum of powers of resolvents. Then it follows that  $r(\frac{t}{n}A_\varepsilon)^n \rightarrow r(\frac{t}{n}A)^n$  as  $\varepsilon \rightarrow 0$ . Hence  $r(\frac{t}{n}A)^n$  is stable. Analogous remarks apply to statements (ii) and (iii) of Theorem 1.8.  $\square$

## 1.2 Stabilization Methods

The main drawback of rational approximation schemes of the exponential is that the norm estimates, such as those of Theorems 1.5 and 1.8, are valid only for sufficiently smooth initial data unless  $A$  generates an analytic semigroup and  $|r(\infty)| < 1$  (see Theorem 1.8). To overcome this problem, stabilization techniques

were introduced in the 1970's by M. Luskin and R. Rannacher in [LuR], who analyzed a method in Hilbert space that couples the smoothing properties of the Backward Euler method with the higher order accuracy of the Crank-Nicolson method. In 1999, A. Hansbo extended this result to Banach space for generators of analytic semigroups with strictly negative spectral bound [Ha]. Indeed, if  $A$  generates an analytic semigroup, then unstable rational approximation schemes can be stabilized by employing first a lower order approximation scheme with  $|r(\infty)| = 0$ . This technique can be used to obtain optimal convergence estimates that are valid for all initial data  $x \in X$ .

**Theorem 1.9 (Hansbo).** *Let  $(A, D(A))$  be an operator on a Banach space  $X$ . Suppose that  $s(A) < 0$  and that  $(A, D(A))$  generates an analytic semigroup  $T(\cdot)$  on  $X$ . Let  $r_a(\cdot)$  be an  $A$ -stable rational approximation of the exponential of approximation order  $m \geq 2$  and let  $r_s(\cdot)$  be an  $A$ -stable scheme of order  $m - 1$  with  $r_s(\infty) = 0$ . Define*

$$r_n(z) := \begin{cases} r_s(z)^n & \text{if } n < m, \\ r_a(z)^{n-m} r_s(z)^m & \text{if } n \geq m. \end{cases}$$

Then,

$$\|r_n(\frac{t}{n}A)x - T(t)x\| \leq C \frac{1}{n^m} \|x\| \text{ for all } x \in X$$

In order to stabilize the Crank-Nicolson scheme using the above theorem, one would first apply two steps of the Backward Euler scheme. That is,

$$\|V_{CN}(\frac{t}{n})^{n-2} V_{BE}(\frac{t}{n})^2\| \leq M \text{ (} n \in \mathbb{N}, t \in [0, \delta] \text{ with } nt \in [0, \tau])$$

and

$$\|V_{CN}(\frac{t}{n})^{n-2} V_{BE}(\frac{t}{n})^2 x - T(t)x\| \leq C \frac{1}{n^2} \|x\|$$

for every  $x \in X$  and  $n \geq 2$ .

The main tool used in the proof of the Hansbo result is the Dunford Functional calculus for analytic semigroups; that is,

$$f(A) := \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda, A) d\lambda,$$

where  $\Gamma$  is the boundary of a sector  $G := \{\lambda : |\arg(\lambda)| < \frac{\pi}{2} + \alpha\}$  for some  $0 < \alpha \leq \frac{\pi}{2}$  that is contained in the resolvent set of  $A$  and where  $f$  is analytic on an open neighborhood of the complement of  $G$ . In particular, the spectrum of  $A$  must be contained in a sector  $\{\lambda : |\arg(-\lambda)| \leq \beta\}$  for some  $0 < \beta < \frac{\pi}{2}$ . Therefore the methods of Hansbo can not be applied to nonanalytic semigroups. In Chapter 2, we present a new stabilization method technique that works for arbitrary strongly continuous semigroups, that is, neither analyticity nor the spectral condition  $s(A) < 0$  are required.

## Chapter 2. Stabilized Approximation Methods

The Lax Equivalence Theorem and the Trotter-Kato Theorem play an important role in the mathematical analysis of approximation methods for semigroups. One of the main ingredients in both of these fundamental results is the *stability* of the approximation method under consideration. In this chapter we discuss variants of these theorems that cover stabilization techniques for intrinsically unstable approximation methods. First, let us recall the statement of the Lax Equivalence Theorem. Suppose that  $A$  generates a strongly continuous semigroup  $T(\cdot)$  on a Banach space  $X$  and let  $\{V(t); t \in [0, \tau]\}$  be an approximation scheme of bounded linear operators with  $V(0) = I$  that satisfies the *consistency* condition

$$\lim_{t \rightarrow 0} \frac{V(t)x - x}{t} = Ax$$

for all  $x$  in a set  $D \subset D(A)$  that is dense in  $X$ . Then it was shown by Lax and Richtmyer in 1956 [LR] (with a stronger consistency condition) and in final form by Chernoff in 1974 [Ch] that the following statements are equivalent.

- (i)  $V(0) = I$  is *stable*; that is there exist  $\omega, M \geq 0$  such that  $\|V(t)^n\| \leq Me^{\omega nt}$  for each  $n \in \mathbb{N}_0$  and for each  $t \in [0, \tau]$ ,
- (ii)  $\lim_{n \rightarrow \infty} V(\frac{t}{n})^n x = T(t)x$  for all  $t \geq 0$  and  $x \in X$ .

As mentioned before, there are two main shortcomings of the Lax-Chernoff Theorem, which we shall fix in this chapter. For arbitrary semigroups (that is, for semigroups that are not necessarily analytic), widely used high order consistent schemes (such as the Crank-Nicolson scheme) become unstable. We have also seen that in the absence of analyticity, all A-stable rational approximation schemes

converge no faster than  $\frac{1}{\sqrt{n}}$  on  $D(A)$ , the set of initial data for which we get classical solutions. We shall show that in principle, the speed of convergence can be improved by a magnitude close to  $\frac{1}{2}$ . That is for initial data  $x \in D(A)$ , one may obtain convergence with speed close to that of  $\frac{1}{n}$ . Confer also part (ii) of Theorem 1.8, where we observe a similar phenomenon without stabilization if we assume that the semigroup is analytic.

## 2.1 Error Estimates for Arbitrary Initial Data

We consider *rational approximation schemes*  $V(t) := r(tA)$  of the strongly continuous semigroup  $T(t)$ , where  $r : \mathbb{C} \rightarrow \mathbb{C}$  is a rational function whose MacLaurin series coincides with the exponential series for the first  $m$  terms; i.e.,

$$r(z) - e^z = O(z^{m+1}) \tag{2.1.1}$$

for some  $m \in \mathbb{N}$  and for all  $z$  of ‘sufficiently small’ modulus. In the above situation, we say that  $r$  is a *rational approximation of the exponential of order  $m$* . A rational approximation  $r$  of the exponential with the property that  $|r(z)| \leq 1$  for  $\operatorname{Re} z \leq 0$  is said to be *A-stable*. Notice that the Backward Euler approximation

$$r_{BE}(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

is A-stable and of approximation order 1 and the Crank-Nicolson approximation

$$r_{CN}(z) = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z} = 1 + z + \frac{z^2}{2} + \frac{z^3}{8} + \dots$$

is A-stable and of approximation order 2. As we saw in the Introduction, the Backward Euler scheme and the Crank-Nicolson scheme are both consistent.

If a consistent approximation scheme  $V(t) = r(tA)$  is not stable, then the Lax Equivalence Theorem tells us that there exists  $x \in X$  such that  $V(\frac{t}{n})^n x$  does not converge to  $T(t)x$ . However, the following proposition shows that such approximation schemes can be stabilized by taking first  $m+1$  modified Backward Euler steps

$(I - (\frac{t}{n})^\alpha A)^{-1}$ . Moreover, by stabilizing the scheme, the speed of convergence can be improved if  $x \in D(A)$  and  $m \geq 3$ .

**Theorem 2.1.** *Let  $A$  be the generator of a strongly continuous semigroup  $T(\cdot)$  with  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . Let  $r(\cdot)$  be an  $A$ -stable rational approximation of the exponential of approximation order  $m$ . Define  $V(t) := r(tA)$  and*

$$W(t) := \frac{1}{t^\alpha} R\left(\frac{1}{t^\alpha}, A\right) = (I - t^\alpha A)^{-1},$$

where  $\alpha := \frac{m}{m+1}(1 - \varepsilon)$  for some  $0 < \varepsilon < 1$ . Then

$$\lim_{n \rightarrow \infty} V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x = T(t)x$$

for all  $x \in X$ . Furthermore, if  $x \in D(A)$  then for all  $\tau > 0$  there exists a constant  $M_\tau$  such that the error estimate

$$\|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x - T(t)x\| \leq M_\tau \left(\frac{1}{n^{m\varepsilon}} + \frac{1}{n^\alpha}\right) (\|x\| + \|Ax\|)$$

holds for all  $t \in [0, \tau]$  and all  $n \geq m + 1$ . Moreover, if  $\varepsilon = \frac{1}{m+2}$ , then  $\alpha = \frac{m}{m+2}$  and for all  $\tau > 0$ , the error estimate

$$\|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x - T(t)x\| \leq 2M_\tau \left(\frac{1}{n}\right)^{\frac{m}{m+2}} (\|x\| + \|Ax\|)$$

holds for all  $t \in [0, \tau]$  and all  $n \geq m + 1$ .

*Proof.* Since  $AR(\lambda, A) = \lambda R(\lambda, A) - I$ , it follows that  $AW(t) = \frac{1}{t^\alpha} (\frac{1}{t^\alpha} R(\frac{1}{t^\alpha}, A) - I)$ . By the Hille-Yosida Theorem (cf. [ABHN]) we have that  $\|\lambda R(\lambda, A)\| \leq \frac{M\lambda}{\lambda - \omega}$  for all  $\lambda > \omega$ . Moreover, if  $\lambda > \omega_0 := \max\{0, 2\omega\}$  then  $\frac{\lambda}{\lambda - \omega} \leq 2$ . Thus,  $\|\lambda R(\lambda, A)\| \leq 2M$  for all  $\lambda > \omega_0$ ,  $\|W(\frac{t}{n})\| \leq 2M$  and  $\|AW(\frac{t}{n})\| \leq \frac{n^\alpha}{t^\alpha} (2M + 1)$  for all  $n > t\omega_0^{1/\alpha}$ . We proceed to estimate

$$\begin{aligned} & \|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x - T(t)x\| \\ & \leq \|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x - T(t)W\left(\frac{t}{n}\right)^{m+1} x\| + \|T(t)W\left(\frac{t}{n}\right)^{m+1} x - T(t)x\|, \end{aligned} \tag{2.1.2}$$

and observe that  $\lim_{n \rightarrow \infty} W(\frac{t}{n})x = x$  for all  $x \in X$  since  $A$  generates a strongly continuous semigroup (see [ABHN], Proposition 3.1.9 (a)). We estimate the second term of (2.1.2) using the binomial formula; that is,

$$\begin{aligned} & \|T(t)W(\frac{t}{n})^{m+1}x - T(t)x\| = \|T(t)(W(\frac{t}{n})^{m+1}x - x)\| \\ \leq & \|T(t)\| \left\| \sum_{j=0}^m W(\frac{t}{n})^j \right\| \|W(\frac{t}{n})x - x\|. \end{aligned}$$

Since the family  $W(\frac{t}{n})$  is uniformly bounded, we may conclude that

$$\lim_{n \rightarrow \infty} \|T(t)W(\frac{t}{n})^{m+1}x - T(t)x\| = 0.$$

By equation (1.1.2), we have that there exist constants  $C, c > 0$  such that

$$\begin{aligned} & \|V(\frac{t}{n})^n W(\frac{t}{n})^{m+1}x - T(t)W(\frac{t}{n})^{m+1}x\| \\ \leq & CMte^{c\omega t} (\frac{t}{n})^m \| (AW(\frac{t}{n}))^{m+1}x \| \\ \leq & CMte^{c\omega t} (\frac{t}{n})^m (\frac{n^\alpha}{t^\alpha})^{m+1} (2M+1)^{m+1} \|x\| \\ = & CMte^{c\omega t} (\frac{t}{n})^m (\frac{n}{t})^{m(1-\varepsilon)} (2M+1)^{m+1} \|x\| \\ = & CMte^{c\omega t} (\frac{t}{n})^{m\varepsilon} (2M+1)^{m+1} \|x\|. \end{aligned}$$

Therefore, for every  $\tau > 0$  there exists a constant  $K_\tau > 0$  such that

$$\|V(\frac{t}{n})^n W(\frac{t}{n})^{m+1}x - T(t)W(\frac{t}{n})^{m+1}x\| \leq K_\tau \frac{1}{n^{m\varepsilon}} \|x\| \quad (2.1.3)$$

for all  $t \in [0, \tau]$ . This shows that  $\lim_{n \rightarrow \infty} V(\frac{t}{n})^n W(\frac{t}{n})^{m+1}x = T(t)x$  for all  $x \in X$ .

Now suppose that  $x \in D(A)$ . Then for every  $\tau > 0$  there exists a constant  $N_\tau > 0$  such that

$$\begin{aligned}
& \|T(t)W\left(\frac{t}{n}\right)^{m+1}x - T(t)x\| \leq \|T(t)\| \left\| \sum_{j=0}^m W\left(\frac{t}{n}\right)^j \right\| \|W\left(\frac{t}{n}\right)x - x\| \\
& = \|T(t)\| \sum_{j=0}^m \|W\left(\frac{t}{n}\right)\|^j \|R\left(\frac{n^\alpha}{t^\alpha}, A\right)Ax\| \leq \|T(t)\| \sum_{j=0}^m \|W\left(\frac{t}{n}\right)\|^j \frac{M}{\left(\frac{n^\alpha}{t^\alpha} - \omega\right)} \|Ax\| \\
& \leq M^2 \sum_{j=0}^m (2M)^j e^{\omega t} \frac{1}{\left(\frac{n^\alpha}{t^\alpha} - \omega\right)} \|Ax\| \\
& = M^2 \frac{1 - (2M)^{m+1}}{1 - 2M} e^{\omega t} \frac{1}{\left(\frac{n^\alpha}{t^\alpha} - \omega\right)} \|Ax\| \\
& \leq 2M^2 \sum_{j=0}^m (2M)^j e^{\omega t} \frac{t^\alpha}{n^\alpha} \|Ax\| \\
& \leq N_\tau \frac{1}{n^\alpha} \|Ax\| \tag{2.1.4}
\end{aligned}$$

for  $\frac{n^\alpha}{t^\alpha} > \omega_0$  and for all  $t \in [0, \tau]$ . Therefore for each  $\tau > 0$  there exists a constant

$$\begin{aligned}
M_\tau & := \max(K_\tau, N_\tau) \\
& = \max\left(\sup_{t \in [0, \tau]} CMte^{c\omega t} t^{m\varepsilon} (2M + 1)^{m+1}, \sup_{t \in [0, \tau]} 2M^2 \frac{(2M)^{m+1} - 1}{2M - 1} e^{\omega t} t^\alpha\right)
\end{aligned}$$

such that if  $t \in [0, \tau]$  and  $x \in D(A)$ , then

$$\|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1}x - T(t)x\| \leq M_\tau \left(\frac{1}{n^{m\varepsilon}} + \frac{1}{n^\alpha}\right) (\|x\| + \|Ax\|).$$

Define  $f_1(\varepsilon) := \frac{1}{n^{m\varepsilon}} + \frac{1}{n^\alpha} = e^{-m\varepsilon \ln(n)} + e^{-\frac{m}{m+1}(1-\varepsilon)\ln(n)}$ . Then

$$f_1'(\varepsilon) = -m \ln(n) e^{-m\varepsilon \ln(n)} + \frac{m}{m+1} \ln(n) e^{-\frac{m}{m+1}(1-\varepsilon)\ln(n)}.$$

Observe that  $f_1''$  is always nonnegative and that  $f_1'(\varepsilon_1) = 0$  when evaluated at  $\varepsilon_1 := \frac{\ln(m+1) + \frac{m}{m+1}\ln(n)}{\left(\frac{1}{m+1} + 1\right)m \ln(n)}$ . Hence by evaluating  $f_1(\varepsilon_1)$ , we see that  $f_1$  attains an absolute minimum value of

$$f_1(\varepsilon_1) = \left(\frac{1}{n}\right)^{\frac{m}{m+2}} \left[\left(\frac{1}{m+1}\right)^{\frac{m+1}{m+2}} + (m+1)^{\frac{1}{m+2}}\right].$$

Notice that  $\varepsilon_1$  depends on the time-step  $n$ , so it is not practical to take  $\varepsilon = \varepsilon_1$ ; instead we analyze the behavior of  $\varepsilon_1$  for large  $n$ . Now,

$$\varepsilon_1 := \frac{\ln(m+1) + \frac{m}{m+1} \ln(n)}{\left(\frac{1}{m+1} + 1\right)m \ln(n)} = \frac{(m+1) \ln(m+1) + m \ln(n)}{(m+2)m \ln(n)} = \frac{\frac{(m+1) \ln(m+1)}{\ln(n)} + m}{(m+2)m}.$$

Hence  $\varepsilon_1 \rightarrow \frac{1}{m+2}$  as  $n \rightarrow \infty$ . Therefore, if we take  $\varepsilon = \frac{1}{m+2}$ , in which case  $\alpha = \frac{m}{m+1}(1 - \varepsilon) = \frac{m}{m+1} \frac{m+2-1}{m+2} = \frac{m}{m+2}$  and  $f(\varepsilon) = \frac{1}{n^{m\varepsilon}} + \frac{1}{n^\alpha} = \frac{2}{n^{\frac{m}{m+2}}}$ . In conclusion, if  $\varepsilon = \frac{1}{m+2}$  then for each  $\tau > 0$  there exists  $M_\tau > 0$  such that the error estimate

$$\|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x - T(t)x\| \leq 2M_\tau \left(\frac{1}{n}\right)^{\frac{m}{m+2}} (\|x\| + \|Ax\|)$$

holds for all  $t \in [0, \tau]$  and all  $n \geq m+1$ .  $\square$

The following table compares the results of the Brenner-Thoméé theorem to the error estimates obtained by stabilizing a rational approximation scheme of order  $m$  for  $m = 1, 2, \dots, 8$ . Notice that for initial data  $x \in D(A)$ , the rate of convergence  $\frac{m}{m+2}$  (for a stabilized scheme) approaches 1 for large  $m$ . The letter c indicates convergence of unknown speed.

Table 2.1. Comparison of Theorem 2.1 to Theorem 1.5

	$x \in X$ unstabilized	$x \in X$ stabilized	$x \in D(A)$ unstabilized	$x \in D(A)$ stabilized
$m = 1$	$n^{\frac{1}{2}}$	c	$\left(\frac{1}{n}\right)^{\frac{1}{2}}$	$\left(\frac{1}{n}\right)^{\frac{1}{3}}$
$m = 2$	$n^{\frac{1}{2}}$	c	$\left(\frac{1}{n}\right)^{\frac{1}{2}}$	$\left(\frac{1}{n}\right)^{\frac{1}{2}}$
$m = 3$	$n^{\frac{1}{2}}$	c	$\left(\frac{1}{n}\right)^{\frac{1}{2}}$	$\left(\frac{1}{n}\right)^{\frac{3}{5}}$
$m = 4$	$n^{\frac{1}{2}}$	c	$\left(\frac{1}{n}\right)^{\frac{1}{2}}$	$\left(\frac{1}{n}\right)^{\frac{2}{3}}$
$m = 5$	$n^{\frac{1}{2}}$	c	$\left(\frac{1}{n}\right)^{\frac{1}{2}}$	$\left(\frac{1}{n}\right)^{\frac{5}{7}}$
$m = 6$	$n^{\frac{1}{2}}$	c	$\left(\frac{1}{n}\right)^{\frac{1}{2}}$	$\left(\frac{1}{n}\right)^{\frac{3}{4}}$
$m = 7$	$n^{\frac{1}{2}}$	c	$\left(\frac{1}{n}\right)^{\frac{1}{2}}$	$\left(\frac{1}{n}\right)^{\frac{7}{9}}$
$m = 8$	$n^{\frac{1}{2}}$	c	$\left(\frac{1}{n}\right)^{\frac{1}{2}}$	$\left(\frac{1}{n}\right)^{\frac{4}{5}}$

## 2.2 Higher Order Estimates for Smooth Initial Data

Proposition 2.1 may be generalized by taking stabilizers of the form

$$W(t) := (-1)^{k+1} t^{k\alpha} A^k \prod_{i=1}^k (b_i I - t^\alpha A)^{-1} + I,$$

for natural numbers  $1 \leq k \leq m$  and for pairwise distinct positive real numbers  $b_i, i = 1, 2, \dots, k$ . The stabilizers  $W(t)$  may also be written as a sum of modified Backward Euler steps

$$W(t) = \sum_{i=1}^k \frac{a_i}{t^\alpha} R\left(\frac{b_i}{t^\alpha}, A\right).$$

Indeed, by the Fundamental Theorem of Algebra, there exist real numbers  $c_l$ , where  $0 \leq l \leq k - 1$  such that

$$\begin{aligned} W(t) &:= (-1)^{k+1} t^{k\alpha} A^k \prod_{i=1}^k (b_i I - t^\alpha A)^{-1} + I \\ &= [ -(-t^\alpha A)^k + \prod_{i=1}^k (b_i I - t^\alpha A) ] \prod_{i=1}^k (b_i I - t^\alpha A)^{-1} \\ &= [c_0 I + c_1 t^\alpha A + \dots + c_{k-1} (t^\alpha A)^{k-1}] \prod_{i=1}^k (b_i I - t^\alpha A)^{-1}. \end{aligned} \quad (2.2.1)$$

Expanding (2.2.1) by partial fractions, we obtain

$$W(t) = a_1 (b_1 I - t^\alpha A)^{-1} + \dots + a_k (b_k I - t^\alpha A)^{-1} \quad (2.2.2)$$

for some  $a_1, a_2, \dots, a_k \in \mathbb{R}$ . For more details on the coefficients  $a_i, 1 \leq i \leq k$ , see the remarks of Section 2.3. The following lemma is crucial in finding the final error estimates for initial data  $x \in D(A^k)$ .

**Lemma 2.2.** *Let  $\alpha := \frac{m}{m+1}(1 - \varepsilon)$  and define  $f_k(\varepsilon) := \frac{1}{n^{m\varepsilon}} + \frac{1}{n^{k\alpha}}$  for  $1 \leq k \leq m$ .*

*Then  $f_k$  achieves a minimum value of*

$$\left(\frac{1}{n}\right)^{\frac{km}{k+m+1}} \left[ \left(\frac{k}{m+1}\right)^{\frac{m+1}{k+m+1}} + \left(\frac{m+1}{k}\right)^{\frac{k}{k+m+1}} \right]$$

at

$$\varepsilon_k := \frac{(m+1)\ln(\frac{m+1}{k}) + km\ln(n)}{(m+k+1)m\ln(n)}.$$

Moreover  $\varepsilon_k = \frac{\frac{(m+1)\ln(\frac{m+1}{k})}{\ln(n)} + km}{m(m+k+1)} \longrightarrow \frac{k}{m+k+1}$  as  $n \longrightarrow \infty$ .

*Proof.* Observe that  $f_k(\varepsilon) := \frac{1}{n^{m\varepsilon}} + \frac{1}{n^{k\varepsilon}} = e^{-m\varepsilon\ln(n)} + e^{-k\frac{m}{m+1}(1-\varepsilon)\ln(n)}$ . Hence

$$f'_k(\varepsilon) = -m\ln(n)e^{-m\varepsilon\ln(n)} + \frac{km}{m+1}\ln(n)e^{-k\frac{m}{m+1}\ln(n)(1-\varepsilon)}.$$

Furthermore

$$f''_k(\varepsilon) = (m\ln(n))^2 e^{-m\ln(n)\varepsilon} + \left(\frac{km}{m+1}\ln(n)\right)^2 e^{-k\frac{m}{m+1}\ln(n)(1-\varepsilon)}$$

is always nonnegative. Now,  $f'_k(\varepsilon) = 0$  if and only if  $\frac{k}{m+1}e^{-k\frac{m}{m+1}\ln(n)(1-\varepsilon)} = e^{-m\varepsilon\ln(n)}$ .

Multiplying through by  $\frac{m+1}{k}$  and taking the natural logarithm of both sides, we obtain

$$\frac{-km}{m+1}\ln(n)(1-\varepsilon) = \ln\left(\frac{m+1}{k}\right) - m\varepsilon\ln(n).$$

Therefore

$$\frac{km\varepsilon}{m+1}\ln(n) + m\varepsilon\ln(n) = \ln\left(\frac{m+1}{k}\right) + \frac{km}{m+1}\ln(n),$$

and so  $\varepsilon = \frac{\ln(\frac{m+1}{k}) + \frac{km}{m+1}\ln(n)}{(\frac{k}{m+1} + 1)m\ln(n)}$ . Observe furthermore that  $f''_k(\varepsilon) > 0$ . We have proven

that  $f_k$  attains a minimum at

$$\begin{aligned} \varepsilon_k &:= \frac{\ln(\frac{m+1}{k}) + \frac{km}{m+1}\ln(n)}{\left(\frac{k+m+1}{m+1}\right)m\ln(n)} \\ &= \frac{\ln([\frac{m+1}{k}]^{m+1}) + km\ln(n)}{(k+m+1)m\ln(n)} \\ &= \frac{(m+1)\ln(\frac{m+1}{k}) + km\ln(n)}{(m+k+1)m\ln(n)}. \end{aligned}$$

Recall that  $f_k(\varepsilon) := \frac{1}{n^{m\varepsilon}} + \frac{1}{n^{k\varepsilon}} = g_k(\varepsilon) + h_k(\varepsilon)$  for  $k = 1, 2, \dots, m$ . We proceed to calculate

$$\begin{aligned}
g_k(\varepsilon_k) &:= e^{-m\varepsilon_k \ln(n)} \\
&= e^{-\left(\frac{\ln\left(\left(\frac{m+1}{k}\right)^{m+1}\right) + km \ln(n)}{k+m+1}\right)} \\
&= e^{-\ln\left(\left(\frac{m+1}{k}\right)^{\frac{m+1}{k+m+1}}\right) - \ln\left(n^{\frac{km}{k+m+1}}\right)} \\
&= \left(\frac{k}{m+1}\right)^{\frac{m+1}{k+m+1}} \cdot \left(\frac{1}{n}\right)^{\frac{km}{k+m+1}}.
\end{aligned}$$

Now, we have that

$$\begin{aligned}
1 - \varepsilon_k &= 1 - \frac{\ln\left(\frac{m+1}{k}\right) + \frac{km}{m+1} \ln(n)}{\frac{k+m+1}{m+1} m \ln(n)} \\
&= \frac{\frac{k+m+1}{m+1} m \ln(n) - \ln\left(\frac{m+1}{k}\right) - \frac{km}{m+1} \ln(n)}{\frac{k+m+1}{m+1} m \ln(n)} \\
&= \frac{1}{m \ln(n)} \left[ m \ln(n) - \frac{1}{k+m+1} \left[ (m+1) \ln\left(\frac{m+1}{k}\right) + km \ln(n) \right] \right].
\end{aligned}$$

Let  $\phi_{k,m,n}(x) := e^{\frac{-k}{m+1} m \ln(n)x}$ . Then  $\phi_{k,m,n}(1-\varepsilon) = h_k(\varepsilon)$  by definition. Furthermore,

$$\begin{aligned}
\phi_{k,m,n}(1 - \varepsilon_k) &= e^{\frac{-k}{m+1} \left[ m \ln(n) - \frac{1}{k+m+1} \left[ (m+1) \ln\left(\frac{m+1}{k}\right) + km \ln(n) \right] \right]} \\
&= e^{\frac{-k}{m+1} \left[ m \ln(n) - (m+1) \left[ \ln\left(\frac{m+1}{k}\right) \right]^{\frac{1}{k+m+1}} - \ln(n)^{\frac{km}{k+m+1}} \right]} \\
&= e^{\frac{-k}{m+1} m \ln(n) + k \left[ \ln\left(\frac{m+1}{k}\right) \right]^{\frac{1}{k+m+1}} + \frac{k}{m+1} \ln(n)^{\frac{km}{k+m+1}}} \\
&= e^{-\ln(n)^{\frac{km}{m+1}} + \left[ \ln\left(\frac{m+1}{k}\right) \right]^{\frac{k}{k+m+1}} + \ln(n)^{\frac{k^2 m}{(m+1)(k+m+1)}}}.
\end{aligned}$$

So

$$\begin{aligned}
h_k(\varepsilon_k) &:= e^{-k \frac{m}{m+1} (1-\varepsilon_k) \ln(n)} = \phi_{k,m,n}(1 - \varepsilon_k) \\
&= e^{-\ln(n)^{\frac{km}{m+1}} + \ln\left(\frac{m+1}{k}\right)^{\frac{k}{k+m+1}} + \ln(n)^{\frac{k^2 m}{(m+1)(k+m+1)}}} \\
&= \left(\frac{1}{n}\right)^{\frac{km}{m+1}} \cdot \left(\frac{m+1}{k}\right)^{\frac{k}{k+m+1}} \cdot \left(\frac{1}{n}\right)^{\frac{-k^2 m}{(m+1)(k+m+1)}} \\
&= \left(\frac{m+1}{k}\right)^{\frac{k}{k+m+1}} \cdot \left(\frac{1}{n}\right)^{\frac{km(k+m+1) - k^2 m}{(m+1)(k+m+1)}}.
\end{aligned}$$

Now,

$$\begin{aligned} \frac{km(k+m+1) - k^2m}{(m+1)(k+m+1)} &= \frac{k^2m + km^2 + km - k^2m}{(m+1)(k+m+1)} \\ &= \frac{km(m+1)}{(m+1)(k+m+1)} = \frac{km}{k+m+1}. \end{aligned}$$

Let  $\theta(k, m) := \left(\frac{k}{m+1}\right)^{\frac{m+1}{k+m+1}} + \left(\frac{m+1}{k}\right)^{\frac{k}{k+m+1}}$ . We have shown that

$$\begin{aligned} f_k(\varepsilon_k) &= g_k(\varepsilon_k) + h_k(\varepsilon_k) \\ &= \left(\frac{k}{m+1}\right)^{\frac{m+1}{k+m+1}} \cdot \left(\frac{1}{n}\right)^{\frac{km}{k+m+1}} + \left(\frac{m+1}{k}\right)^{\frac{k}{k+m+1}} \cdot \left(\frac{1}{n}\right)^{\frac{km}{k+m+1}} \\ &= \left(\frac{1}{n}\right)^{\frac{km}{k+m+1}} \left[ \left(\frac{k}{m+1}\right)^{\frac{m+1}{k+m+1}} + \left(\frac{m+1}{k}\right)^{\frac{k}{k+m+1}} \right] \\ &= \left(\frac{1}{n}\right)^{\frac{km}{k+m+1}} \theta(k, m). \end{aligned}$$

Finally,

$$\begin{aligned} \varepsilon_k &:= \frac{(m+1) \ln\left(\frac{m+1}{k}\right) + km \ln(n)}{(m+k+1)m \ln(n)} \\ &= \frac{\frac{(m+1) \ln\left(\frac{m+1}{k}\right)}{\ln(n)} + km}{m(m+k+1)} \longrightarrow \frac{k}{m+k+1} \end{aligned}$$

as  $n \longrightarrow \infty$ . □

The following table shows approximate values (rounded to the nearest hundredth) of  $\theta(k, m)$  for  $1 \leq k \leq m \leq 5$ . This will be used when we discuss the estimate (2.2.4) of the next theorem.

Table 2.2. Approximate values of the coefficient  $\theta(k, m)$

	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
$k = 1$	1.89	1.75	1.65	1.57	1.51
$k = 2$		1.96	1.89	1.90	1.75
$k = 3$			1.98	1.94	1.89
$k = 4$				1.99	1.96
$k = 5$					1.99

We are now equipped to prove the main result of this chapter.

**Theorem 2.3.** *Let  $A$  be the generator of a strongly continuous semigroup  $T(\cdot)$  with  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . Let  $r(\cdot)$  be an  $A$ -stable rational approximation of the exponential of approximation order  $m$ . Let  $1 \leq k \leq m$  be a natural number and let  $0 < \varepsilon < 1$  be given. Let  $\tau > 0$  and  $\alpha := \frac{m}{m+1}(1 - \varepsilon)$ . Choose pairwise distinct real numbers  $b_i$  such that  $b_i > \omega\tau^\alpha$ . For  $t \in [0, \tau]$  define  $V(t) := r(tA)$  and*

$$W(t) := -(-1)^k t^{k\alpha} A^k \prod_{i=1}^k (b_i I - t^\alpha A)^{-1} + I. \quad (2.2.3)$$

Then

$$\lim_{n \rightarrow \infty} V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x = T(t)x$$

for all  $x \in X$ . Furthermore, if  $x \in D(A^k)$  then for all  $\tau > 0$ , there exists  $M_\tau > 0$  such that the error estimate

$$\|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x - T(t)x\| \leq M_\tau \left(\frac{1}{n^{m\varepsilon}} + \frac{1}{n^{k\alpha}}\right) (\|x\| + \|A^k x\|)$$

holds for all  $t \in [0, \tau]$  and all  $\{n : n \geq m + 1 \text{ and } \frac{n^\alpha}{t^\alpha} > \omega_0\}$ , where  $\omega_0 := \max\{0, 2\omega\}$ . Moreover, if  $\varepsilon = \frac{km}{k+m+1}$  then  $\alpha = \frac{m}{m+k+1}$  and if  $x \in D(A^k)$  then the error estimate

$$\|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x - T(t)x\| \leq 2M_\tau \left(\frac{1}{n}\right)^{\frac{km}{k+m+1}} (\|x\| + \|A^k x\|)$$

holds for all  $t \in [0, \tau]$  and all  $\{n : n \geq m + 1 \text{ and } \frac{n^\alpha}{t^\alpha} > \omega_0\}$ . Furthermore, if  $x \in D(A^q)$  where  $1 \leq q \leq k - 1$  is a natural number, then for all  $\tau > 0$  there exists  $\hat{M}_\tau > 0$  such that the error estimate

$$\|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x - T(t)x\| \leq 2\hat{M}_\tau \left(\frac{1}{n}\right)^{\frac{qm}{q+m+1}} (\|x\| + \|A^q x\|) \quad (2.2.4)$$

holds for all  $t \in [0, \tau]$  and all  $\{n : n \geq m + 1 \text{ and } \frac{n^\alpha}{t^\alpha} > \omega_0\}$ .

*Proof.* Fix a natural number  $1 \leq k \leq m$ . Let  $b_i > 0, 1 \leq i \leq k$  be pairwise distinct natural numbers. Define  $W(t) := -(-1)^k t^{k\alpha} A^k \prod_{i=1}^k (b_i I - t^\alpha A)^{-1} + I$ . Let  $x \in X$ . Then  $\|(W(t) - I)x\| = \|t^{k\alpha} A^k \prod_{i=1}^k (b_i I - t^\alpha A)^{-1} x\| = \|t^{k\alpha} A^k \prod_{i=1}^k \frac{1}{t^\alpha} R(\frac{b_i}{t^\alpha}, A)x\| = \|A^k \prod_{i=1}^k R(\frac{b_i}{t^\alpha}, A)x\|$ . Furthermore, by (2.2.2) there exist real numbers  $a_1, a_2, \dots, a_k$  such that  $W(t) = \sum_{i=1}^k \frac{a_i}{t^\alpha} R(\frac{b_i}{t^\alpha}, A)$ .

By the Hille-Yosida Theorem (cf. [ABHN]) we have that  $\|\lambda R(\lambda, A)\| \leq |\frac{M\lambda}{\lambda - \omega}|$  for all  $\lambda > \omega$ . Moreover, if  $\lambda > \omega_0 := \max\{0, 2\omega\}$  then  $|\frac{\lambda}{\lambda - \omega}| \leq 2$ . Thus,  $\|\lambda R(\lambda, A)\| \leq 2M$  for all  $\lambda > \omega_0$ . Hence if  $\frac{b_i}{t^\alpha} > \omega_0$  for all natural numbers  $i$  such that  $1 \leq i \leq k$ , then

$$\begin{aligned} \|AW(t)\| &= \left\| \sum_{i=1}^k \frac{a_i}{t^\alpha} AR(\frac{b_i}{t^\alpha}, A) \right\| \\ &= \left\| \sum_{i=1}^k \frac{a_i}{t^\alpha} \left[ \frac{b_i}{t^\alpha} R(\frac{b_i}{t^\alpha}, A) - I \right] \right\| \\ &\leq \sum_{i=1}^k \left| \frac{a_i}{t^\alpha} \right| \left\| \frac{b_i}{t^\alpha} R(\frac{b_i}{t^\alpha}, A) - I \right\| \\ &\leq (2M + 1) \sum_{i=1}^k \left| \frac{a_i}{t^\alpha} \right| = (2M + 1) \frac{1}{t^\alpha} \sum_{i=1}^k |a_i|. \end{aligned} \quad (2.2.5)$$

We proceed to estimate

$$\begin{aligned} &\left\| V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x - T(t)x \right\| \\ &\leq \left\| V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1} x - T(t)W\left(\frac{t}{n}\right)^{m+1} x \right\| + \left\| T(t)W\left(\frac{t}{n}\right)^{m+1} x - T(t)x \right\|, \end{aligned}$$

and observe that  $\lim_{n \rightarrow \infty} W(\frac{t}{n})x = x$  for all  $x \in X$  since  $A$  generates a strongly continuous semigroup (see [ABHN], Proposition 3.1.9 (a)). We estimate the second term using the binomial formula; that is,

$$\|T(t)(W(\frac{t}{n})^{m+1}x - x)\| \leq \|T(t)\| \left\| \sum_{j=0}^m W(\frac{t}{n})^j \right\| \|W(\frac{t}{n})x - x\|.$$

Since the family  $W(\frac{t}{n})$  is uniformly bounded, we may conclude that

$$\lim_{n \rightarrow \infty} \|T(t)W(\frac{t}{n})^{m+1}x - T(t)x\| = 0.$$

We focus now on estimating  $\|V(\frac{t}{n})^n W(\frac{t}{n})^{m+1}x - T(t)W(\frac{t}{n})^{m+1}x\|$ . By equation (1.1.2), we have that there exist constants  $C, c > 0$  such that

$$\begin{aligned}
& \|V(\frac{t}{n})^n W(\frac{t}{n})^{m+1}x - T(t)W(\frac{t}{n})^{m+1}x\| \\
& \leq CMte^{c\omega t}(\frac{t}{n})^m \|(AW(\frac{t}{n}))^{m+1}x\| \\
& \leq CMte^{c\omega t}(\frac{t}{n})^m (\frac{n^\alpha}{t^\alpha})^{m+1} (2M+1)^{m+1} (\sum_{i=1}^k |a_i|)^{m+1} \|x\| \\
& = CMte^{c\omega t}(\frac{t}{n})^m (\frac{n}{t})^{m(1-\varepsilon)} (2M+1)^{m+1} (\sum_{i=1}^k |a_i|)^{m+1} \|x\| \\
& = CMte^{c\omega t}(\frac{t}{n})^{m\varepsilon} (2M+1)^{m+1} (\sum_{i=1}^k |a_i|)^{m+1} \|x\|. \tag{2.2.6}
\end{aligned}$$

Therefore, for all  $\tau > 0$  there exists  $K_\tau > 0$  such that

$$\|V(\frac{t}{n})^n W(\frac{t}{n})^{m+1}x - T(t)W(\frac{t}{n})^{m+1}x\| \leq K_\tau \frac{1}{n^{m\varepsilon}} \|x\| \tag{2.2.7}$$

for all  $t \in [0, \tau]$ . This shows that  $\lim_{n \rightarrow \infty} V(\frac{t}{n})^n W(\frac{t}{n})^{m+1}x = T(t)x$  for all  $x \in X$ .

Furthermore

$$\begin{aligned}
\|W(\frac{t}{n})\| & = \left\| \sum_{i=1}^k \frac{a_i n^\alpha}{t^\alpha} R(\frac{b_i n^\alpha}{t^\alpha}, A) \right\| \\
& \leq \sum_{i=1}^k |a_i| \left\| \frac{n^\alpha}{t^\alpha} R(\frac{b_i n^\alpha}{t^\alpha}, A) \right\| \\
& = \sum_{i=1}^k \frac{|a_i|}{b_i} \left\| \frac{b_i n^\alpha}{t^\alpha} R(\frac{b_i n^\alpha}{t^\alpha}, A) \right\| \leq 2M \sum_{i=1}^k \frac{|a_i|}{b_i}. \tag{2.2.8}
\end{aligned}$$

Now suppose that  $x \in D(A^k)$ . Then for every  $\tau > 0$  there exists  $N_\tau > 0$  such that

$$\begin{aligned}
& \|T(t)W(\frac{t}{n})^m x - T(t)x\| \\
& \leq \|T(t)\| \sum_{j=0}^m \|W(\frac{t}{n})\|^j \|W(\frac{t}{n})x - x\| \\
& \leq Me^{\omega t} \sum_{j=0}^m (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^j \|t^{k\alpha} \prod_{l=1}^k \frac{1}{t^\alpha} R(\frac{b_l n^\alpha}{t^\alpha}, A) A^k x\| \tag{2.2.9}
\end{aligned}$$

$$\begin{aligned}
& \leq Me^{\omega t} \frac{1 - (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^{m+1}}{1 - 2M \sum_{i=1}^k \frac{|a_i|}{b_i}} \frac{t^{k\alpha}}{t^{k\alpha}} \prod_{l=1}^k \|R(\frac{b_l n^\alpha}{t^\alpha}, A)\| \cdot \|A^k x\| \\
& \leq Me^{\omega t} \frac{1 - (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^{m+1}}{1 - 2M \sum_{i=1}^k \frac{|a_i|}{b_i}} \left( \prod_{l=1}^k \frac{M}{\frac{b_l n^\alpha}{t^\alpha} - \omega} \right) \|A^k x\| \\
& \leq M^{k+1} 2^k e^{\omega t} \left( \prod_{l=1}^k \frac{1}{b_l} \right) \frac{(1 - (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^{m+1}) t^{k\alpha}}{1 - 2M \sum_{i=1}^k \frac{|a_i|}{b_i}} \frac{1}{n^{k\alpha}} \|A^k x\| \tag{2.2.10} \\
& \leq N_\tau \frac{1}{n^{k\alpha}} \|A^k x\|
\end{aligned}$$

for  $\frac{n^\alpha}{t^\alpha} > \omega_0$  and for all  $t \in [0, \tau]$ . Indeed,  $\frac{1}{|\lambda - \omega|} \leq \frac{2}{|\lambda|}$ , whenever  $\lambda > \omega_0$ . Hence we have shown that for each  $\tau > 0$  there exists  $N_\tau > 0$  such that

$$\|T(t)W(\frac{t}{n})^{m+1} x - T(t)x\| \leq N_\tau \frac{1}{n^{k\alpha}} \|A^k x\|, \tag{2.2.11}$$

whenever  $\frac{n^\alpha}{t^\alpha} > \omega_0$ , where  $\omega_0 := \max\{0, 2\omega\}$ . Therefore for each  $\tau > 0$  there exists a constant

$$M_\tau := \max(K_\tau, N_\tau) > 0 \tag{2.2.12}$$

such that if  $t \in [0, \tau]$  and  $x \in D(A^k)$ , then  $\|V(\frac{t}{n})^n W(\frac{t}{n})^{m+1} x - T(t)x\| \leq M_\tau (\frac{1}{n^{m\varepsilon}} + \frac{1}{n^{k\alpha}})(\|x\| + \|A^k x\|)$ .

Define

$$f_k(\varepsilon) = \frac{1}{n^{m\varepsilon}} + \frac{1}{n^{k\alpha}}.$$

Then by Lemma 2.2, we have that  $f_k$  achieves an absolute minimum value of

$$\left(\frac{1}{n}\right)^{\frac{km}{k+m+1}} \left[ \left(\frac{k}{m+1}\right)^{\frac{m+1}{k+m+1}} + \left(\frac{m+1}{k}\right)^{\frac{k}{k+m+1}} \right]$$

at  $\varepsilon_k := \frac{(m+1)\ln(\frac{m+1}{k})+km\ln(n)}{(m+k+1)m\ln(n)}$ . Notice that, as in the case  $k = 1$ ,  $\varepsilon_k$  depends on the time-step  $n$ , so it is not practical to take  $\varepsilon = \varepsilon_k$ ; instead we analyze the behavior of  $\varepsilon_k$  for large  $n$ . Now,  $\varepsilon_k := \frac{(m+1)\ln(\frac{m+1}{k})+km\ln(n)}{(m+k+1)m\ln(n)} = \frac{\frac{(m+1)\ln(\frac{m+1}{k})+km}{\ln(n)}}{m(m+k+1)} \longrightarrow \frac{k}{m+k+1}$ , as  $n \longrightarrow \infty$ . Therefore we take  $\varepsilon = \frac{k}{m+k+1}$ , in which case  $\alpha = \frac{m}{m+1}(1 - \varepsilon) = \frac{m}{m+1}(1 - \frac{k}{m+k+1}) = \frac{m}{m+k+1}$  and  $f_k(\varepsilon) = \frac{1}{n^{m\varepsilon}} + \frac{1}{n^{k\alpha}} = \frac{2}{n^{\frac{km}{m+k+1}}}$ . Thus, for the optimal  $\varepsilon_k$  we obtain that  $f_k = \frac{1}{n^{\frac{km}{k+m+1}}}\theta(k, m)$ , where  $\theta(k, m)$  is defined as in Table 2.2. For the non-optimal  $\varepsilon = \frac{k}{m+k+1}$  we have that  $f_k(\varepsilon) = \frac{2}{n^{\frac{km}{k+m+1}}}$  which is quite close to the optimal estimate. In conclusion, for each  $\tau > 0$  there exists a constant  $M_\tau > 0$  such that if  $t \in [0, \tau]$  and  $x \in D(A^k)$ , then

$$\|V(\frac{t}{n})^n W(\frac{t}{n})^{m+1}x - T(t)x\| \leq 2M_\tau (\frac{1}{n})^{\frac{km}{k+m+1}} (\|x\| + \|A^k x\|).$$

Finally suppose that  $x \in D(A^q)$ , where  $q$  is a natural number with  $1 \leq q \leq k-1$ . Then by (2.2.9) we have that for all  $\tau > 0$  there exists  $\hat{N}_\tau > 0$  such that

$$\begin{aligned} & \|T(t)W(\frac{t}{n})^m x - T(t)x\| \\ & \leq \|T(t)\| \sum_{j=0}^m \|W(\frac{t}{n})\|^j \|W(\frac{t}{n})x - x\| \\ & \leq M e^{\omega t} \sum_{j=0}^m (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^j \prod_{l=1}^k R(\frac{b_l n^\alpha}{t^\alpha}, A) A^k x\| \\ & \leq M e^{\omega t} \frac{1 - (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^{m+1}}{1 - 2M \sum_{i=1}^k \frac{|a_i|}{b_i}} \prod_{l=1}^k R(\frac{b_l n^\alpha}{t^\alpha}, A) A^k x\|. \end{aligned}$$

Moreover,

$$\begin{aligned}
& Me^{\omega t} \frac{1 - (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^{m+1}}{1 - 2M \sum_{i=1}^k \frac{|a_i|}{b_i}} \left\| \prod_{l=1}^k R\left(\frac{b_l n^\alpha}{t^\alpha}, A\right) A^k x \right\| \\
&= Me^{\omega t} \frac{1 - (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^{m+1}}{1 - 2M \sum_{i=1}^k \frac{|a_i|}{b_i}} \left\| A^{k-q} \prod_{l=q+1}^k R\left(\frac{b_l n^\alpha}{t^\alpha}, A\right) \prod_{p=1}^q R\left(\frac{b_p n^\alpha}{t^\alpha}, A\right) A^q x \right\| \\
&\leq Me^{\omega t} \frac{1 - (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^{m+1}}{1 - 2M \sum_{i=1}^k \frac{|a_i|}{b_i}} \prod_{l=q+1}^k \|AR\left(\frac{b_l n^\alpha}{t^\alpha}, A\right)\| \prod_{p=1}^q \|R\left(\frac{b_p n^\alpha}{t^\alpha}, A\right) A^q x\| \\
&\leq Me^{\omega t} \frac{1 - (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^{m+1}}{1 - 2M \sum_{i=1}^k \frac{|a_i|}{b_i}} \\
&\quad \cdot \prod_{l=q+1}^k \left\| \frac{b_l n^\alpha}{t^\alpha} R\left(\frac{b_l n^\alpha}{t^\alpha}, A\right) - I \right\| \prod_{p=1}^q \|R\left(\frac{b_p n^\alpha}{t^\alpha}, A\right) A^q x\| \\
&\leq Me^{\omega t} \frac{1 - (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^{m+1}}{1 - 2M \sum_{i=1}^k \frac{|a_i|}{b_i}} (2M + 1)^{k-q} \prod_{p=1}^q \|R\left(\frac{b_p n^\alpha}{t^\alpha}, A\right)\| \cdot \|A^q x\| \\
&\leq Me^{\omega t} \frac{1 - (2M \sum_{i=1}^k \frac{|a_i|}{b_i})^{m+1}}{1 - 2M \sum_{i=1}^k \frac{|a_i|}{b_i}} (2M + 1)^{k-q} M^q 2^q \left(\prod_{p=1}^q \frac{1}{b_p}\right) \frac{t^{q\alpha}}{n^{q\alpha}} \|A^q x\| \\
&\leq \hat{N}_\tau \frac{1}{n^{q\alpha}} \|A^q x\|.
\end{aligned}$$

Hence we have shown that for each  $\tau > 0$  there exists  $\hat{N}_\tau > 0$  such that

$$\|T(t)W\left(\frac{t}{n}\right)^{m+1}x - T(t)x\| \leq \hat{N}_\tau \frac{1}{n^{q\alpha}} \|A^q x\|, \quad (2.2.13)$$

whenever  $\frac{n^\alpha}{t^\alpha} > \omega_0$ , where  $\omega_0 := \max\{0, 2\omega\}$ . Therefore for each  $\tau > 0$  there exists a constant  $\hat{M}_\tau := \max(K_\tau, \hat{N}_\tau) > 0$  such that if  $t \in [0, \tau]$  and  $x \in D(A^q)$ , then  $\|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1}x - T(t)x\| \leq M_\tau \left(\frac{1}{n^{m\varepsilon}} + \frac{1}{n^{q\alpha}}\right) (\|x\| + \|A^q x\|)$ . Take  $\varepsilon = \frac{q}{m+q+1}$ , in which case  $\alpha = \frac{m}{m+1}(1 - \varepsilon) = \frac{m}{m+1}\left(1 - \frac{q}{m+q+1}\right) = \frac{m}{m+q+1}$  and  $f_q(\varepsilon) = \frac{1}{n^{m\varepsilon}} + \frac{1}{n^{q\alpha}} = \frac{2}{n^{\frac{qm}{m+q+1}}}$ . Thus, for the optimal  $\varepsilon_q$  we obtain that  $f_q = \frac{1}{n^{\frac{qm}{q+m+1}}}\theta(q, m)$ , where  $\theta(q, m)$  is defined as in Table 2.2. For the non-optimal  $\varepsilon = \frac{q}{m+q+1}$  we have that  $f_q(\varepsilon) = \frac{2}{n^{\frac{qm}{q+m+1}}}$  which is quite close to the optimal estimate. In conclusion, for each  $\tau > 0$  there exists a constant  $\hat{M}_\tau > 0$  such that if  $t \in [0, \tau]$  and  $x \in D(A^q)$ , then

$$\|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^{m+1}x - T(t)x\| \leq 2M_\tau \left(\frac{1}{n}\right)^{\frac{qm}{q+m+1}} (\|x\| + \|A^q x\|).$$

□

We refer to the method of applying stabilizers of the form (2.2.3) described in Theorem 2.3 as *k-stabilization*. We have shown that *k-stabilizing* an A-stable rational approximation scheme of order  $m$  yields convergence of order  $\frac{km}{k+m+1}$ . Note that if  $x \in D(A^k)$  and  $m \geq 2k^2 + k - 1$ , then *k-stabilizing* improves upon the rate of convergence predicted by the Brenner-Thomée Theorem 1.5. Indeed  $\beta(k) = k - \frac{1}{2}$  if  $0 \leq k < \frac{m+1}{2}$ , which is certainly true for  $m \geq 2k^2 + k - 1$ . Now, *k-stabilizing* yields convergence on all of  $X$  and convergence of order  $\frac{qm}{q+m+1}$  on  $D(A^q)$  for  $1 \leq q \leq k$ . Observe that  $m \geq 2k^2 + k - 1$  if and only if  $\frac{1}{2}m \geq k^2 + \frac{1}{2}k - \frac{1}{2}$  if and only if  $km \geq km + k^2 + k - \frac{1}{2}m - \frac{1}{2}k - \frac{1}{2} = (k - \frac{1}{2})(m + k + 1)$ . Therefore  $m \geq 2k^2 + k - 1$  if and only if  $\frac{km}{k+m+1} \geq k - \frac{1}{2} = \beta(k)$ .

Moreover if  $m = 2k^2 + k - 1 + p$  for  $p = 0, 1, 2, \dots$ , then *k-stabilizing* improves the order of convergence by a magnitude of  $\frac{p}{2} \cdot \frac{1}{2k^2+2k+p}$ . That is,  $\frac{mk}{k+m+1} = \beta(k) + \frac{p}{2} \cdot \frac{1}{2k^2+2k+p}$ . Indeed,

$$\begin{aligned}
\beta(k) + \frac{p}{2} \cdot \frac{1}{2k^2 + 2k + p} &= k - \frac{1}{2} + \frac{p}{2} \cdot \frac{1}{2k^2 + 2k + p} \\
&= \frac{(2k - 1)(2k^2 + 2k + p) + p}{2(2k^2 + 2k + p)} \\
&= \frac{(2k - 1)(2k^2 + 2k) + 2kp}{2(2k^2 + 2k + p)} \\
&= \frac{(2k - 1)(k^2 + k) + 2kp}{2k^2 + 2k + p} \\
&= \frac{2k^3 + k^2 - k + kp}{2k^2 + 2k + p} \\
&= \frac{k(2k^2 + k - 1 + p)}{2k^2 + 2k + p} \\
&= \frac{km}{m + k + 1}.
\end{aligned}$$

Observe that the rate of convergence for a *k-stabilized* scheme of order  $m$  is given by  $\frac{km}{k+m+1}$ , which approaches  $k$  as  $m$  approaches infinity. On the other hand,  $\beta(k) =$

$k - \frac{1}{2}$  for  $k < \frac{m+1}{2}$ . Therefore, in principle, stabilization yields convergence on  $X$  and improves the speed of convergence on  $D(A)$  by a magnitude of about  $\frac{1}{2}$ .

## 2.3 Examples

In this section, we discuss Theorem 2.3 for bounded strongly continuous semigroups (that is with  $\omega = 0$ ) in the cases  $k = 2, 3, 4$ . We begin with the case  $k = 2$ .

**Example 2.3.1 (2-stabilization).** Suppose that  $A$  generates a bounded strongly continuous semigroup  $T(t)$  with  $\|T(t)\| \leq M$  and let  $b_1, b_2 > 0$  be distinct positive real numbers. Define

$$W_2(t) := -(-A^2)t^{2\alpha} \prod_{i=1}^2 (b_i I - t^\alpha A)^{-1} + I,$$

where  $\alpha = \frac{m}{m+3}$  and  $m \geq 2$ . Then

$$\begin{aligned} W_2(t) &= \frac{1}{t^{2\alpha}} R\left(\frac{b_1}{t^\alpha}, A\right) R\left(\frac{b_2}{t^\alpha}, A\right) [-t^{2\alpha} A^2 + t^{2\alpha} \left(\frac{b_1}{t^\alpha} I - A\right) \left(\frac{b_2}{t^\alpha} I - A\right)] \\ &= \frac{1}{t^{2\alpha}} R\left(\frac{b_1}{t^\alpha}, A\right) R\left(\frac{b_2}{t^\alpha}, A\right) [-t^{2\alpha} A^2 + t^{2\alpha} \left(\frac{b_1 b_2}{t^{2\alpha}} I - \left(\frac{b_1 + b_2}{t^\alpha}\right) A + A^2\right)] \\ &= \frac{1}{t^{2\alpha}} R\left(\frac{b_1}{t^\alpha}, A\right) R\left(\frac{b_2}{t^\alpha}, A\right) [b_1 b_2 I - (b_1 + b_2) t^\alpha A] \\ &= a_1 (b_1 I - t^\alpha A)^{-1} + a_2 (b_2 I - t^\alpha A)^{-1} \end{aligned} \tag{2.3.1}$$

for some constants  $a_1, a_2$  if and only if  $b_1 b_2 I - (b_1 + b_2) t^\alpha A = a_1 (b_2 I - t^\alpha A) + a_2 (b_1 I - t^\alpha A)$ . Equivalently, we obtain the matrix equation

$$\begin{pmatrix} b_2 & b_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 b_2 \\ b_1 + b_2 \end{pmatrix}.$$

Therefore

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} b_2 & b_1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} b_1 b_2 \\ b_1 + b_2 \end{pmatrix} = \frac{1}{b_2 - b_1} \begin{pmatrix} 1 & -b_1 \\ -1 & b_2 \end{pmatrix} \begin{pmatrix} b_1 b_2 \\ b_1 + b_2 \end{pmatrix} \\ &= \frac{1}{b_2 - b_1} \begin{pmatrix} b_1 b_2 - b_1^2 - b_2 b_1 \\ -b_1 b_2 + b_2^2 + b_2 b_1 \end{pmatrix} = \frac{1}{b_2 - b_1} \begin{pmatrix} -b_1^2 \\ b_2^2 \end{pmatrix}. \end{aligned}$$

By (2.3.1), we have that

$$W_2(t) = -\frac{b_1}{b_2 - b_1} \left(I - \frac{t^\alpha}{b_1} A\right)^{-1} + \frac{b_2}{b_2 - b_1} \left(I - \frac{t^\alpha}{b_2} A\right)^{-1},$$

where  $\alpha = \frac{m}{m+3}$  and  $m \geq 2$ .

To determine the size of the constant  $M_\tau$  defined in (2.2.12), we specialize further and assume that  $T(t)$  is a contraction semigroup; that is, we assume that  $\omega = 0$  and  $M = 1$ . Then by the proof of the previous theorem for  $k = 2$ , we obtain

$$V\left(\frac{t}{n}\right)^n - W_2\left(\frac{t}{n}\right)^{m+1}x - T(t)x \leq 2M_\tau \left(\frac{1}{n}\right)^{\frac{2m}{m+3}} (\|x\| + \|A^k x\|),$$

where  $M_\tau := \max(K_\tau, N_\tau)$  for  $K_\tau = C\tau\tau^{\frac{2m}{m+3}}3^{m+1}\left(\frac{b_1^2+b_2^2}{b_2-b_1}\right)^{m+1}$  (see (2.2.6)) and  $N_\tau = 4\frac{1}{b_1b_2}\frac{1-(2\frac{b_1+b_2}{b_2-b_1})^{m+1}}{1-2\frac{b_1+b_2}{b_2-b_1}}\tau^{\frac{2m}{m+3}}$  (see (2.2.10)). Therefore the choice of the constants  $b_i, i = 1, 2$  effects the size of  $M_\tau$ . In order to further analyze the constants, assume that  $\tau = 1, b_1 = b > 0$  and  $b_2 = \delta$  for some  $\delta > 1$ . Then  $K_1 = C3^{m+1}(b)^{m+1}\left(\frac{1+\varepsilon^2}{\varepsilon-1}\right)^{m+1}$  and  $N_1 = \frac{4}{\varepsilon b^2}\frac{1-(2\frac{1+\varepsilon}{\varepsilon-1})^{m+1}}{1-2\frac{1+\varepsilon}{\varepsilon-1}}$ . If we take  $\varepsilon = 5$ , then  $N_1 = \frac{2}{5b^2}(3^{m+1} - 1) \leq \frac{2}{5b^2}3^{m+1}$  and  $K_\tau \leq C3^{m+1}b^{m+1}\left(\frac{26}{4}\right)^{m+1}$ . If we now take  $b = \frac{4}{26}$ , then  $N_1 \leq 17 \cdot 3^{m+1}$  and  $K_1 \leq C3^{m+1}$ . Therefore, if  $\tau = 1, b_1 \approx \frac{2}{13}$  and  $b_2 \approx \frac{10}{13}$ , then  $M_\tau \approx C3^{m+1}$ . However, if one takes  $b_1 = 1$  and  $b_2 = 2$ , then  $K_1 \leq C15^{m+1}$  and  $N_1 \leq \frac{2}{25} \cdot 6^{m+1}$  so that  $M_\tau \approx C15^{m+1}$  for large  $m$ .

If  $x \in D(A)$ , then we may estimate as follows: for all  $\tau > 0$  there exists  $\hat{N}_\tau > 0$  such that

$$\begin{aligned}
& \|T(t)W_2(\frac{t}{n})^{m+1}x - T(t)x\| \leq \|T(t)\| \left\| \sum_{j=0}^m W_2(\frac{t}{n})^j \right\| \|W_2(\frac{t}{n})x - x\| \\
& = \|T(t)\| \sum_{j=0}^m \|W_2(\frac{t}{n})\|^j \|R(\frac{b_1 n^\alpha}{t^\alpha}, A)R(\frac{b_2 n^\alpha}{t^\alpha}, A)A^2x\| \\
& \leq M e^{\omega t} \frac{1 - (2\frac{b_1+b_2}{b_2-b_1}M)^{m+1}}{1 - (2\frac{b_1+b_2}{b_2-b_1}M)} \|R(\frac{b_1 n^\alpha}{t^\alpha}, A)R(\frac{b_2 n^\alpha}{t^\alpha}, A)A^2x\| \\
& \leq M e^{\omega t} \frac{1 - (2\frac{b_1+b_2}{b_2-b_1}M)^{m+1}}{1 - (2\frac{b_1+b_2}{b_2-b_1}M)} \|AR(\frac{b_1 n^\alpha}{t^\alpha}, A)\| \cdot \|R(\frac{b_2 n^\alpha}{t^\alpha}, A)Ax\| \\
& \leq (2M + 1)M^2 e^{\omega t} \frac{1 - (2\frac{b_1+b_2}{b_2-b_1}M)^{m+1}}{1 - (2\frac{b_1+b_2}{b_2-b_1}M)} \frac{t^\alpha}{n^\alpha} \|Ax\| \\
& \leq \hat{N}_\tau \frac{1}{n^\alpha} \|Ax\|
\end{aligned}$$

for all  $t \in [0, \tau]$ . Therefore 2-stabilizing for initial data  $x \in D(A)$  yields the same rate of convergence as 1-stabilizing for initial data  $x \in D(A)$ . That is, if we define  $\hat{M} := \max(K_\tau, \hat{N}_\tau)$ , then

$$\|V(\frac{t}{n})^n W(\frac{t}{n})^{m+1}x - T(t)x\| \leq 2\hat{M}_\tau (\frac{1}{n})^{\frac{2m}{m+3}} (\|x\| + \|Ax\|). \quad (2.3.2)$$

The following table compares the convergence results of the Brenner-Thomée Theorem 1.5 for the Crank-Nicolson scheme to the error estimates obtained by using Theorem 2.3 to  $k$ -stabilize the Crank-Nicolson scheme for  $k = 1, 2$  in the case  $m = 2$ .

Table 2.3. Comparison of Theorem 2.3 to Theorem 1.5 for  $V_{CN}(t)$

	$x \in X$	$x \in D(A)$	$x \in D(A^2)$
Unstabilized	$n^{\frac{1}{2}}$	$(\frac{1}{n})^{\frac{1}{2}}$	$(\frac{1}{n})^{\frac{4}{3}}$
1-stabilized	converges	$(\frac{1}{n})^{\frac{1}{2}}$	
2-stabilized	converges	$(\frac{1}{n})^{\frac{1}{2}}$	$(\frac{1}{n})^{\frac{4}{5}}$

If one stabilizes the Crank-Nicolson scheme with  $W_2(t)$ , then one must compute  $V_{CN}(\frac{t}{n})^n W_2(\frac{t}{n})^3$ , where  $W_2(\frac{t}{n}) = a_1(b_1 I - z)^{-1} + a_2(b_2 I - z)^{-1}$  for  $z := (\frac{t}{n})^\alpha A$ ,  $\alpha = \frac{2}{5} = 0.4$ ,  $a_1 = \frac{-b_1^2}{b_2 - b_1}$  and  $a_2 = \frac{b_2^2}{b_2 - b_1}$ . Hence there exist constants  $c_i$ ,  $1 \leq i \leq 6$  such that

$$\begin{aligned} W_2\left(\frac{t}{n}\right)^3 &= c_1(b_1 I - z)^{-3} + c_2(b_1 I - z)^{-2} + c_3(b_1 I - z)^{-1} \\ &\quad + c_4(b_2 I - z)^{-3} + c_5(b_2 I - z)^{-2} + c_6(b_2 I - z)^{-1}. \end{aligned}$$

Therefore, 2-stabilizing the Crank-Nicolson scheme is, in principle, a weighted sum of six first order stabilizers.

**Example 2.3.2 (3-stabilization).** Define

$$W_3(t) := t^{3\alpha} A^3 (b_1 I - t^\alpha A)^{-1} (b_2 I - t^\alpha A)^{-1} (b_3 I - t^\alpha A)^{-1} + I$$

for some fixed  $b_i$ ,  $i = 1, 2, 3$  such that  $b_i > 0$  for all  $i$  and that  $b_i \neq b_j$  whenever  $i \neq j$ . Observe that

$$\begin{aligned} \prod_{i=1}^3 (b_i I - t^\alpha A) &= t^{3\alpha} \prod_{i=1}^3 \left(\frac{b_i}{t^\alpha} I - A\right) \\ &= t^{3\alpha} \left(\frac{b_1 b_2}{t^{2\alpha}} I - \left(\frac{b_1 + b_2}{t^\alpha}\right) A + A^2\right) \left(\frac{b_3}{t^\alpha} I - A\right) \\ &= t^{3\alpha} \left[\frac{b_1 b_2 b_3}{t^{3\alpha}} I - \frac{b_3(b_1 + b_2)}{t^{2\alpha}} A + \frac{b_3}{t^\alpha} A^2 - \frac{b_1 b_2}{t^{2\alpha}} A + \left(\frac{b_1 + b_2}{t^\alpha}\right) A^2 - A^3\right] \\ &= t^{3\alpha} [b_1 b_2 b_3 I - b_3(b_1 + b_2) t^\alpha A + b_3 t^{2\alpha} A^2 - b_1 b_2 t^\alpha A + (b_1 + b_2) t^{2\alpha} A^2 - t^{3\alpha} A^3] \\ &= t^{3\alpha} \left[\frac{b_1 b_2 b_3}{t^{3\alpha}} I - \left(\frac{b_1 b_3 + b_2 b_3 + b_1 b_2}{t^{2\alpha}}\right) A + \left(\frac{b_1 + b_2 + b_3}{t^\alpha}\right) A^2 - A^3\right]. \end{aligned}$$

Moreover,

$$\begin{aligned}
W_3(t) &= t^{3\alpha} A^3 (b_1 I - t^\alpha A)^{-1} (b_2 I - t^\alpha A)^{-1} (b_3 I - t^\alpha A)^{-1} + I \\
&= A^3 R\left(\frac{b_1}{t^\alpha}, A\right) R\left(\frac{b_2}{t^\alpha}, A\right) R\left(\frac{b_3}{t^\alpha}, A\right) + I \\
&= \frac{1}{t^{3\alpha}} R\left(\frac{b_1}{t^\alpha}, A\right) R\left(\frac{b_2}{t^\alpha}, A\right) R\left(\frac{b_3}{t^\alpha}, A\right) \\
&\quad \cdot [t^{3\alpha} A^3 + t^{3\alpha} \left(\frac{b_1}{t^\alpha} I - A\right) \left(\frac{b_2}{t^\alpha} I - A\right) \left(\frac{b_3}{t^\alpha} I - A\right)] \\
&= \frac{1}{t^{3\alpha}} \prod_{i=1}^3 R\left(\frac{b_i}{t^\alpha}, A\right) \\
&\quad \cdot [t^{3\alpha} A^3 + \frac{b_1 b_2 b_3}{t^{3\alpha}} I - \left(\frac{b_1 b_3 + b_2 b_3 + b_1 b_2}{t^{2\alpha}}\right) A + \left(\frac{b_1 + b_2 + b_3}{t^\alpha}\right) A^2 - A^3] t^{3\alpha} \\
&= \frac{1}{t^{3\alpha}} \\
&\quad \cdot \prod_{i=1}^3 R\left(\frac{b_i}{t^\alpha}, A\right) [b_1 b_2 b_3 I - (b_1 b_3 + b_2 b_3 + b_1 b_2) t^\alpha A + (b_1 + b_2 + b_3) t^{2\alpha} A^2] \\
&= a_1 (b_1 I - t^\alpha A)^{-1} + a_2 (b_2 I - t^\alpha A)^{-1} + a_3 (b_3 I - t^\alpha A)^{-1}.
\end{aligned}$$

Therefore, in order to obtain the coefficients  $a_1, a_2, a_3$ , such that

$$W_3(t) = a_1 (b_1 I - t^\alpha A)^{-1} + a_2 (b_2 I - t^\alpha A)^{-1} + a_3 (b_3 I - t^\alpha A)^{-1},$$

we must solve

$$\frac{b_1 b_2 b_3 - (b_1 b_3 + b_2 b_3 + b_1 b_2)z + (b_1 + b_2 + b_3)z^2}{(b_1 - z)(b_2 - z)(b_3 - z)} = \sum_{i=1}^3 \frac{a_i}{b_i - z},$$

where  $z = t^\alpha A$ . This leads to the system of equations

$$a_1 b_2 b_3 + a_2 b_1 b_3 + a_3 b_1 b_2 = b_1 b_2 b_3$$

$$a_1 b_2 + a_1 b_3 + a_2 b_1 + a_2 b_3 + a_3 b_1 + a_3 b_2 = b_1 b_3 + b_2 b_3 + b_1 b_2$$

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3.$$

This is equivalent to the matrix equation

$$\begin{pmatrix} b_2 b_3 & b_1 b_3 & b_1 b_2 \\ b_2 + b_3 & b_1 + b_3 & b_1 + b_2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1 b_2 b_3 \\ b_1 b_3 + b_2 b_3 + b_1 b_2 \\ b_1 + b_2 + b_3 \end{pmatrix}.$$

Now,

$$\begin{aligned}
& \det \begin{pmatrix} b_2b_3 & b_1b_3 & b_1b_2 \\ b_2 + b_3 & b_1 + b_3 & b_1 + b_2 \\ 1 & 1 & 1 \end{pmatrix} = -\det \begin{pmatrix} 1 & 1 & 1 \\ b_2 + b_3 & b_1 + b_3 & b_1 + b_2 \\ b_2b_3 & b_1b_3 & b_1b_2 \end{pmatrix} \\
&= -\det \begin{pmatrix} b_1 + b_3 & b_1 + b_2 \\ b_1b_3 & b_1b_2 \end{pmatrix} + \det \begin{pmatrix} b_2 + b_3 & b_1 + b_2 \\ b_2b_3 & b_1b_2 \end{pmatrix} \\
&\quad - \det \begin{pmatrix} b_2 + b_3 & b_1 + b_3 \\ b_2b_3 & b_1b_3 \end{pmatrix} \\
&= -[b_1b_2(b_1 + b_3) - b_1b_3(b_1 + b_2)] + [b_1b_2(b_2 + b_3) - b_2b_3(b_1 + b_2)] \\
&\quad - [b_1b_3(b_2 + b_3) - b_2b_3(b_1 + b_3)] \\
&= (b_2 - b_1)(b_3 - b_1)(b_3 - b_2) =: d.
\end{aligned}$$

Therefore the matrix  $\begin{pmatrix} b_2b_3 & b_1b_3 & b_1b_2 \\ b_2 + b_3 & b_1 + b_3 & b_1 + b_2 \\ 1 & 1 & 1 \end{pmatrix}$  is invertible and

$$\begin{aligned}
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} &= \begin{pmatrix} b_2b_3 & b_1b_3 & b_1b_2 \\ b_2 + b_3 & b_1 + b_3 & b_1 + b_2 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} b_1b_2b_3 \\ b_1b_2 + b_1b_3 + b_2b_3 \\ b_1 + b_2 + b_3 \end{pmatrix} \\
&= \frac{1}{d} \begin{pmatrix} b_3 - b_2 & -b_1(b_3 - b_2) & b_1^2(b_3 - b_2) \\ -(b_3 - b_1) & b_2(b_3 - b_1) & -b_2^2(b_3 - b_1) \\ b_2 - b_1 & -b_3(b_2 - b_1) & b_3^2(b_2 - b_1) \end{pmatrix} \begin{pmatrix} b_1b_2b_3 \\ b_1b_2 + b_1b_3 + b_2b_3 \\ b_1 + b_2 + b_3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{b_1^3}{(b_2 - b_1)(b_3 - b_1)} \\ \frac{-b_2^3}{(b_2 - b_1)(b_3 - b_2)} \\ \frac{b_3^3}{(b_3 - b_1)(b_3 - b_2)} \end{pmatrix}.
\end{aligned}$$

Recall that for  $k = 2$  we found  $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{-b_1^2}{b_2-b_1} \\ \frac{b_2^2}{b_2-b_1} \end{pmatrix}$ . Now we have for  $k = 3$  that

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \frac{b_1^3}{(b_2-b_1)(b_3-b_1)} \\ \frac{-b_2^3}{(b_2-b_1)(b_3-b_2)} \\ \frac{b_3^3}{(b_3-b_1)(b_3-b_2)} \end{pmatrix}. \text{ Hence for arbitrary } k \geq 2 \text{ we claim that}$$

$$a_i = \frac{(-1)^{k+i} b_i^k}{\prod_{\substack{j=1 \\ j \neq i}}^k |b_j - b_i|}. \quad (2.3.3)$$

This formula can be proved by direct computation; the proof is omitted.

**Remark 2.4.** *Let  $A$  be the generator of a strongly continuous semigroup  $T(t)$  with  $\|T(t)\| \leq M$  for all  $t \geq 0$ . Let  $r(\cdot)$  be a rational approximation of the exponential of approximation order  $m$  and let  $1 \leq k \leq m$ . Let  $\alpha = \frac{m}{m+k+1}$  and let  $b_i > 0$  ( $1 \leq i \leq k$ ) be pairwise distinct positive real numbers. Define  $V(t) := r(tA)$  and*

$$\begin{aligned} W_k(t) &:= -(-t^\alpha A)^k \prod_{i=1}^k (b_i I - t^\alpha A)^{-1} + I \\ &= \sum_{i=1}^k a_i (b_i I - t^\alpha A)^{-1} \end{aligned}$$

where  $a_i = \frac{(-1)^{k+i} b_i^k}{\prod_{\substack{j=1 \\ j \neq i}}^k |b_j - b_i|}$ . Let  $1 \leq q \leq k$  and  $n \geq m+1$  be natural numbers. Then  $\lim_{n \rightarrow \infty} V(\frac{t}{n})^n W_k(\frac{t}{n})^{m+1} x = T(t)x$  for all  $x \in X$ . Furthermore, if  $x \in D(A^q)$  and if  $t \in [0, \tau]$ , then

$$\|V(\frac{t}{n})^n W_k(\frac{t}{n})^{m+1} x - T(t)x\| \leq M_\tau \left(\frac{1}{n}\right)^{\frac{qm}{q+m+1}} (\|x\| + \|A^q x\|),$$

where the magnitude of the constant  $M_\tau$  is determined by  $M, \tau$  and the choice of the numbers  $b_i$ .

Moreover,  $k$ -stabilizing a rational approximation scheme with  $W_k(t)$  is equivalent to taking a weighted sum of  $(m + 1)k$  first order stabilizers; that is

$$\begin{aligned} W_k(t)^{m+1} &= \left( \sum_{i=1}^k a_i (b_i I - (\frac{t}{n})^\alpha A)^{-1} \right)^m \\ &= \sum_{j=1}^{m+1} \sum_{i=1}^k c_{i,j} (b_i I - (\frac{t}{n})^\alpha A)^{-j}. \end{aligned}$$

The following table compares the results of the Brenner-Thomée Theorem 1.5 to the error estimates obtained by using Theorem 2.3 to  $k$ -stabilize an A-stable rational approximation scheme of order 3 for  $k = 1, 2, 3$  in the case  $m = 3$ . For example, we may take  $V(t) := r(tA)$ , where  $r(z) := \frac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}$ . The symbol  $\star$  means that  $\beta(k)$  is undefined for the given values of  $k$  and  $m$ .

Table 2.4. Comparison of Theorem 2.3 to Theorem 1.5 for  $m = 3$

	$x \in X$	$x \in D(A)$	$x \in D(A^2)$	$x \in D(A^3)$
Unstabilized	$n^{\frac{1}{2}}$	$(\frac{1}{n})^{\frac{1}{2}}$	$\star$	$(\frac{1}{n})^{\frac{9}{4}}$
1-stabilized	converges	$(\frac{1}{n})^{\frac{3}{5}}$		
2-stabilized	converges	$(\frac{1}{n})^{\frac{3}{5}}$	$\frac{1}{n}$	
3-stabilized	converges	$(\frac{1}{n})^{\frac{3}{5}}$	$\frac{1}{n}$	$(\frac{1}{n})^{\frac{9}{7}}$

**Example 2.3.3 (4-stabilization).** Let  $A$  be the generator of a bounded strongly continuous semigroup and let  $V(t) := r(tA)$ , where  $r$  is an A-stable rational approximation of the exponential of approximation order  $m \geq 4$ . Let  $0 < b_1 < b_2 < b_3 < b_4$

and  $\alpha = \frac{m}{m+5}$ . Define

$$\begin{aligned}
W_4(t) &:= \frac{-b_1^4}{(b_4 - b_1)(b_3 - b_1)(b_2 - b_1)}(b_1I - t^\alpha A)^{-1} \\
&\quad + \frac{b_2^4}{(b_4 - b_2)(b_3 - b_2)(b_2 - b_1)}(b_2I - t^\alpha A)^{-1} \\
&\quad - \frac{b_3^4}{(b_4 - b_3)(b_3 - b_2)(b_3 - b_1)}(b_3I - t^\alpha A)^{-1} \\
&\quad + \frac{b_4^4}{(b_4 - b_3)(b_4 - b_2)(b_4 - b_1)}(b_4I - t^\alpha A)^{-1}.
\end{aligned}$$

Then  $W_4(t)$  stabilizes  $V(t)$ . The following table compares the rates of convergence given by the Brenner-Thomée Theorem 1.5 to the rates of convergence given by using Theorem 2.3 to  $k$ -stabilize an A-stable rational approximation scheme of order 4 for  $k = 1, 2, 3, 4$  in the case  $m = 4$ . For example, we may take the Hammer-Hollingsworth scheme  $V(t) := r(tA)$ , where  $r(z) := \frac{1 + \frac{1}{2}z + \frac{1}{12}z^2}{1 - \frac{1}{2}z + \frac{1}{12}z^2}$ . The letter c indicates convergence of unknown speed.

Table 2.5. Comparison of Theorem 2.3 to Theorem 1.5 for  $m = 4$

	$x \in X$	$x \in D(A)$	$x \in D(A^2)$	$x \in D(A^3)$	$x \in D(A^4)$
Unstabilized	$n^{\frac{1}{2}}$	$(\frac{1}{n})^{\frac{1}{2}}$	$(\frac{1}{n})^{\frac{3}{2}}$	$(\frac{1}{n})^{\frac{12}{5}}$	$(\frac{1}{n})^{\frac{16}{5}}$
1-stabilized	c	$(\frac{1}{n})^{\frac{3}{5}}$			
2-stabilized	c	$(\frac{1}{n})^{\frac{3}{5}}$	$(\frac{1}{n})^{\frac{8}{7}}$		
3-stabilized	c	$(\frac{1}{n})^{\frac{3}{5}}$	$(\frac{1}{n})^{\frac{8}{7}}$	$(\frac{1}{n})^{\frac{3}{2}}$	
4-stabilized	c	$(\frac{1}{n})^{\frac{3}{5}}$	$(\frac{1}{n})^{\frac{8}{7}}$	$(\frac{1}{n})^{\frac{3}{2}}$	$(\frac{1}{n})^{\frac{16}{9}}$

In summary, using Theorem 2.3 in order to  $k$ -stabilize a rational approximation scheme of order  $m$  improves upon the Brenner Thomée Theorem 1.5 in two ways. First, given any strongly continuous semigroup and arbitrary initial data  $x \in X$ , Theorem 2.3 guarantees the convergence of intrinsically unstable schemes, whereas

the Brenner-Thomé Theorem 1.5 does not. Furthermore, if  $x \in D(A^k)$  then  $k$ -stabilizing improves the rate of convergence for schemes of sufficiently large order  $m$ . Indeed, 1-stabilizing improves the speed of convergence for  $m \geq 3$ , whereas 2-stabilizing improves the speed of convergence for  $m \geq 10$ . Moreover, 3-stabilizing improves the speed of convergence for  $m \geq 21$  and 4-stabilizing improves the speed of convergence for  $m \geq 36$ .

## 2.4 Abstract Stabilization Results

The main result of this section is a stabilized version of the Lax-Chernoff Theorem obtained via an abstract version of the Luskin-Rannacher stabilization techniques for analytic semigroups. The proof requires a stabilized version of the Trotter-Kato Theorem. These results were proven first by Y. Zhuang using different methods for the case  $j = 1$  in [Zh1]. In the following we will assume that  $(A, D(A))$  generates a strongly continuous semigroup  $T(\cdot)$  with  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . In [Fl], S. Flory used convolution with certain sequences of strongly continuous operator families in order to obtain the more generalized form of the stabilized Trotter-Kato theorem.

**Theorem 2.5 (Stabilized Lax-Chernoff).** *Suppose  $A$  generates a strongly continuous semigroup  $(T(t))_{t \geq 0}$ . Let  $M, \omega > 0$  be constants such that  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . Let  $\{V(t) : t \in [0, \tau]\}$  and  $\{W(t) : t \in [0, \tau]\}$  be two strongly continuous commuting families of bounded linear operators on  $X$  satisfying  $W(0) = V(0) = I$  and  $V$  is  $D(A^\infty)$ -consistent; that is,  $\lim_{t \rightarrow 0} \frac{V(t)x - x}{t} = Ax$  for all  $x \in D(A^\infty)$ . Consider the following statements.*

(I)  $\|V(t)^{n-j}W(t)^j\| \leq Me^{\omega nt}$  for all  $n \geq j$  and  $t \in [0, \tau]$ .

(II)  $\lim_{n \rightarrow \infty} V(\frac{t}{n})^{n-j}W(\frac{t}{n})^j x = T(t)x$  for all  $x \in X$  uniformly for  $t$  in compact intervals.

Then statement (I) implies statement (II).

Since the proof of our main result in this section requires a stabilized extension of the Trotter-Kato Theorem, we first recall the classical Trotter-Kato Theorem; for a proof see [ABHN], Theorem 3.6.1.

**Theorem 2.6 (Trotter-Kato).** *For each natural number  $n$ , let  $(T_n(t))_{t \geq 0}$  be a strongly continuous semigroup generated by an operator  $(A_n, D(A_n))$  on a Banach space  $X$ . Suppose that  $\|T_n(t)\| \leq M$  for each  $t \geq 0$  and  $n \in \mathbb{N}$ . Let  $(A, D(A))$  be a densely defined operator on  $X$ . Suppose that there exists  $\omega \geq 0$  such that  $(\omega, \infty) \subset \rho(A)$  and*

$$\lim_{n \rightarrow \infty} R(\lambda, A_n)x = R(\lambda, A)x \quad (2.4.1)$$

for all  $x \in X, \lambda > \omega$ . Then  $(A, D(A))$  generates a strongly continuous semigroup  $(T(t))_{t \geq 0}$  and

$$T(t)x = \lim_{n \rightarrow \infty} T_n(t)x \quad (2.4.2)$$

uniformly on  $[0, \tau]$  for all  $\tau > 0$  and all  $x \in X$ .

The following example, which may be found in [EN, p.205-6], shows that the stability assumption  $\|T_n(t)\| \leq M$  is crucial. Consider the multiplication operator  $A(x_k) := (ikx_k)$  on the Banach space  $c_0$ , with domain  $D(A) := \{(x_k) \in c_0 : (ikx_k) \in c_0\}$ . Then  $(A, D(A))$  generates the strongly continuous semigroup  $T(t)(x_k) = (e^{ikt}x_k)$  for  $t \geq 0$ . We perturb  $A$  by the bounded operators  $P_n(x_k) := (0, \dots, nx_n, 0, \dots)$  to obtain the sequence of new operators  $A_n := A + P_n$ . Then each  $(A_n, D(A_n))$  generates a strongly continuous semigroup  $(T_n(t))_{(t \geq 0)}$  and for each  $x = (x_k) \in D(A)$ , we have

$$\|A_n x - Ax\| = \|P_n x\| = n|x_n| \rightarrow 0.$$

However, the semigroups  $T_n(t)$  do not converge. In fact, one has that  $\|T_n(t)\| \geq e^{nt}$  for all  $n \in \mathbb{N}$  and  $t \geq 0$  since

$$T_n(t)x = (e^{it}x_1, e^{2it}x_2, \dots, e^{(in+n)t}x_n, e^{(N+1)it}x_{n+1}, \dots).$$

Hence by the Principle of Uniform Boundedness, there exists  $x \in X$  such that  $(T_n(t)x)_{n \in \mathbb{N}}$  does not converge.

The following two lemmas are needed for the proof of the stabilized Lax-Chernoff Theorem.

**Lemma 2.7.** *Let  $V(\cdot)$  and  $W(\cdot)$  be as in the statement of the Stabilized Lax-Chernoff Theorem above. Define  $A_s x := \frac{V(s)x-x}{s}$  for all  $s \in (0, \tau]$  and  $x \in X$ . Then for every  $\omega' > \omega$  there exists an  $s' > 0$  so that for all  $s \in (0, s')$ ,*

$$\|e^{tA_s}W(s)^j\| \leq Me^{\omega'(js+t)}.$$

*Proof.* Fix  $s \in (0, \tau]$ . Let  $\gamma$  be a smooth closed path around  $\sigma(A_s)$  and the point  $-\frac{1}{s}$ . Since  $A_s$  is bounded, we may use the Dunford-Schwartz calculus to make the representation

$$e^{tA_s} = \frac{1}{2\pi i} \int_{\gamma} e^{tz} R(z, A_s) dz.$$

Hence we have the equality

$$\begin{aligned} \|e^{tA_s}W(s)^j\| &= \left\| \frac{1}{2\pi i} \int_{\gamma} e^{tz} R(z, A_s) dz W(s)^j \right\| \\ &= \left\| \frac{1}{2\pi i} \int_{\gamma} e^{tz} \left( zI - \frac{V(s) - I}{s} \right)^{-1} dz W(s)^j \right\| \\ &= \left\| \frac{1}{2\pi i} \int_{\gamma} e^{tz} \left( \left( z + \frac{1}{s} \right) I - \frac{1}{s} V(s) \right)^{-1} dz W(s)^j \right\|. \end{aligned}$$

Choose the path  $\gamma$  of integration so that  $z$  has sufficiently large modulus to guarantee that  $\|\frac{\frac{1}{s}V(s)}{z+\frac{1}{s}}\| < 1$ . Then we can use the Von Neumann series to obtain

$$\begin{aligned}\|e^{tA_s}W(s)^j\| &= \left\|\frac{1}{2\pi i}\int_{\gamma}e^{tz}\frac{1}{z+\frac{1}{s}}\sum_{n=0}^{\infty}\left(\frac{\frac{1}{s}V(s)}{z+\frac{1}{s}}\right)^n dzW(s)^j\right\| \\ &= \left\|\sum_{n=0}^{\infty}V(s)^n\frac{1}{s}\frac{1}{2\pi i}\int_{\gamma}e^{tz}\frac{1}{(z+\frac{1}{s})^{n+1}}dzW(s)^j\right\| \\ &\leq \sum_{n=0}^{\infty}\|V(s)^nW(s)^j\|\cdot\left|\frac{1}{s}\frac{1}{2\pi i}\int_{\gamma}e^{tz}\frac{1}{(z+\frac{1}{s})^{n+1}}dz\right|.\end{aligned}$$

Thus using assumption (I) of Theorem 2.5 and the Cauchy formula  $\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-a)^{n+1}}dz$  we obtain

$$\begin{aligned}\|e^{tA_s}W(s)^j\| &\leq \sum_{n=0}^{\infty}Me^{(j+n)\omega s}\frac{1}{s^n}\frac{t^n}{n!}e^{-\frac{t}{s}} \\ &= Me^{j\omega s}e^{-\frac{t}{s}}\sum_{n=0}^{\infty}\frac{(\frac{t}{s}e^{\omega s})^n}{n!} \\ &= Me^{j\omega s}e^{\frac{t}{s}(e^{\omega s}-1)}\end{aligned}$$

Now let  $\omega_0 > \omega$ . Then there exists  $s_0 > 0$  such that  $e^{\omega s} - 1 \leq \omega_0 s$  for all  $s \in (0, s_0)$ .

Hence,

$$\|e^{tA_s}W(s)^j\| \leq Me^{j\omega s}e^{\omega_0 t} \leq Me^{\omega_0(js+t)}.$$

□

**Lemma 2.8.** *Let  $V(\cdot), W(\cdot)$  and  $A_s$  be as in the previous Lemma. Furthermore, suppose that statement (I) of the Stabilized Lax-Chernoff Theorem holds. Then for every  $\tau > 0$  and  $x \in X$ , we have*

$$\lim_{n \rightarrow \infty} \left\|V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^j x - e^{tA_{\frac{t}{n}}} W\left(\frac{t}{n}\right)^j x\right\| = 0$$

uniformly for  $t \in (0, \tau]$ .

*Proof.* Hypothesis (I) of the Stabilized Lax-Chernoff Theorem 2.5 and the previous Lemma 2.7 yield that  $\|V(\frac{t}{n})^n W(\frac{t}{n})^j x - e^{tA_{\frac{t}{n}}} W(\frac{t}{n})^j x\|$  is exponentially bounded. By

the Banach Convergence Theorem, it is enough to show that the statement holds for  $x \in D(A^\infty)$ . We use the Dunford-Schwartz calculus to make the following representation of  $e^{tA\frac{t}{n}}$ :

$$e^{tA\frac{t}{n}} = e^{nV(\frac{t}{n})}e^{-n} = \sum_{k=0}^{\infty} \frac{n^k V(\frac{t}{n})^k}{k!} e^{-n}.$$

Thus we may calculate

$$\begin{aligned} & \|V(\frac{t}{n})^n W(\frac{t}{n})^j x - e^{tA\frac{t}{n}} W(\frac{t}{n})^j x\| \\ &= \|e^{-n} [e^n V(\frac{t}{n})^n - \sum_{k=0}^{\infty} \frac{1}{k!} n^k V(\frac{t}{n})^k] W(\frac{t}{n})^j x\| \\ &= \|e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!} [V(\frac{t}{n})^n - V(\frac{t}{n})^k] W(\frac{t}{n})^j x\| \\ &= \|e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!} [V(\frac{t}{n})^n W(\frac{t}{n})^j x - V(\frac{t}{n})^k W(\frac{t}{n})^j x]\|. \end{aligned}$$

For  $k > n$  we perform the following estimation using a telescoping series argument and condition (I) for  $i - 1 + j$ :

$$\begin{aligned} & \|V(\frac{t}{n})^k W(\frac{t}{n})^j x - V(\frac{t}{n})^n W(\frac{t}{n})^j x\| \\ &\leq \sum_{i=n+1}^k \|V(\frac{t}{n})^i W(\frac{t}{n})^j x - V(\frac{t}{n})^{i-1} W(\frac{t}{n})^j x\| \\ &\leq \sum_{i=n+1}^k \|V(\frac{t}{n})^{i-1} W(\frac{t}{n})^j\| \|V(\frac{t}{n})x - x\| \\ &\leq \sum_{i=n+1}^k M e^{(i-1+j)\omega\frac{t}{n}} \|V(\frac{t}{n})x - x\| \\ &\leq M e^{(k-1+j)\omega\frac{t}{n}} |k - n| \|V(\frac{t}{n})x - x\| \\ &\leq M e^{(n+k-1+j)\omega\frac{t}{n}} |k - n| \|V(\frac{t}{n})x - x\|. \end{aligned}$$

The argument for  $n > k$  is similar. Thus, we have

$$\begin{aligned}
& \left\| V\left(\frac{t}{n}\right)^k W\left(\frac{t}{n}\right)^j x - V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^j x \right\| \\
& \leq \left\| e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!} M e^{(n+k-1+j)\omega \frac{t}{n}} |k-n| \left\| V\left(\frac{t}{n}\right) x - x \right\| \right. \\
& = M e^{-n} e^{(n+j-1)\omega \frac{t}{n}} \left\| V\left(\frac{t}{n}\right) x - x \right\| \sum_{k=0}^{\infty} \frac{n^k}{k!} e^{k\omega \frac{t}{n}} |k-n|.
\end{aligned}$$

We use the Cauchy-Schwartz inequality to estimate the sum

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{n^k}{k!} e^{k\omega \frac{t}{n}} |k-n| &= \sum_{k=0}^{\infty} \sqrt{\left(\frac{(ne^{2\omega \frac{t}{n}})^k}{k!}\right)} \sqrt{\left(\frac{n^k(k-n)^2}{k!}\right)} \\
&\leq \sqrt{\sum_{k=0}^{\infty} \frac{(ne^{2\omega \frac{t}{n}})^k}{k!}} \sqrt{\sum_{k=0}^{\infty} \frac{n^k(k-n)^2}{k!}} \\
&= e^{\frac{ne^{2\omega \frac{t}{n}}}{2}} \sqrt{\sum_{k=0}^{\infty} \frac{n^k}{k!} k^2 - 2n \sum_{k=0}^{\infty} \frac{n^k}{k!} k + n^2 \sum_{k=0}^{\infty} \frac{n^k}{k!}} \\
&= e^{\frac{ne^{2\omega \frac{t}{n}}}{2}} \sqrt{\sum_{k=1}^{\infty} \frac{n^k}{(k-1)!} k - 2n \sum_{k=1}^{\infty} \frac{n^k}{(k-1)!} + n^2 e^n} \\
&= e^{\frac{ne^{2\omega \frac{t}{n}}}{2}} \sqrt{\sum_{k=0}^{\infty} \frac{n^{k+1}}{k!} (k+1) - 2n \sum_{k=0}^{\infty} \frac{n^{k+1}}{k!} + n^2 e^n} \\
&= e^{\frac{ne^{2\omega \frac{t}{n}}}{2}} \sqrt{n \sum_{k=0}^{\infty} \frac{n^k}{k!} k + n \sum_{k=0}^{\infty} \frac{n^k}{k!} - 2n^2 e^n + n^2 e^n} \\
&= e^{\frac{ne^{2\omega \frac{t}{n}}}{2}} \sqrt{n^2 e^n + ne^n - n^2 e^n} \\
&= e^{\frac{ne^{2\omega \frac{t}{n}}}{2}} \sqrt{ne^n}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\left\| V\left(\frac{t}{n}\right)^k W\left(\frac{t}{n}\right)^j x - V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^j x \right\| &\leq M e^{-n} e^{(n+j-1)\omega \frac{t}{n}} \left\| V\left(\frac{t}{n}\right) x - x \right\| e^{\frac{ne^{2\omega \frac{t}{n}}}{2}} \sqrt{ne^n} \\
&= M \sqrt{n} e^{(n+j-1)\omega \frac{t}{n}} e^{(e^{2\omega \frac{t}{n}} - 1)\frac{n}{2}} \left\| V\left(\frac{t}{n}\right) x - x \right\|.
\end{aligned}$$

On the other hand, for each  $\omega_0 > \omega$  there exists an  $n_0$  depending on  $\tau$  such that  $e^{2\omega\frac{t}{n}} - 1 < 2\omega_0\frac{t}{n}$  for all  $t \in (0, \tau]$  and  $n > n_0$ . Therefore, for each  $n > n_0$ ,

$$\begin{aligned} \left\| V\left(\frac{t}{n}\right)^k W\left(\frac{t}{n}\right)^j x - V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^j x \right\| &\leq M\sqrt{n}e^{(j-1)\omega\frac{t}{n}}e^{\omega t}e^{\omega_0 t}\left\| V\left(\frac{t}{n}\right)x - x \right\| \\ &\leq M\frac{t}{\sqrt{n}}e^{(j-1)\omega\frac{t}{n}}e^{2\omega_0 t}\frac{\left\| V\left(\frac{t}{n}\right)x - x \right\|}{\frac{t}{n}}. \end{aligned}$$

By the  $D(A^\infty)$ -consistency condition,  $\lim_{n \rightarrow \infty} \frac{\|V(\frac{t}{n})x - x\|}{\frac{t}{n}} = \|Ax\|$  for all  $x \in D(A^\infty)$ , so for each  $x \in D(A^\infty)$

$$\lim_{n \rightarrow \infty} \left\| V\left(\frac{t}{n}\right)^k W\left(\frac{t}{n}\right)^j x - V\left(\frac{t}{n}\right)^n W\left(\frac{t}{n}\right)^j x \right\| = 0$$

uniformly on  $(0, \tau]$ . □

**Lemma 2.9.** *Let  $F : [0, \infty) \rightarrow X$  be a function and let  $W(\cdot)$  be a strongly continuous operator family.*

(i) *If  $F$  is continuous, then  $s \rightarrow W(s)F(t-s)$  is continuous on  $[0, t]$  for every  $t > 0$  and furthermore,*

$$t \rightarrow (W * F)(t) = \int_0^t W(s)F(t-s)ds$$

*is continuous for every  $t \geq 0$ .*

(ii) *If  $F$  is continuously differentiable, then  $W * F$  is continuously differentiable for  $t \geq 0$  and furthermore*

$$(W * F)'(t) = (W * F')(t) + W(t)F(0)$$

*Proof.* (i): Suppose  $F$  is continuous. The strong continuity of the operator family  $W(\cdot)$  implies that for every  $x \in X$  and each  $\tau > 0$  there exists a constant  $M_x > 0$

such that  $\|W(t)x\| \leq M_x$  for all  $t \in [0, \tau]$ . By the Principle of Uniform Boundedness, there exists an  $M > 0$  such that  $\|W(t)\| \leq M$  for all  $t \in [0, \tau]$ . Fix  $s \geq 0$  and  $s_0 \leq t \leq \tau$ . Then by the continuity of  $F$  and the strong continuity of  $W(\cdot)$ , we have

$$\begin{aligned}
& \|W(s)F(t-s) - W(s_0)F(t-s_0)\| \\
& \leq \|W(s)F(t-s) - W(s)F(t-s_0)\| \\
& \quad + \|W(s)F(t-s_0) - W(s_0)F(t-s_0)\| \\
& \leq M\|F(t-s) - F(t-s_0)\| \\
& \quad + \|[W(s) - W(s_0)]F(t-s_0)\|,
\end{aligned}$$

which converges to zero and hence  $s \mapsto W(s)F(t-s)$  is continuous on  $[0, t]$  for all  $t > 0$ .

In order to see the continuity of  $t \mapsto (W * F)(t)$ , we consider first the case  $t > 0$ . Let  $\varepsilon > 0$  be arbitrary. Define  $C := \sup_{\sigma \in [0, 1]} \|F(\sigma)\|$  and fix  $M > 0$  such that  $\|W(s)\| \leq M$  for all  $s \in [0, t+1]$ . Since  $F$  is uniformly continuous of compact intervals, we may choose  $h$  with  $|h| < \min\{\frac{\varepsilon}{2MC}, t, 1\}$  such that  $\|F(t+h-s) - F(t-s)\| < \frac{\varepsilon}{2tM}$  for all  $0 \leq s \leq t$ . Then we may estimate

$$\begin{aligned}
& \|W * F)(t+h) - W * F)(t)\| \\
& = \left\| \int_0^{t+h} W(s)F(t+h-s)ds - \int_0^t W(s)F(t-s)ds \right\| \\
& \leq \left\| \int_0^t W(s)[F(t+h-s) - F(t-s)]ds \right\| \\
& \quad + \left\| \int_0^{t+h} W(s)F(t+h-s)ds \right\| \\
& \leq \frac{\varepsilon}{2} + |h|MC < \varepsilon
\end{aligned}$$

The argument for  $t = 0$  is similar.

(ii): Now suppose that  $F$  is continuously differentiable. In order to show the differentiability of  $W * F$  for  $t \geq 0$  (similarly, the right-differentiability for  $t = 0$ ), we first perform the following calculation:

$$\begin{aligned}
& \frac{1}{h} \left[ \int_0^{t+h} W(s)F(t+h-s)ds - \int_0^t W(s)F(t-s)ds \right] \\
&= \int_0^t W(s) \frac{1}{h} [F(t+h-s) - F(t-s)] ds \\
& \quad + \frac{1}{h} \int_t^{t+h} W(s)F(t+h-s)ds
\end{aligned} \tag{2.4.3}$$

Since  $F$  is continuously differentiable, the difference quotients  $\frac{F(t+h)-F(t)}{h}$  converge uniformly on compact subsets of  $[0, \infty)$ . Therefore, we may interchange the integral and the limit in the first integral of line (2.4.3) above to obtain

$$\lim_{h \rightarrow 0} \int_0^t W(s) \frac{1}{h} [F(t+h-s) - F(t-s)] ds = (W * F')(t).$$

Furthermore, for the second integral we have

$$\begin{aligned}
& \frac{1}{h} \int_t^{t+h} W(s)F(t+h-s)ds \\
&= \frac{1}{h} \int_t^{t+h} W(s)F(t-s)ds \\
& \quad + \int_t^{t+h} W(s) \frac{1}{h} [F(t+h-s) - F(t-s)] ds,
\end{aligned}$$

which converges to  $W(t)F(0) + 0$  as  $h \rightarrow 0$  since the difference quotients

$$\frac{F(t+h-\cdot) - F(t-\cdot)}{h}$$

are bounded. Therefore  $W * F$  is differentiable for  $t \geq 0$  and  $t \mapsto \frac{d}{dt}(W * F)(t) = (W * F')(t) + W(t)F(0)$ . Furthermore,  $t \mapsto (W * F')(t) + W(t)F(0)$  is continuous according to part (i) of the Lemma.  $\square$

It is shown in Section 1.4 of [ABHN] that the Laplace transform  $(\mathcal{L}f)(\lambda) := \int_0^\infty e^{-\lambda t} F(t) dt$  of a Bochner integrable function  $F : [0, \infty) \rightarrow X$  exists for some

$\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > \omega \geq 0$  if and only if the exponential growth bound of the antiderivative  $t \mapsto \int_0^t F(s) ds$  is at most  $\omega$ . The following definition ensures that for  $x \in D$  the continuous functions  $t \mapsto W_n(t)x$  are Laplace transformable on the joint domain  $\{\lambda : \operatorname{Re}(\lambda) > \omega_x\}$ . A sequence  $W_n(\cdot)$  of strongly continuous operator families is said to be *uniformly Laplace transformable* on  $D \subset X$  if for all  $x \in D$  there exist  $C_x, \omega_x > 0$  such that

$$\left\| \int_0^t W_n(s)x ds \right\| \leq C_x e^{\omega_x t} \text{ for all } n \in \mathbb{N} \text{ and } t \geq 0.$$

The preceding definition guarantees that if  $x \in D$ , then the continuous functions  $t \mapsto W_n(t)x$  are Laplace transformable on the joint domain  $\{\lambda : \operatorname{Re}(\lambda) > \omega_x\}$ .

**Theorem 2.10 (Stabilized Trotter-Kato).** *For each natural number  $n$ , let  $(T_n(t))_{t \geq 0}$  be a strongly continuous semigroup generated by an operator  $A_n$  on a Banach space  $X$ . Let  $W_n(\cdot)$  be a sequence of strongly continuous operator families that is uniformly Laplace transformable on  $D := \bigcap_{n=1}^{\infty} D(A_n) \cap D(A)$  (or let  $W_n(\cdot) = \delta_0(\cdot)W_n$ , where  $\delta_0(\cdot)$  is the Dirac delta function and  $W_n \in \mathcal{L}(X)$ ). Assume that for every  $x \in D$  there exist constants  $M_x > 0$  and  $\omega_x > \omega$  such that*

$$\left\| \int_0^t W_n(s)T_n(t-s)A_n x ds \right\| \leq M_x e^{\omega_x t} \quad (2.4.4)$$

for all  $n \in \mathbb{N}$  and  $t \geq 0$ . Then for  $x \in D$  the following are equivalent.

- (a)  $\lim_{n \rightarrow \infty} \mathcal{L}(W_n * T_n x)(\lambda) = R(\lambda, A)x$  for all  $\lambda > \omega_x$ .
- (b)  $\lim_{n \rightarrow \infty} (W_n * T_n x)(\cdot) = T(\cdot)x$  uniformly on compact intervals of  $[0, \infty)$ .

*Proof.* By Lemma 2.9,

$$\begin{aligned} \int_0^t W_n(s)T_n(t-s)x ds &=: F(t) = \int_0^t F'(y)dy + F(0) \\ &= \int_0^t \left[ \int_0^\sigma W_n(s)T_n(\sigma-s)A_n x ds + W_n(\sigma)x \right] d\sigma. \end{aligned} \quad (2.4.5)$$

The Laplace transform  $\mathcal{L}(W_n * T_n x)(\lambda)$  is well defined for each  $n \in \mathbb{N}$  and  $\lambda > \omega'_x > \omega_x$  because for any such  $n, \omega'_x$ , and  $\omega_x$ , one may make an appropriate choice of constants  $M_x, C_x, G_x > 0$  such that

$$\begin{aligned} & \left\| \int_0^t W_s(t-s)T_n(t-s)x ds \right\| \\ \leq & \int_0^t \left\| \int_0^\sigma W_n(s)T_n(\sigma-s)A_n x ds \right\| d\sigma + \left\| \int_0^t W_n(\sigma)x d\sigma \right\| \\ & \leq M_x t e^{\omega_x t} + C_x e^{\omega_x t} \\ & \leq G_x e^{\omega'_x t}. \end{aligned}$$

Furthermore, it can be checked that the sequence  $(W_n * T_n x)_{n \in \mathbb{N}}$  is equicontinuous. Then the equivalence (a) if and only if (b) follows from the following Laplace Transform result.

**Theorem 2.11.** *For each  $n \in \mathbb{N}$  pick  $f_n \in C([0, \infty), X)$  such that  $\|f_n(t)\| \leq M e^{\omega t}$  for some  $M > 0$  and  $\omega \in \mathbb{R}$ . Let  $\lambda_0 > \omega$ . Then the following are equivalent.*

- (a) *The Laplace transforms  $\hat{f}_n$  converge pointwise on  $(\lambda_0, \infty)$  and the sequence  $(f_n)$  is equicontinuous on  $[0, \infty)$ .*
- (b) *The functions  $f_n$  converge uniformly on compact subsets of  $[0, \infty)$ .*

Moreover, if (a) holds, then  $\hat{f}(\lambda) = \lim_{n \rightarrow \infty} \hat{f}_n(\lambda)$  for all  $\lambda > \lambda_0$ , where  $f(t) := \lim_{n \rightarrow \infty} f_n(t)$ .

□

**Corollary 2.12.** *Suppose that  $A$  generates a strongly continuous semigroup  $T(t)$  with  $\|T(t)\| \leq M e^{\omega t}$ . Let  $W_n(\cdot)$  be a strongly continuous operator family and let  $T_n$  be strongly continuous semigroups with generators  $(A_n, D(A_n))$  for  $D(A_n) \supset D(A)$ .*

*If*

$$(i) \|T_n(t)W_n\| \leq Me^{\omega t},$$

$$(ii) \lim_{n \rightarrow \infty} A_n x = Ax \text{ for all } x \in D(A^\infty), \text{ and}$$

$$(iii) \lim_{n \rightarrow \infty} W_n x = x \text{ for all } x \in D(A),$$

then  $\lim_{n \rightarrow \infty} T_n(t)W_n x = T(t)x$  for all  $x \in X$  uniformly for  $t$  in compact subintervals of  $[0, \infty)$ .

*Proof.* For every  $x \in D(A^\infty)$ , hypothesis (ii) yields the existence of a constant  $M_x > 0$  such that  $\|A_n x\| \leq M_x$  for all  $n \in \mathbb{N}$ . This fact in conjunction with hypothesis (i) yields the following inequality: for every  $x \in D := \bigcap_{n=1}^{\infty} D(A_n) \cap D(A)$ , there exist constants  $M_x > 0$  and  $\omega_x > \omega$  such that

$$\|T_n(t)W_n A_n x\| \leq M_x M e^{\omega_x t}$$

for all  $n \in \mathbb{N}$  and  $t \geq 0$ . Observe that this is exactly the condition (2.4.4) of the Stabilized Trotter Kato Theorem 2.10 if we define  $W_n(\cdot) = \delta_0(\cdot)W_n$ .

Now fix  $f \in X$  and  $\lambda > \omega$ . Define  $g := R(\lambda, A)f \in D(A)$  (so  $f = (\lambda I - A)g$ ).

Then

$$\begin{aligned} & \|\mathcal{L}(T_n W_n f)(\lambda) - R(\lambda, A)f\| \\ &= \|\lambda \mathcal{L}(T_n W_n g)(\lambda) - \mathcal{L}(T_n W_n A g)(\lambda) - g\| \\ &= \|\lambda \mathcal{L}(T_n W_n g)(\lambda) - \mathcal{L}(T_n W_n A_n g)(\lambda) \mathcal{L}(T_n W_n (A_n - A)g)(\lambda) - g\|. \end{aligned}$$

By the Fundamental Theorem of Calculus and integration by parts,

$$\begin{aligned}
\lambda \mathcal{L}(T_n W_n g)(\lambda) &= \int_0^\infty \lambda e^{-\lambda t} T_n(t) W_n g \, dt \\
&= \int_0^\infty \lambda e^{-\lambda t} \left[ g + \int_0^t T_n(s) W_n A_n g \, ds \right] dt \\
&= W_n g + \int_0^\infty \lambda e^{-\lambda t} \left( \int_0^t T_n(s) W_n A_n g \, ds \right) dt \\
&= W_n g + \int_0^\infty \lambda e^{-\lambda t} T_n(t) W_n A_n g \, dt \\
&= W_n g + \mathcal{L}(T_n W_n A_n g)(\lambda)
\end{aligned}$$

It follows immediately that

$$\begin{aligned}
&\| \mathcal{L}(T_n W_n f)(\lambda) - R(\lambda, A) f \| \\
&= \| W_n g + \mathcal{L}(T_n W_n (A_n - A) g)(\lambda) - g \| \\
&\leq \| W_n g - g \| + \| \mathcal{L}(T_n W_n (A_n - A) g)(\lambda) \|.
\end{aligned}$$

The first term converges to zero by hypothesis (iii). We estimate

$$\begin{aligned}
\| \mathcal{L}(T_n W_n (A_n - A) g)(\lambda) \| &\leq \int_0^\infty e^{-\lambda t} \| T_n(t) W_n - (A_n - A) g \| \, dt \\
&\leq M \int_0^\infty e^{(\omega - \lambda)t} \, dt \| (A_n - A) g \| \\
&= \frac{M}{\lambda - \omega} \| (A_n - A) g \|,
\end{aligned}$$

which converges to 0 as  $n$  approaches infinity by hypothesis (ii). Therefore

$$\lim_{n \rightarrow \infty} \| \mathcal{L}(T_n W_n f)(\lambda) - R(\lambda, A) f \| = 0$$

and so we may apply the Stabilized Trotter-Kato Theorem 2.10 for  $x \in D = D(A)$  to obtain

$$\lim_{n \rightarrow \infty} (T_n W_n x)(\cdot) = T(\cdot)x$$

uniformly in compact subintervals of  $[0, \infty)$ . By Theorem 2.7 of [Paz], we have that  $D(A^\infty)$  is dense in  $X$ . Hence the Banach Convergence Theorem yields

$$\lim_{n \rightarrow \infty} (T_n W_n x)(\cdot) = T(\cdot)x$$

for all  $x \in X$  uniformly in compact subintervals of  $[0, \infty)$ .  $\square$

We are now ready to prove the Stabilized Lax-Chernoff Theorem.

*Proof.* Now, for  $x \in X$ ,

$$\begin{aligned} \|V(\frac{t}{n})^{n-j}W(\frac{t}{n})^jx - T(t)x\| &\leq \|V(\frac{t}{n})^{n-j}W(\frac{t}{n})^jx - V(\frac{t}{n})^nW(\frac{t}{n})^jx\| \\ &\quad + \|V(\frac{t}{n})^nW(\frac{t}{n})^jx - e^{tA\frac{t}{n}}W(\frac{t}{n})^jx\| + \|e^{tA\frac{t}{n}}W(\frac{t}{n})^jx - T(t)x\| \\ &=: B_1 + B_2 + B_3. \end{aligned}$$

It follows from Lemma 2.8 that  $B_2 \rightarrow 0$  uniformly in  $t \in (0, \tau]$  as  $n \rightarrow \infty$ . To estimate  $B_3 = \|e^{tA\frac{t}{n}}W(\frac{t}{n})^jx - T(t)x\|$ , we employ Corollary 2.12 for  $A_n := A_{s_n}$ ,  $W_n := W(s_n)j$ ,  $T_n(t) := e^{tA_{s_n}}$  and  $s_n \rightarrow 0$  as  $n \rightarrow \infty$ . Condition (iii) follows from the strong continuity of  $W$  and condition (i) follows from Lemma 2.7. Furthermore, condition (ii) follows from  $\lim_{s_n \rightarrow 0} A_{s_n} = \lim_{s_n \rightarrow 0} \frac{V(s_n) - I}{s_n} = A$  for  $x \in D(A^\infty)$ . Hence by Corollary 2.12,  $B_3 \rightarrow 0$  uniformly in  $t \in (0, \tau]$  as  $n \rightarrow \infty$ . Finally,

$$\begin{aligned} B_1 &= \|V(\frac{t}{n})^{n-j}[V(\frac{t}{n})^jW(\frac{t}{n})^jx - W(\frac{t}{n})^jx]\| \leq \|V(\frac{t}{n})^jx - x\| \|V(\frac{t}{n})^{n-j}W(\frac{t}{n})^j\| \\ &\leq Me^{\omega t} \|V(\frac{t}{n})^jx - x\| = Me^{\omega t} \left\| \sum_{i=0}^{j-1} V(\frac{t}{n})^i [V(\frac{t}{n})x - x] \right\| \\ &\leq Me^{\omega t} \sum_{i=0}^{j-1} \|V(\frac{t}{n})^i\| \|V(\frac{t}{n})x - x\| \leq Me^{\omega t} M^{j-1} j \|V(\frac{t}{n})x - x\| \end{aligned}$$

tends to zero uniformly in  $t \in (0, \tau]$  as  $n \rightarrow \infty$ .  $\square$

The following corollary to the stabilized Lax-Chernoff Theorem provides a convergence result akin to that of Theorem 2.1, but does not provide information about the speed of convergence for smooth initial data  $x \in D(A)$ .

**Corollary 2.13.** *Let  $A$  be the generator of a strongly continuous semigroup  $T(\cdot)$  with  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . Let  $r(\cdot)$  be an  $A$ -stable rational approximation of*

the exponential of approximation order  $m$  and define  $V(t) := r(tA)$ . Suppose that  $V$  is  $D(A^\infty)$ -consistent. Define

$$W(t) = \frac{1}{t^\alpha} R\left(\frac{1}{t^\alpha}, A\right)$$

for  $\alpha = \frac{m}{m+1}$ . Then  $\lim_{n \rightarrow \infty} V\left(\frac{t}{n}\right)^{n-j} W\left(\frac{t}{n}\right)^j x = T(t)x$  for all  $x \in X$  uniformly for  $t$  in compact intervals.

*Proof.* Define  $W(t)$  as in the statement of the Corollary above. Then  $AW(t) = \frac{1}{t^\alpha} \left( \frac{1}{t^\alpha} R\left(\frac{1}{t^\alpha}, A\right) - I \right)$  and so for each  $x \in X$ , we have  $\|AW(t)x\| \leq \frac{1}{t^\alpha} (M+1)\|x\|$ . We estimate

$$\begin{aligned} & \|V(t)^{n-(m+1)} W(t)^{m+1} x\| \\ & \leq \|V(t)^{n-(m+1)} W(t)^{m+1} x - T(nt) W(t)^{m+1} x\| + \|T(nt) W(t)^{m+1} x\| \end{aligned}$$

We may treat the second term in the following way:

$$\begin{aligned} \|T(nt) W(t)^{m+1} x\| & \leq \|T(nt)\| \|W(t)\|^{m+1} \|x\| \\ & \leq M^{m+2} e^{\omega nt} \|x\|. \end{aligned}$$

We may estimate the first term by using the Brenner-Thomée Theorem 1.5 as follows:

$$\begin{aligned} & \leq CMnte^{c\omega nt} \left(\frac{t}{n}\right)^m \|(AW(t))^{m+1} x\| \\ & \leq CMnte^{c\omega nt} \left(\frac{t}{n}\right)^m \left(\frac{1}{t^\alpha}\right)^{m+1} (M+1)^{m+1} \|x\| \\ & \leq M(M+1)^{m+1} nte^{c\omega nt} \frac{1}{n^m} \|x\| \end{aligned}$$

Thus we have

$$\begin{aligned} & \|V(t)^{n-(m+1)} W(t)^{m+1} x\| \\ & \leq M(M+1)^{m+1} nte^{c\omega nt} \frac{1}{n^m} \|x\| + M^{m+2} e^{\omega nt} \|x\| \\ & \leq \mathcal{M} e^{\omega nt} \|x\| \end{aligned}$$

for all  $n > m + 1$  and  $t \in [0, \tau]$  We apply the Stabilized Lax Chernoff Theorem in order to obtain the result. □

# Chapter 3. Semigroup Approximation via Numerical Integration

In this chapter, we present an outline for an approximation method for operators  $f(A)$  defined via the Hille-Phillips functional calculus, that is, operators of the form  $f(A)x := \int_0^\infty T(s)x d\alpha(s)$ , where  $A$  generates a strongly continuous semigroup  $T(s)$  and  $f(\lambda) = \int_0^\infty e^{\lambda s} d\alpha(s)$ ,  $\operatorname{Re}(\lambda) \leq 0$  for a function  $\alpha : [0, \infty) \rightarrow \mathbb{C}$  of bounded total variation on  $[0, \infty)$ . Our approach is based on the approximation of the semigroup  $T(s)$  and circumvents the need for a spatial discretization of the operator  $f(A)$  and its resolvent. The results of this chapter are a promising first step towards the approximation of semigroups  $f_{t,\gamma}(A)x := e^{-t(-A)^\gamma}x$ ,  $0 < \gamma < 1$  generated by fractional powers of  $-A$ , or more generally, the approximation of semigroups  $f_t(A)x = e^{tg(A)}x$  for suitable analytic functions  $g$ . In particular, we approximate the semigroup

$$e^{-t\sqrt{-A}}x = \int_0^\infty T(s)x h_t(s) ds,$$

where

$$h_t(s) := \frac{t}{2\sqrt{\pi}} e^{\frac{-t^2}{4s}} s^{-\frac{3}{2}}. \tag{3.0.1}$$

We denote by  $\mathcal{L}(X)$  the space of all bounded linear operators on  $X$  and by  $NBV_0$  the Banach algebra of all normalized functions  $\alpha : [0, \infty) \rightarrow \mathbb{C}$  of bounded variation with multiplication  $(\alpha * \beta)(t) := \int_0^\infty \alpha(t-s) d\beta(s)$ . The norm on  $NBV_0$  is given by  $\|\alpha\|_0 = \int_0^\infty |d\alpha(t)| = V_0^\infty(\alpha)$ , where  $V_0^\infty(\alpha)$  denotes the total variation of  $\alpha$  on the interval  $[0, \infty)$ .

### 3.1 Quadrature Methods for Vector-Valued Functions

The classical Newton-Cotes Formulas for complex valued functions and, in particular, the error estimates for all of these numerical integration methods extend to Banach space valued functions. We use the following classical theorem on the error analysis for the closed Newton-Cotes formulas in order to demonstrate the method by which this extension can be made. The  $(n + 1)$ -point closed Newton Cotes formula is so-called because it approximates an integral using the function values at the endpoints  $r_0 := a$  and  $r_n := b$  as well as interior nodes  $r_i := a + i \frac{(b-a)}{n}$  for  $i = 1, 2, \dots, n - 1$ . Newton-Cotes formulas are numerical quadrature methods, that is, methods of the form

$$\int_a^b f(r) dr = \sum_{i=0}^n a_i f(r_i),$$

with coefficients  $a_i$  given by  $a_i := \int_a^b L_i(r) dr$ , where  $L_i(r) := \prod_{\substack{j=0 \\ j \neq i}}^n \frac{(r-r_j)}{r_i-r_j}$ .

**Theorem 3.1.** *Let  $n \in \mathbb{N}$  and suppose that  $f : [a, b] \rightarrow \mathbb{C}$  is  $n + 2$  times continuously differentiable if  $n$  is even and that  $f : [a, b] \rightarrow \mathbb{C}$  is  $n + 1$  times continuously differentiable if  $n$  is odd. Suppose that  $\sum_{i=0}^n a_i f(r_i)$  is the  $n + 1$ -point closed Newton-Cotes formula. Then there exists  $\xi \in (a, b)$  such that*

$$\int_a^b f(r) dr = \sum_{i=0}^n a_i f(r_i) + \frac{(b-a)^{n+3}}{(n+2)!} f^{(n+2)}(\xi) \int_0^n t^2(t-1) \cdots (t-n) dt \quad (3.1.1)$$

*if  $n$  is even, and*

$$\int_a^b f(r) dr = \sum_{i=0}^n a_i f(r_i) + \frac{(b-a)^{n+2}}{(n+1)!} f^{(n+1)}(\xi) \int_0^n t(t-1) \cdots (t-n) dt \quad (3.1.2)$$

*if  $n$  is odd.*

The error estimates given by (3.1.1) and (3.1.2) may no longer be valid for Banach space valued functions (due to the lack of a Mean Value Theorem). The

following inequalities, however, are true for Banach space valued functions. If  $f : [a, b] \longrightarrow X$  is  $n + 2$  times continuously differentiable, if  $\sum_{i=0}^n a_i f(r_i)$  is the  $n + 1$ -point closed Newton-Cotes formula, then

$$\left\| \int_a^b f(r) dr - \sum_{i=0}^n a_i f(r_i) \right\| \leq \frac{(b-a)^{n+3}}{(n+2)!} \|f^{(n+2)}\|_{L^\infty(a,b)} \left| \int_0^n t^2(t-1) \cdots (t-n) dt \right| \quad (3.1.3)$$

if  $n$  is even, and

$$\left\| \int_a^b f(r) dr - \sum_{i=0}^n a_i f(r_i) \right\| \leq \frac{(b-a)^{n+2}}{(n+1)!} \|f^{(n+1)}\|_{L^\infty(a,b)} \left| \int_0^n t(t-1) \cdots (t-n) dt \right| \quad (3.1.4)$$

if  $n$  is odd. To see why these inequalities hold, let  $X^*$  denote the dual space of a given Banach space  $X$  and let  $\mu \in X^*$  with  $\|\mu\| = 1$ . Then there exists  $\xi_\mu$  such that

$$\begin{aligned} & \left| \left\langle \int_a^b f(r)x dr - \sum_{i=0}^n a_i f(r_i)x, \mu \right\rangle \right| \\ &= \left| \int_a^b \langle f(r)x, \mu \rangle dr - \sum_{i=0}^n a_i \langle f(r_i)x, \mu \rangle \right| \\ &\leq \left| \frac{(b-a)^{n+3}}{(n+2)!} \langle f^{(n+2)}(\xi_\mu), \mu \rangle \right| \cdot \left| \int_0^n t^2(t-1) \cdots (t-n) dt \right| \\ &\leq \frac{(b-a)^{n+3}}{(n+2)!} \|f^{(n+2)}\| \cdot \left| \int_0^n t^2(t-1) \cdots (t-n) dt \right|. \end{aligned}$$

Since  $\sup_{\mu \in X^*, \|\mu\|=1} |\langle x, \mu \rangle| = \|x\|$ , the statement follows. We may prove (3.1.4) in a similar fashion.

In the case  $n = 1$ , Theorem 3.1 becomes the familiar trapezoidal rule. That is,

$$\int_a^b f(r) dr = \frac{b-a}{2} (f(a) + f(b)) - \frac{(b-a)^3}{12} f''(\xi)$$

for some  $\xi \in (a, b)$  if  $f$  is numerically valued, and

$$\left\| \int_a^b f(r) dr - \frac{b-a}{2} (f(a) + f(b)) \right\| \leq \frac{(b-a)^3}{12} \|f''\|_{L^\infty}$$

if  $f$  is Banach space valued.

Unfortunately, the derivatives of the functions  $h_{\gamma,t}$  that appear in the study of the semigroup generated by  $(-A)^\gamma$  for  $0 < \gamma < 1$  have unruly supremum norms. Therefore, we shall present  $L^1$ -norm estimates for the trapezoidal rule. It is not clear at the moment whether or not these  $L^1$ -norm estimates hold for all Newton-Cotes formulas.

**Lemma 3.2.** *If  $f : [a, b] \rightarrow X$  is twice continuously differentiable on  $(a, b)$ , then*

$$\int_a^b f(r) dr = (b-a) \frac{f(b) + f(a)}{2} + \int_a^b \frac{(r-a)(r-b)}{2} f''(r) dr.$$

*Proof.* We first prove the case  $a = 0$  and  $b = 1$

$$\begin{aligned} & \int_0^1 f(r) dr - \frac{f(1) + f(0)}{2} = \int_0^1 \frac{f(r) - f(1)}{2} + \frac{f(r) - f(0)}{2} dr \\ &= \frac{1}{2} \int_0^1 - \int_r^1 f'(s) ds dr + \frac{1}{2} \int_0^1 \int_0^r f'(s) ds dr \\ &= -\frac{1}{2} \int_0^1 \int_0^s f'(s) dr ds + \frac{1}{2} \int_0^1 \int_s^1 f'(s) dr ds \\ &= -\frac{1}{2} \int_0^1 s f'(s) ds + \frac{1}{2} \int_0^1 (1-s) f'(s) ds = \frac{1}{2} \int_0^1 (1-2s) f'(s) ds \\ &= -\frac{1}{2} \int_0^1 (s-s^2) f''(s) ds = \int_0^1 \frac{s(s-1)}{2} f''(s) ds. \end{aligned}$$

That is,

$$\int_0^1 f(r) dr - \frac{f(1) + f(0)}{2} = \int_0^1 \frac{r(r-1)}{2} f''(r) dr. \quad (3.1.5)$$

Now, let  $t = \frac{r-a}{b-a}$ , or  $r = a + t(b-a)$ . Then, by equation(3.1.5),

$$\begin{aligned} \int_a^b f(r) dr - (b-a) \frac{f(b) + f(a)}{2} &= (b-a) \int_0^1 f(a + t(b-a)) dt \\ &= (b-a)^3 \int_0^1 \frac{s(s-1)}{2} f''(a + s(b-a)) ds \\ &= \int_a^b \frac{(r-a)(r-b)}{2} f''(r) dr. \end{aligned}$$

□

Observe that the error estimate for the usual trapezoidal rule is given by the formula  $\frac{(b-a)^3}{12N^2} \|f''\|_\infty$ ; its proof applies the Mean Value Theorem to the result of Lemma 3.2 above.

**Lemma 3.3.** *Let  $a, b \in \mathbb{R}$ ,  $N \in \mathbb{N}$  and define  $r_j := a + \frac{(b-a)j}{N}$  for  $j = 0, \dots, N$ . Define  $c_N := \frac{1}{2}$  and  $c_j := 1$  for  $j \neq N$ . Let  $X$  be a Banach space with norm  $\|\cdot\|$ . If  $f : [a, b] \rightarrow X$  is twice continuously differentiable on  $(a, b)$  then*

$$\left\| \int_a^b f(r) dr - \frac{b-a}{N} \sum_{j=1}^N c_j f(r_j) \right\| \leq \frac{(b-a)^2}{8N^2} \|f''\|_{L^1(a,b)}.$$

*Proof.* Observe that by Lemma 3.2

$$\begin{aligned} \int_a^b f(r) dr &= \sum_{j=1}^N \int_{r_{j-1}}^{r_j} f(r) dr \\ &= \sum_{j=1}^N \frac{f(r_j) + f(r_{j-1})}{2} + \sum_{j=1}^N \int_{r_{j-1}}^{r_j} \frac{(r-r_j)(r-r_{j-1})}{2} f''(r) dr. \end{aligned}$$

Observe also that  $\sum_{j=1}^N (r_j - r_{j-1}) \frac{f(r_j) + f(r_{j-1})}{2} = \frac{b-a}{2N} (f(r_0) + 2f(r_1) + \dots + 2f(r_{N-1}) + f(r_N))$ . That is,

$$\int_a^b f(r) dr - \frac{b-a}{N} \sum_{j=1}^N c_j f(r_j) = \sum_{j=1}^N \int_{r_{j-1}}^{r_j} \frac{(r-r_j)(r-r_{j-1})}{2} f''(r) dr.$$

So

$$\begin{aligned} &\left\| \int_a^b f(r) dr - \frac{b-a}{N} \sum_{j=1}^N c_j f(r_j) \right\| \\ &= \left\| \sum_{j=1}^N \int_{r_{j-1}}^{r_j} f''(r) \frac{(r-r_j)(r-r_{j-1})}{2} dr \right\| \\ &\leq \sum_{j=1}^N \int_{r_{j-1}}^{r_j} \|f''(r)\| \frac{|(r-r_j)(r-r_{j-1})|}{2} dr \end{aligned}$$

Now,  $r \mapsto |(r - r_j)(r - r_{j-1})|$  attains a maximum value of  $\frac{|r_j - r_{j-1}|^2}{4}$  at  $r = \frac{r_j + r_{j-1}}{2}$ .

Therefore,

$$\begin{aligned}
& \sum_{j=1}^N \int_{r_{j-1}}^{r_j} \|f''(r)\| \frac{|(r - r_j)(r - r_{j-1})|}{2} dr \\
& \leq \sum_{j=1}^N \int_{r_{j-1}}^{r_j} \|f''(r)\| \max_{r_{j-1} \leq r \leq r_j} \left| \frac{(r - r_j)(r - r_{j-1})}{2} \right| dr \\
& \leq \frac{1}{8} \sum_{j=1}^N \int_{r_{j-1}}^{r_j} \|f''(r)\| \cdot |r_j - r_{j-1}|^2 dr \\
& \leq \frac{1}{8} \sum_{j=1}^N |r_j - r_{j-1}|^2 \int_{r_{j-1}}^{r_j} \|f''(r)\| dr \\
& = \frac{1}{8} \frac{(b-a)^2}{N^2} \sum_{j=1}^N \int_{r_{j-1}}^{r_j} \|f''(r)\| dr \\
& = \frac{1}{8} \frac{(b-a)^2}{N^2} \int_a^b \|f''(r)\| dr \\
& = \frac{1}{8} \frac{(b-a)^2}{N^2} \|f''\|_{L^1(a,b)}.
\end{aligned}$$

This completes the proof of the fact that

$$\left\| \int_a^b f(r) dr - \frac{b-a}{N} \sum_{j=1}^N c_j f(r_j) \right\| \leq \frac{(b-a)^2}{8N} \int_a^b \|f''(r)\| dr.$$

□

**Lemma 3.4.** *Let  $N \in \mathbb{N}$  and define  $s_j := a + \frac{(b-a)j}{N}$  for  $j = 0, \dots, N$ . If  $f : [a, b] \longrightarrow X$  is continuously differentiable, then*

$$\left\| \int_a^b f(s) ds - \frac{(b-a)}{N} \sum_{j=1}^N f(s_j) \right\| \leq \frac{(b-a)}{N} \|f'\|_{L^1(a,b)}.$$

*Proof.* Let  $\mu \in X^*$  with  $\|\mu\| = 1$ . By the Mean Value Theorem, there exist  $c_j \in (s_{j-1}, s_j)$  such that

$$\begin{aligned}
& \left| \left\langle \int_a^b f(s) ds - \frac{b-a}{N} \sum_{j=1}^N f(s_j), \mu \right\rangle \right| \\
&= \left| \sum_{j=1}^N \int_{s_{j-1}}^{s_j} \langle f(s), \mu \rangle - \langle f(s_j), \mu \rangle ds \right| \\
&= \left| \sum_{j=1}^N (\langle f(c_j), \mu \rangle - \langle f(s_j), \mu \rangle)(s_j - s_{j-1}) \right| \\
&\leq \frac{b-a}{N} \sum_{j=1}^N |\langle f(c_j), \mu \rangle - \langle f(s_j), \mu \rangle| \\
&\leq \frac{b-a}{N} \|\langle f, \mu \rangle\|_{BV} \\
&\leq \frac{b-a}{N} \|\langle f', \mu \rangle\|_{L^1(a,b)} \\
&\leq \frac{b-a}{N} \|f'\|_{L^1(a,b)}.
\end{aligned}$$

Since  $\sup_{\mu \in X^*, \|\mu\|=1} |\langle x, \mu \rangle| = \|x\|$ , the statement follows.  $\square$

### 3.2 Approximation of Semigroups Generated by Fractional Powers of a Closed Operator

If  $\phi$  is continuous and if  $\alpha : [0, S] \rightarrow \mathbb{R}$  is of bounded variation on  $[0, S]$  with  $\alpha' \in L^1[0, S]$ , then  $\int_0^S \phi(s) d\alpha(s) = \int_0^S \phi(s) \alpha'(s) ds$  and  $\|\alpha\|_0 = \|\alpha'\|_1$  (see Section 1.6 in [Wi] and Section 516 in [Ol]). Let  $\alpha(s) := \int_0^s h(r) dr$ . Then  $\alpha \in NBV_0(0, \infty)$ , or equivalently  $h \in L^1(0, \infty)$ . Furthermore,  $\|\alpha\|_0 = \|h\|_1 = 1$ .

Suppose that  $A$  generates a strongly continuous semigroup  $T(s)$  with  $\|T(s)\| \leq Me^{-\omega t}$  for some  $M > 0$  and  $\omega \geq 0$ . Suppose that an operator  $f(A)$  is given by the Hille-Phillips functional calculus, that is,

$$f(A)x := \int_0^\infty T(s)xh(s) ds,$$

where  $\alpha(s) := \int_0^s h(r) dr$  is in  $NBV_0(0, \infty)$ , or equivalently  $h \in L^1(0, \infty)$ . We avoid rescaling the interval  $(0, \infty)$  onto  $(0, 1)$  since doing so introduces an unbounded

term into the integrand. Let  $\varepsilon > 0$  be given. Then there exists  $S > 0$  such that

$$\left\| \int_S^\infty T(s)x h_t(s) ds \right\| \leq M e^{-\omega S} \|x\| \int_S^\infty |h_t(s)| ds \leq \varepsilon \|x\|.$$

Fix  $N \in \mathbb{N}$  and define  $c_N := \frac{1}{2}$ ,  $c_j := 1$  for  $j \neq N$ ,  $s_j := \frac{Sj}{N}$ . Then

$$\begin{aligned} \left\| f(A)x - \frac{S}{N} \sum_{j=1}^N c_j h(s_j) T(s_j)x \right\| &\leq \left\| \int_0^S T(s)x h(s) ds - \frac{S}{N} \sum_{j=1}^N c_j h(s_j) T(s_j)x \right\| \\ &\quad + \varepsilon \|x\|. \end{aligned}$$

Define  $g(s)x := h(s)T(s)x$ . Then  $g'(s)x = h(s)AT(s)x + h'(s)T(s)x$  and so  $g''(s)x = T(s)(h(s)A^2x + 2h'(s)Ax + h''(s)x)$ . Now, the trapezoidal rule for Banach space valued functions yields

$$\begin{aligned} &\left\| \int_0^S h(s)T(s)x ds - \frac{S}{N} \sum_{j=1}^N c_j h(s_j) T(s_j)x \right\| \\ &\leq \frac{S^2}{8N^2} \|g''(s)x\|_{L^1[0,S]} \\ &= \frac{S^2}{8N^2} \|T(s)(h(s)A^2x + 2h'(s)Ax + h''(s)x)\|_{L^1[0,S]} \\ &\leq \frac{S^2}{8N^2} \int_0^S \|T(s)\| \|h(s)A^2x + 2h'(s)Ax + h''(s)x\| ds \\ &\leq \frac{S^2}{8N^2} \int_0^S e^{-\omega s} [h(s)\|A^2x\| + 2h'(s)\|Ax\| + h''(s)\|x\|] ds \\ &\leq \frac{S^2}{8N^2} (\|h\|_{L^1[0,S]}\|A^2x\| + 2\|h'\|_{L^1[0,S]}\|Ax\| + \|h''\|_{L^1[0,S]}\|x\|). \end{aligned}$$

Therefore, for all  $\varepsilon > 0$  there exists  $S > 0$  such that

$$\begin{aligned} &\left\| f(A)x - \frac{S}{N} \sum_{j=1}^N c_j h(s_j) T(s_j)x \right\| \tag{3.2.1} \\ &\leq \frac{S^2}{8N^2} (\|h\|_{L^1[0,S]}\|A^2x\| + 2\|h'\|_{L^1[0,S]}\|Ax\| + \|h''\|_{L^1[0,S]}\|x\|) \\ &\quad + \varepsilon \|x\|. \end{aligned}$$

For example, in order to approximate the semigroup  $\{e^{-t(-A)^\gamma} : t \geq 0\}$  we investigate the numerical function  $\lambda \mapsto e^{-t(-\lambda)^\gamma}$  on the domain  $\{\lambda : \operatorname{Re}(\lambda) \leq 0\}$ ,

where  $t \geq 0$  is a fixed time parameter for the semigroup generated by  $-(-A)^\gamma$ . The proof of the following important lemma may be found in section IX.11 of the classic text by Yosida [Yo].

**Lemma 3.5.** *Let  $0 < \gamma < 1$  and fix a branch of  $z \mapsto z^\gamma$  so that  $\operatorname{Re}(z^\gamma) > 0$  for  $\operatorname{Re}(z) > 0$ . If  $\operatorname{Re}(\lambda) < 0$  and  $t > 0$ , then*

$$e^{-t(-\lambda)^\gamma} = \int_0^\infty e^{\lambda s} h_{\gamma,t}(s) ds,$$

where

$$h_{\gamma,t}(s) := \begin{cases} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{sz} e^{-tz^\gamma} dz & \text{if } s \geq 0, \\ 0 & \text{if } s < 0 \end{cases}$$

and  $\sigma > 0$ . Furthermore, the function  $h_{\gamma,t}$  is a probability density on  $(0, \infty)$ . That is,  $h_{\gamma,t}(s) \geq 0$  for all  $s \in (0, \infty)$  and  $\int_0^\infty h_{\gamma,t}(s) ds = 1$ .

The density function  $h_{\gamma,t}$  enjoys several other useful properties, which are presented in the following lemma.

**Lemma 3.6.** *Let  $0 < \gamma < 1$  and  $t > 0$ . Then  $h_{\gamma,t}(s) = t^{\frac{1}{\gamma}} h_{\gamma,1}(\frac{s}{t^{\frac{1}{\gamma}}})$  and the norm estimates  $\int_0^\infty |h'_{\gamma,t}(s)| ds = \frac{2}{t^{\frac{1}{\gamma}}} \|h_{\gamma,1}\|_\infty$  and  $\int_0^\infty |h''_{\gamma,t}(s)| ds \leq \frac{4}{t^{\frac{2}{\gamma}}} \|h'_{\gamma,1}\|_\infty$  hold.*

*Proof.* By Lemma 3.5 we have that  $e^{-t(-\lambda)^\gamma} = \int_0^\infty e^{\lambda s} h_{\gamma,t}(s) ds$ . On the other hand if  $u = t^{\frac{1}{\gamma}} s$ , then

$$\begin{aligned} e^{-t(-\lambda)^\gamma} &= e^{-1(-t^{\frac{1}{\gamma}}\lambda)^\gamma} \\ &= \int_0^\infty e^{t^{\frac{1}{\gamma}}\lambda s} h_{\gamma,1}(s) ds \\ &= \int_0^\infty e^{\lambda u} h_{\gamma,1}\left(\frac{u}{t^{\frac{1}{\gamma}}}\right) \frac{1}{t^{\frac{1}{\gamma}}} du. \end{aligned}$$

Therefore  $h_{\gamma,t}(s) = t^{\frac{1}{\gamma}} h_{\gamma,1}(\frac{s}{t^{\frac{1}{\gamma}}})$ . Since  $H_{\gamma,t}(r) := \int_0^r h_{\gamma,t}(s) ds$  is unimodal (see [Zo]), and since  $h_{\gamma,t}^{(n)}(0) = h_{\gamma,t}^{(n)}(\infty) = 0$ , there exists  $s_0 \in (0, \infty)$  such that  $h'_{\gamma,t}(s_0) = 0$  and  $h'_{\gamma,t}(s) > 0$  for all  $s \in (0, s_0)$  and  $h'_{\gamma,t}(s) < 0$  for all  $s \in (s_0, \infty)$ . Hence we may

calculate  $\int_0^\infty |h'_{\gamma,t}(s)| ds = \int_0^{s_0} h'_{\gamma,t}(s) ds - \int_{s_0}^\infty h'_{\gamma,t}(s) ds = 2h_{\gamma,t}(s_0) = 2\|h_{\gamma,t}\|_\infty = \frac{2}{t^{\frac{1}{\gamma}}}\|h_{\gamma,1}\|_\infty$ . Furthermore, since  $H_{\gamma,t}$  is unimodal, there exists  $s_1 \in (0, s_0)$  such that  $h''_{\gamma,t}(s) > 0$  for all  $s \in (0, s_1)$  and  $h'_{\gamma,t}(s) < 0$  for all  $s \in (s_1, s_0)$ . Moreover, there exists  $s_2 \in (s_0, \infty)$  such that  $h''_{\gamma,t}(s) < 0$  for all  $s \in (s_1, s_2)$  and  $h'_{\gamma,t}(s) > 0$  for all  $s \in (s_2, \infty)$ . Therefore we may calculate  $\int_0^\infty |h''_{\gamma,t}(s)| ds = \int_0^{s_1} h''_{\gamma,t}(s) ds - \int_{s_1}^{s_2} h''_{\gamma,t}(s) ds + \int_{s_2}^\infty h''_{\gamma,t}(s) ds = 2h'_{\gamma,t}(s_1) + 2h'_{\gamma,t}(s_2) \leq 4\|h'_{\gamma,t}\|_\infty \leq \frac{4}{t^{\frac{2}{\gamma}}}\|h'_{\gamma,1}\|_\infty$ .  $\square$

Suppose that  $A$  generates a strongly continuous semigroup  $T(s)$  with  $\|T(s)\| \leq Me^{-\omega t}$  for some  $M > 0$  and  $\omega \geq 0$  and consider

$$e^{-t(-A)^\gamma} x = \int_0^\infty T(s)x h_{\gamma,t}(s) ds.$$

We avoid rescaling the interval  $(0, \infty)$  onto  $(0, 1)$  since doing so introduces an unbounded term into the integrand. Let  $\varepsilon > 0$  be given and let  $u = \frac{s}{t^{\frac{1}{\gamma}}}$ . Choose  $S > 0$  such that

$$\begin{aligned} \left\| \int_S^\infty T(s)x h_{\gamma,t}(s) ds \right\| &\leq \left\| \frac{1}{t^{\frac{1}{\gamma}}} \int_S^\infty T(s)x h_{\gamma,1}\left(\frac{s}{t^{\frac{1}{\gamma}}}\right) ds \right\| \\ &= \left\| \int_{\frac{S}{t^{\frac{1}{\gamma}}}}^\infty T(t^{\frac{1}{\gamma}}u)x h_{\gamma,1}(u) du \right\| \\ &\leq Me^{-\omega S} \|x\| \int_{\frac{S}{t^{\frac{1}{\gamma}}}}^\infty h_{\gamma,1}(u) du \leq \varepsilon \|x\|. \end{aligned}$$

Fix  $N \in \mathbb{N}$  and define  $c_N := \frac{1}{2}$ ,  $c_j := 1$  for  $j \neq N$ ,  $s_j := \frac{Sj}{N}$  and  $k_j := \frac{S}{N}c_j h_t(s_j)$  for  $j = 1, 2, \dots, N$ . Then by (3.2.1), we have that

$$\begin{aligned} &\left\| e^{-t(-A)^\gamma} x - \frac{S}{N} \sum_{j=1}^N c_j t^{\frac{1}{\gamma}} h_{\gamma,1}\left(\frac{s_j}{t^{\frac{1}{\gamma}}}\right) T(s_j)x \right\| \\ &\leq \frac{MS^2}{8N^2} (\|A^2x\| + \frac{4}{t^{\frac{1}{\gamma}}}\|h_{\gamma,1}\|_\infty \|Ax\| + \frac{2}{t^{\frac{2}{\gamma}}}\|h'_{\gamma,1}\|_\infty \|x\|) \\ &\quad + \varepsilon \|x\|. \end{aligned} \tag{3.2.2}$$

We have shown that error analysis of an application of the trapezoidal rule to our situation requires sup-norm estimates for  $h_{\gamma,1}$  and its derivative  $h'_{\gamma,1}$ . In

the case  $\gamma = \frac{1}{2}$ , the density function  $h_t(s) = h_{\frac{1}{2},t}(s)$  is given by the formula  $h_t(s) := \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} ds$ . In order to compute  $h_{\gamma,t}$  explicitly for arbitrary  $\gamma$ , numerical inversion methods for the Laplace transform are required.

### 3.3 Error Estimates for the Approximation of Semigroups Generated by the Square Root of a Closed Operator

In order to approximate the semigroup  $\{e^{-t\sqrt{-A}} : t \geq 0\}$  we investigate the numerical function  $\lambda \mapsto e^{-t\sqrt{-\lambda}} = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{\lambda s} e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} ds$  on the domain  $\{\lambda : \operatorname{Re}(\lambda) \leq 0\}$ , where  $t \geq 0$  is a fixed time parameter for the semigroup generated by  $-\sqrt{-A}$ . If  $\operatorname{Re}(\lambda) < 0$  and  $t > 0$ , then

$$e^{-t\sqrt{-\lambda}} = \int_0^\infty e^{\lambda s} h_t(s) ds,$$

where  $h_t(s) := \frac{t}{2\sqrt{\pi}} e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}}$ . In particular,  $h_t(s) \geq 0$  and  $\int_0^\infty h_t(s) ds = 1$ . This example is discussed in [Yo, p.268] and section 1.6 of [ABHN]. The probability density function  $h_t(s)$  is a special case of the density function associated with the Inverse Gaussian distribution. The density function for the Inverse Gaussian distribution is given by

$$IG(x) := \sqrt{\frac{\lambda}{2\pi x^3}} e^{-\frac{\lambda(x-m)^2}{2m^2x}},$$

where  $x \in (0, \infty)$ . This can be readily seen by making the substitutions  $\lambda = \frac{t^2}{2}$ ,  $x = s$  and  $m = \frac{s}{2}$ . The term ‘‘Inverse Gaussian’’ was coined by M.C.K. Tweedie, who in 1945 observed that the cumulant generating function of the Inverse Gaussian distribution is exactly the inverse of that of the Gaussian distribution (see [Se], [Tw], or [Wa] for more details). Physically, the Inverse Gaussian distribution represents the first passage time distribution of Brownian motion with positive drift. The interested reader is referred to Appendices A and B.

**Theorem 3.7.** *Suppose that  $A$  generates a strongly continuous semigroup  $T(s)$  such that  $\|T(s)\| \leq Me^{-\omega s}$  for all  $s \geq 0$  and some  $M, \omega \geq 0$ . Fix  $\varepsilon > 0$  and  $t > 0$ .*

*Define*

$$S = S(t, \varepsilon, \omega) := \begin{cases} \frac{1}{\omega} |\ln(\frac{\varepsilon}{M})| & \text{if } \omega > 0, \\ \frac{t^2 M^2}{\varepsilon^2 \pi} & \text{if } \omega = 0. \end{cases}$$

*Define  $h_t(s) := \frac{t}{2\sqrt{\pi}} e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}}$ . Fix  $N \in \mathbb{N}$  and define  $c_N := \frac{1}{2}$ ,  $c_j := 1$  for  $j \neq N$ ,  $s_j := \frac{Sj}{N}$  and  $k_j := \frac{S}{N} c_j h_t(s_j)$  for  $j = 1, 2, \dots, N$ . Then for every  $x \in D(A^2)$ ,*

$$\|e^{-t(-A)^{\frac{1}{2}}} x - \sum_{j=1}^N k_j T(s_j) x\| \leq \frac{MS^2}{N^2} \left( \frac{1}{8} \|A^2 x\| + \frac{1}{2t^2} \|Ax\| + \frac{67}{t^4} \|x\| \right) + \varepsilon \|x\|.$$

*Proof.* Fix  $t > 0$  and let  $\varepsilon > 0$  be given. Suppose that  $\operatorname{Re}(\lambda) \leq 0$ . Choose

$$S = S(t, \varepsilon, \omega) = \begin{cases} \frac{1}{\omega} |\ln(\frac{\varepsilon}{M})| & \text{if } \omega > 0, \\ \frac{t^2 M^2}{\varepsilon^2 \pi} & \text{if } \omega = 0. \end{cases}$$

Notice that if  $\omega > 0$  then  $S(t, \varepsilon, \omega) = \frac{1}{\omega} S(t, \varepsilon, 1)$ . Since the trapezoidal rule only applies to finite intervals, it is tempting to rescale our integrals from  $(0, \infty)$  onto  $(0, 1)$ . We avoid doing so, since rescaling introduces an unbounded term into the estimates of approximation error. Suppose that  $A$  generates a strongly continuous semigroup  $T(s)$  on a Banach space  $X$  and that  $\|T(s)\| \leq Me^{-\omega s}$ , where  $\omega \geq 0$ .

Fix  $x \in X$ . Then

$$\begin{aligned} \|e^{-t(-A)^{\frac{1}{2}}} x - \sum_{j=1}^N k_j T(s_j) x\| &= \left\| \int_0^\infty T(s) x h_t(s) ds - \sum_{j=1}^N k_j T(s_j) x \right\| \\ &\leq \left\| \int_0^S T(s) x h_t(s) ds - \sum_{j=1}^N k_j T(s_j) x \right\| + \left\| \int_S^\infty T(s) x h_t(s) ds \right\|. \end{aligned}$$

Let  $0 < \varepsilon < 1 \leq M$  be given. To analyze the truncation error we consider the cases  $\omega > 0$  and  $\omega = 0$  separately. First suppose that  $\omega > 0$  and choose  $S = \frac{1}{\omega} \ln(\frac{\varepsilon}{M})$ .

Then by Lemma 3.8, we may calculate

$$\begin{aligned} \left\| \int_S^\infty T(s)x h_t(s) ds \right\| &\leq \int_S^\infty \|T(s)\| \|x\| h_t(s) ds \leq M e^{-\omega S} \|x\| \int_S^\infty h_t(s) ds \\ &\leq M e^{-\omega S} \|x\| \int_0^\infty h_t(s) ds = M e^{-\omega S} \|x\| = \varepsilon \|x\|. \end{aligned}$$

Now suppose that  $\omega = 0$  and choose  $S = \frac{t^2 M^2}{\varepsilon^2 \pi}$ . Then

$$\begin{aligned} \left\| \int_S^\infty T(s)x h_t(s) ds \right\| &\leq \int_S^\infty \|T(s)\| \|x\| h_t(s) ds \\ &\leq \frac{t}{2\sqrt{\pi}} M \|x\| \int_S^\infty e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} ds \\ &\leq \frac{t}{2\sqrt{\pi}} M \|x\| \int_S^\infty t^{-\frac{3}{2}} ds \\ &= \frac{t}{2\sqrt{\pi}} M \|x\| \left[ -2 \frac{1}{\sqrt{s}} \right]_S^\infty = \frac{t}{2\sqrt{\pi}} M \|x\| \left( \frac{2}{\sqrt{S}} \right) \\ &= \frac{tM}{\sqrt{S}\pi} \|x\| = \frac{tM\sqrt{\pi}\varepsilon}{\sqrt{\pi}tM} \|x\| = \varepsilon \|x\|. \end{aligned}$$

Define  $g_t(s)x := h_t(s)T(s)x$ . By inequality (3.2.2),

$$\begin{aligned} &\left\| \int_0^S h_t(s)T(s)x ds - \frac{S}{N} \sum_{j=1}^N c_j h_t(s_j)T(s_j)x \right\| \\ &\leq \frac{MS^2}{8N^2} (\|A^2x\| + \frac{4}{t^2} \|h_1\|_\infty \|Ax\| + \frac{4}{t^4} \|h_1'\|_\infty \|x\|). \end{aligned}$$

We must find  $\|h_1\|_\infty$  and  $\|h_1'\|_\infty$ . Now,  $\|h_1\|_\infty = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4s}} s^{-\frac{3}{2}}$ . On the other hand,

$$h_t'(s) = h_t(s) \left( \frac{t^2 - 6s}{4s^2} \right), \quad (3.3.1)$$

so  $h_1'(s) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4s}} \left( \frac{1}{4} s^{-\frac{7}{2}} - \frac{3}{2} s^{-\frac{5}{2}} \right)$ .

Now,  $s \mapsto e^{-\frac{t^2}{4s}} s^{-K}$  is nonnegative for all  $s \geq 0$  with derivative  $s \mapsto s^{-K-2} e^{-\frac{t^2}{4s}} \left[ \frac{t^2}{4} - Ks \right]$ . Thus,  $s \mapsto e^{-\frac{t^2}{4s}} s^{-K}$  achieves an absolute maximum value of  $M_K(t)$  at  $s = \frac{t^2}{4K}$  which is given by  $M_K(t) := \left( \frac{4K}{t^2 e} \right)^K$ . In particular,  $s \mapsto e^{-\frac{1}{4s}} s^{-\frac{3}{2}}$  achieves an absolute maximum value of  $M_{\frac{3}{2}}(1) = \left( \frac{6}{e} \right)^{\frac{3}{2}}$ . Therefore,  $\|h_1\|_\infty \leq \frac{1}{2\sqrt{\pi}} \left( \frac{6}{e} \right)^{\frac{3}{2}} \leq \frac{3.3}{2\sqrt{\pi}}$ . Moreover,  $s \mapsto e^{-\frac{1}{4s}} s^{-\frac{5}{2}}$  achieves an absolute maximum value of  $M_{\frac{5}{2}}(1) = \left( \frac{10}{e} \right)^{\frac{5}{2}}$

and  $s \mapsto e^{-\frac{1}{4s}} s^{-\frac{7}{2}}$  achieves an absolute maximum value of  $M_{\frac{3}{2}}(1) = (\frac{14}{e})^{\frac{7}{2}}$ . Hence  $\|h'_1\|_\infty \leq \frac{1}{2\sqrt{\pi}}(\frac{1}{4}(\frac{10}{e})^{\frac{5}{2}} + \frac{3}{2}(\frac{14}{e})^{\frac{7}{2}}) \leq \frac{1}{2\sqrt{\pi}}(\frac{1}{4}26 + \frac{3}{2}311) \leq \frac{236.5}{\sqrt{\pi}}$ . Therefore,

$$\begin{aligned}
& \left\| \int_0^S h_t(s)T(s)x \, ds - \frac{S}{N} \sum_{j=1}^N c_j h_t(s_j)T(s_j)x \right\| \\
& \leq \frac{MS^2}{8N^2} (\|A^2x\| + \frac{4}{t^2} \|h_1\|_\infty \|Ax\| + \frac{4}{t^4} \|h'_1\|_\infty \|x\|) \\
& \leq \frac{MS^2}{8N^2} (\|A^2x\| + \frac{4}{t^2} \frac{3.3}{2\sqrt{\pi}} \|Ax\| + \frac{4}{t^4} \frac{236.5}{\sqrt{\pi}} \|x\|) \\
& = \frac{MS^2}{8N^2} (\|A^2x\| + \frac{1}{t^2} \frac{6.6}{\sqrt{\pi}} \|Ax\| + \frac{1}{t^4} \frac{946}{\sqrt{\pi}} \|x\|) \\
& \leq \frac{MS^2}{8N^2} (\|A^2x\| + \frac{4}{t^2} \|Ax\| + \frac{534}{t^4} \|x\|) \\
& = \frac{MS^2}{N^2} (\frac{1}{8} \|A^2x\| + \frac{1}{2t^2} \|Ax\| + \frac{67}{t^4} \|x\|).
\end{aligned}$$

□

Recall the following statement from the Brenner Thomée Theorem 1.5. Let  $A$  be the generator of a strongly continuous semigroup  $T(\cdot)$  with  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . Let  $r(\cdot)$  be an  $A$ -stable rational approximation of the exponential of approximation order  $m$  and define  $V(t) := r(tA)$ . Then there exist constants  $C, c > 0$  such that

$$\|V(\frac{t}{n})^n x - T(t)x\| \leq CMte^{c\omega t} (\frac{t}{n})^m \|A^{m+1}x\|$$

for all  $t \geq 0, n \in \mathbb{N}$ , and  $x \in D(A^{m+1})$ . In the following theorem, we employ Theorem 3.7 in order to obtain estimates for the error incurred in the approximation of  $\{e^{-t\sqrt{-A}} : t \geq 0\}$  by the classical rational schemes. The following lemma is required to calculate the error term.

**Lemma 3.8.** *If  $n$  is a natural number with  $n \geq 1$ , then*

$$\int_0^\infty h_t(s) s^{1-n} \, ds = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4s}} s^{-\frac{2n+1}{2}} \, ds = \frac{2^{n-1}}{t^{2(n-1)}} (2n-3)(2n-5) \cdots 1.$$

*Proof.* If  $n$  is a natural number with  $n \geq 1$ , then  $\Gamma(\frac{2n-1}{2}) = \frac{(2n-3)(2n-5)\cdots 1}{2^{n-1}}\sqrt{\pi}$ .

Indeed, this follows by induction, since it is well known that  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$  and that

$\Gamma(x+1) = x\Gamma(x)$  for  $x > 0$ . Making the substitution  $u = \frac{1}{s}$ , we may write

$$\frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4s}} s^{-\frac{2n+1}{2}} ds = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4}u} u^{\frac{2n-3}{2}} du.$$

Since  $\int_0^\infty e^{-\lambda s} s^\alpha ds = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}$ , we have that

$$\frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4}u} u^{\frac{2n-3}{2}} du = \frac{t}{2\sqrt{\pi}} \frac{\Gamma(n-\frac{1}{2})}{(\frac{t^2}{2})^{\frac{2n-1}{2}}} = \frac{t}{2\sqrt{\pi}} \frac{\Gamma(n-\frac{1}{2})}{(\frac{t}{2})^{2n-1}} = \frac{2^{2n-2} \Gamma(n-\frac{1}{2})}{t^{2n-2} \sqrt{\pi}}.$$

That is,

$$\frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4s}} s^{-\frac{2n+1}{2}} ds = \frac{2^{2n-2} \Gamma(n-\frac{1}{2})}{t^{2n-2} \sqrt{\pi}}.$$

By part (ii), we have that  $\Gamma(n-\frac{1}{2}) = \frac{2^{n-1}}{t^{2(n-1)}}(2n-3)(2n-5)\cdots 1$ . Therefore,

$$\frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4s}} s^{-\frac{2n+1}{2}} ds = \frac{2^{n-1}}{t^{2(n-1)}}(2n-3)(2n-5)\cdots 1.$$

□

**Theorem 3.9.** *Suppose that  $A$  generates a strongly continuous semigroup  $T(s)$  with  $\|T(s)\| \leq Me^{-\omega s}$  for all  $s \geq 0$ . Let  $r(\cdot)$  be an  $A$ -stable rational approximation of the exponential of approximation order  $m$  and define  $V(t) := r(-sA)$ . Fix  $\varepsilon > 0$*

*and  $t > 0$ . Define  $S = S(t, \varepsilon, \omega) = \begin{cases} |\frac{1}{\omega} \ln(\frac{\varepsilon}{M})| & \text{if } \omega < 0, \\ \frac{t^2 M^2}{\varepsilon^2 \pi} & \text{if } \omega = 0. \end{cases}$  Fix  $N \in \mathbb{N}$  and define*

*$c_N = \frac{1}{2}$  and  $c_j = 1$  for  $j = 1, 2, \dots, N-1$ . Define  $h_t(s) := \frac{t}{2\sqrt{\pi}} e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}}$ ,  $s_j := \frac{Sj}{N}$*

*and  $k_j := \frac{t}{2\sqrt{\pi}} \frac{S}{N} c_j h_t(s_j)$ . Then there exists a constant  $C_S > 0$  such that for each*

*$x \in D(A^{m+1})$ ,*

$$\begin{aligned} & \|e^{-t(-A)^{\frac{1}{2}}} x - \sum_{j=1}^N k_j V(\frac{s_j}{n})^n x\| \\ & \leq \frac{MS^2}{N^2} \left( \frac{1}{8} \|A^2 x\| + \frac{1}{2t^2} \|Ax\| + \frac{67}{t^4} \|x\| \right) + \varepsilon \|x\| \\ & \quad + \frac{MC_S}{n^m} \|(-A)^{m+1} x\| \left[ \frac{S^2}{N^2} \frac{38}{t^4} + 1 \right]. \end{aligned}$$

*Proof.* Let  $A$  be the generator of a strongly continuous semigroup  $T(s)$  with  $\|T(s)\| \leq Me^{-\omega s}$  for all  $s \geq 0$ . Let  $r(\cdot)$  be an  $A$ -stable rational approximation of the exponential of order  $m$  and define  $V(s) := r(-sA)$ . Fix  $t > 0$  as choose  $S > 0$  as in the proof of Theorem 3.7. By the Brenner-Thomée Theorem 1.5, there exists a constant  $C_S > 0$  such that  $\|V(\frac{s}{n})^n x - T(s)x\| \leq \frac{MC_S}{n^m} \|(-A)^{m+1}x\|$  for every  $s \in [0, S]$ . Fix  $n \in \mathbb{N}$ . We observe that  $s_j \in [0, S]$  for every  $j = 1, 2, \dots, N$  so by the Brenner-Thomée Theorem 1.5, we may estimate

$$\|T(s_j)x - V(\frac{s_j}{n})^n x\| \leq \frac{MC_S}{n^m} \|(-A)^{m+1}x\|.$$

Hence by Theorem 3.7, we have

$$\begin{aligned} & \|e^{-t(-A)^{\frac{1}{2}}}x - \sum_{j=1}^N k_j V(\frac{s_j}{n})^n x\| \\ & \leq \|e^{-t(-A)^{\frac{1}{2}}}x - \sum_{j=1}^N k_j T(s_j)x\| + \|\sum_{j=1}^N k_j T(s_j)x - \sum_{j=1}^N k_j V(\frac{s_j}{n})^n x\| \\ & \leq \frac{MS^2}{N^2} (\frac{1}{8}\|A^2x\| + \frac{1}{2t^2}\|Ax\| + \frac{67}{t^4}\|x\|) \\ & \quad + \varepsilon\|x\| + \sum_{j=1}^N k_j \|T(s_j)x - V(\frac{s_j}{n})^n x\| \\ & \leq \frac{MS^2}{N^2} (\frac{1}{8}\|A^2x\| + \frac{1}{2t^2}\|Ax\| + \frac{67}{t^4}\|x\|) \\ & \quad + \varepsilon\|x\| + \frac{MC_S}{n^m} \|(-A)^{m+1}x\| \sum_{j=1}^N k_j. \end{aligned}$$

We must estimate  $\sum_{j=1}^N k_j$ . Now,

$$\begin{aligned} |\sum_{j=1}^N k_j| & \leq |\sum_{j=1}^N k_j - \int_0^S h_t(s) ds| + |\int_0^S h_t(s) ds| \\ & \leq |\int_0^S h_t(s) ds - \sum_{j=1}^N k_j| + |\int_0^\infty h_t(s) ds| \\ & \leq |\int_0^S h_t(s) ds - \frac{S}{N} \sum_{j=1}^N c_j h_t(s_j)| + 1 \\ & \leq |\tilde{E}| + 1, \end{aligned}$$

where the trapezoidal error  $|\tilde{E}| = \frac{S^2}{8N^2} \|h_t''\|_{L^1}$ . Recall from equation (3.3.1) that  $h_t'(s) = h_t(s)(\frac{t^2-6s}{4s^2})$ , so  $h_t''(s) = h_t'(s)(\frac{t^2-6s}{4s^2}) + h_t(s)(\frac{3}{2s^2} - \frac{t^2}{2s^3})$ . Hence

$$\begin{aligned} h_t''(s) &= h_t(s)\left(\frac{t^2-6s}{4s^2}\right)^2 + h_t(s)\left(\frac{3s-t^2}{2s^3}\right) \\ &= h_t(s)\left(\frac{t^4-12t^2s+36s^2}{16s^4} + \frac{8s(3s-t^2)}{16s^4}\right) \\ &= h_t(s)\frac{t^4-20t^2s+60s^2}{16s^4}. \end{aligned} \tag{3.3.2}$$

By equation (3.3.2), we have that  $h_t''(s) = h_t(s)\frac{60s^2-20t^2s+t^4}{16s^4} = \frac{t}{32\sqrt{\pi}}e^{-\frac{t^2}{4s}}s^{-\frac{11}{2}}(60s^2-20t^2s+t^4)$ . Therefore,

$$\begin{aligned} |\tilde{E}| &\leq \frac{t}{2\sqrt{\pi}}\frac{S^2}{8N^2}\left\|\frac{1}{16}e^{-\frac{t^2}{4s}}s^{-\frac{11}{2}}(60s^2-20t^2s+t^4)\right\| \\ &\leq \frac{t}{2\sqrt{\pi}}\frac{S^2}{8N^2}\left\|e^{-\frac{t^2}{4s}}\left(\frac{15}{4}s^{-\frac{7}{2}} - \frac{5}{4}t^2s^{-\frac{9}{2}} + \frac{t^4}{16}s^{-\frac{11}{2}}\right)\right\|_{L^1}. \end{aligned}$$

Applying Lemma 3.8 for  $n = 3, 4, 5$  we find that  $\frac{t}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4s}}s^{-\frac{7}{2}}ds = \frac{12}{t^4}$  and  $\frac{t}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4s}}s^{-\frac{9}{2}}ds = \frac{120}{t^6}$  and  $\frac{t}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4s}}s^{-\frac{11}{2}}ds = \frac{16\cdot 105}{t^8}$ . Therefore

$$\begin{aligned} |\tilde{E}| &\leq \frac{t}{2\sqrt{\pi}}\frac{S^2}{8N^2}\left\|e^{-\frac{t^2}{4s}}\left(\frac{15}{4}s^{-\frac{7}{2}} - \frac{5}{4}t^2s^{-\frac{9}{2}} + \frac{t^4}{16}s^{-\frac{11}{2}}\right)\right\|_{L^1} \\ &\leq \frac{S^2}{8N^2}\left(\frac{15}{4}\frac{12}{t^4} + \frac{5}{4}t^2\frac{120}{t^6} + \frac{t^4}{16}\frac{16\cdot 105}{t^8}\right) \\ &= \frac{S^2}{8N^2}\frac{300}{t^4} \leq \frac{S^2}{N^2}\frac{38}{t^4}. \end{aligned}$$

We have shown that

$$\begin{aligned}
& \|e^{-t(-A)^{\frac{1}{2}}}x - \sum_{j=1}^N k_j V\left(\frac{s_j}{n}\right)^n x\| \\
& \leq \frac{MS^2}{8N^2} \left( \frac{1}{8} \|A^2x\| + \frac{1}{2t^2} \|Ax\| + \frac{67}{t^4} \|x\| \right) + \varepsilon \|x\| \\
& \quad + \frac{MC_S}{n^m} \|(-A)^{m+1}x\| \sum_{j=1}^N k_j \\
& \leq \frac{MS^2}{8N^2} \left( \frac{1}{8} \|A^2x\| + \frac{1}{2t^2} \|Ax\| + \frac{67}{t^4} \|x\| \right) + \varepsilon \|x\| \\
& \quad + \frac{MC_S}{n^m} \|(-A)^{m+1}x\| (|\tilde{E}| + 1) \\
& \leq \frac{MS^2}{8N^2} \left( \frac{1}{8} \|A^2x\| + \frac{1}{2t^2} \|Ax\| + \frac{67}{t^4} \|x\| \right) + \varepsilon \|x\| \\
& \quad + \frac{MC_S}{n^m} \|(-A)^{m+1}x\| \left[ \frac{S^2}{N^2} \frac{38}{t^4} + 1 \right].
\end{aligned}$$

□

# References

- [ABHN] W. Arendt, C. Batty, M. Hieber and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*. Boston: Birkhäuser Verlag, 2001.
- [AS] A. Ashyralyev and P. Sobolevskii, *Well-posedness of Parabolic Difference Equations*. Boston: Birkhäuser Verlag, 1994.
- [Ba] R. Bartle, *Elements of Integration and Lebesgue Measure*. New York: John Wiley and Sons, 1995.
- [BT] P. Brenner and V. Thomée, On rational approximations of semigroups. *SIAM J. Numer. Anal.* 16 (1979), no. 4, 683-694.
- [Ch] P. R. Chernoff, *Product Formulas, Nonlinear Semigroups, and Addition of Unbounded Operators*. Memoirs of the American Math. Soc. 140. Providence: American Mathematical Society, 1974.
- [CHMM] A. Chorin, T. Hughes, M. McCracken and J. Marsden, Product formula and numerical algorithms. *Comm. Pure Appl. Math.* 31 (1978), 605-656.
- [CLPT] M. Crouzeix, S. Larsson, S. Piskarev and V. Thomeée, The Stability of Rational Approximations of Semigroups. *BIT* 33 (1993), 74-84.
- [DS] N. Dunford and J. T. Schwartz, *Linear Operators: Part I. General Theory*. Interscience Publ., 1958.
- [Ei] A. Einstein, *Investigations on the Theory of Brownian Movement*. New York: E.P. Dutton and Company, 1926.
- [EN] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*. New York: Springer-Verlag, 1999.
- [Fl] S. Flory, *On the Stabilization and Regularization of Approximation Schemes for Semigroups*, Dissertation, Louisiana State University, 2002.
- [FNW] S. Flory, F. Neubrander, and L. Weis, Consistency and Stabilization of Padé Approximation Schemes for  $C_0$ -semigroups, Preprint 2002.
- [FNZ] S. Flory, F. Neubrander and Y. Zhuang, On the regularization and stabilization of approximation schemes for  $C_0$ -semigroups. *Partial Differential Equations and Spectral Theory*, edited by M. Demuth and B.-W. Schulze, OT 126. Boston: Birkhäuser Verlag, 2001.
- [Ga] G. Gamow, *Mr. Tompkins in Paperback*. Cambridge, England: Cambridge University Press, 1998.

- [Gi] J. W. Gibbs, *The Collected Works of J. Willard Gibbs in two volumes*, vol. II. New York: Longmans, Green, and company, 1931.
- [HW] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II - Stiff and Differential-Algebraic Problems*. New York: Springer-Verlag, 1991.
- [Ha] A. Hansbo, Nonsmooth data error estimates for damped single step methods for parabolic equations in Banach spaces. *Calcolo* 36 (1999), 75-101.
- [HK] R. Hersh and T. Kato, High-accuracy stable difference schemes for well-posed initial value problems. *SIAM J. Numer. Anal.* 16 (1979), no. 4, 670-682.
- [Hid] T. Hida, *Brownian Motion*. Trans. T. Hida and T.S. Speed. New York: Springer-Verlag, 1980.
- [Hil] F. Hildebrand, *Introduction to Numerical Analysis, 2nd ed.* New York: McGraw-Hill, 1973.
- [HP] E. Hille and S. Phillips, *Functional Analysis and Semi-Groups*. Providence: American Mathematical Society, 1957.
- [Ko] M. Kovács, *On the Qualitative Properties and Convergence of Time-discretization Methods for Semigroups*, Dissertation, Louisiana State University, 2004.
- [Ku] H-H. Kuo, *Gaussian Measures in Banach Spaces*. New York: Springer-Verlag, 1975.
- [La] J. Lamperti, *Probability: A Survey of the Mathematical Theory*. New York: W.A. Benjamin, incorporated, 1966.
- [LR] P. Lax and R. Richtmyer, Survey of stability of linear finite difference equations. *Comm. Pure Appl. Math.* 9 (1956), 267-293.
- [LTW] S. Larsson, V. Thomée and L. B. Wahlbin, Finite-element methods for a strongly damped wave equation. *IMA J. Numer. Anal.* II (1991), 115-142.
- [LuR] M. Luskin and R. Rannacher, On the smoothing property of the Crank-Nicolson scheme. *Applicable Analysis*, Vol. 14 (1982), 117-135.
- [Mc] H. McKean, *Stochastic Integrals*. New York: Academic Press, 1969.
- [Ol] J. Olmsted, *Intermediate Analysis: an Introduction to the Theory of Functions of One Real Variable*. New York: Appleton-Century-Crofts, Inc., 1956.

- [Pad] M. H. Padé, Sur représentation approchée d'une fonction par des fractionnelles. *Ann. de l'Ecole Normale Supérieure* 9 (1892).
- [Paz] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. New York: Springer-Verlag 1983.
- [Pe] J. Perrin, *Brownian movement and Molecular Reality*. London: Taylor and Francis, 1910.
- [PT] G. Phillips and P. Taylor, *Theory and Applications of Numerical Analysis*. New York: Academic Press, 1973.
- [Se] V. Seshadri, *The Inverse Gaussian Distribution: A Case Study in Exponential Families*. Oxford: Clarendon Press, 1993.
- [Th] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*. New York: Springer-Verlag, 1997.
- [Tw] M.C.K. Tweedie, Inverse Statistical Variates, *Nature*, Vol. 155, p. 453.
- [Wa] M. T. Wasan, *First Passage Time Distribution of Brownian Motion with Positive Drift (Inverse Gaussian Distribution)*. Queen's Paper in Pure and Applied Mathematics no. 19. Kingston, ON: Queen's University, 1969.
- [Wid] D. Widder, *The Laplace Transform*. Princeton: Princeton University Press, 1941.
- [Wie1] N. Wiener, *I am a Mathematician*. Garden City, NY: Doubleday and Company, Inc., Garden City, 1956.
- [Wie2] N. Wiener, *Selected Papers of Norbert Wiener*. Society for Industrial and Applied Mathematics, and the M.I.T. Press, 1964.
- [WSRM] N. Wiener, A. Siegel, B. Rankin, and W.T. Martin, *Differential Space, Quantum Systems, and Prediction*. Cambridge: M.I.T. Press, 1966.
- [Yo] K. Yosida, *Functional Analysis*. New York: Springer-Verlag, 1965.
- [Zh] Y. Zhuang, *Classically Unstable Approximations for Linear Evolution Equations and Applications*. Dissertation, Department of Mathematics, Louisiana State University, 2000.
- [Zo] V. M. Zolotarev, *One-dimensional Stable Distributions*. Providence: American Mathematical Society, 1986.

# Appendix A. Brownian Motion: Constructing the Wiener Space

We saw in Chapter 3 that the inverse Laplace transform of  $e^{-t\sqrt{-\lambda}}$  is given by the density function  $h_t(s)$ , which is a special case of the probability density function associated with the Inverse Gaussian distribution. The Inverse Gaussian distribution may be interpreted physically as the first passage time distribution of Brownian motion with positive drift. A rigorous mathematical treatment of Brownian motion led N. Wiener to his celebrated definition of the celebrated Wiener measure, a probability measure on the space  $C[0, 1]$  of continuous functions. In this appendix, we begin with a brief history of the study of Brownian motion, including a biographical sketch of N. Wiener. We conclude this appendix with a thorough construction and existence proof of the Wiener measure.

First described by the botanist Robert Brown in 1828 as the movement of pollen particles suspended in fluid, Brownian motion is a phenomenon of erratic molecular movement [Ei, p.86]. Brown himself

proved that the movement was not due to living animalculae, and recognized that the particles in suspension are agitated the more briskly the smaller they are [Pe,p. 2].

The classical Wiener space was constructed in 1923 by N. Wiener in connection with his mathematical study of Brownian motion. In order to detect Brownian motion, the particle must be large enough to be visible, yet not so large that the influence of the fluid molecules averages out so much that no motion can be detected. If such a balance of scale is attained, then the Brownian motion is manifest as a continuous irregular motion of the particle.

For decades, physicists had ignored Brownian motion, falsely attributing the source of the motion to external influences. The first precise treatment of Brownian motion is attributed to Gouy's 1888 publication on the subject [Ei,p.87]. By 1888, it had been observed that Brownian motion is not due to vibration since it persists equally everywhere, nor is it due to currents in a fluid approaching thermal equilibrium since it does not desist after adequate time has been given for equilibrium to be attained [Pe, p. 5]. Furthermore, the Brownian movement is so erratic that no thermal currents are detected in the movement.

This is a truly remarkable discovery! Indeed, Jean Perrin, one of the main contributors to the physical study of Brownian motion remarked "what is really strange and new in the Brownian motion is precisely that it never stops" [Pe, p.7]. Brownian motion is a perpetual motion of the second type, that is, the energy required for the motion is extracted from surrounding heat [Ga, p.101]. Brownian motion stood in direct opposition to the traditionally held belief in the impossibility of perpetual motion of the second type [Pe, p.1]. Thus, Brownian motion, so long ignored, was direct evidence of a previously unknown fundamental property of matter: perpetual motion of the second type exists in every fluid on the small scale! This demanded a kinetic conception of the fluid state. Formulating such a kinetic conception is enough to lead to the postulation of the existence of molecules. Indeed, if fluid is in perpetual motion, yet to the human eye appears static, then fluid cannot be infinitely subdivisible. This implies that the fluid itself is composed of "molecules which can assume all possible motions relative to one another" [Pe, p.8].

Albert Einstein studied the displacement of particles in Brownian motion from the point of view of statistical mechanics in 1905. The first complete mathematical description of Brownian motion was given by N. Wiener in his 1923 paper

*Differential Space* [WSRM]. Einstein, unlike Wiener, did not consider the mathematical properties of the path traced by a single particle exhibiting the Brownian movement. Indeed,

without further troubling about the infinitely tangled trajectory which the granule describes in a given time, Einstein considered simply its displacement during this time [Pe, p.51].

Einstein's treatment of Brownian motion was the first theoretical and quantitative approach to the topic, and was published in 1905, the same year as his theory of special relativity [Hi, p.ix].

The mathematical investigation of Brownian motion led N. Wiener to define a probability measure on the space  $C[0, 1]$  of continuous functions on the time interval  $[0, 1]$ . Although Brownian motion was rather well understood before Wiener began to consider it mathematically,

there was little indication of the mathematical properties that would result from a rigorous description of the physical phenomenon [WSRM, p.15].

The formulation of a mathematical description of Brownian motion was the motivating force that led Wiener to define the Wiener measure on an infinite dimensional function space. Wiener essentially originated the analysis of infinite dimensional function space, which at his time was a new branch of mathematics.

The main result that will be considered in this Appendix is known as the Wiener Theorem. The Wiener Theorem proves the existence of a probability measure on  $C[0, 1]$ , the set of all continuous real valued functions defined on the interval  $[0, 1]$  that vanish at zero. It is well known that  $C[0, 1]$  may be regarded as an infinite dimensional vector space. Lebesgue measure has been put to much effective use

in probability theory when some integration is to occur over a finite dimensional space. However, as is shown in [Ku, p.1], there is no ‘infinite dimensional Lebesgue measure’; that is to say, if  $H$  is an infinite dimensional Hilbert space then there is no translation invariant Borel measure  $\mu$  on  $H$  such that  $0 < \mu(B) < \infty$  for every nonempty open ball  $B$  in  $H$ . Hence Wiener’s construction of a probability measure on  $C[0, 1]$  was a crucial step in the advancement of the theory of infinite dimensional probability spaces.

In Appendix B, we shall show that when  $C[0, 1]$  is equipped with the Wiener measure  $w$ , almost every continuous function in  $C[0, 1]$  represents the path traced by some particle in Brownian motion. We will then discuss the amazing result that with respect to  $w$ , almost every (path of a particle in) Brownian motion is nowhere differentiable. Hence with respect to the Wiener measure, almost every continuous function is everywhere jagged! Before we can be justified in asserting the validity of the above statements, we must prove the Wiener measure exists.

## A.1 A Short Biography of N. Wiener

Norbert Wiener found the motivation for many of his creative contributions to the knowledge process in the frontiers between mathematics and various different scientific disciplines. As Wiener himself wrote regarding many of the problems he considered,

...I saw, as was my habit, a physical and even an engineering application, and my sense of this often determined the images I formed and the tools by which I sought to solve my problems [Wie1, p.168].

Wiener obtained his Ph.D. from Harvard at the age of 18. During Wiener’s time at Harvard and also as a recent Ph.D., mathematician-philosopher Bertrand Russell supplied Wiener with the role of primary mentor. Soon after obtaining his Ph.D.,

Wiener traveled to Europe. He spent much time in Cambridge, England and later in Göttingen, Germany. Wiener's penchant for considering problems from various perspectives was strengthened by his mentor: Wiener writes that at Cambridge, the need for a "physical sense" was impressed upon him by Russell [Wie1, p.25]. Wiener spent time in Göttingen during a time when the town enjoyed an atmosphere well charged by the burgeoning ideas of quantum physics. While in Göttingen, Wiener studied with Edmund Landau and David Hilbert.

After a certain amount of traveling, and of moving from job to job (including a position as a hack writer for a newspaper), Wiener settled in 1919 at the Massachusetts Institute of Technology, where he was to hold a position for over 36 years. Wiener's interest in the delicate interplay between physics and mathematics solidified to a large extent at M.I.T.

When he arrived at M.I.T., Wiener's interests included a study of the Lebesgue integral, to which he had been introduced as a student by G.H. Hardy. Although it is well known that the Lebesgue integral bears intimate connections to the theory of probability and statistics, this fact was not always known. Wiener introduced the Wiener measure in his 1923 paper entitled *Differential Space*, which was the first paper to clearly unite measure theory with probability theory [WSRM, p.1],[Wei1, p.39]. One of Wiener's colleagues at M.I.T. was Henry Bayard Phillips, whom Wiener attributes with contributing the most to his (Wiener's) awareness of the physical facets of mathematics and their importance to any mathematician, whether pure or applied. Through Phillips, Wiener became aware of Willard Gibbs' work on statistical mechanics. Gibbsian statistical mechanics "contributed physical motivation for Wiener's measure theoretic approach to probability" [WSRM, p.9].

This description, published in his 1923 paper *Differential Space*, provided Wiener with a natural link between physics, measure theory, and probability. It was in the frontier between physics and mathematics that Wiener found the inspiration for his work in probability. In reference to probability and statistics, Wiener reflects:

They stood approximately in the middle ground where physics and mathematics meet and this middle ground was just where I was eventually to do my best work, for such work seemed to be in harmony with a basic aspect of my personality [Wie1, p.23].

In his autobiography, Wiener describes the process by which he realized that he was seeking a mathematical tool that could be used to describe nature, and that moreover it was within nature herself that he should find this tool and from nature that he should extract his problems of interest. He became interested in the motion of the River Charles, which he could observe from an M.I.T. building. The movement of the water is continual and given any particular direction, it is almost certain that at some point in time, some molecule of the fluid will have travelled in that direction. However, the water is not so unstable that it might spontaneously collapse or dissipate. Even in a gaseous state, water molecules tend to group together in a certain sense. That is to say, the movement of water molecules in opposing directions has a tendency to average itself out on the large scale. Specifically, the greater the number of molecules we observe, and the greater the duration of time over which we observe them, the more improbable it becomes that the molecular movement will co-ordinate itself spontaneously [Pe, p.7]. Indeed, the fluid equilibrium commonly perceived by humankind is an illusion; this equilibrium is a statistical equilibrium which exists only as an average for large

masses (relative to the molecular scale) over large periods of time (relative to the time between molecular collisions) [Pe, p.5-6]. Wiener claims to have wondered:

How could one bring to a mathematical regularity the study of the mass of ever shifting ripples and waves, for was not the highest destiny of mathematics the discovery of order among disorder?...the problem of the waves was clearly one for averaging statistics [Wie1, p.33].

If we consider the River Charles from the point of view of Newtonian determinism, then complete data about the condition of the river at a single point in time would be enough to predict everything about the movement of the water for all future time. However, such complete and precise data can never be humanly attained. The work of Gibbs deals not with a fixed predetermined universe, but a collection of universes, which Gibbs refers to as systems. Gibbs writes: “Enlarge this collection so as to include every possible condition that the system could ever reach” [Gi, p. vii]. Whereas the Newtonian approach is to follow one system (given by a configuration at a fixed time) through its various configurations over time, Gibbs formulated an entirely new question. Given information at a fixed time about how the many ‘phases’ of the possible systems are distributed, Gibbs sought to follow how the ‘phases’ of the possible systems are distributed over time. This is clearly a probabilistic approach.

The beginning of Wiener’s career marked an era in the history of mathematics when a predominant trend in research was the progression from classical finite dimensional analysis to the analysis of infinite dimensional function spaces. This point of view necessitates the theory of functionals, which are sometimes replaced by an average. This average is obtained by some form of integration over an infinite dimensional space. Such an integral requires careful definition. Shortly after tak-

ing his position at M.I.T., Wiener requested of Dr. Irving Barnett that he suggest some new and interesting unresolved problems. Barnett impressed upon Wiener that a very active topic at the time was the generalization of the classical finite dimensional probability theory to a probability space whose elements are themselves continuous functions. Such a generalization was provided by Wiener in the form of the Wiener measure. Wiener's successful construction of an infinite dimensional probability space followed directly from Wiener's previous study of the Lebesgue integral, and was one of the primary accomplishments of his incipient career as a creative mathematician [Wie1, p.22]. The main inspiration for Wiener's construction arose from his mathematical study of Brownian motion, a well known physical phenomenon.

In the spirit of Gibbs, we would like to assign a probability to one particular path that could possibly be traced by a particle in Brownian motion. We idealize this path as being given by a continuous function on the interval  $[0, 1]$ . Hence, we seek a probability measure on  $C[0, 1]$ . Such a probability measure would provide a generalized integration on  $C[0, 1]$ . This generalized integral was crucial to Wiener's formal theory of Brownian motion. With the Brownian motion, Wiener found

an ideal proving ground for... ideas concerning the Lebesgue integral in a space of curves, and it had the abundantly physical texture of the work of Gibbs [Wie1, p.38].

In order to construct the Wiener measure, and thereby lay the foundation for integration on the space  $C[0, 1]$  and hence for the theory of Brownian motion, we present the definitions and notations necessary for a rigorous discussion.

## A.2 The Wiener Measure

Recall that a field of subsets of a given nonempty set  $X$  is a family  $\mathcal{F}$  of subsets of  $X$  such that  $\emptyset \in \mathcal{F}$  and  $\mathcal{F}$  is closed under complements and finite unions. Recall also that a  $\sigma$ -field on a given set  $X$  is a field of subsets of  $X$  which is closed under countable unions. The *Borel field* of a topological space  $X$  is the smallest  $\sigma$ -field which contains all the open subsets of  $X$ . For the remainder of this paper, we will write  $\mathcal{B}$  to mean the Borel field of  $C$ . Likewise, we will henceforth write  $\mathcal{B}(\mathbb{R}^n)$  to mean the Borel field of  $\mathbb{R}^n$  with respect to the standard topology.

Fix any nonempty set  $X$  and a field  $\mathcal{F}$  of subsets of  $X$ . We say that an extended real valued map  $\mu : \mathcal{F} \rightarrow [0, \infty]$  is a *measure* if

- (i)  $\mu(\emptyset) = 0$
- (ii)  $\mu(E) \geq 0$  for every  $E \in \mathcal{F}$ , and
- (iii)  $\mu$  is  $\sigma$ -additive, that is, if  $(E_n)_{n \in \mathbb{N}}$  is a sequence of disjoint sets in  $\mathcal{F}$  whose union  $\bigcup_{n=1}^{\infty} E_n$  belongs to  $\mathcal{F}$ , then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Let  $\Omega$  be a nonempty set and let  $\mathcal{F}$  be a  $\sigma$ -field of subsets of  $\Omega$ . A map  $P : \mathcal{F} \rightarrow [0, \infty)$  is called a *probability measure* if the following conditions hold:

- (i)  $\mathcal{F}$  is a Borel field of subsets of some nonempty set  $\Omega$
- (ii)  $P(\Omega) = 1$ , and
- (iii)  $P$  is  $\sigma$ -additive.

In this case, the triple  $(\Omega, \mathcal{F}, P)$  is called a *probability space*. Moreover, the set  $\Omega$  is called the *sample space*, the elements of  $\Omega$  are called *outcomes*, and elements of

$\mathcal{F}$  are called *events*. We say that a map  $X : \Omega \rightarrow \mathbb{R}$  is a *random variable* if for each Borel set  $A$  in  $\mathbb{R}$ , its inverse image  $X^{-1}(A)$  is an event. In other words, a random variable is nothing more and nothing less than a *measurable function* on the probability space  $(\Omega, \mathcal{F}, P)$ .

Recall also that if we have a nonempty set  $X$ , a field  $\mathcal{F}$ , and a measure  $\mu$  defined on  $\mathcal{F}$ , then we may define the *outer measure* generated by  $\mu$  as follows: for an arbitrary subset  $B$  of  $X$ , the outer measure  $\mu^*$  of  $B$  is given by  $\mu^*(B) = \inf \sum_{j=1}^{\infty} \mu(E_j)$ , where the infimum is extended over all sequences  $(E_j)_{j \in \mathbb{N}}$  in  $\mathcal{F}$  such that  $B \subseteq \bigcup_{j=1}^{\infty} E_j$ .

A subset  $E$  of  $X$  is said to be  $\mu^*$ -measurable if  $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$  for every  $A \subseteq X$ .

We call a subset  $I$  of  $C$  a *cylinder set* if and only if  $I$  is of the form

$$\{x \in C; (x(t_1), \dots, x(t_n)) \in E\} \quad (\text{A.2.1})$$

where  $E \in \mathcal{B}(\mathbb{R}^n)$  and  $0 < t_1 < t_2 < \dots < t_n \leq 1$ . For the remainder of our discussion,  $\mathcal{R}$  will denote the collection of all cylinder subsets of  $C$ . It is easy to see that  $\mathcal{R}$  is a field of subsets but not a  $\sigma$ -field.

For notational convenience, let  $t_0 = 0$  and  $u_0 = 0$  for the remainder of this exposition. Define  $I$  by (A.2.1) for some fixed  $E \in \mathcal{B}(\mathbb{R}^n)$  and  $0 < t_1 < t_2 < \dots < t_n \leq 1$ . The expression

$$\frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n (t_i - t_{i-1})}} \int_E e^{-\frac{1}{2}(\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}})} du_1 \dots du_n \quad (\text{A.2.2})$$

defines the *Wiener measure* of  $I$ , which is written  $w(I)$ . It is easy to see that  $w$  is finitely additive in  $\mathcal{R}$ . It is highly nontrivial to show that  $w$  is indeed  $\sigma$ -additive. For reference purposes, we now state, without proof, the Carathéodory Extension Theorem. The Carathéodory Extension Theorem is a well known result of real analysis; for a formal proof, the interested reader is referred to Bartle [Ba, p.101].

**Theorem A.1 (Carathéodory Extention).** *Let  $\mu$  be a measure on a field of subsets  $\mathcal{F}$  of a given nonempty set  $X$ . Let  $\mu^*$  be the outer measure associated to  $\mu$  and let  $\mathcal{A}^*$  be the collection of all  $\mu^*$ -measurable sets. Then  $\mathcal{A}^*$  is a  $\sigma$ -field containing  $\mathcal{A}$ . Moreover, if  $(E_n)_{n \in \mathbb{N}}$  is a sequence of disjoint sets which are elements of  $\mathcal{A}^*$ , then  $\mu^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu^*(E_n)$ . That is,  $\mu^*$  is  $\sigma$ -additive on the  $\sigma$ -field  $\mathcal{A}^*$ .*

With this equipment in hand, let us reconsider  $w$  given by (A.2.2). Supposing that we already know that  $w$  is  $\sigma$ -additive, we then know that  $w$  is in fact a measure on the field  $\mathcal{R}$  of cylinder sets and so  $w$  has an associated outer measure  $w^*$ . The Carathéodory Extension Theorem guarantees that  $w$  can be extended to a measure on a  $\sigma$ -field containing the collection  $\mathcal{R}$  of cylinder sets (particularly, the  $\sigma$ -field generated by  $\mathcal{R}$ ). Furthermore, we will show that the  $\sigma$ -field generated by  $\mathcal{R}$  is precisely the Borel field  $\mathcal{B}$ . Once we have established this result and the result concerning the  $\sigma$ -additivity of  $w$ , we will have constructed a Borel measure on the infinite dimensional normed vector space  $C[0, 1]$ . We will also see that  $w(C[0, 1]) = 1$ . That is, we will construct a probability measure on an infinite dimensional space of continuous functions! By abuse of notation, the extension of  $w$  to  $\mathcal{B}$  will also be denoted by  $w$ . When these results have been proven, we may ultimately define the *Wiener integral* on  $C[0, 1]$ . If  $f$  is a Wiener integrable function then we may write  $E_w(f) = \int_{C[0,1]} f(x)w(dx)$ . If  $X$  is a random variable, then the Wiener integral  $E_w(X)$  defines the expected value of  $X$ .

We say that a random variable  $X$  on a probability space  $(\Omega, \mathcal{F}, P)$  has normal (Gaussian) distribution with mean  $m$  and variance  $\sigma^2$  if

$$P\{x; a \leq X \leq b\} = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx.$$

**Example A.2.** Suppose  $x \in C[0, 1]$ . Fix numbers  $0 < s < t \leq 1$ . Then  $x(t) - x(s)$  is a normally distributed random variable with respect to the Wiener measure  $w$ . Moreover, as a random variable,  $x(t) - x(s)$  has mean 0 and variance  $t - s$ .

*Proof.* Indeed, if we define  $E = \{(x, y); a \leq x - y \leq b\}$  and observe that  $E$  is a cylinder set, then we may compute

$$\begin{aligned}
w(\{x; a \leq x(t) - x(s) \leq b\}) &= w(\{x; (x(t), x(s)) \in E\}) \\
&= \frac{1}{\sqrt{(2\pi)^2 s(t-s)}} \int_E e^{-\frac{1}{2}\left(\frac{v^2}{s} + \frac{(u-v)^2}{t-s}\right)} dudv \\
&= \frac{1}{\sqrt{(2\pi)^2 s(t-s)}} \int_{-\infty}^{\infty} \int_a^b e^{-\frac{1}{2}\left(\frac{v^2}{s} + \frac{(u-v)^2}{t-s}\right)} dudv. \tag{A.2.3}
\end{aligned}$$

Setting  $\tau_1 = u - v$  and  $\tau_2 = u$ , we obtain the following from (A.2.3):

$$\begin{aligned}
&w(\{x; a \leq x(t) - x(s) \leq b\}) \\
&= \frac{1}{\sqrt{(2\pi)^2 s(t-s)}} \int_{-\infty}^{\infty} \int_a^b e^{-\frac{1}{2}\left(\frac{\tau_2^2}{s} + \frac{\tau_1^2}{t-s}\right)} d\tau_1 d\tau_2 \\
&= \left(\frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{1}{2}\left(\frac{\tau_1^2}{t-s}\right)} d\tau_1\right) \left(\frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{\tau_2^2}{s}\right)} d\tau_2\right) \\
&= \frac{\sqrt{2\pi s}}{\sqrt{(2\pi)^2 s(t-s)}} \int_a^b e^{-\frac{1}{2}\left(\frac{\tau_1^2}{t-s}\right)} d\tau_1 \tag{A.2.4}
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{1}{2}\left(\frac{\tau^2}{t-s}\right)} d\tau. \tag{A.2.5}$$

The computation that results in (A.2.4) is due to the fact that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{A.2.6}$$

□

### A.3 The Wiener Theorem

Let us construct the Wiener measure; we follow a proof given by H.-H. Kuo in [Ku].

**Theorem A.3 (Wiener).** *Let  $t_0 = 0$ . Fix a Borel set  $E \in \mathcal{B}$  and fix  $0 < t_1 < t_2 < \dots < t_n \leq 1$ . Let a cylinder subset  $I$  be given by (A.2.1). Define  $w(I)$  by (A.2.2). Then  $w$  is  $\sigma$ -additive in the Borel field  $\mathcal{B}$ .*

*Proof.* We follow a proof given by Kuo in [Ku]. The proof is of a very technical nature. We begin by fixing notations for several sets which will be of use during the main argument. First we shall let  $S$  denote the set of all binary rationals in  $[0, 1]$ , that is, the set of rational numbers in  $[0, 1]$  which may be written as  $\frac{k}{2^n}$  for some  $k$  odd and some  $n \in \mathbb{N}$ . We write

$$C_\alpha = \{x \in C; \exists a_x, |x(t) - x(s)| \leq a_x |t - s|^\alpha \forall t, s\}$$

$$B_\alpha = \{x \in C; \exists a_x, |x(t) - x(s)| \leq a_x |t - s|^\alpha \forall t, s \in S\}$$

$$H_\alpha[a] = \{x \in C; \exists s, t \in S, |x(s) - x(t)| > a |s - t|^\alpha\}$$

$$H_\alpha = \{x \in C; \forall a > 0, \exists s, t \in S, |x(s) - x(t)| > a |s - t|^\alpha\}$$

and

$$I_{\alpha, a, k, n} = \{x \in C; |x(\frac{k}{2^n}) - x(\frac{k-1}{2^n})| > a (\frac{1}{2^n})^\alpha\} \quad (\text{A.3.1})$$

for  $k \in \{1, 2, \dots, 2^n\}$ . Finally, we define  $w^*$  to be the outer measure associated to  $w$ .

The following are easy consequences of the above set equalities:

- (a)  $0 < \alpha < \beta$  implies  $C_\alpha \subseteq C_\beta \subseteq C[0, 1]$ ,
- (b)  $C_\alpha = B_\alpha$  whenever  $\alpha > 0$ ,
- (c) if  $(a_n)_{n \in \mathbb{N}}$  is an increasing unbounded sequence of positive real numbers then

$$H_\alpha = \bigcap_{\alpha > 0} H_\alpha[a] = \bigcap_{n=1}^{\infty} H_\alpha[a_n],$$

and

(d)  $H_\alpha = C[0, 1] - B_\alpha$ .

Next we state two technical lemmas which will be useful during the proof of the main result. Proofs of each of these lemmas will be provided at the end of this section.

**Lemma A.4.** *Fix positive real numbers  $\alpha$  and  $a > 0$ . Then*

$$w^*(H_\alpha[2a \frac{1}{1-2^{-\alpha}}]) \leq \sqrt{\frac{2}{\pi}} \frac{1}{a} \sum_{n=0}^{\infty} 2^{n(\alpha-\frac{1}{2})} e^{-\frac{a^2}{2} 2^{n(1-2\alpha)}}$$

**Lemma A.5.** *Fix  $a > 0$  and  $0 < \alpha < \frac{1}{2}$ . If  $I$  is any cylinder set contained in  $H_\alpha[2a \frac{1}{1-2^{-\alpha}}]$  then*

$$w(I) \leq \sqrt{\frac{2}{\pi}} \frac{1}{a} \frac{1}{(1 - 2^{1-\delta} e^{-\frac{1}{2} a^2 \delta})},$$

where  $\delta = \frac{1}{2} - \alpha$ .

Notice that

$$\lim_{a \rightarrow \infty} \frac{1}{a} \frac{1}{(1 - 2^{1-\delta} e^{-\frac{1}{2} a^2 \delta})} = 0.$$

The statement that  $w$  is  $\sigma$ -additive in the  $\sigma$ -field generated by  $\mathcal{R}$  is equivalent to the condition that if  $I_n$  is a decreasing sequence in  $\mathcal{R}$  with empty intersection, then  $\lim_{n \rightarrow \infty} w(I_n) = 0$ . We strive to show the latter. Fix  $\varepsilon > 0$  and let  $I_n$  be a decreasing sequence in  $\mathcal{R}$  with empty intersection. Write

$$\begin{aligned} I_n &= I_n(t_1^{(n)}, \dots, t_{s_n}^{(n)}; E_n) \\ &= \{x \in C; (x(t_1^{(n)}), \dots, x(t_{s_n}^{(n)})) \in E_n\}, \end{aligned}$$

where  $E_n \in \mathcal{B}(\mathbb{R}^{s_n})$ .

Now, for each  $n \in \mathbb{N}$ , we have that  $E_n$  is Borel measurable so  $E_n$  may be approximated by compact subsets. For each  $n \in \mathbb{N}$ , choose a compact subset  $G_n$  of  $E_n$  such that  $w(I_n - K_n) < \frac{\varepsilon}{2^{n+1}}$  where  $K_n = K_n(t_1^{(n)}, \dots, t_{s_n}^{(n)}; G_n)$ . Clearly each  $K_n$  is a closed cylinder set.

Define  $L_n = \bigcap_{j=1}^n K_j$ . Then  $L_n$  is a closed cylinder set and  $L_n \subseteq K_n \subseteq I_n$  for each  $n \in \mathbb{N}$ . We have

$$w(I_n) = w(I_n - L_n) + w(L_n)$$

since  $w$  is finitely additive in the field  $\mathcal{R}$  of cylinder sets. Observe that

$$I_n - L_n = I_n - \bigcap_{j=1}^n K_j = \bigcup_{j=1}^n (I_n - K_j) \subseteq \bigcup_{j=1}^n (I_j - K_j)$$

so

$$w(I_n - L_n) \leq w\left(\bigcup_{j=1}^n (I_j - K_j)\right) \leq \sum_{j=1}^n \frac{\varepsilon}{2^{j+1}} \leq \frac{\varepsilon}{2}.$$

Hence we have

$$w(I_n) = w(I_n - L_n) + w(L_n) \leq \frac{\varepsilon}{2} + w(L_n) \tag{A.3.2}$$

for each  $n \in \mathbb{N}$ .

We next endeavor to show that there is a natural number  $N$  such that  $w(L_n) < \frac{\varepsilon}{2}$  for every  $n > N$ . Once this has been established, we will have  $w(I_n) < \varepsilon$  for every  $n > N$ ; that is,  $\lim_{n \rightarrow \infty} w(I_n) = 0$ .

Fix numbers  $a$  and  $0 < \alpha < \frac{1}{2}$  and define  $b = 2a \frac{1}{1-2^{-\alpha}}$ . By Lemma A.4 and Lemma A.5, we may choose  $b$  large enough so that

**A.6.**  $w(I) < \frac{\varepsilon}{2}$  whenever  $I \subseteq H_\alpha[b]$  is a cylinder set.

It follows immediately from the definition of  $H_\alpha[b]$  that its complement

$$\tilde{H}_\alpha[b] = \{x \in C; \forall s, t \in S, |x(t) - x(s)| \leq |t - s|^\alpha\}. \tag{A.3.3}$$

Notice that  $\tilde{H}_\alpha[b]$  is closed: indeed, if  $x_n$  is a sequence in  $\tilde{H}_\alpha[b]$  with  $x_n \rightarrow x_0$ , then we have

$$|x_0(t) - x_0(s)| \leq |x_0(t) - x_n(t)| + |x_n(t) - x_n(s)| + |x_n(s) - x_0(s)|. \tag{A.3.4}$$

Now, the first and last terms of (A.3.4) can be made arbitrarily small with sufficiently large  $n$ , and  $x_n \in \tilde{H}_\alpha[b]$  for every  $n \in \mathbb{N}$ , so we have the estimate on  $|x_0(t) - x_0(s)|$  required by the definition given in (A.3.3) of  $\tilde{H}_\alpha[b]$ .

Furthermore,  $\tilde{H}_\alpha[b]$  is pointwise bounded and equicontinuous and so is compact by Ascoli's Theorem. Define  $M_n = L_n \cap \tilde{H}_\alpha[b]$ . Then  $M_n$  is compact, since  $\tilde{H}_\alpha[b]$  is compact and  $L_n$  is closed. It is enough to show that there is a natural number  $N$  such that  $M_n = \emptyset$  for every  $n \geq N$ . Indeed, if  $M_n = \emptyset$ , then  $L_n \subseteq H_\alpha[b]$  and so  $w(L_n) \leq \frac{\varepsilon}{2}$  by line **A.6**, as  $L_n$  would be a cylinder set contained entirely within  $H_\alpha[b]$ . Notice that  $M_n \subseteq L_n \subseteq I_n$ , so  $M_n \searrow$  since  $L_n \searrow$  and that

$$\bigcap_{n=1}^{\infty} M_n = \emptyset. \quad (\text{A.3.5})$$

since  $\bigcap_{n=1}^{\infty} I_n = \emptyset$ .

Suppose for the sake of an eventual contradiction that  $M_n \neq \emptyset$  for every natural number  $n$ . For every  $n \in \mathbb{N}$ , select an element  $x_n \in M_n$  and consider the sequence  $(x_n)_{n \in \mathbb{N}}$  in  $C[0, 1]$ . Observe that  $(x_n)_{n \in \mathbb{N}}$  is equicontinuous because  $x_n \in \tilde{H}_\alpha[b]$  for every  $n \in \mathbb{N}$ . Furthermore,  $\{x_n(t); n = 1, 2, \dots\} \subseteq \mathbb{R}$  is bounded for each  $t \in [0, 1]$  because  $|x_n(t)| = |x_n(t) - x_n(0)| \leq b|t - 0|^\alpha$ . Thus, by Ascoli-Arzelà's theorem,  $(x_n)_{n \in \mathbb{N}}$  is precompact. This means that there is a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  of  $(x_n)_{n \in \mathbb{N}}$  such that  $x_{n_k} \rightarrow x_0 \in C$  uniformly as  $k \rightarrow \infty$ . Observe that for  $(x_{n_k})_{k \in \mathbb{N}}$ , the corresponding  $(M_{n_k})_{k \in \mathbb{N}}$  are a decreasing sequence of sets because  $M_n$  is a decreasing sequence of sets, and the sequence  $(n_k)_{k \in \mathbb{N}}$  is necessarily a strictly increasing sequence of positive integers. We have that  $x_0 \in \tilde{H}_\alpha[b]$  since  $\tilde{H}_\alpha[b]$  is closed.

Fix some  $N \in \mathbb{N}$ . Then  $x_{n_k} \in M_N$  for every  $k \geq N$  since  $M_{n_k}$  is decreasing. Furthermore,  $M_N$  is compact, so  $x_0 \in M_N$ . Hence  $x_0 \in M_n$  for every  $n \in \mathbb{N}$ . This shows that  $\bigcap_{n=1}^{\infty} M_n \neq \emptyset$ , which contradicts our previous observation (A.3.5) that

$\bigcap_{n=1}^{\infty} M_n = \emptyset$ . Thus, there is a natural number  $N_0$  such that  $M_{N_0} = \emptyset$ , from which it follows that  $M_n = \emptyset$  for every  $n \geq N_0$  by nonincreasing monotonicity of the set-valued sequence  $(M_n)_{n \in \mathbb{N}}$ .

By previous remarks, this is enough to conclude that  $w(L_n) < \frac{\varepsilon}{2}$  for every  $n \geq N_0$ . With this information, together with (A.3.2), we obtain an  $\varepsilon$ -estimate of  $w(I_n)$  for  $n \geq N_0$ . This proves that  $w$  is  $\sigma$ -additive in the  $\sigma$ -field generated by the collection  $\mathcal{R}$  of cylinder subsets of  $C[0, 1]$ .

We lastly show that the  $\sigma$ -field generated by  $\mathcal{R}$  is exactly the Borel field  $\mathcal{B}$ . Now, the Stone-Weierstrass Theorem tells us that the polynomial functions with rational coefficients are dense in  $(C[0, 1], \|\cdot\|_{\infty})$ . There is an obvious bijection between the set of all polynomials with rational coefficients and the set  $\bigcup_{i=1}^{\infty} \mathbb{Q}^i$ . The set  $\bigcup_{i=1}^{\infty} \mathbb{Q}^i$  is countable as a countable union of finite products of countable sets. Thus,  $(C[0, 1], \|\cdot\|_{\infty})$  is separable. Any open subset of a separable metric space is a countable union of open balls. Indeed, it is a well known result of general topology that any separable metric space is Lindelöf, that is, if we cover any open set  $U$  with open balls then we may extract a countable subcovering of  $U$ . Hence it is enough to show that the closed unit ball  $\overline{B}(0, 1)$  in  $C[0, 1]$  is an element of the  $\sigma$ -field generated by  $\mathcal{R}$ . However, it is easy to see that

$$\overline{B}(0, 1) = \bigcap_{n=1}^{\infty} \left\{ x : |x(t)| \leq \frac{k}{2^n}, k = 1, 2, \dots, 2^n \right\}.$$

Our proof of the Wiener Theorem will be complete once we have established the validity of the two technical lemmas which were employed in the calculations that arose in the main argument. However, Lemma A.4 is most efficiently proven with the aid of yet two more sizeable estimations. We establish these useful estimations with the following two propositions.

**Proposition A.7.** Fix positive real numbers  $\alpha$  and  $a$ . Suppose  $x \in C$  satisfies  $|x(\frac{k}{2^n}) - x(\frac{k-1}{2^n})| \leq a(\frac{1}{2^n})^\alpha$  for every  $k \in \{1, 2, \dots, 2^n\}$  and for every  $n \in \mathbb{N}$ . Then  $|x(t) - x(s)| \leq 2a\frac{1}{1-2^{-\alpha}}|t-s|^\alpha$  for each choice of  $s, t \in S$ .

*Proof.* Suppose  $s = 0$  and  $t = 1$ . In this case, the conclusion of Proposition A.7 reads as follows:

$$|x(1)| = |x(1) - x(0)| \leq 2a\frac{1}{1-2^{-\alpha}}.$$

However, we have by hypothesis that  $|x(1)| \leq a$ . It is clear that  $a \leq 2a\frac{1}{1-2^{-\alpha}}$  since  $1 - 2^{-\alpha} \leq 2$ .

The case  $s = 0$  and  $t = 1$  having already been established, we may assume henceforth that  $s < t$  and  $[s, t] \neq [0, 1]$ . Observe that each binary rational  $s \in S$  may be expressed uniquely as  $\frac{k}{2^n}$  for some odd number  $k$ . Furthermore, note that there is a unique  $x_0 \in S$  with  $s \leq x_0 \leq t$  such that  $x_0 = \frac{q}{2^m}$  ( $q$  odd) has smallest possible  $m$ .

Now either  $x_0 \neq s$  or  $x_0 \neq t$  since  $s \neq t$ . First suppose that  $x_0 \neq s$ . Then  $x_0 - s = \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_j}}$  for some  $m_1 < \dots < m_j$ . Similarly, if  $x_0 \neq t$ , then  $t - x_0 = \frac{1}{2^{n_1}} + \dots + \frac{1}{2^{n_k}}$  for some  $n_1 < \dots < n_k$ . We may now consider the intervals  $[s, s + \frac{1}{2^{m_j}}], [s + \frac{1}{2^{m_j}}, s + \frac{1}{2^{m_{j-1}}} + \frac{1}{2^{m_j}}], \dots, [x_0 - \frac{1}{2^{m_1}}, x_0]$ , and  $[x_0, x_0 + \frac{1}{2^{n_1}}], \dots, [x_0 + \sum_{j=1}^{k-1} \frac{1}{2^{n_j}}, t]$ . Define  $p$  to be the minimum of  $(m_1, n_1)$  and  $q$  to be the maximum of  $(m_j, n_k)$ . We may estimate

$$\begin{aligned} |x(t) - x(s)| &\leq 2a \sum_{k=p}^q \left(\frac{1}{2^k}\right)^\alpha = 2a \frac{\left(\frac{1}{2^p}\right)^\alpha}{1 - 2^{-\alpha}} \\ &\leq 2a \frac{1}{1 - 2^{-\alpha}} (t - s)^\alpha. \end{aligned}$$

□

**Proposition A.8.** The inequality  $w(I_{\alpha, a, k, n}) \leq \sqrt{\frac{2}{\pi}} \frac{1}{a} 2^{n(\alpha - \frac{1}{2})} e^{-\frac{a^2}{2} 2^{n(1-2\alpha)}}$  holds for every  $a > 0, \alpha > 0, k \in 1, \dots, 2^n$ , and  $n \in \mathbb{N} \cup 0$ .

*Proof.* Recall the definition (A.3.1) of the set  $I_{\alpha,a,k,n}$  and observe that this is a cylinder set. Recall from Example A.2 that if  $x$  in  $C[0, 1]$ , then  $x(t) - x(s)$  is a normally distributed random variable (with respect to  $w$ ) with mean 0 and variance  $t - s$ . This means that

$$w(\{x; a \leq x(t) - x(s) \leq b\}) = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{1}{2}(\frac{\tau^2}{t-s})} d\tau. \quad (\text{A.3.6})$$

With (A.3.6) in mind, we perform the following computation:

$$\begin{aligned} w(I_{\alpha,a,k,n}) &= w(\{x \in C : |x(\frac{k}{2^n}) - x(\frac{k-1}{2^n})| > a(\frac{1}{2^n})^\alpha\}) \\ &= \frac{2}{\sqrt{2\pi\frac{1}{2^n}}} \int_{a(\frac{1}{2^n})^\alpha}^\infty e^{-\frac{\tau^2}{2\frac{1}{2^n}}} d\tau \\ &= \sqrt{\frac{2}{\pi}} \sqrt{2^n} \int_{a(\frac{1}{2^n})^\alpha}^\infty e^{-\frac{\tau^2}{2\frac{1}{2^n}}} d\tau. \end{aligned} \quad (\text{A.3.7})$$

Let  $u = \sqrt{2^n}\tau$  and  $\beta = a(\frac{1}{2^n})^{\alpha-\frac{1}{2}}$ . Then (A.3.7) becomes

$$\sqrt{\frac{2}{\pi}} \int_\beta^\infty e^{-\frac{u^2}{2}} du.$$

Observe that  $\beta > 0$  and  $1 \leq \frac{u}{\beta}$  for  $\beta \leq u$ . Let  $v = \frac{u^2}{2}$  and calculate

$$\begin{aligned} w(I_{\alpha,a,k,n}) &= \sqrt{\frac{2}{\pi}} \int_\beta^\infty e^{-\frac{u^2}{2}} du \leq \sqrt{\frac{2}{\pi}} \int_\beta^\infty \left(\frac{u}{\beta}\right) e^{-\frac{u^2}{2}} du \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\beta} \int_{\frac{\beta^2}{2}}^\infty \frac{1}{\beta} e^{-v} dv = \sqrt{\frac{2}{\pi}} \frac{1}{\beta} \frac{1}{\beta} e^{-v} \Big|_{v=\frac{\beta^2}{2}}^{v=\infty} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\beta} e^{-\frac{\beta^2}{2}} = \sqrt{\frac{2}{\pi}} \frac{1}{a} 2^{n(\alpha-\frac{1}{2})} e^{-\frac{a^2}{2} 2^{n(1-2\alpha)}} \end{aligned}$$

□

We are now ready to prove the Lemmas which were used during the proof of the Wiener Theorem A.3. We first prove Lemma A.4.

*Proof.* Observe that Proposition A.7 implies that

$$\bigcap_{n=0}^\infty \bigcap_{k=1}^{2^n} \tilde{I}_{\alpha,a,k,n} \subseteq \tilde{H}_\alpha \left[ 2a \frac{1}{1-2^{-\alpha}} \right].$$

Indeed, if  $x \in \bigcap_{n=0}^{\infty} \bigcap_{k=1}^{2^n} \tilde{I}_{\alpha,a,k,n}$  then

$$x \in \bigcap_{n=0}^{\infty} \bigcap_{k=1}^{2^n} \{f \in C; |f(\frac{k}{2^n}) - f(\frac{k-1}{2^n})| \leq a(\frac{1}{2^n})^\alpha\}.$$

This means exactly that

$$|x(\frac{k}{2^n}) - x(\frac{k-1}{2^n})| \leq a(\frac{1}{2^n})^\alpha$$

for every  $k = 1, \dots, 2^n$  and  $n \in \mathbb{N}$ , which is exactly the hypothesis of Proposition A.7. Applying the Proposition, we have

$$|x(s) - x(t)| \leq 2a \frac{1}{1 - 2^{-\alpha}} |s - t|^\alpha$$

for every  $s, t \in S$ , which means exactly that  $x \in \tilde{H}_\alpha[2a \frac{1}{1-2^{-\alpha}}]$  by (A.3.3). Taking complements, we see that  $H_\alpha[2a \frac{1}{1-2^{-\alpha}}] \subseteq \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{2^n} I_{\alpha,a,k,n}$ . Hence we may calculate

$$\begin{aligned} w^*(H_\alpha[2a \frac{1}{1-2^{-\alpha}}]) &\leq \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} w(I_{\alpha,a,k,n}) \\ &\leq \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \sqrt{\frac{2}{\pi}} \frac{1}{a} 2^{n(\alpha-\frac{1}{2})} e^{-\frac{a^2}{2} 2^{n(1-2\alpha)}} \quad (\text{A.3.8}) \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{a} \sum_{n=0}^{\infty} 2^n 2^{n(\alpha-\frac{1}{2})} e^{-\frac{a^2}{2} 2^{n(1-2\alpha)}} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{a} \sum_{n=0}^{\infty} 2^{n(\alpha+\frac{1}{2})} e^{-\frac{a^2}{2} 2^{n(1-2\alpha)}} \end{aligned}$$

where (A.3.8) follows directly from Proposition A.8.  $\square$

Next we prove Lemma A.5.

*Proof.* Let  $I$  be a cylinder set contained in  $H_\alpha[2a \frac{1}{1-2^{-\alpha}}]$ . We have

$$\begin{aligned} w(I) &\leq w^*(H_\alpha[2a \frac{1}{1-2^{-\alpha}}]) \\ &\leq \sqrt{\frac{2}{\pi}} \frac{1}{a} \sum_{n=0}^{\infty} 2^{n(\alpha+\frac{1}{2})} e^{-\frac{a^2}{2} 2^{n(1-2\alpha)}} \quad (\text{A.3.9}) \end{aligned}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{a} \sum_{n=0}^{\infty} 2^{n(1-\delta)} e^{-\frac{a^2}{2} \delta n} \quad (\text{A.3.10})$$

Where (A.3.9) follows from Lemma A.4 and (A.3.10) follows by replacing  $\frac{1}{2} - \alpha$  by  $\delta$ . Next, we employ the estimate  $2^z \geq \frac{z}{2}$  to obtain  $2^{2\delta n} \geq \delta n$ . This tells us that

$$\begin{aligned} \sqrt{\frac{2}{\pi}} \frac{1}{a} \sum_{n=0}^{\infty} 2^{n(1-\delta)} e^{-\frac{a^2}{2}} 2^{2\delta n} &\leq \sqrt{\frac{2}{\pi}} \frac{1}{a} \sum_{n=0}^{\infty} 2^{n(1-\delta)} e^{-\frac{a^2}{2} \delta n} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{a} \sum_{n=0}^{\infty} (2^{1-\delta} e^{-\frac{a^2 \delta}{2}})^n \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{a} \frac{1}{1 - 2^{1-\delta} e^{-\frac{1}{2} a^2 \delta}}. \end{aligned}$$

□

We have finally given a complete proof of the  $\sigma$ -additivity of the Wiener measure!

□

Brownian motion is an example that demonstrates that the continuous functions are not nearly so regular as geometrical diagrams would lead one to believe. Furthermore, the mathematical formulation of Brownian motion is a description of a naturally occurring, albeit highly nonsmooth, molecular movement and as such is not an obscure construction. A rigorous treatment of the Brownian motion not only clarifies the nature of the continuous functions, but also serves as a natural and illuminating link between physics, chemistry, probability, statistics, measure and integration theory, and functional analysis. Indeed, it would be self-defeating to attempt to describe the Wiener measure without also considering the physicality of the Brownian motion. Here we have a beautiful example of an extremely applied topic which yields fundamental results of purely mathematical interest.

# Appendix B. The Brownian Motion Process

This appendix describes the first passage time distribution of Brownian motion with positive drift. We begin with a description of Brownian motion with zero drift.

## B.1 Brownian Motion with Zero Drift

In what follows,  $P$  always refers to a probability measure on the Borel field  $\mathcal{B}$  of some nonempty set  $\Omega$ . We commence to make rigorous some of the earlier discussion of Brownian motion. In this spirit, we endure the following definitions: In the probability space  $(\Omega, \mathcal{B}, P)$ , a collection of events  $A_1, A_2, \dots, A_n$  is called *independent* if every subcollection  $A_{i_1}, A_{i_2}, \dots, A_{i_k}$  satisfies

$$P(A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}) = P(A_{i_1})P(A_{i_2}) \dots P(A_{i_k}) \quad (\text{B.1.1})$$

Random variables  $X_1, X_2, \dots, X_n$  on  $(\Omega, \mathcal{B}, P)$ , are called *independent* if

$$A_i := \{\omega; X_i(\omega) \in S_i\}$$

are independent for every choice of Borel sets  $S_1, S_2, \dots, S_n$ . A *stochastic process* is a function  $X(\cdot, \cdot) : \Lambda \times \Omega \rightarrow \mathbb{R}$  such that for each fixed  $t$  in the index set  $\Lambda$ ,  $X(t, \cdot)$  is a random variable on a probability space  $(\Omega, \mathcal{F}, P)$ .

We are now ready to construct a mathematical definition of Brownian motion. Recall that Brownian motion is explained physically in the following way: a particle

suffers innumerable collisions with the ‘randomly’ moving molecules of the surrounding fluid; each collision has individually a negligible effect but cumulatively they produce the observable motion [La, p.98].

Since Brownian motion is observed as motion over time, and since these underlying molecular movements can only be known statistically, we realize that our mathematical definition of Brownian motion must allow us to regard Brownian motion both as a function of time and as a random variable. Hence we seek to define a specific stochastic process. We seek a stochastic process  $X(t, \omega)$  such that for almost every  $\omega$ , the path  $x(t) := X(t, \omega)$  has properties characteristic of the movement of a small particle suspended in fluid.

A particle, suspended in fluid, which exhibits Brownian motion is not subject to perpetual influence by the neighboring fluid molecules because there is a duration of time between molecular collisions. These intervals, however, are essentially imperceptible to us and our ordinary methods of measurement. In his mathematical description of Brownian motion, Wiener idealized it as if molecules are infinitesimal in size and that the motion induced by their collisions is continuous. Hence it is reasonable to think of such motion as a continuous function of time, and to suppose that the particle assumes position zero at time zero. We proved in Example A.2 that if  $0 < s < t \leq 1$ , then  $x(t) - x(s) = X(t, \omega) - X(s, \omega)$  is a Gaussian random variable with mean 0 and variance  $t - s$ . Moreover, Brownian movement is highly erratic. More precisely,

the most striking feature of the Brownian motion is the absolute independence of the displacement of the neighboring particles, so near together that they pass by one another [Pe, p.5].

In reference to this property, we say that Brownian motion has independent increments; we make the following formal definition.

**Definition B.1.** *A stochastic process  $X(t, \omega)$ , where  $0 \leq t \leq 1$  and  $\omega \in \Omega$ , is called a Brownian motion process if it satisfies the following conditions:*

(1)  $P\{\omega \in \Omega; X(\cdot, \omega) \text{ is a continuous function}\} = 1,$

(2)  $P\{\omega \in \Omega; X(0, \omega) = 0\} = 1,$

(3) *the random variable  $X(t) - X(s)$  is normally distributed with mean 0 and variance  $t - s$  for every  $0 \leq s < t \leq 1$ , and*

(4) *the random variables  $X(t_1), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$  are independent for every  $0 < t_1 < t_2 < \dots < t_n \leq 1$ .*

Let  $X(\cdot, \cdot) : [0, 1] \times \Omega \rightarrow \mathbb{R}$  be a Brownian motion. Condition (4) of Brownian motion means that the displacement of a particle exhibiting Brownian motion during a given increment of time can be regarded as independent of its entire history. In particular, Brownian motion is independent of starting time. Physically speaking, this means that a particle exhibiting Brownian motion is moving in a highly irregular way; that is, where the particle has been has little to do with where it will go in the future. In his autobiography, Wiener analogizes this behavior to that of the path of a drunken man, the direction of whose previous step has no relation to the direction of his next step [Wie1, p.35, 37]. The keen reader will have noticed that we have been discussing not the position of the drunken man or particle at a fixed time, but the displacement incurred over some interval of time.

Note also that condition (1) of Definition B.1 tells us that for almost every  $\omega$ , the sample path  $X(\cdot, \omega)$  of a Brownian motion is continuous. With this in mind, we seek a characterization of the path given by fixing an outcome  $\omega_0 \in \Omega$  and considering  $X(\cdot, \omega_0)$  as a function on  $[0, 1]$ ; this will be done in the next theorem. Before stating and proving this theorem, however, we discuss the relationship of a mathematical description of Brownian motion to the construction of the Wiener measure.

The equation (A.2.2) that defines the Wiener measure may seem monstrous at first glance, and the motivation for the original formulation of this definition may seem to be obscure. If, however, one thinks as Wiener did in terms of Einstein’s work on Brownian motion, then the definition arises naturally from the physical reality.

In his study of Brownian motion, Einstein assumed that the position and velocity of fluid molecules are randomly distributed and that the intervals of time under consideration are large compared with the intervals of time between molecular collisions [NSRM, p. 34]. Under these customary physical assumptions, Einstein considered an interval of time divided into subintervals of equal length and observed that the total displacement of a particle over an interval is ‘very nearly’ a Gaussian distribution. The total error is the sum of the errors incurred over the subintervals [Wie2, p.57-58].

Recall that for a random variable  $X$  with Gaussian distribution, the probability that  $x_1 \leq X \leq x_2$  is given by

$$\mathfrak{P} = \int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi a}} e^{-\frac{x^2}{2a}} dx. \quad (\text{B.1.2})$$

For a Brownian motion  $X(t, \omega)$ , we seek a probability measure which yields a distribution similar to (B.1.2), where the quantity  $a$  depends on  $t$ . We remark that  $a$  should be a linear function of time because Brownian motion has independent increments and is independent of starting time. Now consider (B.1.2) as a function of time whose value at a certain time  $t_0$  represents the position of a particle at that time. Starting at  $t_0$ , consider the displacement of the particle at a second time. In this case, the probability density is given by

$$\frac{1}{\sqrt{2\pi a}} e^{-\frac{x^2}{2a}} \frac{1}{\sqrt{2\pi b}} e^{-\frac{(y-x)^2}{2b}}.$$

The  $y - x$  term appears because we are considering displacement. Consider  $a$  and  $b$  as linear functions of time and normalize to obtain

$$\frac{1}{\sqrt{(2\pi)^2 t(s-t)}} e^{-\frac{x^2}{2t}} e^{-\frac{(y-x)^2}{2(s-t)}}.$$

This led Wiener to define the probability that

$$x \in I := \{z \in C; \lambda_j < x(t_j) \leq \mu_j, 1 \leq j \leq n\}$$

by the expression

$$\frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n (t_i - t_{i-1})}} \int_{\lambda_1}^{\mu_1} \cdots \int_{\lambda_n}^{\mu_n} e^{-\frac{1}{2}(\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}})} du_1 \cdots du_n \quad (\text{B.1.3})$$

where  $u_0 := 0$  and  $t_0 := 0$ . The definition (A.2.2) of the Wiener measure is a straightforward generalization of (B.1.3). Indeed, to define the Wiener measure we simply take  $I$  to be a cylinder set of the form (A.2.1) and mimic (B.1.3), performing the necessary integration over the Borel set  $E$ . The bulk of the above discussion may be found in [NSRM, p.36-38].

**Theorem B.2.** *The stochastic process  $X(t, x) = x(t)$  on the sample space  $C[0, 1]$  is a Brownian motion with respect to the Wiener measure  $w$ .*

*Proof.* Fix  $x \in C[0, 1]$  and set  $X(t, x) := x(t)$ . Observe that conditions (1) and (2) of Definition B.1 hold automatically. Condition (3) of Definition B.1 was shown in Example A.2. We demonstrate the validity of condition (4). Choose any  $0 < t_1 < t_2 < \dots < t_n \leq 1$ . We need to show that

$$\begin{aligned} & w(\{x : x(t_1) \leq a_1, x(t_2) - x(t_1) \leq a_2, \dots, x(t_n) - x(t_{n-1}) \leq a_n\}) \\ &= \prod_{i=1}^n w(\{x(t_i) - x(t_{i-1}) \leq a_i\}). \end{aligned} \quad (\text{B.1.4})$$

Define  $t_0 = 0$  and  $u_0 = 0$ . Now, (B.1.4) equals

$$w(\{(x(t_1), x(t_2), \dots, x(t_n)) \in A\})$$

where  $A$  is the Borel set

$$A = \{(u_1, \dots, u_n) : u_i - u_{i-1} \leq a_i, \forall 1 \leq i \leq n\}.$$

We have

$$\begin{aligned}
& w(\{(x(t_1), x(t_2), \dots, x(t_n)) \in A\}) \\
& := \frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n (t_i - t_{i-1})}} \int_E e^{-\frac{1}{2}(\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}})} du_1 \cdots du_n \\
& = \frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n (t_i - t_{i-1})}} \int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_n} e^{-\frac{1}{2}(\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}})} du_1 \cdots du_n \\
& = \prod_{i=1}^n \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \int_{-\infty}^{a_i} e^{-\frac{1}{2}(\frac{y^2}{t_i - t_{i-1}})} dy \tag{B.1.5} \\
& = \prod_{i=1}^n w(\{x(t_i) - x(t_{i-1}) \leq a_i\}).
\end{aligned}$$

The last equality follows from Example (A.2). Line (B.1.5) follows from a well known theorem of advanced calculus.  $\square$

**Remark B.3.** *Considering the above calculations, together with (A.2.6), we see that indeed  $w(C[0, 1]) = 1$ .*

The reader may already be aware that in some sense, ‘most’ continuous functions are nowhere differentiable, although this statement requires some justification. This superficially counterintuitive result is well illustrated by the characterization of Brownian motion in terms of continuous functions.

For a formal proof of nowhere differentiability, the interested reader is referred to Kuo [Ku, p.45-46] or to McKean [Mc, p.9]. We indicate an intuitive argument. Observe that in the situation of Brownian motion,  $B(t + \varepsilon, \cdot) - B(t, \cdot)$  is a random variable with variance  $t + \varepsilon - t = \varepsilon$ . Hence

$$E|B(t + \varepsilon, \cdot) - B(t, \cdot)|^2 = \varepsilon$$

and so the variance of the differential quotient is given by

$$E\left|\frac{B(t + \varepsilon, \cdot) - B(t, \cdot)}{\varepsilon}\right|^2 = \frac{1}{\varepsilon}.$$

Now  $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} = \infty$  so we know that the variances of the difference quotients approach infinity as  $\varepsilon$  approaches zero. This makes it intuitively clear the pointwise derivative of a sample path of Brownian motion should not exist.

## B.2 Brownian Motion with Positive Drift

Let  $\{B(t) : t \geq 0\}$  be a Brownian motion process, as defined in section (B.1). Let  $a \geq 0$  be any nonnegative real constant. We are concerned with the probability density function of the first time when the Brownian motion process attains (or passes) the value  $a$ . We suppose that the Brownian motion is superimposed on a fluid with drift of constant velocity  $\nu$ . The classical Brownian motion, that is, Brownian motion with zero drift is exactly the case where  $\nu = 0$ .

The Brownian motion with positive drift  $\nu > 0$  and variance  $\sigma^2 > 0$  is given by

$$X(t) := \nu t + \sigma B(t)$$

for  $t \geq 0$ . Moreover, the first passage time process of Brownian motion with positive drift is given for  $a \geq 0$  by

$$M(a) := \inf\{s \geq 0 : X(s) \geq a\}.$$

The distribution of  $M(a)$  is exactly

$$f_a(x) = \frac{ax^{-\frac{3}{2}}}{\sigma 2\sqrt{\pi}} e^{-\frac{(\nu x - a)^2}{2\sigma^2 x}}$$

for  $x \in (0, \infty)$ . Putting  $\frac{a}{\sigma} = \sqrt{\lambda}$  and  $\frac{a}{\nu} = m$ , we obtain the Inverse Gaussian distribution as defined in Chapter 3. See, for example, [Se] or [Wa] for more details.

# Vita

Sarah Campbell McAllister was born on March 25, 1982, in Kingston, New York. She began her undergraduate studies in August 1995 at SUNY Ulster County Community College, where she earned an Associate of Science degree in January 1999. She finished her undergraduate studies in August 2000, when she earned a Bachelor of Arts degree in mathematics from Empire State College. In August 2000 she journeyed to Louisiana State University to pursue graduate studies in mathematics. She earned a Master of Science degree in mathematics from Louisiana State University in December 2001. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2005.