IMPULSIVE SYSTEMS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Stanislav Žabić
B.A. in Mathematics, University of Belgrade, 1998
M.S. in Mathematics, Louisiana State University, 2002
August 2005

Acknowledgments

My sincere thanks goes out to my advisor, Dr. Peter R. Wolenski, who has been my main guide for the past five years through the complicated yet delightful world of graduate mathematics. He is a fine educator with a great gift, a collaborator with great passion for mathematics and a dear friend. I would also like to express my deepest gratitude to Dr. Fernando Lobo Pereira from the University of Porto in Portugal. Dr. Pereira was my host for a fruitful study semester abroad at his department and he greatly influenced my work. Let me also mention that my success would be impossible without the support and inspiring atmosphere at the LSU Department of Mathematics.

I would also like to thank my friend Steven L. Bujenović for his words of encouragement and for introducing me to cancer image processing, which became my secondary research interest. My colleague, Vinicio Ríos, is also deserving of a heartfelt acknowledgment for many discussions about mathematics daily in our shared office.

Through my years of graduate school, my wife Sarah has been incredibly patient, constantly offering her love and support. This thesis is dedicated to her, my brilliant sister Snežana, and my loving parents, Ivica and Mira, who were my strength in the face of adversity and taught me how to defy the odds. I lost both of my grandmothers while working on my PhD and my deepest regret is that they will not see me receiving my diploma, though their inspiration is in all that I do.

Table of Contents

Notation

$\mathbb{R}^d$	set of real vectors with dimension d.
$\mathcal{M}_{n\times m}$	set of $n \times m$ matrices with real entries.
$\vert \cdot \vert$	Eucledean norm in $\mathbb{R}^d$.
$\Vert \cdot \Vert$	Norm in $\mathcal{M}_{n\times m}$.
C^1	set of continuously differentiable functions.
$\mathrm{d}_S(x)$	distance from point x to set S.
$\mathrm{proj}_S(x)$	set of projections of point x to the set S.
$\mathrm{co}\, S$	convex hull of set S.
B_r	closed ball with radius r and center in origin.
S_r	sphere with radius r and center in origin.
$\mathcal{B}_K([0,T])$	set of Borel measures defined on $[0,T]$ with values in set K.
$\overline{S}$	closure of the set S.
$L_1^1([0,T])$	set of all real valued measurable functions on $[0,T]$.
$\mathrm{dist}_{\mathrm{H}}(A_1, A_2)$	Hausdorff distance between sets A_1 and A_2.

Abstract

Impulsive systems arise when dynamics produce discontinuous trajectories. Discontinuities occur when movements of states happen over a small interval that resembles a point-mass measure. We adopt the formalism in which the controlled dynamic inclusion is the sum of a slow and a fast time velocities belonging to two distinct vector fields. Fast time velocities are controlled by a vector valued Borel measure.

The trajectory of impulsive systems is a function of bounded variation. To give a definition of solutions, a notion of graph completion of the control measure is needed. In the nonimpulsive case, a solution can be defined as a limit of a sequence of approximate arcs which converge to an absolutely continuous arc. Even in simple cases it shows that this is not a good way to define solutions of the impulsive systems. The key point is that the approximate controls converge to two different graph completions.

Introduction contains examples in which we discuss the need for the impulsive systems, their relation to the hybrid systems. A paradox related to the convergence of approximate arcs is illustrated. Chapter 1 contains preliminary results in nonimpulsive systems and mathematical analysis in general. Chapter 2 precisely defines impulsive systems, discusses two different solution concepts and proves properties of graph completions.

Chapter 3 is entirely dedicated to adaptation of the Euler approximating schemes to the impulsive system. Two different schemes are offered. For one of them a measure which drives the system needs to be specified. We used it to show that

the approximate trajectories graph-converge to a solution. The other sampling technique constructs a measure along with the solution. We use it in Chapter 4.

Chapter 4 deals with issues when a trajectory remains within a closed set. This property is called invariance. Notions of weak and strong invariance for the impulsive systems are introduced and proximal characterizations are proved. In the case of weak invariance, two proofs are offered: one based on a sampling technique from Chapter 3 and other based on selections theory.

The final chapter of this thesis discusses directions in future research.

Introduction

Many problems in physics, geometry and mechanics throughout history, have involved finding an extremum of a function or functional. The interest in such problems has further increased in recent years due to the demands in technology and economics. Typically, one wants to minimize a certain function by controlling a space vector via dynamical systems. The following is a formulation that captures this need.

$$\min_u \int_0^T L_0\big(t, x(t), u(t)\big)\,dt \tag{1}$$
$$\dot{x}(t) = f\big(t, x(t), u(t)\big).$$

We require the control function $u(\cdot)$ to live in a certain subset $\mathcal{U}$ of $I\!R^m$, $u \in \mathcal{U} \subseteq I\!R^m$. The data $L_0 : [0,T] \times I\!R^n \times I\!R^m \to I\!R$, and $f : [0,T] \times I\!R^n \times I\!R^m \to I\!R^n$ are given.

An important issue is the class of functions over which to minimize. The following is a simple example which does not have a continuously differentiable solution and motivates us to extend the search for solution beyond C^1.

Example 1. Consider the following problem in calculus of variations.

$$\min \int_0^1 (1 - \dot{x}^2)^2 dt,$$
$$x(0) = 0, x(1) = 0.$$

By inspection, the minimum of the cost function is zero and it is attained for functions such that $|\dot{x}| = 1$ almost everywhere. They form a zig zag pattern going from $x(0) = 0$ to $x(1) = 0$ switching slopes from 1 to -1. If we were looking for continuously differentiable solutions, the value of the cost function could approach zero infinitely close, but it would never attain zero.

Moreover, the following example illustrates that there are simple systems which fail to have solutions in class of absolutely continuous functions. This is a motivation to allow the state vector to be a function of bounded variation.

Example 2.

$$\min \int_0^1 t^2 \dot{x}^2(t) dt$$

$$x(0) = 0, \quad x(1) = 1$$

The following sequence of continuous trajectories minimize the given problem.

$$x_n(t) = \begin{cases} 0, & 0 \le t \le 1 - 1/n \\ 1 + n(t-1), & 1 - 1/n < t \le 1. \end{cases}$$

Indeed,

$$\int_0^1 t^2 \dot{x}_n^2(t) dt = \int_{1-1/n}^1 n^2 t^2 dt = \frac{n^2 t^3}{3}\bigg|_{1-1/n}^1 = \frac{1}{3n} \longrightarrow 0 \quad (n \to \infty).$$

Sequence $x_n(\cdot)$ converges pointwise everywhere to $x = 0$ except in $t = 1$. However, function $x = 0$ does not satisfy the end point conditions. The following function represents the solution of this example. It is not absolutely continuous but it has a bounded variation.

$$x^*(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1. \end{cases}$$

This function has a jump discontinuity of size 1 at $t = 1$.

Systems in which we allow "jumps", like in the previous example, we call *impulsive systems*. They arise in a variety of applications where states can move at different time scales. Optimality conditions can induce optimal strategy with impulsive nature. The "slow" movement can be thought of as the usual time progression infinitesimally incremented by dt, and the "fast" movement occurs over a small interval that resembles the effect of a point-mass measure.

This mixture of continuous and discontinuous dynamics has been considered by many researchers. In particular, there is a vast literature on the *hybrid systems* with many different solution concepts. Let us consider the formulation of hybrid systems from [13, 14]. There, hybrid systems are given by two dynamical equations, one defined on each interval of continuity,

$$\dot{x} = f\big(x\big), \quad x \in C$$

and the other, discontinuous dynamics at jump times determined by $x \in D$, relating to $x(\cdot)$ through

$$x^{+} = g\big(x\big). \tag{2}$$

Here $f(\cdot)$ and $g(\cdot)$ are upper semicontinuous functions with linear growth and C and D are closed subsets of R^n. A notion of *hybrid time domain* is introduced as a subset of $[0, +\infty) \times I\!N_0$, given as a union of finitely or infinitely many intervals $[t_j, t_{j+1}] \times \{j\}$, where the numbers t_j represent jump times. In this framework, a solution to the hybrid system is a function defined on a hybrid time domain such that

$$\dot{x}(t,j) = f\big(x(t,j)\big) \text{ and } x(t,j) \in C \text{ on } (t_j, t_{j+1}) \text{ and}$$

$$\forall j, \quad x(t_{j+1}, j+1) = g\big(x(t_{j+1}, j)\big), \text{ and } x(t_{j+1}, j) \in D.$$

This notion of solution is well defined even in in the case of infinite number of jumps. However, in that case, hybrid time domain blows up even if its first component is bounded. Such behavior of trajectory is known as *Zeno behavior*. To illustrate this behavior, let us look at the following example.

Example 3. Behavior of a ball bouncing on a hard floor in presence of gravity is modeled by the following two dimensional hybrid system, with x_1 being the ball's height above the floor and x_2 being the ball's velocity (γ is a constant which

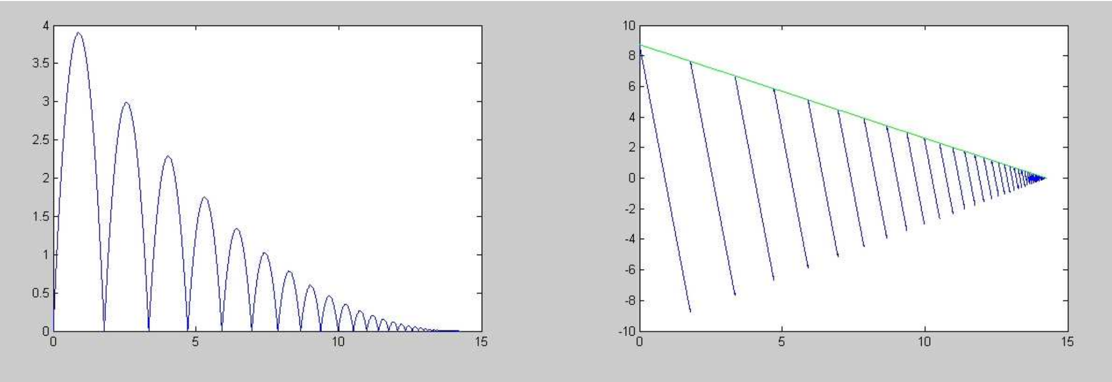

FIGURE 1. Trajectory of a bouncing ball. The figure on the left represents height as a function of time and the figure on the right represents velocity as a function of time. The green line on the right figure is here to visualize how velocities converge to 0 at a finite time.

represents acceleration due to gravity):

$$\dot{x} = \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} =: f(x), \tag{3}$$

whenever $x \in C := \{x_1 > 0, \text{ or } x_1 = 0 \text{ and } x_2 > 0\}$. When $x \in D := \{x_1 = 0 \text{ and } x_2 \leq 0\}$, the flow is described by the jump equation

$$x_2^+ = -\mu x_2. \tag{4}$$

Function $g(\cdot)$ from (2) is in this case

$$g(x) := (x_1, -\mu x_2).$$

Figure 1 represents a sample trajectory for $x_1(0) = 0$ m, $x_2(0) = 70/8$ m/s, Earth's gravity $\gamma = 9.80665$ m/s^2 and $\mu = 7/8$. The trajectories tend to origin and eventually vanish. For sample trajectory presented in Figure 1, trajectories vanish at time

$$T := \frac{2x_2(0)}{(1-\mu)\gamma} \approx 14.28.$$

Therefore, this is a model of Zeno behavior. As both velocity and height of the ball reach a constant zero position at finite time (after infinitely many jumps), it is desirable to be able to continue this model after time $t = T$. In the solution concept for hybrid systems, it is not possible to do so as the hybrid time domain actually blows up.

However, the setting introduced in this thesis is able to handle this situation, as shown in the Chapter 2. The way the time reparameterization is chosen and ability of controls to take continuous singularities in the setting of impulsive systems makes an essential difference. The next example is another illustration of the role of singular continuous dynamics. In this example the optimality condition force continuous singular decisions.

Example 4. Suppose that we wish to minimize the same integral as in Example 2, with the same end point conditions. However, let us make another invariance-type condition. Suppose that we wish the trajectory to remain in closed set $C \subset I\!R^2$,

$$C := \left\{ (t,x) \,\middle|\, t \le x \le \sqrt{1-(1-t)^2}, t \in [0,1] \right\}.$$

Again, the minimum value of the given problem is achieved when $\dot{x}(t) = 0$ almost everywhere. As we saw in the previous example, absolutely continuous solutions that give the minimum value does not exist. Moreover, solutions with jumps or which stay constant for a positive time in the neighborhood of times $t = 0$ and $t = 1$ are also not acceptable, because in the "corners" of set C there is "no room" such behavior. At time $t = 0$ and $t = 1$, the only option is to travel along a strictly increasing singularly continuous trajectory such as the Riesz-Nagy function described in [15] on page 278. Figure 2 shows the set C and an approximation of the singular-continuous trajectory for this example. On this figure, the entire trajectory is singular-continuous, although once we moved away from the corners of set C, we could have a trajectory with discontinuous singularities and continuous piecewise-horizontal graph.

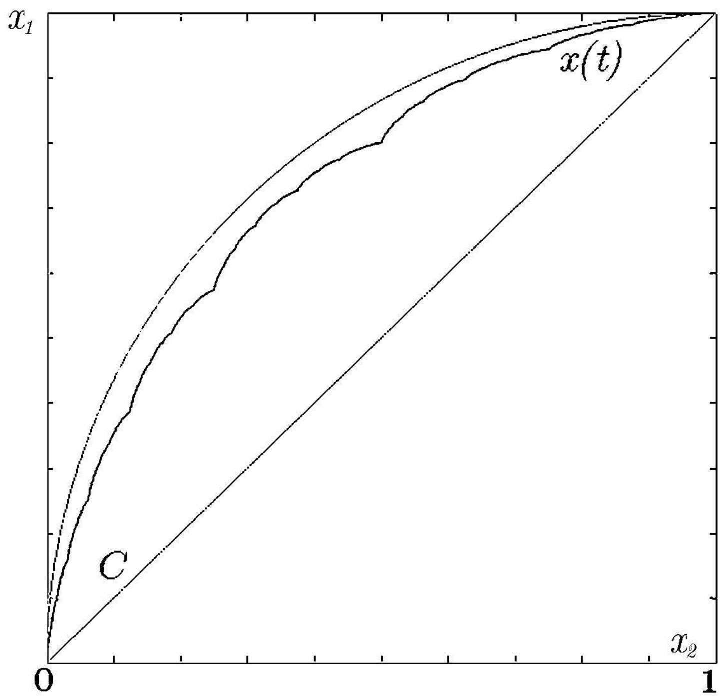

FIGURE 2. Optimality conditions forced singular continuous solutions in Example 4. This trajectory in particular is entirely singular continuous.

In this thesis, the considered dynamical system is the sum of a slow-time velocity belonging to a set $F(x)$ and a fast-time contribution coming from another set $G(x)d\mu$, where μ is a vector-valued measure. Additionally, dynamics are represented with a differential inclusion, rather than a differential equation.

In particular, the following differential form describes the impulsive systems considered in this thesis

$$\begin{cases} dx & \in & F\big(x(t)\big)\,dt + G\big(x(t)\big)\,d\mu(dt) \\ x(0-) & = & x_0. \end{cases}$$

Here, measure μ belongs to the set of vector-valued Borel measures defined on the interval $[0, T] \subset I\!R$ with values in a closed convex cone $K \subseteq I\!R^m$. Multifunctions $F : I\!R^n \rightrightarrows I\!R^n$ and $G : I\!R^n \rightrightarrows \mathcal{M}_{n\times m}$ (where $\mathcal{M}_{n\times m}$ denotes the $n \times m$ dimensional matrices with real entries) are with closed graph and convex values, and satisfy the linear growth condition.

Issues arise to the nature of a "solution" in nonlinear systems when the fast dynamic velocities are affected by multiplicative state dependence (i.e. G depends nontrivially on x), and that the solution concept will only be well defined if a realization of the measure is also prescribed through a graph completion. We illustrate this problem on a concrete example, which is borrowed from [3]. Although the method of approximate non-impulsive trajectories did produce a unique solution in Example 2, the following example shows that this procedure yields ambiguous trajectories in general.

Example 5. Consider the impulsive system on $\mathbb{R}^2$:

$$(\dot{x}_1, \dot{x}_2) = \begin{bmatrix} 1 & 0 \\ 0 & x_1 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}.$$

$$x(0) = (0, 0).$$

Consider the discontinuous control function $u : [0, 2] \to \mathbb{R}^2$:

$$(u_1, u_2) = \begin{cases} (0, 0) & \text{if } t < 1, \\ (1, 1) & \text{if } t > 1. \end{cases}$$

The control $u(\cdot)$ can be approximated by the sequence of continuous, piecewise linear functions $v_n(\cdot)$:

$$(v_1^{(n)}, v_2^{(n)})(t) = \begin{cases} (0, 0) & \text{if } t \in [0, 1 - 1/n], \\ (0, 1 + n(t-1)) & \text{if } t \in [1 - 1/n, 1], \\ (n(t-1), 1) & \text{if } t \in [1, 1 + 1/n], \\ (1, 1) & \text{if } t \in [1 + 1/n, 2]. \end{cases}$$

The corresponding approximate solutions are

$$(x_1^{(n)}, x_2^{(n)})(t) = \begin{cases} (0, 0) & \text{if } t \in [0, 1], \\ (n(t-1), 0) & \text{if } t \in [1, 1 + 1/n], \\ (1, 0) & \text{if } t \in [1 + 1/n, 2]. \end{cases}$$

As $n \to \infty$, x_n converges to

$$(x_1, x_2)(t) = \begin{cases} (0,0) & \text{if } t < 1, \\ (1,0) & \text{if } t > 1. \end{cases} \tag{5}$$

Now, consider another approximating sequence $w_n(\cdot)$

$$(w_1^{(n)}, w_2^{(n)})(t) = \begin{cases} (0,0) & \text{if } t \in [0, 1-1/n], \\ (1+n(t-1), 0) & \text{if } t \in [1-1/n, 1], \\ (1, n(t-1)) & \text{if } t \in [1, 1+1/n], \\ (1,1) & \text{if } t \in [1+1/n, 2]. \end{cases}$$

For these control, we have a new sequence of approximate solutions:

$$(x_1^{(n)}, x_2^{(n)})(t) = \begin{cases} (0,0) & \text{if } t \in [0, 1], \\ (1+n(t-1), 0) & \text{if } t \in [1-1/n, 1], \\ (1, n(t-1)) & \text{if } t \in [1, 1+1/n], \\ (1,1) & \text{if } t \in [1+1/n, 2]. \end{cases}$$

As $n \to \infty$, $x_n(\cdot)$ converges to

$$(x_1, x_2)(t) = \begin{cases} (0,0) & \text{if } t < 1, \\ (1,1) & \text{if } t > 1. \end{cases}$$

Although both sequences of approximate controls $v^{(n)}(\cdot)$ and $w^{(n)}(\cdot)$ converge to the same control function $u(\cdot)$, the key difference between them is that they converge to two different graphs of $u(\cdot)$. Indeed, the limiting graph of the sequence $v^{(n)}(\cdot)$ connects points $u(1-)$ and $u(1+)$ first along the second component, keeping the first one constant at value $u_1(1-) = 0$, and then approximates the first component, keeping the second one at $u_2(1+) = 1$. The limiting graph of $w^{(n)}(\cdot)$ connects $u(1-)$ and $u(1+)$ first along the second component until $u_2(1+)$ is reached, keeping the first fixed, and then it approximates along the first component until $u_1(1+)$ is reached. Italian mathematician Bressan and Rampazzo showed [3, 4, 5, 17] that the method of approximate trajectories produces a unique solution only when the dimension m is equal to 1 and when columns of matrix G commute as vector fields.

Monograph by Miller and Rubinovich [16] overviews optimal control problems with impulsive controls. This book is related to the work of Rishel [20]. Systems considered in this book have a similar form to what is presented in this thesis, but the notion of solution is fundamentaly different. Namely, systems there are limited to those in which approximate trajectories give unique solutions. Authors also show that it is possible to consider the hybrid setting as a special case of impulsive systems and they also discuss problems such as the Zeno behavior that we described earlier in this introduction.

In this thesis a new solution concept is proposed to an impulsive system which is defined in the original time, rather than the time reparameterization. This novel concept requires a direct correlation respectively of the absolutely continuous, continuous singular, and atomic parts of the bounded variation solution and the given measure. However, the main result is a sampling method that is analogous to the classical Euler one-step method for the non-impulsive systems. In addition to the sampling method, this thesis also offers a modified method of approximate trajectories that actually covers the general case.

New solution concept and new sampling technique is foundation to other research results presented in this thesis. These results are related to the proximal theory, in particular weak and strong invariance properties.

Chapter 1
Preliminaries

This thesis uses tools of nonsmooth analysis and differential inclusions. In this preliminary chapter, we offer an overview of these tools which represents foundations of this thesis. Moreover, a few facts from real analysis are also covered. Here, we will just state the standard definitions and theorems often without proofs, encouraging the reader to seek them in more detail in sources such as [1, 6, 7, 12, 15, 22, 26, 27, 29, 30, 31].

1.1 Proximal Normal Cone

In this section we offer a definition of proximal normal cone, which represents a generalization of tangential normals for nonsmooth functions. This object plays an important role in characterization of invariance properties. Let us first illustrate the proximal normal cone less formally.

Suppose that a nonempty set $S \subset \mathbb{R}^n$ is given. For all $x \in \mathbb{R}^n$ we can define the *distance function form set* S, $\mathrm{d}_S : \mathbb{R}^n \to \mathbb{R}$ as

$$\mathrm{d}_S(x) := \inf\{|x - s| : s \in S\}, \quad \text{for all } x \in \mathbb{R}^n.$$

For a point $x \notin S$, let us consider a set of all points $s \in S$ whose distance $\mathrm{d}_S(x)$ to x is minimal. Since S is a closed set, such points exist. The point s is called a *projection of* x onto S. The set of all such closest points is denoted by $\mathrm{proj}_S(x)$:

$$\mathrm{proj}_S(x) = \{s \in S \mid \mathrm{d}_S(x) = |x - s|\}.$$

Vectors $x - s$ form a proximal normal direction to S at x. A proximal normal to S at s, $\zeta = t(x - s)$, $t \geq 0$, is any nonnegative multiple of $x - s$. The set of all proximal normals is called the proximal normal cone to S, denoted $N_S^P(s)$. We use the following formula to precisely define the proximal normal cone:

$$N_S^P(s) := \{\zeta \mid \exists t > 0 \text{ so that } \mathrm{d}_S(s + t\zeta) = t|\zeta|\}.$$

The *proximal normal inequality* (Proposition 1.5 in [7]) is a useful characterization of proximal normals: a vector ζ belongs to $N_S^P(s)$ if and only if there exists $\sigma = \sigma(\zeta, s) \geq 0$ such that

$$\langle \zeta, s' - s \rangle \leq \sigma |s' - s|^2 \quad \text{for all } s' \in S.$$

Moreover, for any $\varepsilon > 0$, we have $\zeta \in N_S^P(s)$ if and only if there exists $\sigma = \sigma(\zeta, s) \geq 0$ such that

$$\langle \zeta, s' - s \rangle \leq \sigma |s' - s|^2 \quad \text{for all } s' \in S \cap (s + \varepsilon B),$$

where B is a unit ball.

Let us finish this section by defining two important objects. A set $K \subset I\!\!R^n$ is called a *conus* (with vertex at the origin) if it is closed with respect to multiplication with a positive scalar. That is, for all $k \in K$ and all $\lambda > 0$, $\lambda k \in K$ as well.

The intersection of all convex sets containing a set $S \in I\!\!R^n$, is called the *convex hull* of S. We also use notation co S.

1.2 Multifunctions and Measurable Selections

In our research, a dynamical system is represented by a differential inclusion, a generalization of differential equation, and in the next section of the preliminary chapter we briefly give foundations of the differential inclusion theory. Objects of crucial interest will be measurable multifunctions and their measurable selections. In this section we spend time defining and characterizing these mathematical objects. A *multifunction* $F : S \rightrightarrows \mathbb{R}^n$ is a mapping from $S \subset \mathbb{R}^m$ to the subsets of

$\mathbb{R}^n$. A multifunction is *measurable* if sets

$$\{x \in S : F(x) \cap C \neq \emptyset\},$$

are Lebesgue measurable for any closed set C in $\mathbb{R}^n$. A multifunction F is *close-valued* (or just *closed*) if all sets $F(x)$ are closed sets. A multifunction F is *non-empty* if all sets $F(x)$ are not empty.

Several important sets are defined as:

$$\begin{aligned} &\mathrm{dom}F = \{s \in S \mid F(s) \neq \emptyset\}, && \textit{domain} \text{ of multifunction } F, \\ &\mathrm{graph}\, F = \{(s,x) \mid x \in F(s)\}, && \textit{graph} \text{ of multifunction } F \text{ and} \\ &F(T) = \cup_{s\in T} F(s) && \text{where } T \subset S \end{aligned}$$

The inverse multifunction $F^{-1} : X \rightrightarrows S$ is the multifunction obtained by reversing the pairs in F so that

$$F^{-1}(x) = \{s \in S \mid x \in F(s)\},$$

$$F^{-1}(C) = \cup_{x\in C} F^{-1}(x) = \{s \in S \mid F(s) \cap C \neq \emptyset\}.$$

Our definition of measurable multifunction is now equivalent to the following: for each closed set $C \subset \mathbb{R}^n$ the set $F^{-1}(C)$ is measurable.

We next give some practical tests for the measurability of a multifunction.

Proposition 1.2.1. *For a closed-valued multifunction $F : S \rightrightarrows \mathbb{R}^n$, the following properties are equivalent:*

(a) F is measurable;

(b) $F^{-1}(C)$ is measurable for all open sets C;

(c) $F^{-1}(C)$ is measurable for all compact sets C;

(d) $d_{F(s)}(\zeta))$ is a measurable function of $s \in S$ for each $\zeta \in \mathbb{R}^n$.

Proof. (a) $\Rightarrow$ (b). Let C be open. Then $C = \cup_{k=1}^{\infty} C_k$, where each C_k is a closed ball. Thus

$$F^{-1}(C) = \cup_{k=1}^{\infty} F^{-1}(C_k).$$

holds with $F^{-1}(C_k)$ measurable, and we conclude $F^{-1}(C)$ is measurable.

(b) $\Rightarrow$ (c). Given a compact set C, let

$$C_k = \{z \in \mathbb{R}^n \mid \mathrm{d}_C(\zeta) < k^{-1}\} \quad \text{for } k = 1, 2, ...$$

Then C_k is open, $\mathrm{cl}\, C_k$ is compact, and $C_k \supset \mathrm{cl}\, C_{k+1}$. We have $F(s) \cap C_k \neq \emptyset$ for all k if and only if $F(s) \cap \mathrm{cl}\, C_k \neq \emptyset$ for all k. Since $F(s)$ is closed, the latter is equivalent by compactness to

$$\emptyset \neq \cap_{k=1}^{\infty} F(s) \cap \mathrm{cl}\, C_k = F(s) \cap C.$$

Therefore

$$F^{-1}(C) = \cap_{k=1}^{\infty} F^{-1}(C_k),$$

and since each $F^{-1}(C_k)$ is measurable by assumption, it follows that $F^{-1}(C)$ is measurable.

(c) $\Rightarrow$ (a). Let C be any closed set in $\mathbb{R}^n$. Then $C = \cup_{k=1}^{\infty} C_k$, where each C_k is compact, and hence

$$F^{-1}(C) = \cup_{k=1}^{\infty} F^{-1}(C_k).$$

We have each $F^{-1}(C_k)$ measurable, hence so is $F^{-1}(C)$.

(b) $\Leftrightarrow$ (d). We have $\mathrm{d}_{F(s)}(\zeta) \leq \alpha$ iff $F(s)$ meets the ball $\zeta + \alpha B_1$ (B_1 - closed unit ball, $\alpha > 0$). Thus

$$\{s \mid \mathrm{d}_{F(s)}(\zeta) \leq \alpha\} = \{s \mid F(s) \cap [\zeta + \alpha B_1] \neq \emptyset\} = F^{-1}(\zeta + \alpha B_1 \backslash S_1). \qquad (1.1)$$

Condition (d) means that all the sets of the form on the left in (1.1) are measurable, while (b) means all those on the right are measurable. $\square$

We will see now how we can find a measurable function which lives in a measurable multifunction. Selection of such function is an essential tool throughout the thesis.

Theorem 1.2.2. ***(Measurable Selection Theorem.)*** *Let F be measurable, closed and nonempty multifunction on S. Then there exists a measurable function $f : S \to \mathbb{R}^n$ such that $f(x) \in F(x)$ for all $x \in S$.*

Proof. This is a proof by construction. We will construct $f(s)$ as a pointwise limit of measurable functions $f_i(s)$. For that matter, let $\{\zeta_i\}$ be a countable dense subset of $\mathbb{R}^n$. For $s \in S$, let us define:

$$f_0(s) = \zeta_i \quad \text{s.t.} \quad \mathrm{d}_{F(s)}(\zeta_i) \leq 1 \quad \mathrm{d}_{F(s)}(\zeta_j) > 1 \quad \text{for } j = 1, ..., i-1.$$

Recall, the function $s \mapsto \mathrm{d}_{F(s)}(\zeta)$ is measurable by Theorem 1.2.1. The function $s \mapsto f_0(s)$ is measurable because

$$\{s \mid f_0(s) = \zeta_i\} = \bigcap_{j=1}^{i-1} \{s \mid \mathrm{d}_{F(s)}(\zeta_j) > 1\} \cap \{s \mid \mathrm{d}_{F(s)}(\zeta_i) \leq 1\}.$$

We now define the sequence f_i, i =0, 1, 2,... by induction. Define $f_{i+1}(s)$ to be the first ζ_j such that

$$|\zeta_j - \zeta_i| \leq \frac{2}{3}\mathrm{d}_{F(s)}(f_i(s)) \quad \text{and} \quad \mathrm{d}_{F(s)}(\zeta_j) \leq \frac{2}{3}\mathrm{d}_{F(s)}(\zeta_i).$$

Such an f_{i+1} is measurable.

Suppose that $f_k(s)$ is measurable for all k =0, ..., i. Then

$$\{s \mid \mathrm{d}_{F(s)}(f_k(s)) > \alpha\} = \bigcup_{j \in \mathbb{N}} [\{s \mid f_k(s) = \zeta_j\} \cap \{s \mid \mathrm{d}_{F(s)}(\zeta_j) > \alpha\}],$$

and we conclude that also $s \mapsto \mathrm{d}_{F(s)}(f_0(s))$ is measurable for all k =0, ..., i.

Now let us prove that f_{k+1} is a measurable function. By definition of $f_{j+1}(s)$ we have

$$\Big\{s \ \Big| \ f_{j+1}(s) = \zeta_j \ \Big\} = \bigcap_{k=1}^{j-1} \left[\left\{ s \mid \mathrm{d}_{F(s)}(f_i(s)) < \frac{3}{2}|\zeta_k - f_i(s)| \right\} \cup \left\{ s \mid \mathrm{d}_{F(s)}(f_i(s)) < \frac{3}{2}\mathrm{d}_{F(s)}(\zeta_k) \right\} \right] \cap$$
$$\cap \left\{ s \mid \mathrm{d}_{F(s)}(f_i(s)) \geq \frac{3}{2}|\zeta_j - f_i(s)| \right\} \cap \left\{ s \mid \mathrm{d}_{F(s)}(f_i(s)) \geq \frac{3}{2}\mathrm{d}_{F(s)}(\zeta_j) \right\}.$$

Since the right hand side represents a countable intersection of measurable sets, we conclude that $\{s \mid f_{j+1}(s) = \zeta_j\}$ is also measurable. Hence, $f_{i+1}(s)$ is a measurable function.

Also,

$$\mathrm{d}_{F(s)}(f_{i+1}(s)) \leq \left(\frac{2}{3}\right)^i \mathrm{d}_{F(s)}(f_0(s)) \leq \left(\frac{2}{3}\right)^i,$$

and

$$|f_{i+1}(s) - f_i(s)| \leq \left(\frac{2}{3}\right)^{i+1}.$$

So, $f_i(s)$ is a Cauchy sequence converging to a value $f(s)$ for each s, a measurable selection for F. $\square$

1.3 Differential Inclusions

Let us now define basic properties of differential inclusions, which will be used in this thesis. Refer to [2] and [26] for more detailed presentation on this vast topic.

Given a multifunction F mapping $I\!R^n$ to the subsets of $I\!R^n$, and a time interval $[0, T]$, the *differential inclusion* is

$$\begin{aligned} \dot{x}(t) &\in F\big(x(t)\big) \quad \text{a.e. } t \in [0, T) \\ x(0) &= x_0. \end{aligned}$$

A *solution* (also referred to as *trajectory*) $x(\cdot)$ of the differential inclusion is an absolutely continuous function $x : [0, T] \mapsto I\!R^n$ which satisfies (1.3). This is what is sometimes referred to as a solution in *Carathéodory* sense. This solution concept already represents the first difficulty in properly defining the impulsive systems, as we already hinted in the introduction. In general, we allow impulsive systems to have solutions of bounded variation.

Differential inclusions are clearly differential equations when the multifunction $F(\cdot)$ is single valued, $F(x) = \{f(x)\}$. The real value that the differential inclusions bring is their relation to control systems. The famous Filippov's Lemma (see for example [7] 3.7.20) shows it, and we will illustrate this connection here. Namely, consider a control system given by

$$\dot{x}(t) = f\big(x(t), u(t)\big), \tag{1.2}$$

where $f : I\!R^n \times I\!R^m \mapsto I\!R^n$, and where the control $u(\cdot)$ takes values in a certain set $\mathcal{U} \subset I\!R^m$. One can consider $F(x) = f(x,\mathcal{U})$ and Filippov's Lemma implies that an arc $x(\cdot)$ satisfies (1.3) if and only if there is a measurable selection $u(\cdot)$ of $\mathcal{U}$ such that (1.2) holds.

Let us assume that the following *standard hypotheses* are satisfied:

(a) for every x, $F(x)$ is a nonempty, compact and convex set,

(b) the multifunction $F(\cdot)$ is closed,

(c) there exists a positive constant c, such that the *linear growth condition* holds:

$$f \in F(x) \Rightarrow |f| \leq c(1 + |x|).$$

We shall see that similar assumptions will be hypothesized for the right-hand side of the impulsive dynamics.

The linear growth condition is essential in establishing a priori boundaries of solutions for differential inclusions, just like it is in the case of differential equations.

The a priori bounds are achieved by using the *Gronwall's inequality.* We state it here in its continuous and discrete version, as we use both in the following chapters.

Lemma 1.3.1. ***(Gronwall's lemma)*** *Let $x(\cdot) : [0,T] \to \mathbb{R}^n$ be an absolutely continuous function defined on $[0,T]$ satisfying*

$$|\dot{x}(t)| \leq \gamma |x(t)| + c(t), \quad \text{a.e. } t \in [0,T]$$

for $\gamma \geq 0$ and $c(\cdot) \in L_1^1[0,T]$. Then for all $t \in [0,T]$, the following inequality holds

$$|x(t) - x(0)| \leq (e^{\gamma t} - 1)|x(0)| + \int_0^t e^{\gamma(t-s)} c(s)\, ds.$$

□

A proof of Gronwall's lemma can be found in [7], Proposition 4.1.4, page 179 - thus it will be omitted here. However, the following discrete version and its corollary which is used in Chapter 3 of this thesis, are not so frequently seen in literature and we offer these statements together with their proofs.

Lemma 1.3.2. ***(Discrete Gronwall's lemma)*** *Suppose $x_0, x_1, \ldots x_N$ are elements in $\mathbb{R}^n$ so that*

$$|x_{j+1}| \leq \gamma |x_j| + c, \tag{1.3}$$

where γ and c are scalars. Then,

$$|x_N| \leq c \frac{1 - \gamma^N}{1 - \gamma} + \gamma^N |x_0|. \tag{1.4}$$

Proof. The proof is by mathematical induction. Note that for $j = 1$, from (1.3), inequality (1.4) follows immediately. Indeed,

$$|x_1| \leq c + \gamma |x_0|.$$

Now, suppose

$$|x_j| \leq c \frac{1 - \gamma^j}{1 - \gamma} + \gamma^j |x_0|$$

holds for $j < N$. Combining the latter with (1.3), one obtains

$$
\begin{aligned}
|x_{j+1}| &\leq c + c\gamma\frac{1-\gamma^j}{1-\gamma} + \gamma^{j+1}|x_0| \\
&= \frac{c}{1-\gamma}\big(1-\gamma+\gamma(1-\gamma^j)\big) + \gamma^{j+1}|x_0| \\
&= \frac{c}{1-\gamma}(1-\gamma^{j+1}) + \gamma^{j+1}|x_0|,
\end{aligned}
$$

which completes the proof. □

The following is a simple corollary of the discrete Gronwall's inequality.

Corollary 1.3.3. *If in Lemma 1.3.2, $\gamma = 1 + \frac{\alpha}{N}$ and $c = \frac{\alpha}{N}$, then inequality (1.4) is*

$$|x_N| \leq e^{\alpha}(1 + |x_0|) - 1.$$

Proof. The result follows immediately because $c = \gamma - 1$ and

$$\gamma^N = (1 + \frac{\alpha}{N})^N \leq e^{\alpha}.$$

□

Suppose now that a selection $f(x) \in F(x)$ is taken for all x. Solutions of differential equation $\dot{x} = f(x)$ will not satisfy (1.3) in the general case. The main problem is that a selection with required regularity properties (continuity, for example), may not exist. However, the Euler iterative scheme from the ordinary differential equations comes in handy.

For $N \in I\!N$, let

$$\pi = \{t_0, t_1, ..., t_{N-1}, t_N\}$$

be a partition of $[0, T]$ where $t_0 = 0$, $t_1 = T/N$, $t_2 = t_1 + T/N$,..., $t_{N-1} = t_{N-2} + T/N$, $t_N = T$. A piecewise affine arc $x^N(\cdot)$, called the *Euler polygonal arc*, is defined by the nodes x_0, x_1,..., x_{N-1} from the following sampling scheme:

$$\begin{array}{llll}
\dot{x}(t) = f(x_0) & x(t_0) = x_0 & \text{on } [t_0, t_1], & x_1 := x(t_1) \\
\dot{x}(t) = f(x_1) & x(t_1) = x_1 & \text{on } [t_1, t_2], & x_2 := x(t_2) \\
\vdots & & & \\
\dot{x}(t) = f(x_{i-1}) & x(t_{i-1}) = x_{i-1} & \text{on } [t_{i-1}, t_i], & x_i := x(t_i) \\
\vdots & & & \\
\dot{x}(t) = f(x_{N-1}) & x(t_{N-1}) = x_{N-1} & \text{on } [t_{N-1}, t_N]. &
\end{array}$$

An *Euler solution* to $\dot{x} = f(x)$ is any uniform limit $x(\cdot)$ of Euler polygonal arcs $x^N(\cdot)$ as $N \to 0$.

The following sequential compactness of trajectories property guarantees the existence of a solution of (1.3) under the standard hypotheses.

Theorem 1.3.4. *Let $\{x_i\}$ be a sequence of arcs on $[0, T]$ such that the set $\{x_i(0)\}$ is bounded, and satisfying*

$$\dot{x}_i(t) \in F(x_i(t) + y_i(t)) + r_i(t)B \quad a.e. \ ,$$

where $\{y_i\}$ and $\{r_i\}$ are sequences of measurable functions on $[0, T]$ such that $y_i(\cdot)$ converges to 0 in L^2 and $r_i(t) \geq 0$ converges to 0 in L^2. Then there is a subsequence of $\{x_i\}$ which converges uniformly to an arc x which is a trajectory of F, and whose derivatives converge weakly to $\dot{x}$.

Please refer to [7] for the proof. Here, we will just state that the previous theorem guarantees existence of a solution by taking a selection f of F and letting the limit of sampled arcs (the so-called Euler arcs) to be an *Euler* solution of $\dot{x} = f(x)$, $x(0) = x_0$. Again, this will be essentially different for the impulsive case, which was already illustrated in the introduction.

The following result is known as the Filippov's Theorem. We will be using it in Chapter 3 of this thesis, when we introduce our sampling technique for the impulsive systems. Filippov's Theorem requires the multifunction $F(\cdot)$ to be locally Lipschitz.

Theorem 1.3.5. *Suppose $\varepsilon > 0$ is given. If $y(\cdot)$ is an arc on $[0,T]$ such that*

$$\dot{y}(t) \in F\big(y(t) + \varepsilon B\big) \quad a.e. \ ,$$

and if $\rho_F(y) < \varepsilon/K$, for some $K > 0$, then there exists a trajectory $x(\cdot)$ for (1.3) with $x(0) = y(0)$ and

$$\max\{|x(t) - y(t)| \mid t \in [0,T]\} \leq \int_0^T |\dot{x}(t) - \dot{y}(t)|dt \leq K\rho_F(y) < \varepsilon.$$

Here

$$\rho_F(y) := \int_0^T \inf\{|y(t) - \dot{y}(t)| \mid y \in F(x)\}.$$

Details of this theorem one can find for example in [9] on page 114.

1.4 Invariance Properties

In this section of the preliminary chapter, we offer definitions and characterizations for both weakly and strongly invariant systems of form (1.3). The final chapter of this thesis is dedicated to the same properties, but this time for the impulsive systems. Invariance properties deal with conditions under which a solution starts and remains in a given closed set C.

Definition 1.4.1. The system (1.3) is said to be *weakly invariant* in set C provided that for all $x_0 \in C$, there exists a trajectory $x(\cdot)$ on $[0,\infty)$ such that

$$x(0) = x_0, \quad x(t) \in C \qquad \forall t \geq 0.$$

Definition 1.4.2. The system (1.3) is *strongly invariant* in set C provided that for all $x_0 \in C$, all trajectories $x(\cdot)$ on $[0,\infty)$ with $x(0) = x_0$ satisfy

$$x(t) \in C \qquad \forall t \geq 0.$$

The following theorem characterizes the weak invariance property.

Theorem 1.4.1. *The system (1.3) is weakly invariant if and only if for all $x \in C$ and $\zeta \in N_C^P(x)$, there exists a $v \in F(x)$ such that*

$$\langle v, \zeta \rangle \leq 0.$$

If we assume, in addition to the standard hypotheses, that the multifunction $F(\cdot)$ is also locally Lipschitz, one shows the similar characterization for the strong invariance.

Definition 1.4.3. We say that the multifunction $F(\cdot)$ is *locally Lipschitz* if for every point x there is a neighborhood $U = U(x)$ and a positive constant $L = L(x)$ such that

$$x_1, x_2 \in U \Rightarrow F(x_2) \subseteq F(x_1) + L|x_1 - x_2|B_1.$$

The number L is called in that case the *Lipschitz rank* of $F(\cdot)$ on the set U.

Theorem 1.4.2. *The system (1.3) is strongly invariant if and only if all $x \in C$, all $\zeta \in N_C^P(x)$ and all $v \in F(x)$ are such that*

$$\langle v, \zeta \rangle \leq 0.$$

1.5 Change of Variable

The closing section in the preliminary chapter is a modification of well known Lebesgue-Radon-Nikodým Theorem of real analysis. We use it to justify the change of variable in Chapter 2. This section is borrowed from [15]. We begin by defining so called N-functions, which map sets of measure zero to sets of measure zero. This concept is due to N.N. Luzin (1915), and it is also referred to in literature as a "null condition". Lebesgue measure on $\mathbb{R}^1$ is denoted by $m(\cdot)$.

Definition 1.5.1. Let $g(\cdot)$ be a function with domain $[a, b] \subset I\!\!R$ and range $[\alpha, \beta] \subset I\!\!R$. If $m(E) = 0$ implies $m(g(E))$ for all $E \subset [a, b]$, then $g(\cdot)$ is said to be an N-function.

Theorem 1.5.1. ***(Banach)*** *Let $g(\cdot)$ be a continuous function of bounded variation with domain $[a, b] \subset I\!\!R$ and range $[\alpha, \beta] \subset I\!\!R$. Then, $g(\cdot)$ is an N-function if and only if $g(\cdot)$ is absolutely continuous.*

The original Banach's proof of this theorem is in [15], page 288 and we will skip it here. This characterization shows that the time component of what we will call a graph completion is an N-function.

The following theorems are an application of Lebesgue-Radon-Nikodým Theorem ([15], page 315). The first theorem we just quote (Corollary 20.5 in [15]), and the other one we state and prove for the vector-valued case.

Theorem 1.5.2. *Let $g(\cdot)$ be a monotone continuous N-function with domain $[a, b]$ and range $[\alpha, \beta] \subset I\!\!R$. Then $g(\cdot)$ is absolutely continuous and for an m-integrable function $z(\cdot) : [\alpha, \beta] \to I\!\!R^n$, we have $(z \circ g)(\cdot)\dot{g}(\cdot)$ is m-integrable and*

$$\int_{\alpha}^{\beta} z(t)dt = \int_{a}^{b} z(g(s))\dot{g}(s)ds.$$

Theorem 1.5.3. *Let ν be a measure. Suppose that $g(\cdot) : [a, b] \mapsto [\alpha, \beta]$ is such that*

$$\nu(g(E)) = 0 \quad \text{for all} \quad E \subset [a, b] \quad \text{such that} \quad m(E) = 0.$$

Then there exists a nonnegative, Borel measurable function $h : [a, b] \to I\!\!R^m$ so that whenever $z(\cdot) : [\alpha, \beta] \to \mathcal{M}_{n\times m}$ is ν-integrable, then $(z \circ g)(\cdot)h(\cdot)$ is m-integrable on $[a, b]$, and satisfies

$$\int_{[\alpha,\beta]} z(t)\, d\nu(t) = \int_{0}^{S} z\big(g(s)\big)h(s)\, ds.$$

Proof. When $m = 1$, this Theorem is the known case from [15], Theorem 20.3, page 342. When $m > 1$, we can write columns of $z(\cdot)$ as

$$z(\cdot) = [z_1(\cdot) \quad z_2(\cdot) \quad \cdots \quad z_m(\cdot)],$$

where for each $i \in \{1, ..., m\}$, $z_i : [\alpha, \beta] \to \mathbb{R}^n$ is a ν-integrable function. For each $z_i(\cdot)$, by the scalar case, there exists a nonnegative, Borel measurable function $h_i : [a, b] \mapsto \mathbb{R}$ so that

$$\int_{[\alpha,\beta]} z_i(t)\, d\nu(t) = \int_0^S z_i\big(g(s)\big) h_i(s)\, ds.$$

The result is obtained by letting $h(\cdot) = [h_1(\cdot) \quad h_2(\cdot) \quad \cdots \quad h_m(\cdot)]^T$. □

We end the preliminaries with another result from the real analysis, called Egoroff's Theorem. We use it in Chapter 3. Once again, we offer only the statement of this theorem. Deeper study of this property can be found in [15], page 158 or in [12], page 60.

Theorem 1.5.4. ***(Egoroff's Theorem)*** *Let $f(\cdot) : [0, T] \to \mathbb{R}^n$ and $\{f_j(\cdot)\}$ with $f_j : [0, T] \to \mathbb{R}^n$, be measurable functions that are defined and finite almost everywhere on $[0, T]$. Suppose that $f_j \to f$ almost everywhere on $[0, T]$. Then for each $\varepsilon > 0$ there exists a set $A \subset [0, T]$ such that their measure is less than ε and $f_j \to f$ uniformly on $[0, T]\backslash A$.*

Chapter 2
Impulsive Systems Defined

Impulsive systems are precisely defined in Section 2.2. A new solution concept to an impulsive system is introduced in 2.4, and it is shown that it agrees with the (appropriate modification) of the Bressan-Rampazzo solution. The new concept requires a direct correlation of respectively the absolutely continuous, continuous singular, and atomic parts of the bounded variation solution and the given measure.

2.1 Introduction

Impulsive systems arise in a variety of applications where states can move at different time scales. The "slow" movement can be thought of as the usual time progression infinitesimally incremented by dt, and the "fast" movement occurs over a small interval that resembles the effect of a point-mass measure. We adopt the mathematical formalism introduced in [25, 24, 28], in which the controlled dynamic inclusion is the sum of a slow time velocity belonging to a set $F(x)$ and a fast time contribution coming from another set $G(x)d\mu$, where μ is a vector-valued measure.

Bressan and Rampazzo [5] emphasized that issues arise to the nature of a "solution" in nonlinear systems when the fast dynamic velocities are affected by multiplicative state dependence (i.e. G depends nontrivially on x), and that the solution concept will only be well-defined if a realization of the measure is also prescribed through a graph completion. Murray [18] made an independent and similar discovery by extending integral functionals of generalized variational problems from absolutely continuous functions to ones of bounded variation, where the dynamics are encoded through infinite penalization. An earlier work on measure driven dynamical systems and graph completion can also be found in works of Rishel [20].

Throughout the paper, the following data with accompanying assumptions are given:

(H1) A closed convex cone $K \subseteq I\!R^m$;

(H2) A multifunction $F : I\!R^n \rightrightarrows I\!R^n$ with closed graph and convex values, and satisfying

$$f \in F(x) \implies |f| \leq c(1+|x|) \quad \forall x \in I\!R^n.$$

(where $c > 0$ is a given constant);

(H3) A multifunction $G : I\!R^n \rightrightarrows \mathcal{M}_{n\times m}$ (where $\mathcal{M}_{n\times m}$ denotes the $n \times m$ dimensional matrices with real entries) with closed graph and closed convex values, and satisfying

$$g \in G(x) \implies \|g\| \leq c(1+|x|) \quad \forall x \in I\!R^n.$$

The set of vector-valued Borel measures defined on the interval $[0, T] \subset I\!R$ with values in K is denoted by $\mathcal{B}_K([0, T])$.

2.2 Impulsive Systems and Their Trajectories

Suppose $\mu \in \mathcal{B}_K([0, T])$ is given. The impulsive system considered in this paper is described by a differential inclusion (see [2, 26, 9, 7] basic theory of differential inclusions) of the form

$$\begin{cases} dx & \in & F\big(x(t)\big)\, dt + G\big(x(t)\big)\, d\mu(dt) \\ x(0-) & = & x_0. \end{cases} \tag{2.1}$$

The trajectory $x(\cdot)$ is a function of bounded variation, however further information is required to frame an unambiguous solution concept. Recall that the (right continuous) distribution function $u(\cdot) : [0, T] \to I\!R^m$ of μ is given by $u(t) = \mu([0, t])$. Following [5, 3], the following is a definition of a *graph completion*.

Definition 2.2.1. *Graph completion* of distribution function $u(\cdot) : [0, T] \to I\!R^m$ of measure μ, is a Lipschitz continuous map $(\psi_0, \psi) : [0, S] \to [0, T] \times I\!R^m$ so that

(a) $\psi_0(\cdot)$ is non-decreasing,

(b) for every $t \in [0, T]$ there exists $s \in [0, S]$ so that $\big(\psi_0(s), \psi(s)\big) = \big(t, u(t)\big)$ and

(c) $\dot{\psi}(s) \in K$, for almost all $s \in [0, S]$.

The role of the graph completion is to pin down the behavior of the trajectory $x(\cdot)$ during the "jumps" of $u(\cdot)$ so that multiplication by $G(x)$ during this fast time movement is unambiguous. The function ψ_0 is a reparameterized time variable. Since $\psi(\cdot)$ is a Lipschitz function by definition, there exists a positive number r such that

$$|\dot{\psi}(s)| \leq r. \tag{2.2}$$

Thus, the requirement (c) of the previous definition implies

$$\dot{\psi}(s) \in K \cap B_r.$$

This property is called the *cone adherence*. We point out here that r must be greater or equal to 1, as

$$s_i^+ - s_i^- = \left| \int_{s_i^-}^{s_i^+} \dot{\psi}(s') ds' \right| \leq \int_{s_i^-}^{s_i^+} |\dot{\psi}(s')| ds' \leq r(s_i^+ - s_i^-).$$

Suppose we are now given $\mu \in \mathcal{B}_K([0, T])$ and let $\mathcal{I}$ be an at most countable index set of atoms $\mathcal{T} := \{t_i\}_{i \in \mathcal{I}}$. Consider a three-tuple

$$X_\mu := \Big(x(\cdot), \big(\psi_0(\cdot), \psi(\cdot)\big), \{y_i(\cdot)\}_{i \in \mathcal{I}}\Big) \tag{2.3}$$

with the following constituents: $x(\cdot) : [0,T] \to I\!R^n$ is of bounded variation with its points of discontinuity equal to the set $\mathcal{T}$ of μ's atoms, $\big(\psi_0(\cdot), \psi(\cdot)\big) : [0,S] \to [0,T] \times I\!R^m$ is a graph completion of μ's distribution function $u(\cdot)$, and $\{y_i(\cdot)\}_{i\in\mathcal{I}}$ is a collection of Lipschitz functions, each defined on the nondegenerate interval $I_i := [s_i^-, s_i^+] := \psi_0^{-1}(t_i)$ and satisfying $y_i(s_i^\pm) = x(t_i\pm)$.

The following is a slight modification of a definition given in [5, 3].

Definition 2.2.2 (Bressan-Rampazzo (B-R)). Consider a three-tuple X_μ as in (2.9), and let

$$y(s) = \begin{cases} x(t) & \text{if } s \notin \cup_{i\in\mathcal{I}} I_i, \quad t = \psi_0(s) \\ y_i(s) & \text{if } s \in I_i. \end{cases} \tag{2.4}$$

Then X_μ is a *Bressan-Rampazzo (B-R) solution* of (2.1) provided $y(\cdot)$ is Lipschitz on $[0,S]$ and satisfies

$$\begin{cases} \dot{y}(s) & \in & F\big(y(s)\big)\dot{\psi}_0(s) + G\big(y(s)\big)\dot{\psi}(s) \quad \text{a.e. } s \in [0,S]. \\ y(0) & = & x_0. \end{cases} \tag{2.5}$$

One may observe that $y(\cdot)$ defined by (2.4) is a graph completion of the vector-valued function $x(\cdot)$.

We next introduce a solution concept with the same data structure as in (2.9), but which requires properties stated directly in the original timeframe. Recall that an arc $x(\cdot)$ of bounded variation induces a measure dx that can be decomposed into absolutely continuous, continuous singular, and discrete (that is, purely atomic) parts, and so can be written as

$$dx = \dot{x}(t)\,dt + dx_\sigma + dx_D,$$

where dx_σ is a singular continuous measure and $dx_D := \sum_{i\in\mathcal{I}} \delta_{t_i}^x$ is the discrete part with $\delta_{t_i}^x$ denoting the point mass jump of the vector $x(t_i+) - x(t_i-)$. If 0 is an atom, then the initial point of the jump is denoted by $x(0-)$. Likewise, the measure $\mu \in \mathcal{B}_K([0,T])$ decomposes into $\mu = \dot{u}(t)\,dt + \mu_\sigma + \mu_D$ where $\mu_D = \sum_{i\in\mathcal{I}} \delta^u_{t_i}$.

Definition 2.2.3. The three-tuple X_μ in (2.9) is a *solution* of (2.1) provided

(i) for almost all $t \in [0,T]$,

$$\begin{cases} \dot{x}(t) & \in & F\big(x(t)\big) + G\big(x(t)\big)\dot{u}(t), \\ x(0-) & = & x_0; \end{cases}$$

(ii) there exists a bounded μ_σ-measurable selection $\gamma(t) \in G\big(x(t)\big)$ with

$$dx_\sigma = \gamma(t)\,\mu_\sigma \quad (\text{as measures on } [0,T]); \text{and}$$

(iii) the set of atoms of dx is $\mathcal{T} = \{t_i\}_{i\in\mathcal{I}}$, and for each $i \in \mathcal{I}$, $y_i(s_i^-) = x(t_i-)$, $y_i(s_i^+) = x(t_i+)$, and

$$\dot{y}_i(s) \in G\big(y_i(s)\big)\dot{\psi}(s) \quad \text{a.e. } s \in I_i.$$

The two notions of solution of 2.1 are equivalent, and we will show that in Section 2.4. The fundamental role played by the graph completion in this definition surfaces in the differential inclusions stated in (iii), and in effect circumscribes the fast velocities that are available during that jump in t time. A simple concrete example where different graph completions give different reachable sets is given in the introduction of this thesis and it can also be found in [3].

2.3 Properties of Graph Completions

Perhaps the most natural example of a graph completion is the following *canonical* graph completion. In fact, we will see that we can replace any graph completion by the canonical graph completion without changing the solution.

Definition 2.3.1. For a given $\mu \in \mathcal{B}_K\big([0,T]\big)$, a Lipschitz pair $(\phi_0(\cdot), \phi(\cdot))$: $[0, \bar{S}] \to [0, T]$ is the *canonical graph completion* if

(GC1) $\phi_0(\cdot)$ is a filled-in inverse of $\eta(t) := t + |\mu|([0,t])$.

(GC2) for every $t \in [0,T]$ there exists $s \in [0,\bar{S}]$ so that $\big(\phi_0(s), \phi(s)\big) = \big(t, u(t)\big)$ and

(GC3) $\dot{\phi}(s) \in K$, for all $s \in [0,S]$.

Note, this definition can be obtained from Definition 2.2.1 by replacing condition (a) with (GC1). The gain is that in Definition 2.3.1, the choice for the temporal component is fixed. Condition (CG1) means

$$\phi_0(s) = t \quad \Leftrightarrow \quad \eta(t-) \le s \le \eta(t+), \tag{2.6}$$

where $\eta(t-)$ and $\eta(t+)$ respectively denote the lefthand limit $\lim_{t\nearrow t_0} \eta(t)$ and righthand limit $\lim_{t\searrow t_0} \eta(t)$. These left and righthand limits are equal if and only if t is not an atom of μ. If 0 is an atom of μ, then $\eta(0-) = 0$ by convention.

Obviously, since $\phi_0(\cdot)$ is a non-decreasing Lipschitz continuous function, the canonical graph completion is indeed a graph completion. The following lemma shows even more $-$ all graph completions can be rescaled to the canonical graph completion without changing the solution.

Lemma 2.3.1. *Suppose measure $\mu \in \mathcal{B}_K\big([0,T]\big)$ is given and a three tuple*

$$X_\mu := \Big(x(\cdot), \big(\psi_0(\cdot), \psi(\cdot)\big), \{y_i(\cdot)\}_{i\in\mathcal{I}}\Big)$$

is a Bressan-Rampazzo solution, where the pair $(\phi_0, \phi) : [0,T] \mapsto [0,S] \times I\!R^m$ is a graph completion (Definition 2.2.1). Then, there exists $\bar{S} > 0$ and an absolutely continuous function $\Psi : [0,\bar{S}] \mapsto [0,S]$ such that the pair $(\phi_0, \phi) : [0,T] \times [0,\bar{S}] \times I\!R^m$,

$$(\phi_0, \phi)(s) := (\psi_0, \psi)(\Psi(s)) \tag{2.7}$$

is the canonical graph completion and there exists a Bressan-Rampazzo solution

$$\bar{X}_\mu := \Big(\bar{x}(\cdot), \big(\phi_0(\cdot), \phi(\cdot)\big), \{\bar{y}_i(\cdot)\}_{i\in\mathcal{I}}\Big)$$

is such that $\bar{x}(t) = x(t)$ *and* $\bar{y}_i(s) = y_i(\Psi(s))$.

Proof. Let begin the proof by defining an "inverse" $\Upsilon : [0, T] \mapsto [0, S]$ of $\psi_0(\cdot)$ as $\Upsilon(t) = \psi_0(t+)$. Moreover, let $\phi_0(\cdot)$ satisfy condition (GC1), and let $\bar{I}_i := [\bar{s}_i^-, \bar{s}_i^+] := \phi_0^{-1}(t_i)$ for all $t_i \in \mathcal{T}$. Mapping $\Psi : [0, \bar{S}] \mapsto [0, S]$ so that the (2.7) holds is constructed in the following way;

$$\Psi(s) := \begin{cases} \Phi_i(s), & \text{for } s \in \bar{I}_i, i \in \mathcal{I}, \\ (\Upsilon \circ \phi_0)(s), & \text{for } s \notin \cup_i \bar{I}_i, \end{cases}$$

where for all $i \in \mathcal{I}$, $\Phi_i(\cdot)$ maps linearly $[\bar{s}_i^-, \bar{s}_i^+]$ to $[s_i^-, s_i^+]$:

$$\Phi_i(s) = s_i^+ \frac{s - \bar{s}_i^-}{\bar{s}_i^+ - \bar{s}_i^-} + s_i^- \frac{\bar{s}_i^+ - s}{\bar{s}_i^+ - \bar{s}_i^-}.$$

Indeed,

$$\phi_0(s) = t_i = \psi_0(\Psi(s)), \text{ for } s \in \bar{I}_i, \text{ and}$$

$$\psi_0(\Psi(s)) = \psi_0(\Upsilon(\phi_0(s))) = \phi_0(\psi_0^{-1}(\phi_0(s))) = \phi(s), \text{ for } s \notin \cup_i \bar{I}_i,$$

and, finally, (2.7) is obtained by defining

$$\phi(s) := \psi(\Psi(s)).$$

To show that pair $\big(\phi_0(\cdot), \phi(\cdot)\big)$ is the canonical graph completion we only need to show that (GC3) holds, as (GC1) and (CG2) immediately follow by definition of $\phi_0(\cdot)$ and $\phi(\cdot)$. Condition (GC3) holds almost everywhere on $[0, S]$ because

$$\dot{\phi}(s) = \dot{\psi}(\Psi(s))\dot{\Psi}(s) \in K.$$

From inclusion (2.5), there exist measurable selections $f(s) \in F(y(s))$ and $g(s) \in G(y(s))$ so that, on $[0, S]$

$$\dot{y}(s) = f(s)\dot{\psi}_0(s) + g(s)\dot{\psi}(s). \tag{2.8}$$

For all $s \in [0, \bar{S}]$, let $\bar{y}(s) := y(\Psi(s))$, let $\bar{x}(t) = \bar{y}(\eta(t))$ on $[0, T]$ and for all $i \in \mathcal{I}$, let $\bar{y}_i(s) = \bar{y}(s)$ on $\bar{I}_i$. Also let $\bar{f}(s) = f(\Psi(s))$ and $\bar{g}(s) = g(\Psi(s))$. The proof is now completed because,

$$\begin{aligned}
\dot{\bar{y}}(s) &= \dot{y}(\Psi(s))\dot{\Psi}(s) && \text{(from definition of } \bar{y}\text{)} \\
&= f(\Psi(s))\dot{\psi}_0(\Psi(s))\dot{\Psi}(s) + g(\Psi(s))\dot{\psi}(\Psi(s))\dot{\Psi}(s) && (2.8) \\
&= \bar{f}(s)\dot{\phi}_0(s) + \bar{g}(s)\dot{\phi}(\Psi(s)) && \text{(from definition of } \phi_0\text{)} \\
&\in F(\bar{y}(s))\dot{\phi}_0(s) + G(\bar{y}(s))\dot{\phi}(s).
\end{aligned}$$

□

Previous Lemma gives us the luxury to always specify our graph completion to be canonical, that is $(\psi_0, \psi) = (\phi_0, \phi)$, and to take the three tuple

$$X_\mu := \big(x(\cdot), \phi(\cdot), \{y_i(\cdot)\}_{i \in \mathcal{I}}\big) \tag{2.9}$$

satisfying Definition 2.2.2 as a solution to (2.1), without the loss of generality. Through the remainder of this Chapter and in the Chapter 3, whenever we mention "graph completion", we will always consider the canonical graph completion. Note that $r = 1$ in (2.2) implies a *straight line completion*. This is a completion where the jump from $u(t_i-)$ to $u(t_i+)$ is bridged directly with a smooth straight line of slope 1. However, when r is allowed to take values greater than 1, then the second component of graph completion $(\phi_0, \phi)(\cdot)$ is not uniquely determined. In general, various $\phi(\cdot)$ produce various solutions of (2.1) even if $F(\cdot)$ and $G(\cdot)$ are single valued [3].

We continue by recording two technical results that will be used in the sequel. We also use the following notation. Let supp $\mu_\sigma \subseteq [0,T]$ denote the closed support of μ_σ. Define

$$\Gamma := \eta\big(\text{supp } \mu_\sigma\big) \subseteq [0,S], \tag{2.10}$$

which has Lebesgue measure $\|\mu_\sigma\|$, and set

$$\widetilde{\Gamma} := \Gamma \bigcup \left(\cup_{i \in \mathcal{I}} I_i\right). \tag{2.11}$$

Then $\dot{\phi}_0(s) = 0$ a.e. $s \in \widetilde{\Gamma}$ and $\dot{\phi}_0(s) > 0$ a.e. $s \in [0,S] \setminus \widetilde{\Gamma}$. Lebesgue measure on $I\!R^1$ is denoted by $m(\cdot)$, and $\eta(\cdot)$ and $\phi_0(\cdot)$ are as in Definition 2.3.1.

Lemma 2.3.2. *Let ν be either the measure m or μ_σ.*

(a) If $A \subseteq [0,S]$ with $m(A) = 0$, then $\nu\big(\phi_0(A)\big) = 0$.

(b) If $z(\cdot) : [0,S] \to I\!R^d$ is ν-measurable, then $(z \circ \eta)(\cdot) : [0,T] \to I\!R^d$ is ν-measurable.

Proof. Since $\eta(\cdot)$ is strictly increasing, part (a) is immediate. For part (b), if $O \subseteq I\!R^d$ is open, then $z^{-1}(O)$ is ν-measurable and so differs from a Borel set by a set of ν-measure zero. Now $\phi_0(\cdot)$ is nondecreasing, and so in particular maps Borel sets onto Borel sets. We have by part (a) that $\eta^{-1}\big(z^{-1}(O)\big) = \phi_0\big(z^{-1}(O)\big)$ differs from a Borel set by a set of ν-measure zero, and therefore is ν-measurable. □

The following are "Change of Variable" formulas.

Lemma 2.3.3.

(a) If $z(\cdot) : [0,T] \to I\!R^d$ is m-integrable, then $z \circ \eta$ is m-integrable on $[0,S]$, and satisfies

$$\int_0^T z(t)\,dt = \int_0^S z\big(\phi_0(s)\big)\dot{\phi}_0(s)\,ds.$$

(b) There exists a Borel measurable function $\psi(\cdot) : [0, S] \to I\!R^m$ so that whenever $z(\cdot) : [0, T] \to \mathcal{M}_{n\times m}$ is μ_σ-integrable, then $(z \circ \phi_0)(\cdot)\psi(\cdot)$ is m-integrable on $[0, S]$, and satisfies

$$\int_{[0,T]} z(t)\, d\mu_\sigma(t) = \int_0^S z\big(\phi_0(s)\big)\psi(s)\, ds.$$

Remark: number d in the previous statement is either m or n.

Proof. Part (a) is given in [15], Corollary 20.5; part (b) is the vector-valued case of [15], Theorem 20.3 (page 342); The hypotheses of the cited results are satisfied due to Lemma 2.3.2(a). □

Suppose $X_\mu = \big(x(\cdot), \phi(\cdot), \{y_i(\cdot)\}_{i\in\mathcal{I}}\big)$ is as in (2.9), and $y(\cdot)$ is defined as in (2.4). Let $u_a(\cdot)$, $u_\sigma(\cdot)$, and $u_D(\cdot)$ be the distribution functions of $\dot{u}(t)dt$, μ_σ, and μ_D, respectively, and let $x_a(\cdot)$, $x_\sigma(\cdot)$, and $x_D(\cdot)$ denote the distributions associated to the decomposition of dx. That is,

$$\begin{array}{rclrcl} u_a(t) & = & \int_0^t \dot{u}(t')\, dt', & x_a(t) & = & x_0 + \int_0^t \dot{x}(t')\, dt', \\ u_\sigma(t) & = & \mu_\sigma\big([0,t]\big), & x_\sigma(t) & = & dx_\sigma\big([0,t]\big), \\ u_D(t) & = & \sum_{t_i\in\mathcal{T},\, t_i\le t} \mu\big(\{t_i\}\big), & x_D(t) & = & dx_D\big([0,t]\big). \end{array}$$

for $t \in [0, T]$. Recall $\Gamma = \eta\big(\text{supp } \mu_\sigma\big) \subseteq [0, S]$ and $\widetilde{\Gamma} = \Gamma \bigcup (\cup_{i\in\mathcal{I}} I_i)$. The following related decompositions for $\phi(\cdot)$ and $y(\cdot)$ are defined on $[0, S]$ by

$$\begin{array}{rclrcl} \phi_a(s) & = & \int_{[0,s]\setminus\widetilde{\Gamma}} \dot{\phi}(s')\, ds', & y_a(s) & = & x_0 + \int_{[0,s]\setminus\widetilde{\Gamma}} \dot{y}(s')\, ds', \\ \phi_\sigma(s) & = & \int_{[0,s]\cap\Gamma} \dot{\phi}(s')\, ds', & y_\sigma(s) & = & \int_{[0,s]\cap\Gamma} \dot{y}(s')\, ds', \\ \phi_D(s) & = & \int_{[0,s]\bigcap(\cup_{i\in\mathcal{I}} I_i)} \dot{\phi}(s')\, ds', & y_D(s) & = & \int_{[0,s]\bigcap(\cup_{i\in\mathcal{I}} I_i)} \dot{y}(s')\, ds'. \end{array}$$

It is clear that $\phi = \phi_a + \phi_\sigma + \phi_D$ and $y = y_a + y_\sigma + y_D$ on $[0, S]$. We next show the corresponding parts of the two decompositions match up after composition with η.

Lemma 2.3.4. *If $t \notin \mathcal{T}$, then*

(a) $(\phi_D \circ \eta)(t) = u_D(t)$ *and* $(y_D \circ \eta)(t) = x_D(t)$.

For each $t \in [0, T]$, we have

(b) $(\phi_a \circ \eta)(t) = u_a(t)$ *and* $(y_a \circ \eta)(t) = x_a(t)$, *and*

(c) $(\phi_\sigma \circ \eta)(t) = u_\sigma(t)$ *and* $(y_\sigma \circ \eta)(t) = x_\sigma(t)$.

Proof. We only write in detail the proof involving $\phi(\cdot)$ and $u(\cdot)$, since the corresponding argument involving $y(\cdot)$ and $x(\cdot)$ is the same.

Part (a) is clear from the definitions, since both functions equal $\sum_{t_i<t}\big[u(t_i+) - u(t_i-)\big]$ whenever $t \notin \mathcal{T}$. The definition of a graph completion says that $(\phi \circ \eta)(t) = u(t)$ for these t also, and therefore

$$(\phi_a \circ \eta)(t) = u_a(t) + u_\sigma(t) - (\phi_\sigma \circ \eta)(t). \tag{2.12}$$

The equality in (2.12) actually holds for all $t \in [0, T]$ since every function there is continuous. Now recall supp μ_σ is a closed set of measure zero. If $0 \le t_1 < t_2 \le T$ and $[t_1, t_2] \cap \operatorname{supp} \mu_\sigma = \emptyset$, then $\eta\big([t_1, t_2]\big) \cap \Gamma = \emptyset$, and hence $t \mapsto (\phi_\sigma \circ \eta)(t)$ is constant on $[t_1, t_2]$. This implies $\frac{d}{dt}(\phi_\sigma \circ \eta)(t) = 0$ for almost every $t \in [0, T]$. It follows by differentiating (2.12) at all these $t \in [0, T]$ that

$$\frac{d}{dt}(\phi_a \circ \eta)(t) = \frac{d}{dt}u_a(t) = \dot{u}(t).$$

Part (b) now follows by integration. Finally, part (c) is an immediate consequence of part (b) and (2.12). □

Corollary 2.3.5. *We have*

(a) $\dot{\phi}(s) = \dot{u}\big(\phi_0(s)\big)\dot{\phi}_0(s)$ *a.e.* $s \in [0, S] \setminus \widetilde{\Gamma}$, *and*

(b) $\dot{y}(s) = \dot{x}\big(\phi_0(s)\big)\dot{\phi}_0(s) \quad$ *a.e.* $s \in [0, S] \setminus \widetilde{\Gamma}$.

Proof. We have by Lemma 2.3.4(b) and Lemma 2.3.3(a) that

$$\phi_a(s) = u_a\big(\phi_0(s)\big) = \int_0^{\phi_0(s)} \dot{u}_a(t)\, dt = \int_0^s \dot{u}_a\big(\phi_0(s)\big)\dot{\phi}_0(s)\, ds.$$

The conclusion (a) follows from the definition of ϕ_a and the differentiation theorem. A similar argument proves (b). $\square$

Let us now define another graph completion which will be heavily used in the Chapter 4 of this thesis.

Definition 2.3.2. For a given measure $\mu \in \mathcal{B}_K([0,T])$, normalized graph completion is a Lipschitz continuous map $(\psi_0, \psi) : [0, \bar{S}] \to [0, T] \times I\!R^m$ so that

(NC1) $0 \le \dot{\psi}_0(s) \le 1$ almost everywhere on $[0, \bar{S}]$

(NC2) for every $t \in [0, T]$ there exists $s \in [0, \bar{S}]$ so that $\big(\psi_0(s), \psi(s)\big) = \big(t, u(t)\big)$,

(NC3) $\dot{\psi}(s) = \big(1 - \dot{\psi}_0(s)\big)k(s)$, for almost all $s \in [0, \bar{S}]$, where $k(s) \in K_1 := K \cap S_1$.

Using the Lemma 2.3.1, normalized graph completion can be rescaled to the canonical graph completion. Moreover, the following lemma holds:

Lemma 2.3.6. *Suppose measure $\mu \in \mathcal{B}_K([0,T])$ is given. Suppose the pair $(\phi_0, \phi)(\cdot)$ is a canonical graph completion and X_μ is a solution of (2.1) corresponding to μ and pair $(\phi_0, \phi)(\cdot)$. Then there exists an absolutely continuous non-decreasing function $R : [0, S] \mapsto [0, \bar{S}]$ such that the pair*

$$(\psi_0, \psi)(s) := (\phi_0, \phi)(R^{-1}(s)) \tag{2.13}$$

is the normalized graph completion and there exists a Bressan-Rampazzo solution

$$\bar{X}_\mu := \Big(\bar{x}(\cdot), \big(\psi_0(\cdot), \psi(\cdot)\big), \{\bar{y}_i(\cdot)\}_{i \in \mathcal{I}}\Big)$$

is such that $\bar{x}(t) = x(t)$ and $\bar{y}_i(s) = y_i(R^{-1}(s))$.

Proof. Suppose that $(\phi_0, \phi)(\cdot)$ is a canonical graph completion. Define

$$\bar{k}(s) := \begin{cases} \frac{\dot{\phi}(s)}{1-\dot{\phi}_0(s)} & \text{when } \dot{\phi}_0(s) \neq 1, \\ \frac{\dot{\phi}(s)}{|\dot{\phi}(s)|} & \text{when } \dot{\phi}_0(s) = 1. \end{cases}$$

Note that when s belongs to an interval I on which $\dot{\phi}_0(s) = 1$, then on interval $\phi_0(I)$, measure μ is inactive and its distribution is constant, thus $\dot{\phi}(s) = 0$ on I. Therefore, for $s \in I$, $0 = \dot{\phi}(s) = \bar{k}(s)(1-\dot{\phi}_0(s))$ trivially holds. Using the definition of $\bar{k}(\cdot)$ we conclude $\dot{\phi}(s) = \bar{k}(s)(1 - \dot{\phi}_0(s))$ almost everywhere on $[0, S]$.

Note, $\mu(t) = \dot{\phi}(\eta(t))$ for $t \notin \mathcal{T}$. For almost all $s \in [0, S]\backslash\widetilde{\Gamma}$, $\dot{\phi}_0(s) > 0$ and

$$s - \phi_0(s) = |\mu|([0, \phi_0(s)]) = \int_0^s |\dot{\phi}(s')|ds',$$

which is $1 - \dot{\phi}_0(s) = |\dot{\phi}(s)|$ after differentiation. This implies

$$|\bar{k}(s)| = \frac{|\dot{\phi}(s)|}{1 - \dot{\phi}_0(s)} = 1. \tag{2.14}$$

Moreover, for almost all $s \in [0, S]\backslash\widetilde{\Gamma}$, $\dot{\phi}(s)$ belongs to the cone K and

$$\bar{k}(s) = \frac{\dot{\phi}(s)}{(1 - \dot{\phi}_0(s))} \in K. \tag{2.15}$$

For almost all $s \in \widetilde{\Gamma}$, $\dot{\phi}_0(s) = 0$ and

$$\bar{k}(s) = \dot{\phi}(s) \in \overline{B_r \backslash B_1}. \tag{2.16}$$

Define on $[0, S]$ the function

$$r(s) = \begin{cases} 1 & s \notin \cup_i I_i \\ |\dot{\phi}(s)| & s \in \cup_i I_i. \end{cases}$$

and let $R : [0, S] \mapsto [0, \bar{S}]$ be

$$R(s) = \int_0^s r(s')ds'.$$

Define

$$k(s) = \frac{\bar{k}(R^{-1}(s))}{|\bar{k}(R^{-1}(s))|},$$

$\psi_0(s) := \phi_0(R^{-1}(s))$ and $\psi(s) := \phi(R^{-1}(s))$. Almost everywhere on $[0, S]$,

$$\dot{\psi}_0(s) = \dot{\phi}_0(R^{-1}(s))\frac{dR^{-1}(s)}{ds} = \frac{\dot{\phi}_0(R^{-1}(s))}{|\dot{\phi}_0(R^{-1}(s))|}.$$

Now, for almost all $s \notin \cup_i \bar{I}_i$, where $\bar{I}_i := \psi_0^{-1}(t_i)$, $t_i \in \mathcal{T}$,

$$\dot{\psi}_0(s) = \dot{\phi}_0(R^{-1}(s)), \qquad \text{and}$$

$$\dot{\psi}(s) = \dot{\phi}(R^{-1}(s))\frac{dR^{-1}(s)}{ds} = \frac{\dot{\phi}(R^{-1}(s))}{|\dot{\phi}(R^{-1}(s))|}.$$

Now, for almost all $s \notin \cup_i I_i$, using (2.14) and (2.15),

$$\dot{\psi}(s) = \dot{\phi}(R^{-1}(s)) = (1 - \dot{\phi}_0(R^{-1}(s)))k(R^{-1}(s)) = (1 - \dot{\psi}_0(s))k(s).$$

For almost all $s \in \cup_i I_i$, using (2.16),

$$\dot{\psi}(s) = \dot{\phi}(R^{-1}(s)) = \frac{\dot{\phi}_0(R^{-1}(s))}{|\dot{\phi}_0(R^{-1}(s))|} = \frac{\bar{k}(R^{-1}(s))}{|\bar{k}(R^{-1}(s))|} = k(s).$$

Therefore, the pair $(\psi_0, \psi)(\cdot)$ is the normalized graph completion.

From inclusion (2.5), there exist measurable selections $f(s) \in F(y(s))$ and $g(s) \in G(y(s))$ so that, on $[0, S]$

$$\dot{y}(s) = f(s)\dot{\psi}_0(s) + g(s)\dot{\psi}(s). \tag{2.17}$$

For all $s \in [0, \bar{S}]$, let $\bar{y}(s) := y(R^{-1}(s))$, let $\bar{x}(t) = \bar{y}(\eta(t))$ on $[0, T]$ and for all $i \in \mathcal{I}$, let $\bar{y}_i(s) = \bar{y}(s)$ on $\bar{I}_i$. Moreover, let $\bar{f}(s) = f(R^{-1}(s))$ and $\bar{g}(s) = g(R^{-1}(s))$. The proof is now completed because,

$$\begin{aligned} \dot{\bar{y}}(s) &= \dot{y}(R^{-1}(s))\tfrac{dR^{-1}(s)}{ds} && \text{(from definition of } \bar{y}\text{)} \\ &= \bar{f}(s)\dot{\psi}_0(s) + \bar{g}(s)\dot{\psi}(s), \end{aligned}$$

$\bar{f}(s) \in F(\bar{y}(s))$ and $\bar{g}(s) \in G(\bar{y}(s))$ □

2.4 Equivalence of the Solution Concepts

This section is devoted to proving the following equivalence theorem.

Theorem 2.4.1. *Suppose $\mu \in \mathcal{B}_K([0,T])$ and X_μ is as in (2.9). Then X_μ is a B-R solution of (2.1) if and only if X_μ is a solution of (2.1).*

Proof. Suppose the 3-tuple $X_\mu = \big(x(\cdot), \phi(\cdot), \{y_i(\cdot)\}_{i\in\mathcal{I}}\big)$ is a B-R solution as given in Definition 2.2.2. Thus $y(\cdot)$ as defined in (2.4) satisfies (2.5). We will show that X_μ satisfies the conditions of Definition 2.2.3. The initial condition $x(0-) = x_0$ is immediate, as well as the requirement in Definition 2.2.3(iii).

Since $y(\cdot)$ satisfies (2.5), there exist measurable selections $f(\cdot)$, $g(\cdot)$ of $F\big(y(\cdot)\big)$, $G\big(y(\cdot)\big)$, respectively, that satisfy

$$\dot{y}(s) = f(s)\dot{\phi}_0(s) + g(s)\dot{\phi}(s) \quad \text{a.e. } s \in [0,S]. \tag{2.18}$$

Let $\mathrm{dom}_{\bar{f}}$ consist of those $t \in [0,T] \setminus \mathcal{T}$ for which f is defined at $s = \eta(t)$ and satisfies $f(s) \in F\big(y(s)\big)$, and similarly define $\mathrm{dom}_{\bar{g}}$. Both sets have full measure in $[0,T]$. Define $\bar{f}$ on $\mathrm{dom}_{\bar{f}}$ and $\bar{g}$ on $\mathrm{dom}_{\bar{g}}$ by

$$\bar{f}(t) = (f \circ \eta)(t) \quad \text{and} \quad \bar{g}(t) = (g \circ \eta)(t).$$

The fact that $\bar{f}(\cdot)$ and $\bar{g}(\cdot)$ are both Lebesgue measurable on $[0,T]$ is a consequence of Lemma 2.3.2(b). Moreover, $x(t) = (y \circ \eta)(t)$ for all $t \notin \mathcal{T}$, and so we have that $\bar{f}(t) = (f\circ\eta)(t) \in F\big((y\circ\eta)(t)\big) = F\big(x(t)\big)$ for all $t \in \mathrm{dom}_{\bar{f}}$. Similar considerations apply to $\bar{g}(\cdot)$. Hence we have shown

$$\bar{f}(t) \in F\big(x(t)\big) \text{ and } \bar{g}(t) \in G\big(x(t)\big) \quad \text{a.e. } t \in [0,T]. \tag{2.19}$$

The change of variables formula in Lemma 2.3.3(a) says that

$$x_a(t) = x_0 + \int_0^t \dot{x}(t')\,dt' = x_0 + \int_0^{\eta(t)} \dot{x}\big(\phi_0(s')\big)\dot{\phi}_0(s')\,ds', \tag{2.20}$$

and we also have by definition of $y_a(\cdot)$ and (2.18) that

$$\begin{aligned}(y_a \circ \eta)(t) &= x_0 + \int_{[0,\eta(t)]\setminus\widetilde{\Gamma}} \dot{y}(s')\,ds' \\ &= x_0 + \int_{[0,\eta(t)]\setminus\widetilde{\Gamma}} f(s')\dot{\phi}_0(s') + g(s')\dot{\phi}(s')\,ds' \\ &= x_0 + \int_{[0,\eta(t)]\setminus\widetilde{\Gamma}} f(s')\dot{\phi}_0(s') + g(s')\dot{u}_a\big(\phi_0(s')\big)\dot{\phi}_0(s')\,ds',\end{aligned} \tag{2.21}$$

where the last equality is valid by Corollary 2.3.5(a). Lemma 2.3.4(b) says that (2.20) and (2.21) are equal, and since $\dot{\phi}_0(s) = 0$ on $\widetilde{\Gamma}$, we have

$$\int_{[0,s]\setminus\widetilde{\Gamma}} \dot{x}\big(\phi_0(s')\big)\dot{\phi}_0(s')\,ds' = \int_{[0,s]\setminus\widetilde{\Gamma}} f(s')\dot{\phi}_0(s') + g(s')\dot{u}_a\big(\phi_0(s')\big)\dot{\phi}_0(s')\,ds',$$

for all $s \in [0, S]$. We deduce by differentiating with respect to s that

$$\dot{x}\big(\phi_0(s)\big)\dot{\phi}_0(s) = f(s)\dot{\phi}_0(s) + g(s)\dot{u}_a\big(\phi_0(s)\big)\dot{\phi}_0(s) \quad \text{a.e. } s \in [0, S] \setminus \widetilde{\Gamma}.$$

Recall $\dot{\phi}_0(s) > 0$ for almost all $s \in [0, S] \setminus \widetilde{\Gamma}$, and so

$$\dot{x}\big(\phi_0(s)\big) = f(s) + g(s)\dot{u}_a\big(\phi_0(s)\big) \quad \text{a.e. } s \in [0, S] \setminus \widetilde{\Gamma} \tag{2.22}$$

holds by dividing the previous line by $\dot{\phi}_0(s)$. We next switch over to the t-variable by substituting into (2.22) those $t = \phi_0(s)$ that belong to

$$\mathrm{dom}_{\bar{f}} \,\cap\, \mathrm{dom}_{\bar{g}} \,\cap\, \phi_0\big([0, S] \setminus \widetilde{\Gamma}\big),$$

which is a set of t that constitutes a set of full measure. The conclusion is that

$$\dot{x}(t) = \bar{f}(t) + \bar{g}(t)\dot{u}_a(t) \quad \text{a.e. } t \in [0, T], \tag{2.23}$$

which, in conjunction with (2.19), verifies Definition 2.2.3(a).

Now consider the condition in Definition 2.2.3(ii). Let us first note that the measurable function $\psi(\cdot)$ given in Lemma 2.3.3(b) is none other than $\dot{\phi}(\cdot)\chi_{\Gamma}(\cdot)$, where $\chi_\Gamma(\cdot)$ is the characteristic function of Γ. Indeed, for $s \in [0, S]$, let $z(\cdot)$ equal $\chi_{[0,s]}(\cdot)$. Then Lemma 2.3.4(c) and Lemma 2.3.3(b) say that

$$\phi_\sigma(s) = u_\sigma(\phi_0(s)) = \mu_\sigma\big([0, \phi_0(s)]\big) = \int_0^s \psi(s')\, ds'$$

It follows by differentiation that $\psi(s) = \dot{\phi}(s)\chi_\Gamma(s)$ for almost all $s \in [0, S]$. Lemma 2.3.3(b) now reads as

$$\int_{[0,T]} z(t)\, d\mu_\sigma(t) = \int_{[0,S]\cap\Gamma} z\big(\phi_0(s)\big)\dot{\phi}(s)\, ds \tag{2.24}$$

for all μ_σ-integrable functions $z(\cdot)$ defined from $[0, T]$ into $\mathcal{M}_{n\times m}$.

If $A \subseteq \text{supp}\ \mu_\sigma$ is such that $|\mu_\sigma|(A) > 0$, then $\eta(A)$ has positive Lebesgue measure in Γ, and therefore contains points of $\eta(\text{dom}_{\bar{g}})$. It follows that $\text{dom}_{\bar{g}} \cap \text{supp}\ \mu_\sigma$ has full μ_σ-measure in supp μ_σ. We let $\gamma(\cdot) : [0, T] \to \mathcal{M}_{m\times n}$ be $\bar{g}(\cdot)$ restricted to supp μ_σ and 0 elsewhere; in other words,

$$\gamma(t) := \begin{cases} (g \circ \eta)(t) & \text{if} \quad t \in \text{dom}_{\bar{g}} \cap \text{supp}\ \mu_\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\gamma(\cdot)$ is μ_σ-measurable by Lemma 2.3.2(b), and we have for $s \in [0, S]$ and $t = \phi_0(s)$ that

$$\begin{aligned} x_\sigma(t) &= y_\sigma(s) && \text{(Lemma 2.3.4(c))} \\ &= \int_{[0,s]\cap\Gamma} \dot{y}(s')\, ds' && \text{(Definition of } y_\sigma(\cdot)) \\ &= \int_{[0,s]\cap\Gamma} g(s')\dot{\phi}(s')\, ds' && \text{(by (2.18) and since } \dot{\phi}_0(s) = 0 \text{ on } \Gamma) \\ &= \int_{[0,t]} \gamma(t')\, d\mu_\sigma(t'). && \text{(by (2.24))} \end{aligned}$$

This shows that the distribution function of $\gamma(t)d\mu_\sigma$ is x_σ, and hence Definition 2.2.3(ii) holds. The proof that a B-R solution satisfies the requirements of Definition 2.2.3 is now complete.

To prove the converse, again suppose X_μ is as in (2.9). It is immediate that $y(\cdot)$ defined by (2.4) is Lipschitz. It is also clear that $\dot{y}(s) \in G\big(y(s)\big)\dot{\phi}(s)$ for $s \in \cup_{i\in\mathcal{I}} I_i$, since this is Definition 2.2.3(iii). Recall that $y(s) = x\big(\phi_0(s)\big)$ for all other s. For almost all $s \in [0, S] \setminus \widetilde{\Gamma}$, we have

$$
\begin{aligned}
\dot{y}(s) &= \dot{x}\big(\phi_0(s)\big)\dot{\phi}_0(s) && \text{(Corollary 2.3.5(b))}\\
&\in F\big(x(\phi_0(s))\big)\dot{\phi}_0(s) + G\big(x(\phi_0(s))\big)\dot{u}\big(\phi_0(s)\big)\dot{\phi}_0(s) && \text{(Definition 2.2.3(a))}\\
&= F\big(y(s)\big)\dot{\phi}_0(s) + G\big(y(s)\big)\dot{\phi}(s), && \text{(Corollary 2.3.5(a))}
\end{aligned}
$$

which shows (2.5) holds for these s. The final case to consider is whenever $s \in \Gamma$, in which case we have

$$
\begin{aligned}
y_\sigma(s) &= x_\sigma\big(\phi_0(s)\big) && \text{(Lemma 2.3.4(c))}\\
&= \int_{[0,\phi_0(s)]} \gamma(t)\, d\mu_\sigma(t) && \text{(Definition 2.2.3(ii))}\\
&= \int_0^s \gamma\big(\phi_0(s')\big)\dot{\phi}(s')\, ds' && \text{(by (2.24)).}
\end{aligned}
$$

It follows that for almost all $s \in \Gamma$, we have $\dot{y}(s) = \dot{y}_\sigma(s) = \gamma\big(\phi_0(s)\big)\dot{\phi}(s)$. Since $\dot{\phi}_0(s) = 0$ and $\gamma\big(\phi_0(s)\big) \in G\big(y(s)\big)$ for almost $s \in \Gamma$, then (2.5) holds for these s as well. We conclude that (2.5) holds for almost all $s \in [0, S]$, and so X_μ is a B-R solution. □

2.5 Bouncing Ball Model

In the final section of this chapter we return to the bouncing ball model from Example 3. We will reformulate this model from the hybrid systems setting to the impulsive system setting given by Definition 2.2.3. Moreover, we will see that with an appropriately chosen measure μ and its graph completion, it is not difficult to continue trajectories after Zeno behavior. Suppose that the solution of system (3, 4) is known, T is the time when trajectory vanishes and jumps occur in times $\mathcal{T} := \{t_1, t_2, t_3, ..., t_j, ...\}$ (those are the times when $x_1(t_i) = 0$). We will show that the trajectory is also a solution (with an appropriate measure and graph completions attached to the measure and $x(\cdot)$) of the impulsive system

$$dx \in F(x)dt + G(x)d\mu, \tag{2.25}$$

where $F(x) = f(x)$ and

$$G(x) = \begin{bmatrix} 0 & 0 \\ 0 & \mu + 1 \end{bmatrix}.$$

Most importantly, this solution will not be difficult to continue after the Zeno behavior.

Take the measure whose distribution is

$$u(t) = \begin{cases} 0 & \text{when} \quad 0 \le t < t_1 \\ -\sum_{j=1}^{i} x_2(t_j-) & \text{when} \quad t_i \le t < t_{i+1}. \end{cases}$$

This is also the total variation of this measure since the measure is taking only positive real values. Note that $u(T)$ is not a finite number, so $\eta(t) = t + u(t)$ blows up as t goes to T. In this case, we can consider

$$\bar{\eta}(t) = \begin{cases} t & \text{when} \quad 0 \le t < t_1 \\ t + \sum_{j=1}^{i} \mu^j & \text{when} \quad t_i \le t < t_{i+1}. \end{cases}$$

Now, $S := \bar{\eta}(T) < \infty$. Define the graph completion $\phi_0 : [0, S] \to [0, T]$ to be the graph filled inverse of $\bar{\eta}(\cdot)$, and let $\phi : [0, S] \to I\!R$ be as follows

$$\phi(s) := \begin{cases} 0 & \text{on } [0, s_1^-], \\ -\frac{x_2(t_1-)}{\mu}(s - s_1^-) & \text{on } [s_1^-, s_1^+], \\ -\sum_{j=1}^{i-1} x_2(t_j-) & \text{on } [s_{i-1}^+, s_i^-], i \ge 2, \\ -\sum_{j=1}^{i-1} x_2(t_j-) - \frac{x_2(t_i-)}{\mu^i}(s - s_i^-) & \text{on } I_i, i \ge 2 \end{cases}$$

Note,

$$s_i^+ - s_i^- = \mu^i$$

and

$$\dot{\phi}_0(s) = \begin{cases} 0 & \text{on } I_i, \forall i \in \mathcal{I} \\ 1 & \text{otherwise,} \end{cases} \qquad \dot{\phi}(s) = \begin{cases} -\frac{x_2(t_i-)}{\mu^i} & \text{on } I_i, \forall i \in \mathcal{I} \\ 0 & \text{otherwise.} \end{cases}$$

Also, note that on $[0, S]$

$$u(\phi_0(s)) = \phi(s),$$

therefore the pair $(\phi_0, \phi)(\cdot)$ is indeed a graph completion for the measure generated by distribution $u(\cdot)$.

One inspects that $\dot{u} = 0$ and $\mu_\sigma = 0$ because μ is entirely contained in atoms. Therefore, inclusion (ii) of Definition 2.2.3 becomes obsolete in this case, inclusion (i) of Definition 2.2.3 becomes

$$\dot{x}(t) = F(x(t) = f(x(t)) \quad \text{a.e. } t \in [0, T],$$

(here the trajectory $x(\cdot)$ is the one that solves (3,4) and inclusion (iii) becomes

$$\dot{y}_i(s) = g\big(y_i(s)\big)\dot{\phi}(s) = \begin{cases} \left(0, -\frac{\mu+1}{\mu^i} x_2(t_i-)\right) & \text{a.e. } s \in I_i \\ (0,0) & \text{otherwise} \end{cases}.$$

After integrating, the velocity component of y_i at point s_i^+ becomes

$$x_2(t_i-) - (\mu+1)x_2(t_i-) = -\mu x_2(t_i-).$$

Therefore,

$$y_i(s_i^-) = (0, x_2(t_i-)), \qquad y_i(s_i^+) = (0, -\mu x_2(t_i+),$$

which confirms that

$$X_{\bar{\mu}} = \Big(x(\cdot), \big(\phi_0, \phi\big)(\cdot), \{y_i(\cdot)\}\Big).$$

is solution to (2.25). (Here we denoted by $\bar{\mu}$ the measure generated by $u(\cdot)$.) Once again, the advantage is that S is finite, which allows the continuation of this Zeno behavior if needed.

Chapter 3
Sampling Impulsive Systems

In this chapter we introduce a sampling method that is analogous to the classical Euler one-step method. The discretization essentially takes place in the reparameterized time space but is brought back to the original time; it is shown in Section 3.1 that a limit of the graphs of the sampled trajectories converge to the graph of a solution. The third goal is to approximate impulsive systems by ordinary ones, and relate the corresponding solutions. In the Section 3.2, we show that if absolutely continuous measures "graph converge" to a graph completion of μ, then corresponding trajectories converge, and moreover, under additional Lipschitz hypotheses of the data, all such solutions can be obtained this way. In the final section of this chapter, we introduce another sampling technique, in which the measure is not specified, but rather constructed along with solution. This sampling technique is used in the Chapter 4, as a tool in solving an invariance problem. Most of the results from Chapter 2 and Chapter 3 are gathered in the journal paper [32].

3.1 A Sampling Method

In this section, an Euler-type discretization procedure is introduced that produces approximate discrete solutions (called *sampled trajectories*) when the measure μ and a graph completion are given (see Section 1.3 for the non-impulsive case). The limit of a subsequence of approximations will be shown to graph-converge in the Hausdorff metric to some solution X_μ of (2.1). A sampling method that produces the measure and graph completion together, is presented in Section 3.2.

With X_μ as in (2.9), its graph is defined as the set

$$\text{gr } X_\mu := \{(t, x(t)) : t \in [0,T]\} \cup \{(t_i, y_i(s)) : s \in I_i,\ i \in \mathcal{I}\}.$$

The idea is to discretize the ordinary trajectory $y(\cdot)$ that is defined in (2.4), where the "compactness of trajectories" is known to hold, and to project it down into t-space.

Let N be a positive integer, and let $h := \frac{S}{N}$ be the step-size parameter. Let $s_0 = 0 = t_0$, and for each $j = 1, \ldots, N$, let $s_j = jh$, $t_j = \phi_0(s_j)$, and $\lambda_j = t_j - t_{j-1}$. Sampled points $\{x_j\}_{j=1}^N$ are defined and "velocity" data are selected as follows (the parameter N is suppressed in this notation):

$$
\begin{array}{lll}
x_0 = x_0 & f_0 \in F(x_0) & g_0 \in G(x_0) \\
x_1 = x_0 + \lambda_1 f_0 + g_0\big(\phi(s_1) - \phi(s_0)\big) & f_1 \in F(x_1) & g_1 \in G(x_1) \\
\vdots & \vdots & \vdots \\
x_{j+1} = x_j + \lambda_j f_j + g_j\big(\phi(s_j) - \phi(s_{j-1})\big) & f_{j+1} \in F(x_{j+1}) & g_{j+1} \in G(x_{j+1}) \\
\vdots & \vdots & \vdots \\
x_N = x_{N-1} + \lambda_N f_{N-1} + g_{N-1}\big(\phi(s_N) - \phi(s_{N-1})\big) & &
\end{array}
$$

We denote by Ω^N the graph of a sampled trajectory:

$$\Omega^N := \big\{(t_j, x_j) : j = 0, \ldots, N\big\}. \tag{3.1}$$

Recall that the Hausdorff distance $\text{dist}_\text{H}(A_1, A_2)$ between two compact subsets A_1, A_2 of $I\!R^n$ is defined by

$$\text{dist}_\text{H}(A_1,\ A_2) = \min\big\{\delta \geq 0 : A_1 \subseteq A_2 + \delta B_1 \text{ and } A_2 \subseteq A_1 + \delta B_1\big\},$$

and that any multifunction $M : I\!R^n \rightrightarrows I\!R^m$ with compact values is locally Lipschitz if for every bounded set $C \subset I\!R^n$, there exists a constant c so that

$$\text{dist}_\text{H}\big(M(x),\ M(y)\big) \leq c|x - y| \quad \text{for all } x,\ y \in C.$$

The main result of this section follows.

Theorem 3.1.1. *Suppose $\mu \in \mathcal{B}_K([0,T])$ and a graph completion $\phi(\cdot)$ are given.*

(a) For every sequence $\{\Omega^N\}_N$ of graphs of sampled trajectories, there is a solution X_μ of (2.1) and a subsequence $\{\Omega^{N_k}\}_k$ of $\{\Omega^N\}_N$ such that

$$dist_H\big(\Omega^{N_k},\ gr\ X_\mu\big) \to 0 \quad as \quad k \to \infty.$$

(b) Assume F and G are locally Lipschitz. For every solution X_μ of (2.1), there exists a sequence $\{\Omega^N\}_N$ of graphs of sampled trajectories so that

$$dist_H\big(\Omega^{N},\ gr\ X_\mu\big) \to 0 \quad as \quad N \to \infty.$$

Proof. Suppose the sequences $\{f_j\}$, $\{g_j\}$, $\{x_j\}$ are constructed by the sampling method described above. We first show there exists a constant c_1 independent of N so that

$$\max_j \big\{|x_j|,\ |f_j|,\ \|g_j\|\big\} \le c_1 \tag{3.2}$$

for all j and $N \in I\!N$. Indeed, with r as in (GC3), Definition 2.3.1 (which is the Lipschitz constant of $\phi(\cdot)$) and c as in (H2) and (H3), we have

$$\begin{aligned} |x_{j+1}| &\le |x_j| + h|f_j| + \|g_j\| r h \\ &\le |x_j| + \big[c(1+|x_j|) + c(1+|x_j|r)\big]h \\ &= h\alpha + \big[1 + h\alpha\big]|x_j|, \end{aligned}$$

where $\alpha := c(1+r)$. It follows from the discrete Gronwall inequality (Corollary 1.3.3) that

$$|x_j| \le e^{\alpha S}(1+|x_0|) - 1,$$

and that then (3.2) holds by (H2) and (H3) with $c_1 := c[e^{\alpha S}(1+|x_0|)]$.

Define the multifunction $M : [0, S] \times I\!R^n \rightrightarrows I\!R^n$ by

$$M(s, y) = F(y)\dot{\phi}_0(s) + G(y)\dot{\phi}(s), \tag{3.3}$$

which is $\mathcal{L} \times \mathcal{B}$ measurable, has nonempty compact convex values, and has linear growth. Moreover, $M(s, \cdot)$ has closed graph for almost all $s \in [0, S]$. For each $N \in \mathbb{N}$, let $\tilde{\Omega}^N$ be the sampled trajectory in s-time:

$$\tilde{\Omega}^N := \big\{(s_j, x_j) : j = 0, \ldots, N\big\}. \tag{3.4}$$

Also consider its related polygonal arc $y^N(\cdot)$ defined on $[0, S]$ given by

$$y^N(s) := x_j + \frac{s - s_j}{h}(x_{j+1} - x_j) \quad \text{for } s \in [s_j, s_{j+1}]. \tag{3.5}$$

Note for later use that

$$\mathrm{dist_H}\big(\tilde{\Omega}^N,\ \mathrm{gr}\ \ y^N(\cdot)\big) \leq \max\big\{h, c_1(1 + r)h\big\}. \tag{3.6}$$

We claim there exist the following sequences of

- positive numbers δ_N and r_N so that $\delta_N \to 0$ and $r_N \to 0$, and
- measurable sets $A_N \subseteq [0, S]$ so that $m(A_N) \to 0$

where the limits are as $N \to \infty$, and that satisfy

$$\inf\big\{|\dot{y}^N(s) - v| :\ v \in M\big(s, y^N(s) + \delta_N \overline{I\!B}\big)\big\}\ \leq\ r_N \quad \text{a.e. } s \in A_N. \tag{3.7}$$

To see this, let $\delta_N = \frac{S}{N}c_1(1 + r)$ where c_1 is as in (3.2). Note for each $j = 1, 2, \ldots, N - 1$ and $s \in [s_{j-1}, s_j]$ that

$$\begin{aligned} \big|y^N(s) - x_j\big| &\leq \big|x_{j+1} - x_j\big| \\ &= \big|\lambda_{j+1} f_j + g_j\big(\phi(s_{j+1}) - \phi(s_j)\big)\big| \\ &\leq h\big[|f_j| + \|g_j\|r\big] \\ &\leq \delta_N. \end{aligned}$$

Next, for $s \in [0, S-h]$, define

$$\Phi_0^N(s) := \frac{1}{h}\int_s^{s+h} \dot{\phi}_0(s')ds' \quad \text{and} \quad \Phi^N(s) := \frac{1}{h}\int_s^{s+h} \dot{\phi}(s')ds',$$

and recall that $\Phi_0^N(s) \to \dot{\phi}_0(s)$ and $\Phi_0^N(s) \to \dot{\phi}(s)$ for almost all $s \in [0, S]$ as $N \to \infty$. By Egoroff's Theorem, there exist measurable sets $A_N \subseteq [0, S]$ with $m(A_N) \to 0$ (and for notational simplicity, we may assume $[S-h, S] \subseteq A_N$) and satisfying

$$r_N := c_1 \max_{s \in [0,S]\setminus A_N} \Big\{ \big|\Phi_0^N(s) - \dot{\phi}_0(s)\big|, \ \big|\Phi^N(s) - \dot{\phi}(s)\big| \Big\} \to 0$$

as $N \to \infty$. Now let

$$v^N(s) := f_j \dot{\phi}_0(s) + g_j \dot{\phi}(s) \quad \text{for } s \in [s_j, s_{j+1}],$$

and note that $v^N(s) \in M(s, x_j)$ for almost all $s \in [s_j, s_{j+1}]$. Recall that $\dot{y}^N(s) = \Phi_0^N(s_j) f_j + g_j \Phi^N(s_j)$, and thus

$$\max_{s \in [0,S]\setminus A_N} \big|\dot{y}^N(s) - v^N(s)\big| \leq \max_{\substack{j=1,\ldots,N \\ s \in [s_j, s_{j+1}]\setminus A_N}} \Big| \big(\Phi_0^N(s) - \dot{\phi}_0(s)\big) f_j + g_j \big(\Phi^N(s) - \dot{\phi}(s)\big) \Big| \leq r_N.$$

We have shown that (3.7) holds.

From the compactness of trajectories theorem [7, Theorem 4.1.11], there exists a trajectory $y(\cdot)$ of M and a subsequence (we label as $\{y^{N_k}(\cdot)\}_k$) of $\{y^N(\cdot)\}_N$ so that $y^{N_k}(\cdot) \to y(\cdot)$ uniformly on $[0, S]$. One sees easily that this means

$$\mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\ y^{N_k}(\cdot),\ \mathrm{gr}\ y(\cdot)\big) \to 0 \tag{3.8}$$

as $k \to \infty$. We define the components of a solution X_μ to (2.1) as follows. Let $x(\cdot) : [0, T] \to \mathbb{R}^n$ be given by $x(t) = y\big(\eta(t)\big)$, and define the functions $y_i(\cdot)$ (for each $i \in \mathcal{I}$) as the restriction of $y(\cdot)$ to I_i.

Now recall Ω^N as in (3.1) and $\tilde{\Omega}^N$ as in (3.4), and observe the second coordinates are the same for each $j = 1, \ldots, N$. Similarly, the second coordinates of gr X_μ and gr $y(\cdot) := \big\{\big(s, y(s)\big) : \ s \in [0,S]\big\}$ are the same for each $t \notin \mathcal{T}$, $t = \phi_0(s)$; and when $t \in \mathcal{T}$, the set of projections onto the second coordinate are the same. Thus the difference between the Hausdorff distances of Ω^N and gr X_μ on the one hand, and $\tilde{\Omega}^N$ and gr $y(\cdot)$ on the other is affected by only the first coordinate. It follows that

$$\mathrm{dist}_{\mathrm{H}}\big(\Omega^N, \ \mathrm{gr}\ X_\mu\big) \le \mathrm{dist}_{\mathrm{H}}\big(\tilde{\Omega}^N, \ \mathrm{gr}\ y(\cdot)\big), \tag{3.9}$$

where the righthand is at most h larger than the left side. By the triangle inequality, one has

$$\mathrm{dist}_{\mathrm{H}}\big(\tilde{\Omega}^N, \ \mathrm{gr}\ y(\cdot)\big) \le \mathrm{dist}_{\mathrm{H}}\big(\tilde{\Omega}^N, \ \mathrm{gr}\ y^N(\cdot)\big) + \mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\ y^N(\cdot), \ \mathrm{gr}\ y(\cdot)\big) \tag{3.10}$$

Finally, passing to the subsequence $\{N_k\}$ and starting from (3.9), it follows from (3.10), (3.6), and (3.8) that

$$\mathrm{dist}_{\mathrm{H}}\big(\Omega^{N_k}, \ \mathrm{gr}\ X_\mu\big) \to 0$$

which finishes the proof of part (a).

To prove part (b), assume now that F and G are locally Lipschitz, and X_μ is as in (2.9) and is a solution of (2.1). Let $y(\cdot)$ be defined as in (2.4), and so there exist measurable selections $f(\cdot)$ and $g(\cdot)$ of $F\big(y(\cdot)\big)$ and $G\big(y(\cdot)\big)$ respectively so that

$$\dot{y}(s) = f(s)\dot{\phi}_0(s) + g(s)\dot{\phi}(s) \quad \text{a.e. } s \in [0,S].$$

In a manner similar to proving the discrete bound (3.2), one can show there exists a constant c_2 so that $|y(s)| \le c_2$. Indeed, having in mind the linear growth properties of $F(\cdot)$ and $G(\cdot)$ (see (H2) and (H3)), note that $f(s) \in F\big(y(s)\big)$ and $g(s) \in G\big(y(s)\big)$ imply

$$|f(s)| \le c\big(1 + |y(s)|\big), \qquad \text{and}$$

$$\|g(s)\| \le c\big(1 + |y(s)|\big), \qquad \text{on } [0,S].$$

Also, for almost all $s \in [0, S]$,

$$\begin{aligned} |\dot{y}(s)| &\leq |f(s)|\,|\dot{\phi}_0 s| + \|g(s)\|\,|\dot{\phi}(s)| \\ &\leq c\big(1 + |y(s)|\big) + cr\big(1 + |y(s)|\big) \\ &= c(1+r)\big(1 + |y(s)|\big) \\ &= c(1+r)|y(s)| + c(1+r). \end{aligned}$$

Now by Gronwall's inequality (Lemma 1.3.1),

$$\begin{aligned} |y(s)| &\leq e^{c(1+r)} + \int_0^s c(1+r)e^{c(1+r)(s-s')}ds' \\ &\leq e^{c(1+r)} + e^{Sc(1+r)}. \end{aligned}$$

Taking $c_2 := e^{c(1+r)} + e^{Sc(1+r)}$ one obtains $|y(s)| \leq c_2$ almost everywhere on $[0, S]$.

Observe that for $0 \leq \bar{s} < \hat{s} \leq S$, one has

$$\big|y(\hat{s}) - y(\bar{s})\big| \leq \int_{\bar{s}}^{\hat{s}} |\dot{y}(s)|\,ds \leq (1 + c_2)(1 + r)(\hat{s} - \bar{s}) =: c_3(\hat{s} - \bar{s}). \tag{3.11}$$

Let $L > 0$ be the Lipschitz constant for F and G on $c_2\overline{\mathbb{B}}$, and denote by $\mathrm{proj}_{F(y)}(f)$ the projection of f into $F(y)$ (which is unique since $F(y)$ is convex). If $|y_j| \leq c_2$ ($j = 1, 2$) and $f \in F(y_1)$, then $|f - \mathrm{proj}_{F(y_2)}(f)| \leq L|y_1 - y_2|$. Similar considerations hold with F replaced by G.

We use the notation of the sampling method, and will show there exists a sequence $\{\Omega^N\}$ that graph converges to gr X_μ.

Let $f_0 = \frac{1}{h}\int_0^{s_1} \mathrm{proj}_{F(x_0)}\big(f(s)\big)\,ds$, $g_0 = \frac{1}{h}\int_0^{s_1} \mathrm{proj}_{G(x_0)}\big(g(s)\big)\,ds$, and x_1 defined as in the sampling method. We observe

$$\begin{aligned}
x_1 - y(s_1) &= \frac{\phi_0(s_1)-\phi_0(0)}{h}\int_0^{s_1}\big[\mathrm{proj}_{F(x_0)}\big(f(s)\big) - f(s)\big]\,ds \\
&\quad + \int_0^{s_1}\big[\mathrm{proj}_{G(x_0)}\big(g(s)\big) - g(s)\big]\left(\frac{\phi(s_1)-\phi(0)}{h}\right)ds \\
&\quad + \int_0^{s_1}\left(\frac{\phi_0(s_1)-\phi_0(0)}{h} - \dot{\phi}_0(s)\right) f(s)\,ds \\
&\quad + \int_0^{s_1} g(s)\left(\frac{\phi(s_1)-\phi(0)}{h} - \dot{\phi}(s)\right)ds \\
&=: I + II + III + IV.
\end{aligned}$$

Recall $\phi_0(\cdot)$ is Lipschitz of rank 1, and so by the Lipschitz property of F, we have

$$|I| \le L\int_0^{s_1}\big|y(s) - x_0\big|\,ds \le Lc_3\int_0^{s_1} s\,ds = \frac{Lc_3}{2}h^2,$$

where the second inequality follows from (3.11). In the same way, one can show

$$|II| \le \frac{Lc_3 r}{2}h^2$$

since $\phi(\cdot)$ is Lipschitz of rank r. To estimate III and IV, we re-use earlier notation to redefine $\Phi^N(\cdot)$ on $[0,S]$ by setting

$$\Phi^N(s) := \max\left\{\left|\frac{\phi_0(s_{j+1})-\phi_0(s_j)}{h} - \dot{\phi}_0(s)\right|, \left|\frac{\phi(s_{j+1})-\phi(s_j)}{h} - \dot{\phi}(s)\right|\right\}$$

whenever $s \in [s_j, s_{j+1}]$. Then it follows that both $|III|$ and $|IV|$ are bounded above by $c(1+c_2)\int_0^{s_1}\Phi^N(s)\,ds$. Putting all this together, we have

$$|x_1 - y(s_1)| \le \frac{Lc_3(1+r)}{2}h^2 + 2c(1+c_2)\int_0^{s_1}\Phi^N(s)\,ds.$$

Inductively, one proceeds by setting

$$f_j = \frac{1}{h}\int_{s_j}^{s_{j+1}} \mathrm{proj}_{F(x_j)}\big(f(s)\big)\,ds \quad \text{and}$$

$$g_j = \frac{1}{h} \int_{s_j}^{s_{j+1}} \mathrm{proj}_{G(x_j)}\big(g(s)\big)\, ds,$$

and letting x_{j+1} be as in the sampling method construction. The same argument used above can operate at each iteration, and inductively, one has the following estimate:

$$|x_j - y(s_j)| \leq \frac{Lc_3(1+r)}{2} jh^2 + 2c(1+c_2) \int_0^{s_j} \Phi^N(s)\, ds.$$

Since $\Phi^N(s)$ is bounded above and converges to 0 almost everywhere, it follows that $\tilde{\Omega}^N := \{(s_j, x_j) : j = 1, \ldots, N\}$ satisfies $\mathrm{dist}_{\mathrm{H}}\big(\tilde{\Omega}^N, \mathrm{gr}\ y(\cdot)\big) \to 0$ as $N \to \infty$. The bound in (3.9) is still valid here, and the conclusion of (b) readily follows. □

3.2 Approximate Controls

The original and perhaps most natural approach to defining solutions to the impulsive inclusion (2.1) is to consider limits of a sequence of solutions $x^N(\cdot)$ of an *approximate* control problem of the form

$$\dot{x}^N(t) \in F\big(x(t)\big)\dot{\phi}_0(t) + G\big(x(t)\big)\dot{u}^N(t), \tag{3.12}$$

where $d\mu^N = \dot{u}^N(\cdot)dt$ are absolutely continuous measures that approximate μ in some sense. See, for example, the discussion in [3, 16]. We introduce in this section a concept of "graph convergence" of measures that is appropriate to carry out such an analysis. Graph convergence as defined below is perhaps considerably stronger than would be desirable, but we mention that even when the solutions of (3.12) are unique, (which happens, for example, in the singleton case $F(x) = \{f(x)\}$ and $G(x) = \{g(x)\}$ with $f(\cdot)$ and $g(\cdot)$ smooth functions), the limit arc may not be unique if the measures converge in some weaker sense.

Suppose we are given the following: a measure $\mu \in \mathcal{B}_K([0,T])$, an associated graph completion $\phi(\cdot) : [0,S] \to I\!R^n$ that is Lipschitz of rank r, and a sequence

$\{\mu^N\}$ of absolutely continuous Borel measures belonging to $B_K([0,T])$ whose associated distribution functions $u^N(t) := \mu^N\big([0,t]\big)$ are Lipschitz.

Definition 3.2.1. The sequence $\{\mu^N\}_N$ of absolutely continuous measures *graph-converges* to (μ, ϕ) provided

(i) there exist numbers $S^N > 0$ such that $S^N \to S$;

(ii) for each N, there exists a strictly increasing function $\phi_0^N(\cdot) : [0, S^N] \to [0,T]$ that is onto and Lipschitz of rank at most one, and such that

$$\int_0^{\min\{S,S^N\}} \big|\dot{\phi}_0^N(s) - \dot{\phi}_0(s)\big|\, ds \to 0 \quad \text{as} \quad N \to \infty; \qquad \text{and}$$

(iii) for each N, the sequence of functions defined by $\phi^N(s) := \big(u^N \circ \phi_0^N\big)(s)$ are Lipschitz with $\limsup_{N\to\infty} \|\dot{\phi}^N(\cdot)\|_\infty \le r$, and satisfy

$$\int_0^{\min\{S,S^N\}} \big|\dot{\phi}^N(s) - \dot{\phi}(s)\big|\, ds \to 0 \quad \text{as} \quad N \to \infty.$$

The main result in this section follows.

Theorem 3.2.1. *Suppose the measure $\mu \in \mathcal{B}_K([0,T])$ and an associated graph completion $\phi(\cdot) : [0,S] \to I\!R^n$ are given.*

(a) Suppose $\{\mu^N\}$ is a sequence of absolutely continuous measures that graph-converges to $\big(\mu, \phi(\cdot)\big)$, and $\{x^N(\cdot)\}$ is a sequence of absolutely continuous arcs satisfying

$$\dot{x}^N(t) \in F\big(x^N(t)\big) + G\big(x^N(t)\big)\dot{u}^N(t). \tag{3.13}$$

Then there exists a solution X_μ of (2.1) and a subsequence $\{x^{N_k}(\cdot)\}$ of $\{x^N(\cdot)\}$ such that

$$dist_H\big(gr\ x^{N_k}(\cdot), gr\ X_\mu\big) \to 0 \quad as \quad k \to \infty.$$

(b) Conversely, suppose F and G are locally Lipschitz multifunctions and $X_\mu := \big(x(\cdot), \phi(\cdot), \{y_i(\cdot)\}_{i\in\mathcal{I}}\big)$ is a solution of (2.1). Then there is a sequence $\{\mu^N\}$ of absolutely continuous measures that graph converge to $\big(\mu, \phi(\cdot)\big)$, and a sequence $x^N(\cdot)$ of solutions to (3.13) so that

$$dist_H\big(gr\ x^N(\cdot), gr\ X_\mu\big) \to 0 \quad as \quad N \to \infty.$$

Proof. Suppose we are given the measures $d\mu^N = \dot{u}^N(t)dt$, the functions $\phi_0^N(\cdot)$ and $\phi^N(\cdot)$ satisfying Definition 3.2.1, and solutions $x^N(\cdot)$ of (3.13). Set $\bar{S}^N := \min\{S, S^N\}$. Let $y^N(s) = (x^N \circ \phi_0^N)(s)$, which for almost all $s \in [0, \bar{S}^N]$ satisfies

$$\begin{aligned} \dot{y}^N(s) &= \dot{x}^N\big(\phi_0^N(s)\big)\dot{\phi}_0^N(s) \\ &\in F\big(y^N(s)\big)\dot{\phi}_0^N(s) + G\big(y^N(s)\big)\dot{u}^N\big(\phi_0^N(s)\big)\dot{\phi}_0^N(s) \\ &= F\big(y^N(s)\big)\dot{\phi}_0^N(s) + G\big(y^N(s)\big)\dot{\phi}^N(s), \end{aligned}$$

where the last equality follows since $\dot{\phi}^N(s) = \dot{u}^N\big(\phi_0^N(s)\big)\dot{\phi}_0^N(s)$ almost everywhere. It follows that there exist measurable selections $f^N(s) \in F\big(y^N(s)\big)$ and $g^N(s) \in G\big(y^N(s)\big)$ so that

$$\dot{y}^N(s) = f^N(s)\dot{\phi}_0^N(s) + g^N(s)\dot{\phi}^N(s).$$

Recall Definition 3.2.1 imposes a priori bounds on the Lipschitz rank of $\phi_0^N(\cdot)$ and $\phi^N(\cdot)$, and that $F(\cdot)$ and $G(\cdot)$ satisfy linear growth assumptions. A standard argument involving Gronwall's inequality implies there exists a constant c_4 independent of N that is an upper bound of both $\|f^N(\cdot)\|_\infty$ and $\|g^N(\cdot)\|_\infty$.

Let $M : [0, S] \times I\!R^n \rightrightarrows I\!R^n$ be defined as in (3.3), and define $\dot{z}^N(\cdot) : [0, \bar{S}^N] \to I\!R^n$ by

$$\dot{z}^N(s) := f^N(s)\dot{\phi}_0(s) + g^N(s)\dot{\phi}(s),$$

and define $z^N(\cdot) : [0, \bar{S}^N] \to I\!R^n$ by $z^N(s) := x_0 + \int_0^s \dot{z}^N(s')\, ds'$. It is clear from the definitions that

$$\dot{z}^N(s) \in M\big(s, y^N(s)\big) \quad \text{a.e. } s \in [0, \bar{S}^N]. \tag{3.14}$$

Furthermore, it is readily seen that

$$\sup_{s \in [0, \bar{S}^N]} \big|z^N(s) - y^N(s)\big| \leq c_4 \big\{ \|\dot{\phi}_0^N - \dot{\phi}_0\|_1 + \|\dot{\phi}^N - \dot{\phi}\|_1 \big\}$$

which implies via the assumption of the graph convergence of the measures that $y^N - z^N$ approaches zero uniformly. In view of (3.14) and the compactness of trajectories theorem [7, Theorem 4.1.11], there exists $y(\cdot) : [0, S] \to I\!R^n$ that is a trajectory of M and to which a subsequence of $\{z^N(\cdot)\}$, and hence also of $\{y^N(\cdot)\}$, converges uniformly. That is, there exists a subsequence N_k for which

$$\text{dist}_{\text{H}}\big(\text{gr } y^{N_k}(\cdot), \text{gr } y(\cdot)\big) \to 0 \quad \text{as } k \to \infty. \tag{3.15}$$

We now define X_μ as before - see the paragraph containing (3.8) in the previous section. Similar reasoning as employed there shows also that $\text{dist}_{\text{H}}\big(\text{gr } x^{N_k}(\cdot), \text{gr } X_\mu\big)$ is bounded above by

$$\text{dist}_{\text{H}}\big(\text{gr } y^{N_k}(\cdot), \text{gr } y(\cdot)\big) + \sup_{s \in [0, \bar{S}^{N_k}]} |\phi_0^{N_k}(s) - \phi_0(s)|$$

which goes to zero as $k \to \infty$ by (3.15) and the assumption contained in Definition 3.2.1(ii). This finishes the proof of part (a).

We turn to part (b). Suppose F and G are now locally Lipschitz and X_μ is a solution to (2.1). For $N = 1, \ldots,$ we proceed to construct the absolutely continuous measures μ^N and solutions $x^N(\cdot)$ of (3.13) that will converge in graph to X_μ. Fix $N > 0$ and set $h = \frac{S}{N}$, and for $j = 1, \ldots, N$, set $s_j = jh$ and $t_j = \phi_0(s_j)$. We will first introduce a new partition $\{\bar{t}_j\}$ of $[0, T]$ consisting of N distinct points that resembles the partition $\{t_j\}$ but has repeated nodes "pulled apart" and indexed accordingly. To this end, let $\mathcal{J}_0^N$ be those indices j for which $t_{j-1} < t_j < t_{j+1}$ (to treat the endpoints, by convention, we take $t_{-1} < t_0$ and $t_{N+1} > t_N$; thus $t_0 \in \mathcal{J}_0^N$ if $t_0 < t_1$ and $t_N \in \mathcal{J}_0^N$ if $t_{N-1} < t_N$). We set $\bar{t}_j = t_j$ whenever $j \in \mathcal{J}_0^N$. Let $\mathcal{J}^N$ be those indices j for which $t_{j-1} < t_j = t_{j+1}$ (by convention, then, $t_0 \in \mathcal{J}^N$ if $t_0 = t_1$ and t_N cannot belong to $\mathcal{J}^N$). For these latter j, let $k_j \geq 1$ be such that $t_j = t_{j+1} = \cdots = t_{j+k_j} < t_{j+k_j+1}$, and

$$\lambda_j := \frac{1}{2} \min \left\{ h^2, t_j - t_{j-1}, t_{j+k_j+1} - t_j \right\}.$$

(if $0 \in \mathcal{J}^N$, then $\lambda_0 := \min\{h^2, \frac{t_{j+k_j+1} - t_j}{2}\}$). If $j \notin \mathcal{J}_0^N$, then $j = \bar{j} + k$ where there exists precisely one pair $(\bar{j}, k)$ with $\bar{j} \in \mathcal{J}^N$ and $0 \leq k \leq k_{\bar{j}}$. In this case $\bar{t}_j$ is defined by

$$\bar{t}_j := \begin{cases} t_j + \left[\frac{2k}{k_j} - 1 \right] \lambda_j & \text{if } j \neq 0 \\ \frac{k}{k_0} \lambda_0 & \text{if } j = 0. \end{cases}$$

Thus a new partition $\{\bar{t}_j\}$ of $[0, T]$ has been constructed consisting of N distinct points, and which satisfy

$$|\bar{t}_j - t_j| \leq h^2 \quad \text{for all } j. \tag{3.16}$$

Next, we define $\phi_0^N(\cdot) : [0, S] \to [0, T]$ by

$$\phi_0^N(s) = \bar{t}_j + \frac{s - s_j}{h} (\bar{t}_{j+1} - \bar{t}_j) \quad \text{whenever } s \in [s_j, s_{j+1}]$$

which is onto and Lipschitz of rank at most 1. We claim that $\dot{\phi}_0^N(\cdot)$ converges to $\dot{\phi}_0(\cdot)$ in $L^1[0, S]$. Indeed, let $\tilde{\phi}_0^N(\cdot) : [0, S] \to [0, T]$ be given by

$$\tilde{\phi}_0^N(s) = t_j + \frac{s - s_j}{h} (t_{j+1} - t_j) \quad \text{whenever } s \in [s_j, s_{j+1}].$$

The difference between the linear interpolations $\phi_0^N(\cdot)$ and $\tilde{\phi}_0^N(\cdot)$ is that $\phi_0^N(\cdot)$ maps s_j to $\bar{t}_j$, whereas $\tilde{\phi}_0^N(\cdot)$ maps s_j to t_j. For $s \in [s_j, s_{j+1}]$, we have

$$\left| \dot{\phi}_0^N(s) - \dot{\tilde{\phi}}_0^N(s) \right| = \frac{1}{h} |\bar{t}_{j+1} - \bar{t}_j - t_{j+1} + t_j| \leq 2h, \tag{3.17}$$

where the inequality is justified by (3.16). The Lebesgue differentiation Theorem says that $\dot{\tilde{\phi}}_0^N(s) \to \dot{\phi}_0(s)$ as $N \to \infty$ for almost all $s \in [0, S]$, and since these functions are bounded above by 1, the Dominated Convergence Theorem implies that $\dot{\tilde{\phi}}_0^N(\cdot) \to \dot{\phi}_0(\cdot)$ in $L^1[0, S]$. It follows from this and (3.17) that $\dot{\phi}_0^N(\cdot) \to \dot{\phi}_0(\cdot)$ in $L^1[0, S]$ as claimed.

Now define $u^N(\cdot) : [0, T] \to I\!R^n$ as the piecewise linear interpolation satisfying $u^N(\bar{t}_j) = \phi(s_j)$; that is,

$$u^N(t) = \phi(s_j) + \frac{t - \bar{t}_j}{\bar{t}_{j+1} - \bar{t}_j}\big(\phi(s_{j+1}) - \phi(s_j)\big) \quad \text{whenever } t \in [\bar{t}_j, \bar{t}_{j+1}],$$

Let $\phi^N(\cdot) := (u^N \circ \phi_0^N)(\cdot)$, and note $\phi^N(s_j) = \phi(s_j)$ for all j and for $s \in [s_j, s_{j+1}]$ that

$$\dot{\phi}^N(s) = \dot{u}^N\big(\phi_0^N(s)\big)\dot{\phi}_0^N(s) = \frac{\phi(s_{j+1}) - \phi(s_j)}{\bar{t}_{j+1} - \bar{t}_j}\,\frac{\bar{t}_{j+1} - \bar{t}_j}{h} = \frac{\phi(s_{j+1}) - \phi(s_j)}{h}.$$

Since $\phi(\cdot)$ is Lipschitz of rank r, it follows that each of $\phi^N(\cdot)$ are also of rank at most r. Completely analogous to the proof above showing $\dot{\phi}_0^N(\cdot) \to \dot{\phi}_0(\cdot)$ in $L^1[0, S]$ as $N \to \infty$, one has that $\dot{\phi}^N(\cdot) \to \dot{\phi}(\cdot)$ in $L^1[0, S]$ as $N \to \infty$. Therefore, with μ^N the absolutely continuous measure satisfying $d\mu^N = \dot{u}^N(t)dt$, we have shown that μ^N graph converges to $(\mu, \phi(\cdot))$ as $N \to \infty$ (where $S^N = S$ for all N in Definition 3.2.1).

We now turn to approximating a given a solution X_μ by a solution of (3.13). By Theorem 3.1.1(b), there exists a sequence of sampled trajectories whose graphs converge to gr X_μ. Denote these graphs by

$$\Omega^N := \{(t_j, x_j) \mid j = 1, ..., N\},$$

where $x_{j+1} = x_j + (t_{j+1} - t_j)f_j + (g_j)\big(\phi(s_{j+1}) - \phi(s_j)\big)$, $f_j \in F(x_j)$, and $g_j \in G(x_j)$, and they satisfy

$$\text{dist}_{\text{H}}\big(\Omega^N,\ \text{gr}\ X_\mu\big) \to 0 \quad \text{as} \quad N \to \infty. \tag{3.18}$$

For simplicity of notation, the dependence of x_j, f_j, and g_j on N has been suppressed. A new sampled set of points $\{\bar{x}_j\}$ is defined by replacing the partition $\{t_j\}$ by $\{\bar{t}_j\}$ and "tracking" the given sampled data. This is done as follows. Let $\bar{f}_0 = f_0$ and $\bar{g}_0 = g_0$ and define

$$\bar{x}_1 = \bar{x}_0 + (\bar{t}_1 - \bar{t}_0)\bar{f}_0 + \big(\bar{g}_0\big)\big(\phi(s_1) - \phi(s_0)\big)$$

Having chosen the data at stage $j-i$, inductively let $\bar{f}_j \in F(\bar{x}_j)$ and $\bar{g}_j \in G(\bar{x}_j)$ be the projections of f_j and g_j onto $F(\bar{x}_j)$ and $G(\bar{x}_j)$, respectively. That is, $\bar{f}_j \in F(\bar{x}_j)$ and satisfies

$$|\bar{f}_j - f_j| = \inf_{f \in F(\bar{x}_j)} |f - f_j|,$$

and similarly for $\bar{g}_j$. Define the next node by

$$\bar{x}_{j+1} = \bar{x}_j + (\bar{t}_{j+1} - \bar{t}_j)\bar{f}_j + \big(\bar{g}_j\big)\big(\phi(s_{j+1}) - \phi(s_j)\big).$$

The linear growth assumptions on F and G guarantee that all of the sampled data remains in a bounded set, and let c_1 be as in (3.2) but which also bounds the newly sampled data. With L a Lipschitz constant for both F and G on $c_1 B_1$, one has

$$|\bar{f}_j - f_j| \leq L|\bar{x}_j - x_j|, \quad \text{and} \quad \|\bar{g}_j - g_j\| \leq L|\bar{x}_j - x_j|. \tag{3.19}$$

The estimate between the nodes x_j and $\bar{x}_j$ is calculated by

$$\begin{aligned} \big|\bar{x}_{j+1} - x_{j+1}\big| &\leq |\bar{x}_j - x_j| + |t_{j+1} - t_j - \bar{t}_{j+1} + \bar{t}_j|\,|f_j| \\ &\quad + |\bar{t}_{j+1} - \bar{t}_j|\,|\bar{f}_j - f_j| + \|\bar{g}_j - g_j\|\,|\phi(s_{j+1}) - \phi(s_j)| \\ &\leq |\bar{x}_j - x_j| + 2h^2 c_1 + hL|\bar{x}_j - x_j| + hLr|\bar{x}_j - x_j| \\ &= 2h^2 c_1 + (1 + hL + hLr)|\bar{x}_j - x_j|, \end{aligned}$$

where (3.16), (3.19), and that $\phi(\cdot)$ is Lipschitz of rank r were invoked to deduce the second inequality. Gronwall's inequality implies

$$|\bar{x}_j - x_j| \leq 2hc_1 \frac{e^{LS(1+r)} - 1}{L(1+r)}$$

for each $j = 0, 1, \ldots, N$, and in particular implies that

$$\text{dist}_{\text{H}}(\Omega^N, \bar{\Omega}^N) \to 0, \text{ as } N \to \infty \tag{3.20}$$

where $\bar{\Omega}^N$ is the newly sampled graph:

$$\bar{\Omega}^N := \{(\bar{t}_j, \bar{x}_j) \mid j = 1, ..., N\}.$$

Next, let $\bar{x}^N(\cdot)$ be the piecewise linear arc interpolating the points in $\bar{\Omega}^N(\cdot)$, which specifically means

$$\begin{aligned} \bar{x}^N(t) &= \bar{x}_j + (t - \bar{t}_j)\bar{f}_j + (t - \bar{t}_j)\bar{g}_j \frac{\phi(s_{j+1}) - \phi(s_j)}{\bar{t}_{j+1} - \bar{t}_j} \quad \text{and} \\ \dot{\bar{x}}^N(t) &= \bar{f}_j + \bar{g}_j \dot{u}^N(t) \in F(\bar{x}_j) + G(\bar{x}_j)\dot{u}^N(t) \quad \text{whenever } t \in (\bar{t}_j, \bar{t}_{j+1}). \end{aligned} \tag{3.21}$$

Let $\Gamma^N(\cdot) : [0, T] \times I\!R^n \rightrightarrows I\!R^n$ be given by $\Gamma^N(t, x) := F(x) + G(x)\dot{u}^N(t)$, which is the multifunction appearing in (3.13). It has convex compact values, is measurably Lipschitz (see [9]), and has linear growth in x. We will find a trajectory $x^N(\cdot)$ of Γ^N that is close to $\bar{x}^N(\cdot)$. Following the notation in [9], we have

$$\begin{aligned} \rho_\Gamma\big(\bar{x}^N(\cdot)\big) &:= \int_0^T \text{dist}\Big(\dot{\bar{x}}^N(t), \Gamma^N\big(t, \bar{x}^N(t)\big)\Big)\, dt \\ &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \text{dist}\Big(\dot{\bar{x}}^N(t), \Gamma^N\big(t, \bar{x}^N(t)\big)\Big)\, dt \\ &\le \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \text{dist}_{\text{H}}\Big(\Gamma^N\big(t, \bar{x}_j\big), \Gamma^N\big(t, \bar{x}^N(t)\big)\Big)\, dt \\ &\le L \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \big(1 + |\dot{u}^N(t)|\big)\big|\bar{x}^N(t) - \bar{x}_j\big|\, dt, \end{aligned} \tag{3.22}$$

where (3.21) was used in the first inequality, and the Lipschitz property of F and G in the second. For $t \in [\bar{t}_j, \bar{t}_{j+1}]$, one has

$$\begin{aligned} \big|\dot{\bar{x}}^N(t) - \bar{x}_j\big| &\le \frac{t - \bar{t}_j}{\bar{t}_{j+1} - \bar{t}_j}\big|\bar{x}_{j+1} - \bar{x}_j\big| \\ &\le \frac{t - \bar{t}_j}{\bar{t}_{j+1} - \bar{t}_j}\big[(\bar{t}_{j+1} - \bar{t}_j)|\bar{f}_j| + \|\bar{g}_j\|\,|\phi(s_{j+1}) - \phi(s_j)| \\ &\le c_1 \left[1 + \frac{rh}{\bar{t}_{j+1} - \bar{t}_j}\right](t - \bar{t}_j) \end{aligned}$$

and

$$|\dot{u}^N(t)| = \left|\frac{\phi(s_{j+1}) - \phi(s_j)}{\bar{t}_{j+1} - \bar{t}_j}\right| \le \frac{rh}{\bar{t}_{j+1} - \bar{t}_j}.$$

We thus have

$$\begin{aligned}
&\int_{t_j}^{t_{j+1}} \big(1 + |\dot{u}^N(t)|\big)\big|\bar{x}^N(t) - \bar{x}_j\big|\,dt \\
&\qquad\qquad \le \left[1 + \frac{rh}{\bar{t}_{j+1} - \bar{t}_j}\right] c_1 \left[1 + \frac{rh}{\bar{t}_{j+1} - \bar{t}_j}\right] \int_{t_j}^{t_{j+1}} (t - \bar{t}_j)\,dt \\
&\qquad\qquad = c_1 \left[1 + \frac{rh}{\bar{t}_{j+1} - \bar{t}_j}\right]^2 \frac{(\bar{t}_{j+1} - \bar{t}_j)^2}{2} \\
&\qquad\qquad \le c_6 h^2
\end{aligned}$$

for some constant c_6. Combining with (3.22), this estimate yields that

$$\rho_\Gamma\big(\bar{x}^N(\cdot)\big) \le LSc_6 h,$$

and so by Filippov's Theorem (see [9], Theorem 3.1.6. page 115), for each N there exists a trajectory $x^N(\cdot)$ of Γ^N such that $x^N(0) = x_0$ and for which

$$\mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\ x^N(\cdot), \mathrm{gr}\ \bar{x}^N(\cdot)\big) \to 0 \tag{3.23}$$

as $N \to \infty$. Finally, we have by the triangular inequality

$$\begin{aligned}
\mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\ x^N(\cdot), \mathrm{gr}\ X_\mu\big) \ \le\ & \mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\ x^N(\cdot), \mathrm{gr}\ \bar{x}^N(\cdot)\big) + \mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\ \bar{x}^N(\cdot), \bar{\Omega}^N\big) \\
& + \mathrm{dist}_{\mathrm{H}}\big(\bar{\Omega}^N, \Omega^N\big) + \mathrm{dist}_{\mathrm{H}}\big(\Omega^N, \mathrm{gr}\ X_\mu\big)
\end{aligned}$$

which approaches 0 as $N \to \infty$ by (3.23), (3.20), and (3.18). This finishes the proof. □

3.3 Constructing Measure Via Sampling

In this section another Euler-type discretization procedure is introduced. This sampling technique produces approximate discrete solutions along with the measure μ

and a graph completion, given only a positive number S, multifunctions $F(\cdot)$ and $G(\cdot)$ satisfying the standing hypotheses and a closed cone $K \subset I\!R^m$. The limit of a subsequence of approximations proves to graph-converge to a solution X_μ of (2.1). Like in section (3.1), the idea is to discretize the ordinary trajectory $y(\cdot)$ that is defined in (2.4). In the reparameterized time, the "compactness of trajectories" is known to hold, and the result is obtained be projecting it down into original time. This sampling technique will be used in the following section to prove the weak invariance result.

Again, let $N > 0$ be an integer and let $h := \frac{S}{N}$ be the step size. Let $s_0 = 0$ and for each $j = 1, ..., N$, let $s_j = jh$. Let us now define the sampled points $\{x_j\}_{j=1}^N$ as follow:

$$
\begin{array}{lllll}
x_0 := x_0 & \lambda_0 \in [0,1] & f_0 \in F(x_0) & k_0 \in K_1 & g_0 \in G(x_0) \\
 & & & & \\
x_1 := x_0 + \lambda_0 h f_0 + (1-\lambda_0) h g_0 k_0 & & & & \\
 & \lambda_1 \in [0,1] & f_1 \in F(x_1) & k_1 \in K_1 & g_1 \in G(x_1) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_{j+1} := x_j + \lambda_j h f_j + (1-\lambda_j) h g_j k_j & & & & \\
 & \lambda_{j+1} \in [0,1] & f_{j+1} \in F(x_{j+1}) & k_{j+1} \in K_1 & g_{j+1} \in G(x_{j+1}) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_N := x_{N-1} + \lambda_{N-1} h f_{N-1} + (1-\lambda_{N-1}) h g_{N-1} k_{N-1}, & & & &
\end{array}
$$

where $K_1 = K \cap S_1$.

We reuse the notation of Ω^N, this time as a graph of a sampled trajectory where the first component takes values in the set $\{s_0, s_1, ...\}$:

$$\Omega^N := \big\{(s_j, x_j) : j = 0, \ldots, N\big\}. \tag{3.24}$$

We will show not that the given scheme converges to a solution.

Theorem 3.3.1. *Suppose that $S > 0$ is given. For every sequence $\{\Omega^N\}_N$ of graphs of sampled trajectories, there is a measure $\mu \in \mathcal{B}_K([0,T]$, solution X_μ of (2.1) and a subsequence $\{\Omega^{N_k}\}_k$ of $\{\Omega^N\}_N$ such that*

$$dist_H\big(\Omega^{N_k},\ gr\ y\big) \to 0 \quad as \quad k \to \infty,$$

where $y(\cdot)$ is defined as in (2.4).

Proof. We first show part (a). Suppose the sequences $\{f_j\}$, $\{g_j\}$, $\{x_j\}$ are constructed by the sampling method described above. Similar to the procedure that lead to the bound 3.2 in Theorem 3.1.1, we also find a bound for the sequences involved in this theorem. There exists a constant c_1 independent of N so that

$$\max_j\big\{|x_j|,\ |f_j|,\ \|g_j\|\big\} \le c_1 \tag{3.25}$$

for all j and $N \in I\!N$. Indeed, with c as in (H2) and (H3), we have

$$\begin{aligned} |x_{j+1}| &\le |x_j| + h|f_j| + \|g_j\|h \\ &\le |x_j| + 2c(1+|x_j|)h \\ &= h\alpha + \big[1+h\alpha\big]|x_j|, \end{aligned}$$

where $\alpha := 2c$. It follows from the discrete Gronwall inequality that

$$|x_j| \le e^{\alpha S}(1+|x_0|) - 1,$$

and that then (3.25) holds by (H2) and (H3) with $c_1 := c[e^{\alpha S}(1+|x_0|)]$.

Define $\lambda^N(\cdot)$ and $k^N(\cdot)$ on $[0,S]$ so that

$$\lambda^N(s) := \lambda_j \quad \text{and} \quad k^N(s) := k_j \quad \text{on } [s_j, s_{j+1}].$$

Let $\lambda(\cdot)$ be any uniform limit of $\{\lambda^N(\cdot)\}_N$ and let $k(\cdot)$ be any uniform limit of $\{k^N(\cdot)\}_N$. These limits exist because both the sequence $\{\lambda^N(\cdot)\}_N$ and the sequence

$\{|k^N(\cdot)|\}_N$ are uniformly by 1. Define the multifunction $M : [0,S] \times I\!\!R^n \rightrightarrows I\!\!R^n$ by

$$M(s,y) := F(y)\lambda(s) + \big(1-\lambda(s)\big)G(y)k(s), \tag{3.26}$$

which is $\mathcal{L} \times \mathcal{B}$ measurable, has nonempty compact convex values, and has linear growth. Moreover, $M(s,\cdot)$ has closed graph for almost all $s \in [0,S]$. For each $N \in I\!\!N$, consider a polygonal arc $y^N(\cdot)$ defined on $[0,S]$ related to Ω^N given by

$$y^N(s) := x_j + \frac{s-s_j}{h}(x_{j+1}-x_j) \quad \text{for } s \in [s_j, s_{j+1}]. \tag{3.27}$$

Note for later use that

$$\mathrm{dist_H}\big(\Omega^N,\ \mathrm{gr}\ y^N(\cdot)\big) \le \max\big\{h, 2c_1 h\big\}. \tag{3.28}$$

We claim there exist the sequences of positive numbers δ_N and r_N so that $\delta_N \to 0$ and $r_N \to 0$ where the limits are as $N \to \infty$, that satisfy

$$\inf\big\{|\dot{y}^N(s) - v| :\ v \in M\big(s, y^N(s) + \delta_N B_1\big)\big\} \ \le\ r_N \quad \text{a.e. } s \in [0,S]. \tag{3.29}$$

To see this, let $\delta_N = 2\frac{S}{N}c_1$ where c_1 is as in (3.25). Note for each $j = 1, 2, \ldots, N-1$ and $s \in [s_{j-1}, s_j]$ that

$$\begin{aligned} \big|y^N(s) - x_j\big| &\le \big|x_{j+1} - x_j\big| \\ &= \big|\lambda_j h f_j + (1-\lambda_j) h g_j k_j\big| \\ &\le h\big[|f_j| + \|g_j\|\big] \\ &\le \delta_N. \end{aligned}$$

By definition of $\lambda(\cdot)$ and $k(\cdot)$,

$$r_N := c_1 \max_{s\in[0,S]} \Big\{ \big|\lambda^N(s) - \lambda(s)\big|, \big|\big(1-\lambda^N(s)\big)k^N(s) - \big(1-\lambda(s)\big)k(s)\big| \Big\} \to 0.$$

as $N \to \infty$. Now let

$$v^N(s) := \lambda(s) f_j + \big(1-\lambda(s)\big) g_j k(s).$$

and note that $v^N(s) \in M(s, x_j)$ for almost all $s \in [s_j, s_{j+1}]$. Recall that for all $s \in [s_j, s_{j+1}]$,

$$\dot{y}^N(s) = \lambda_j f_j + (1 - \lambda_j) g_j k_j = \lambda^N(s) f_j + (1 - \lambda^N(s)) g_j k^N(s),$$

and thus

$$\begin{aligned}
&\max_{s\in[0,S]} \left|\dot{y}^N(s) - v^N(s)\right| \\
&\quad \leq \max_{\substack{j=1,\dots,N \\ s\in[s_j,s_{j+1}]}} \left|(\lambda(s) - \lambda^N(s)) f_j + g_j \left(\big(1 - \lambda(s)\big) k(s) - \big(1 - \lambda^N(s)\big) k^N(s) \right)\right| \\
&\quad \leq r_N.
\end{aligned}$$

We have shown that (3.29) holds.

From the compactness of trajectories theorem [7, Theorem 4.1.11], there exists a trajectory $y(\cdot)$ of M and a subsequence (we label as $\{y^{N_k}(\cdot)\}_k$) of $\{y^N(\cdot)\}_N$ so that $y^{N_k}(\cdot) \to y(\cdot)$ uniformly on $[0, S]$. One sees easily that this means

$$\mathrm{dist_H}\big(\mathrm{gr}\ y^{N_k}(\cdot),\ \mathrm{gr}\ y(\cdot)\big) \to 0 \tag{3.30}$$

as $k \to \infty$.

Note that $\lambda(s) \in [0, 1]$ and $k(s) \in K_1$. Define pair $(\psi_0, \psi)(s) : [0, S] \to [0, T] \times I\!R^m$ as

$$\psi_0(s) := \int_0^s \lambda(s')ds', \qquad \psi(s) := \int_0^s k(s')(1 - \lambda(s'))ds'.$$

and functions $\bar{\eta} : [0, T] \to [0, S]$ and $u : [0, T] \to I\!R^m$ as

$$\bar{\eta}(t) := \phi_0^{-1}(t+), \qquad u(t) := \phi\big(\bar{\eta}(t)\big).$$

Also, let $\mu \in \mathcal{B}_K[0, T]$ be such that $u(\cdot)$ is its distribution. The pair $(\psi_0(s), \psi(s))(\cdot)$ is a normalized graph completion of measure μ (see Definition 2.3.2).

We define other components of a solution X_μ to (2.1), as follows. Let $x(\cdot) : [0, T] \to I\!R^n$ be given by $x(t) = y\big(\eta(t)\big)$, and define the functions $y_i(\cdot)$ (for each $i \in \mathcal{I}$) as the restriction of $y(\cdot)$ to I_i.

Now recall Ω^N as in (3.24). By the triangle inequality, one has

$$\mathrm{dist}_{\mathrm{H}}\big(\Omega^N,\ \mathrm{gr}\ y(\cdot)\big) \le \mathrm{dist}_{\mathrm{H}}\big(\Omega^N,\ \mathrm{gr}\ y^N(\cdot)\big) + \mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\ y^N(\cdot),\ \mathrm{gr}\ y(\cdot)\big) \tag{3.31}$$

Finally, passing to the subsequence $\{N_k\}$ the statement of the theorem follows from (3.31), (3.28), and (3.30). □

Chapter 4

Invariance Conditions for Impulsive Systems

The preliminary chapter presented an overview of the non-impulsive invariance theory. In this chapter, notions of the weak and strong invariance are extended to the impulsive systems.

4.1 Weak Invariance

We now consider the problem of characterizing invariance properties of trajectories associated to the system (2.1) on a closed set $C \subseteq \mathbb{R}^n$. Definition of weak invariance, which is introduced here, requires the existence of a trajectory to lie in C over all slow and fast times. Precisely:

Definition 4.1.1. The system is *weak invariant* on C if for every $x_0 \in C$ and $S > 0$, there exists a time $0 \le T \le S$, a measure $\mu \in \mathcal{B}_K[0,T]$, and a three-tuple solution X_μ of (2.1), $x(t\pm) \in C$ for all $t \in [0,T]$ and each fast time arc $\{y_i(\cdot)\}$ satisfies $y_i(s) \in C$ for all $s \in I_i$.

Notice that the definition does not require the trajectory to exist for arbitrarily large slow times, but rather for arbitrarily large sums of slow and fast times, the latter size being proportioned according to the total variation of the measure μ.

The following is a proximal characterization of weak invariance for impulsive systems. Here we use notation $K_1 := K \cap S_1$.

Theorem 4.1.1. *The system (2.1) is weak invariant on a closed set C if and only if for each $x \in C$ and $\zeta \in N_C^P(x)$ (= the proximal normal cone to C at x), there exists $\lambda \in [0,1]$, and $v \in \big[\lambda F(x) + (1-\lambda)G(x)K_1\big]$ so that*

$$\langle v, \zeta \rangle \le 0.$$

Proof. We begin the proof by utilizing the sampling result in Section 3.3.1 to show, that the condition

$$\begin{aligned}&\forall x \in C, \forall \zeta \in N_C^P(x) \Rightarrow \\ &\qquad \exists \lambda \in [0,1] \text{ and } \exists v \in [\lambda F(x) + (1-\lambda)G(x)K_1] \text{ s.t. } \langle v, \zeta \rangle \leq 0\end{aligned} \tag{4.1}$$

implies weak invariance of the system (2.1) on closed set C.

Let $S > 0$, $x_0 \in C$ and $N \in I\!N$. Let h and $\{s_j\}$ be as in the sampling scheme in Section 3.3.1. The condition (4.1) guarantees existence of $\{\lambda_j\}$,$\{k_j\}$, $\{f_j\}$ and $\{g_j\}$ from the sampling scheme so that for a $c(x_j) \in \text{proj}_C(x_j)$

$$\langle \lambda_j h f_j + (1-\lambda_j) h g_j k_j, x_j - c(x_j) \rangle \leq 0. \tag{4.2}$$

By Theorem 3.3.1, there exists a measure $\mu \in \mathcal{B}_K([0,T])$ and solution X_μ of (2.1) so that graph of sampled trajectories converge to the graph of $y(\cdot)$ in Hausdorff metric. We claim that $y(s) \in C$ for all $s \in [0,S]$, where $y(\cdot)$ is defined as in 2.4.

Because $x_0 \in C$,

$$\mathrm{d}_C(x_1) \leq |x_1 - x_0| \leq \lambda_0 h |f_0| + (1-\lambda_0) h \|g_0\| \, |k_0| \leq 2hc_1.$$

where c_1 is from (3.2). Moreover,

$$\begin{aligned}\mathrm{d}_C^2(x_2) &\leq |x_2 - c(x_1)|^2 \qquad (\text{recall, } c(x_1) \in C) \\ &= |x_2 - x_1|^2 + |x_1 - c(x_1)|^2 + 2\langle x_2 - x_1, x_1 - c(x_1) \rangle \\ &\leq 4h^2 c_1^2 + \mathrm{d}_C^2(x_1) + 2\int_{s_1}^{s_2} \langle \dot{y}^N(s), x_1 - c(x_1) \rangle ds \\ &\leq 8h^2 c_1^2 + 2\int_{s_1}^{s_2} \langle \lambda_1 h f_1 + (1-\lambda_1) h g_1, x_1 - c(x_1) \rangle ds \\ &\leq 8h^2 c_1^2,\end{aligned}$$

where the last inequality is justified by (4.2). For general j,

$$\mathrm{d}_C^2(x_j) \leq j 4h^2 c_1^2 \leq 4hc_1^2.$$

It follows that when $N \to \infty$ (that is $h \to 0$), the nodes $\{x_j\}$ of the sampling scheme converge to C. This implies $y(s) \in C$ on $[0, S]$. This direction of the proof is completed.

Let us now find λ and v from the statement assuming that the system (2.1) is weak invariant on C.

Let $x_0 \in C$, and let the three-tuple X_μ as in (2.9) be a solution that lies in C with $x(0) = x_0$ and where measure μ is completed by a normalized graph completion $(\phi_0, \phi)(\cdot)$. For $\zeta \in N_C^P(x_0)$ there is a $\sigma > 0$ with

$$\langle \zeta, x - x_0 \rangle \leq \sigma |x - x_0|^2, \tag{4.3}$$

for all $x \in C$ by Proposition 1.1.5. in [7].

Since $0 \leq \dot{\phi}_0(s) \leq 1$, we have $\phi_0(s) \leq s$, so there exists a sequence $s_j \searrow 0$ and such that the following limit exists:

$$\lambda := \lim_{j \to +\infty} \frac{\phi_0(s_j)}{s_j} = \dot{\phi}_0(0).$$

If time $t = 0$ is an atom with $\eta(0+) = a > 0$ then $\lambda = 0$ and for a large j, $s_j \in \phi_0^{-1}(0) = [0, a]$. Using the Definition 2.2.2, any trajectory $y(\cdot)$ of (2.5) corresponding to the solution X_μ satisfies $\dot{y}(s) \in G(y(s))\dot{\phi}(s)$ and $\dot{\phi}(s) \in K_1$ almost everywhere on $[0, a]$. Moreover,

$$\frac{y(s_j) - x_0}{s_j} = \frac{1}{s_j} \int_0^{s_j} \dot{y}(s) ds \in \frac{1}{s_j} \int_0^{s_j} G(y(s))\dot{\phi}(s) ds \in G(x_0)K_1 + O(j),$$

where $O(j) \to 0$ as $j \to \infty$. Thus, $\left\{ \frac{y(s_j) - x_0}{s_j} \right\}$ has at least one cluster point v as $j \in \infty$ and by passing to a subsequence we can assume it is the only cluster point. Furthermore, it belongs to $G(x_0)K_1$. Since $y(s) \in C$ on $[0, \eta(T)]$, using (4.3),

$$\begin{aligned} \langle \zeta, v \rangle &= \lim_{j \to \infty} \left\langle \zeta, \frac{y(s_j) - x_0}{s_j} \right\rangle \\ &\leq \lim_{j \to \infty} \frac{\sigma}{s_j} |y(s_j) - x_0|^2 = 0, \end{aligned} \tag{4.4}$$

because trajectory $y(\cdot)$ is Lipschitz and $y(s_j) \to x_0$ as $j \to \infty$.

Suppose now that time $t = 0$ is not an atom and let $t_j := \phi_0(s_j)$. For all j any trajectory $y(\cdot)$ of (2.5) corresponding to solution X_μ satisfies

$$\begin{aligned} \frac{y(s_j) - x_0}{s_j} &= \frac{1}{s_j}\int_0^{s_j} f(s)\dot{\phi}_0(s)ds + \frac{1}{s_j}\int_0^{s_j} g(s)\dot{\phi}(s)ds \\ &= \frac{\phi_0(s_j)}{s_j}\frac{1}{t_j}\int_0^{t_j} \bar{f}(t)dt + \frac{1}{s_j}\int_0^{s_j} g(s)\dot{\phi}(s)ds, \end{aligned}$$

where $f(s) \in F(y(s))$ and $g(s) \in G(y(s))$ are selections and $\bar{f}(t) = f(\eta(t+))$ on $[0, t_j]$. A property of normalized graph completion is that $\dot{\phi}(s) = k(s)(1 - \dot{\phi}_0(s))$ with $k(s) \in K_1$. Since,

$$\begin{aligned} \frac{1}{s_j}\int_0^{s_j} g(s)\dot{\phi}(s)ds &= \frac{s_j - \phi_0(s_j)}{s_j}\frac{1}{s_j - \phi_0(s_j)}\int_0^{s_j} g(s)\dot{\phi}(s)ds \\ &= \frac{s_j - \phi_0(s_j)}{s_j}\frac{1}{s_j - \phi_0(s_j)}\int_0^{s_j} g(s)k(s)d(s - \phi_0(s)), \end{aligned}$$

we get

$$\begin{aligned} \frac{y(s_j) - x_0}{s_j} &= \frac{\phi_0(s_j)}{s_j}\frac{1}{t_j}\int_0^{t_j} \bar{f}(t)dt \ + \\ &\quad + \frac{s_j - \phi_0(s_j)}{s_j}\frac{1}{s_j - \phi_0(s_j)}\int_0^{s_j} g(s)k(s)d(s - \phi_0(s)). \end{aligned}$$

Since

$$\frac{1}{t_j}\int_0^{t_j} \bar{f}(t)dt \in F(x_0) + O(j), \quad \text{and}$$

$$\frac{1}{s_j - \phi_0(s_j)}\int_0^{s_j} g(s)k(s)d(s - \phi_0(s)) \in G(x_0)K_1 + O(j),$$

we find clustering points for both

$$\left\{\frac{1}{t_j}\int_0^{t_j} \bar{f}(t)dt\right\} \text{ and } \left\{\frac{1}{s_j - \phi_0(s_j)}\int_0^{s_j} g(s)k(s)d(s - \phi_0(s))\right\}$$

belonging to $F(x_0)$ and $G(x_0)K_1$ respectively. Passing to subsequences we have

$$v := \lim_{j \to +\infty} \frac{y(s_j) - x_0}{s_j} \in [\lambda F(x_0) + (1 - \lambda)G(x_0)K_1],$$

since,

$$\lim_{j \to +\infty} \frac{s_j - \phi_0(s_j)}{s_j} = 1 - \lambda.$$

Knowing that $y(s) \in C$ is Lipschitz on $[0, \eta(T)]$ with $y(s_j) \to x_0$ when $j \to \infty$, we again get (4.4). This completes the proof. □

4.2 Weak Invariance - Alternative Proof

In the previous section we showed that the condition 4.1 implies weak invariance using the sampling technique defined in Section 3.3. Now, we show the Theorem 4.1.1 again, this time with an alternative proof based on selections, which does not use results from the Section 3.3.

Alternative proof to Theorem 4.1.1. We show again that the condition (4.1) implies weak invariance of the system (2.1) on closed set C.

Let $S > 0$ and $x_0 \in C$. Suppose $y \in I\!R^n$ and let $\pi \in \text{proj}_C(y)$. By condition (4.1) there exist $\lambda_\pi \in [0, 1]$ and

$$v \in [\lambda_\pi F(\pi) + (1 - \lambda_\pi)G(\pi)K_1] \quad \text{so that} \quad \langle v, \zeta \rangle \leq 0.$$

Moreover, there exists a $k_\pi \in K_1$, $f \in F(\pi)$ and $g \in G(\pi)$ such that

$$v = \lambda_\pi f + (1 - \lambda_\pi) g k_\pi.$$

Let $w \in [\lambda_\pi F(\pi) + (1 - \lambda_\pi)G(\pi)k_\pi]$ minimize over $[\lambda_\pi F(\pi) + (1 - \lambda_\pi)G(\pi)k_\pi]$ the map

$$w \mapsto \langle w, y - \pi \rangle,$$

and define mapping $f_P : I\!R^n \to [\lambda_\pi F(\pi) + (1 - \lambda_\pi)G(\pi)k_\pi]$ by $f_P(y) := w$. This mapping inherits the linear growth condition from F and G. Indeed, from the linear growth property of $F(\cdot)$ and $G(\cdot)$,

$$|f| \leq c\big(1 + |\pi|\big), \text{ and } \|g| \leq c\big(1 + |\pi|\big). \tag{4.5}$$

Since, $f_P(y) = \lambda_\pi f + (1-\lambda_\pi) g k_\pi$,

$$\begin{aligned} |f_P(y)| &\leq \lambda_\pi c(1+|\pi|) + (1-\lambda_\pi)c(1+|\pi|)k_\pi && (4.5) \\ &\leq c(1+|\pi|) + c(1+|\pi|)r && \text{(bounds on } \lambda_\pi \text{ and } k_\pi) \\ &\leq c(1+r)(1+|\pi|) \end{aligned}$$

and the linear growth of $f_P(\cdot)$ is obtained.

By Proposition 4.2.1 in [7], we have that the Euler solution defined by

$$\dot{y} = f_P(y), \qquad y(0) = x_0 \in C$$

necessarily lie in C on $[0, S]$.

Let us define

$$\bar{F}_C(y) := \text{co } \left\{ [\lambda_\pi F(\pi) + (1-\lambda_\pi) G(\pi) k_\pi] \;\middle|\; \pi \in \text{proj}_C(y) \right\}.$$

Since f_P is a selection for the multifunction $\bar{F}_C$, we have that $\dot{y}(s) \in \bar{F}_C$ a.e. on $[0, S]$. Since for $\pi \in C$ the set $\bar{F}_C(\pi)$ is equal to the set $\lambda_\pi F(\pi) + (1-\lambda_\pi) G(\pi) k_\pi$, and since $y(s) \in C$ on $[0, S]$, we have

$$\dot{y}(s) \in \left[\lambda(s) F(y) + \big(1-\lambda(s)\big) G(y) K_1\right],$$

where $\lambda(s)$ is a measurable selection in $[0, 1]$.

We can take a measurable selections $k(s) \in K_1$, $f(s) \in F\big(y(s)\big)$, and $g(s) \in G\big(y(s)\big)$ so that

$$\dot{y}(s) = \lambda(s) f(s) + \big(1-\lambda(s)\big) g(s) k(s). \tag{4.6}$$

Define now

$$\phi_0(s) := \int_0^s \lambda(s') ds', \quad \text{ on } [0, S] \text{ and}$$
$$\phi(s) := \int_0^s \big(1-\lambda(s')\big) k(s') ds', \quad \text{also on } [0, S].$$

Note $\dot{\phi}_0(s) = \lambda(s)$ and

$$\dot{\phi}(s) = \big(1-\lambda(s)\big) k(s) \tag{4.7}$$

almost everywhere on $[0, S]$. This means,

$$0 \leq \dot{\phi}_0(s) \leq 1, \qquad \text{a.e.} \quad \forall s \in [0, S]. \tag{4.8}$$

If we define, $T := \phi_0(S)$,

$$\eta(t) := \phi_0^{-1}(t+), \qquad u(t) := \phi\big(\eta(t)\big) \quad \text{on } [0, T],$$

then we have $0 \leq T \leq S$ (since $\phi_0(\cdot)$ is nondecreasing, being with a nonnegative derivative). Moreover, from definition of $\eta(\cdot)$ and $u(\cdot)$, for all $t \in [0, T]$ there exists $s = \eta(t) \in [0, S]$ so that

$$\phi(s) = \big(\phi_0(s), \phi(s)\big) = \big(t, u(t)\big). \tag{4.9}$$

Now, taking a Borel measure $\mu \in \mathcal{B}_K([0, T])$ so that $u(\cdot)$ is its distribution, equations (4.9), (4.8) and (4.7) imply that the conditions of Lemma 2.3.6 are satisfied for the pair $(\phi_0, \phi)(\cdot)$, and therefore that pair is a normalized graph completion of measure μ.

Let us define $x(t) := y(\eta(t))$ on $[0, T]$ and, $\forall i \in \mathcal{I}$, let $y_i(s) := y(s)$ on when $s \in I_i := \phi_0^{-1}(t_i)$. The three-tuple $X := \big(x(\cdot), \phi(\cdot), \{y_i(\cdot)\}_{i \in \mathcal{I}}\big)$ is a solution of (2.1) by Definition 2.2.2 on $[0, T]$, because (4.6) implies

$$\dot{y}(s) \in F(y)\dot{\phi}_0(s) + G(y)\dot{\phi}(s).$$

Moreover, $x(t\pm) \in C$ for all $t \in [0, T]$ and each arc $\{y_i(\cdot)\}$ satisfies $y_i(s) \in C$ for all $s \in I_i$ because arc $y(\cdot)$ remains in C on $[0, S]$. □

Example 6. We finish this section with an example which illustrates how trajectory must remain within a closed set for both slow and fast times in order to have the invariance as defined in this Chapter. Moreover, we will see how not just the size of the jump, but also the choice of graph completion keeps the trajectory within a specified set.

Let $K := \mathbb{R}^+ \times \mathbb{R}^+$. Consider the following problem of finding the Borel measure measure $\mu \in \mathcal{B}_K([0,\infty))$, graph completion $(\psi_0, \psi)(\cdot) : [0,\infty) \to [0,\infty)$ and a solution X_μ of the system

$$\begin{cases} \begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^2(t) \end{bmatrix} = F(x^1, x^2), \quad \text{a.e. on } t \in [0,\infty) \\ \mu_\sigma = 0 \\ \begin{bmatrix} \dot{y}_i^1(s) \\ \dot{y}_i^2(s) \end{bmatrix} = G(x^1, x^2)\dot{\psi}(s) \quad \text{a.e. } s \in I_i \ \forall i \in \mathcal{I} \\ (y_i^1, y_i^2)(s_i^\pm) = (x^1, x^2)(t_i \pm) \ \forall i \in \mathcal{I} \\ (x^1, x^2)(0-) = (0,0). \end{cases} \tag{4.10}$$

so that the trajectory remains in set

$$C := \big\{(0, x^2) \mid x^2 \in [0,1]\big\} \cup \big\{(x^1, 1) \mid x^1 \geq 0\big\}.$$

Functions $F(\cdot)$ and $G(\cdot)$ are taken as

$$F(x^1, x^2) := \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{when} \quad x^2 \leq x^1, \\ \begin{bmatrix} 1 + x^1 - x^2 \\ -x^1 + x^2 \end{bmatrix} & \text{elsewhere,} \end{cases}$$

$$G(x^1, x^2) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(identity matrix)}$$

Note, the problem (4.10) is of the form of Definition 2.2.3, with additional assumption that the measure μ can only take discontinuous singularities. This is a mixture of dynamics often seen in hybrid system. It is clear that a jump control needs to be applied at time $t_0 := 0$, because $(\dot{x}^1, \dot{x}^2)(0, 0) = (1, 0)$ pushes trajectory outside of set C. Therefore, the control measure is realized through an atomic part at $t_0 = 0$ of a certain size. Let that size be denoted by $\bar{k} = (\bar{k}^1, \bar{k}^2) \in K$, where $\bar{k}^1 \geq 0$ and $\bar{k}^2 \geq 0$. Let us reparematerize the time of this jump with

$$\psi_0(s) = 0, \qquad s \in [0, 2].$$

Note that the shape of set C admits only movement on the second component of the trajectory $(y_0^1, y_0^2)(\cdot)$, therefore it must be

$$y_0^1(s) = 0$$

on a certain neighborhood around time $s = 0$, which means

$$\dot{y}_0^1(s) = 0,$$

almost everywhere in a open neighborhood around time $s = 0$. This rules out the straight line completion, because then

$$\psi(s) = \frac{s}{2}(\bar{k}^1, \bar{k}^2)$$

has a non-zero derivative in the second component at $s = 0$.

We will find $(\bar{k}^1, \bar{k}^2)$ and the corresponding graph completion that solves this invariance problem by using the sampling method introduced in Section 3.3. Note that

$$N_C^P(0, 0) = \{(x^1, x^2) \mid x^2 \leq 0\}.$$

Let $S = 2$ and $h = 1$. We take $y_0 = (0, 0)$. If $\zeta \in N_C^P(0, 0)$, then it must be $v^2 \geq 0$ and $v^1 = 0$ in order to assure $\langle v, \zeta \rangle \leq 0$, where $v = (v^1, v^2)$. Such choice for

$$v \in \lambda F(y_0) + (1 - \lambda)G(x)K_1$$

is available for taking $k_1 = (0,1) \in K_1$ and $\lambda = 0$. Sampling method in Section 3.3 gives us $y_1 = (1,0)$. Proximal normal cone on set C at point $(1,0)$ is

$$N_C^P(1,0) = \{(x^1, x^2) \mid x^1 \leq 0, x^2 \geq 0\}.$$

We again need to assure $\langle v, \zeta \rangle \leq 0$ for $\zeta \in N_C^P(1,0)$ and $v \in G(x)K_1$. If $k_2 = (k_2^1, k_2^2) \in K_1$, then both k_2^1 and k_2^2 must be greater or equal to 0, because $K_1 = (I\!R^+ \times I\!R^+) \cap S_1$. This leaves only one option the velocity in this step: $v = (1,0)$. We now find $y_2 = (1,1)$ by using the sampling method in Section 3.3.

For this example, we may stop our iterative process as we have already found a solution to our invariance problem. Indeed, control measure at time $t = 0$ jumps from $(0,0)$ to $\bar{k} := k_1 + k_2 = (1,1)$. The jump is realized via the completion obtained as the linear interpolation of points

$$\Big\{\big(0,(0,0)\big),\big(1,k_1\big),\big(2,k_1+k_2\big)\Big\} = \Big\{\big(0,(0,0)\big),\big(1,(1,0)\big),\big(2,(1,1)\big)\Big\} \qquad \text{i.e}$$

$$\psi_0(s) = 0, \qquad s \in [0,2],$$

$$\psi(s) = \begin{cases} (0,s) & s \in [0,1] \\ (s-1,1) & \in [1,2] \end{cases}$$

The latter implies,

$$\dot{\psi}_0(s) = 0, \qquad s \in [0,2],$$

$$\dot{\psi}(s) = \begin{cases} (0,1) & s \in [0,1] \\ (1,0) & \in [1,2] \end{cases}$$

On the other hand, the linear interpolation $(y_0^1, y_0^2)(\cdot)$ of points

$$\Big\{\big(0,(0,0)\big),\big(1,y_1\big),\big(2,y_2\big)\Big\} = \Big\{\big(0,(0,0)\big),\big(1,(1,0)\big),\big(2,(1,1)\big)\Big\}$$

happens to coincide with $\psi(\cdot)$ on $[0,2]$ in this example, and therefore

$$\begin{bmatrix} \dot{y}_i^1(s) \\ \dot{y}_i^2(s) \end{bmatrix} = \dot{\psi}(s) = G(x)\dot{\psi}(s). \quad \text{a.e.} \quad s \in [0,2].$$

Thus, it represents the trajectory of our problem during the jump period at time $t = 0$. We proceed now with non-impulsive dynamics,

$$\begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{a.e. on } t \in [0, \infty),$$

$$(x^1, x^2)(0) = (1, 1),$$

and obtain a trajectory that remains in set C. Solution

$$\Big((x^1, x^2)(\cdot), (\psi_0, \psi)(\cdot), (y_0^1, y_0^2)(\cdot) \Big)$$

is an impulsive solution to the invariance problem (4.10).

4.3 Strong Invariance

The property that *all* solutions of (2.1) remain in a given closed set C for all fast and slow times is called strong invariance. Important assumption is that both F and G are locally Lipschitz with respect to the Hausdorff distance. Precisely:

Definition 4.3.1. We say that the system is *strongly invariant* on C if for every $x_0 \in C$ and any $T > 0$, all measures $\mu \in \mathcal{B}_K[0, T]$ and all corresponding three-tuple solutions X_μ of (2.1) with $x(0) = x_0$, are such that, $x(t\pm) \in C$ for all $t \in [0, T]$ and each fast time arc $\{y_i(\cdot)\}$ satisfies $y_i(s) \in C$ for all $s \in I_i$.

We proceed with the following proximal characterization of strong invariance.

Theorem 4.3.1. *The system (2.1) is strong invariant on a closed set C if and only if for each $x \in C$ and $\zeta \in N_C^P(x)$ we have*

$$\langle v, \zeta \rangle \leq 0 \tag{4.11}$$

for all $\lambda \in [0, 1]$ and every $v \in \big[\lambda F(x) + (1 - \lambda) G(x) K_1\big]$.

Notice that this theorem is in complete consistency with the non-impulsive case. Indeed, in that case $G(x) = \mathbf{0}$, and we get that the condition (4.11) holds for every $\lambda \in [0,1]$ and every $v \in \lambda F(x)$. For $v \in \lambda F(x)$, $v = \lambda f$, where $f \in F(x)$ and $0 \leq \lambda \leq 1$. We get

$$\langle v, \zeta \rangle = \langle \lambda f, \zeta \rangle \leq \langle f, \zeta \rangle \leq 0,$$

which is a known condition for the non-impulsive case. Moreover, if the we have $\langle f, \zeta \rangle \leq 0$ for arbitrary $f \in F(x)$, then multiplying that inequality with any $\lambda \in [0,1]$, we get the condition (4.11). Let us now proceed with the proof of Theorem 4.3.1. It is a careful modification of the non-impulsive case found in [7].

Proof. Suppose that the system (2.1) is strong invariant on a closed set C. any arc $y(\cdot)$ from Definition 2.2.2 corresponding to solution X_μ which satisfies

$$\dot{y}(s) \in F\big(y(s)\big)\dot{\phi}_0(s) + G\big(y(s)\big)\dot{\phi}(s), \tag{4.12}$$

remains within the set C, where pair $(\phi_0, \phi)(\cdot)$ is a normalized graph completion of an arbitrary measure $\mu \in \mathcal{B}_K\big([0,T]\big)$. Let $x \in C$, let λ be any number in $[0,1]$ and let v be arbitrary element in $\lambda F(x) + (1-\lambda)G(x)K_1$. Then $v = \lambda f + (1-\lambda)gk$, where $f \in F(x)$, $g \in G(x)$ and $k \in K_1$.

For any y, let us define $\bar{\nu}(y)$ to be the closest point to $v := \lambda f + (1-\lambda)gk$ in $\lambda F(y) + (1-\lambda)G(y)k$. Notice, $\bar{\nu}(x) = v$. Since both F and G are locally Lipschitz, multifunction $\lambda F + (1-\lambda)Gk$ is also locally Lipschitz. This implies that the multifunction $\mathcal{V}(y) = \{\bar{\nu}(y)\}$ is with close graph and convex values with linear growth properties inherited from F and G. For $S = 1$, consider measure $\mu \in \mathcal{B}_K([0,\lambda])$ to be such that the pair $\phi_0(s) := \lambda s$ and $\phi(s) := (1-\lambda)ks$ represents a normalized graph completion of this measure on $[0,1]$. For such choice of measure μ, the system (4.12) becomes a strongly invariant system

$$\dot{y} \in \lambda F + (1-\lambda)Gk.$$

Any strongly invariant system is also weakly invariant, hence the previous system is also weakly invariant. Moreover, since function $\bar{\nu}(y)$ is a continuous selection of $\lambda F(y)+(1-\lambda)G(y)k$ (see Exercise 4.3.3 in [7]), the system $\dot{y} \in \mathcal{V}(y)$ is also weakly invariant. That is for our $x \in C$, and $v = \bar{\nu}(x) \in \mathcal{V}(x)$ we have

$$\langle v, \zeta \rangle \leq 0, \qquad \text{for all } \zeta \in N_C^P(x). \tag{4.13}$$

Let us show now that if condition (4.11) holds, then the system (2.1) is strong invariant on C. Let $x_0 \in C$ and let the three tuple X_μ as in (2.9) be a solution of (2.1) with $x(0-) = x_0$. Condition (4.11) implies that for all $y \in C$,

$$\max_{\substack{\lambda \in [0,1], \\ v \in \lambda F(y)+(1-\lambda)G(y)K_1}} \langle v, \zeta \rangle \leq 0, \qquad \forall \zeta \in N_C^P(y). \tag{4.14}$$

Now, let $T \geq 0$ and let $\mu \in \mathcal{B}_K([0,T])$ be arbitrary. For any solution X_μ of (2.1) with $x(0-) \in C$, we have

$$\dot{y}(s) \in F(y(s))\dot{\phi}_0(s) + G(y(s))\dot{\phi}(s), \tag{4.15}$$

where $y(\cdot)$ is the one from Definition 2.2.2. Since $\dot{\phi}_0(s) \in [0,1]$ and $\dot{\phi}(s) \in K_1$, by the strong invariance theorem for the non-impulsive systems [7], the system (4.15) is strong invariant. The usage of strong invariance characterization for non-impulsive systems is justified by the condition (4.14). By necessity,

$$y(s) \in C \qquad \text{on } [0, S],$$

where $S := \phi_0^{-1}(T+)$. This in turn means that $x(t\pm) \in C$ for all $t \in [0,T]$ and each fast time arc $\{y_i(\cdot)\}$ satisfies $y_i(s) \in C$ for all $s \in I_i$. Since T and μ are arbitrarily chosen, the proof is completed. □

Chapter 5
Conclusions and Open Problems

This thesis extends the classical existence, sampling theory and invariance properties to the impulsive systems. Having the invariance theory set in Chapter 4, the next thing is to consider the Hamilton-Jacobi theory and the minimal time function for the impulsive systems. This has already been done for the non-impulsive systems [35]. The minimal time control problem consists of a given closed set $C \subset I\!R^n$, and a control system in which the goal is to steer an initial point of the system so that a trajectory of the system reaches the target set in minimal time. Precisely, given a non-impulsive system (1.3), the minimum time function is $T_C : I\!R^n \to [0, \infty]$

$$\begin{aligned} T_C(s) := \inf\{T \mid \quad & \exists x(\cdot) \text{ satisfying } (1.3) \\ & \text{with } x(0) = x_0 \text{ and } x(T) \in C\}. \end{aligned}$$

By convention, if set C can not be reached for any of the trajectories, minimum time function then takes the value ∞.

Having in mind the notion of solution to the impulsive system, which depends on the time raparameterization, it is clear that the notion of minimum time needs to be modified. Namely, during the jump, reparameterized time streams as the original time does not move, and it is not immediately clear if one should consider minimum of the reparameterized time or the minimum of the original time. Just like in invariance results, when it was required that both the slow and the fast time trajectories remain in a given closed set, perhaps reparameterized time should be considered for the minimum time function problem as well. The following function is of interest: $E_C : \mathbb{R}^n \to [0, \infty]$

$$\begin{aligned} E_C(x) := \inf\{S \mid \ & \exists X_\mu \text{ a solution of (2.1)} \\ & \text{with } x(0-) = x \\ & \text{and } x(\phi_0(S)) \in C \text{ or } y_i(S) \in C \text{ for some } i \in \mathcal{I}\}, \end{aligned}$$

where the infimum is taken over all $\mu \in \mathcal{B}_K([0, T])$. This function can hardly be called not the "minimum time" function as it is expressed in the reparameterized time. Perhaps "minimum time and energy" function is a better name for it. Indeed, the infimum of interest is the one of all original times and energies spent during the jumps. In fact, scaling the importance of the energy component of this function might also give new insights: for $\lambda \in [0, 1]$ consider

$$\begin{aligned} E_C^\lambda(x) := \inf\{T + \lambda S \mid \ & \exists X_\mu \text{ a solution of (2.1)} \\ & \text{with } x(0-) = x \text{ and } x(\phi_0(T)) \in C \\ & \text{where } S = \textstyle\sum_{T > t_j \in \mathcal{T}} [\eta(t_j+) + \eta(t_j-)]\}, \end{aligned}$$

In any case, a proximal characterization of the minimum time and energy function will follow by applying the non-impulsive minimum time function characterization [35] to the system (2.5).

Another problem worth considering is relaxing conditions on dynamics $F(\cdot)$ and $G(\cdot)$ from Lipschitz to dissipative Lipschitz conditions for theorems on strong invariance. In [10, 11, 21] invariance theory for non-impulsive systems is extended to the class of dissipative Lipschitz dynamics. It is of interest to extend these results to the impulsive case as well.

Furthermore, more examples that link impulsive systems to hybrid systems and systems with singular peturbations need investigation.

References

[1] J.P. Aubin, *Viability Theory,* Birkhäuser, Boston, (1991).

[2] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin, Germany, (1984).

[3] A. Bressan, *Impulsive control systems*, in "Nonsmooth Analysis and Geometric Methods in deterministic Optimal Control," B. Mordukhovich and H. Sussmann, eds., 1-22.

[4] A. Bressan and F. Rampazzo, *Impulsive systems with commutative vector fields*, J. of Optimization Theory and Applications, v. 71, (1991), 67-83.

[5] A. Bressan and F. Rampazzo, *Impulsive control systems without commutativity assumptions*, J. of Optimization Theory and Applications, v. 81, (1994),435-457.

[6] G. Buttazzo, M. Giaquinta and S. Hildebrandt, *One-Dimensional Variational Problems: An Introduction*, Oxford University Press, Oxford, (1999)

[7] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, New York (1998).

[8] F.H. Clarke, Yu. S. Ledyaev, R.J. Stern and P.R. Wolenski, *Qualitative properties of trajectories of control theory: A survey*, J. Dynamical Control Systems, **1** (1995), 1–48.

[9] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society, Wiley-Interscience, Toronto, (1983).

[10] T. Donchev, V. Ríos and P.R. Wolenski, *A characterization of strong invariance for perturbed dissipative systems, Optimal Control, Stabilization, and Nonsmooth Analysis* in Lecture Notes in Control and Information Sciences Vol. 301, Springer-Verlag, Heidelberg, 2004, 343–349.

[11] T. Donchev, V. Ríos and P.R. Wolenski, *Strong invariance and one-sided Lipschitz multifunctions,* Nonlinear Analysis: Theory, Methods and Applications, **60** (2005), 849862.

[12] G.B. Folland, *Real Analysis : Modern Techniques and Their Applications*, Wiley-Interscience, New York, (1984).

[13] R. Goebel, J. Hespanha, A.R. Teel, *Hybrid Systems: Generalized Solutions and Robust Stability*, Symposium on Nonlinear Control Systems, NOLCOS 2004, Stuttgart, (2004).

[14] R. Goebel and A.R. Teel. *Solutions to hybrid inclusions via set and graphical convergence with stability theory applications* submitted.

[15] E. Hewitt and K.Stromberg (1965). *Real and Abstract Analysis.* Springer. New York.

[16] B.M.Miller and E.Ya. Rubinovich, *Impulsive control in continuous and discrete-continuous systems (Foundations of the hybrid systems theory)*, Kluwer Academic/Plenum Publishers, New York, (2002)

[17] M. Motta and F. Rampazzo, *Dynamic programming for nonlinear systems driven by ordinary and impulsive control*, SIAM J. Control and Optimization, v. 34 (1996), 199-225.

[18] J.M. Murray, *Existence theorems for optimal control and calculus of variations problems where the states can jump*, SIAM J. Control, v. 24 (1986), 412-438.

[19] F. Rampazzo and C. Sartori, *The minimum time function with unbounded controls*, J. of Math. Systems, Estimation , and Control, v. 8 (1998), 1-34.

[20] R. Rishel, *An extended Pontryagin principle for control systems whose control laws contain measures.* SIAM J. Control, v. 3 (1965), 191-205.

[21] V. Ríos and P.R. Wolenski, A characterization of strongly invariant systems for a class of non-Lipschitz multifunctions, *Proceedings of the 42th IEEE Conference on Decision and Control,* Maui, Hawaii USA (2003), 2593–2594.

[22] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, (1970).

[23] R.T. Rockafellar, *Dual problems of Lagrange for arcs of bounded variation*, in Calculus of Variations and Control Theory, D.L. Russell, ed., Academic Press, New York, 1976, 155-192.

[24] G.N. Silva, and R.B. Vinter, *Measure driven differential inclusions.* J. of Mathematical Analysis and Applications, v. 202 (1996), 767–746.

[25] G.N. Silva and R.B. Vinter, *Necessary conditions for optimal impulsive control systems*, SIAM J. of Control and Optimization, v. 35 (1998), 1829-1846.

[26] G.V. Smirnov, *Introduction to the theory of Differential Inclusions*, American Mathematical Society, Providence, RI (2002)

[27] R.B. Vinter, *Optimal control*, Birkhäuser, Boston, (2000).

[28] R.B. Vinter and F.M.F.L. Pereira, *A maximum principle for optiomal processes with discontinuous trajectories.* SIAM J. of Control and Optimization v. 26 (1988), 205–229.

[29] S. Vrećica, *Konveksna analiza*, Matematički faukletet, Beograd, (1993).

[30] P.R. Wolenski, *A uniqueness theorem for differential inclusions*, J. Differential Equations **84**, no.1, (1990), 165–182.

[31] P.R. Wolenski, *The Exponential Formula for the Reachable Set of a Lipschitz Differential Inclusion*, SIAM J. on Control and Optimization **28**, no.5, (1990).

[32] P.R. Wolenski and S. Žabić *A Solution Concept and Sampling Method For Impulsive Systems*, SIAM J. on Control and Optimization (submitted).

[33] P.R. Wolenski and S. Žabić *Direct Solution Concept for Impulsive Systems*, Proceedings of Controlo 2004, Porto (2004).

[34] P.R. Wolenski and S. Žabić *Invariance Properties for Impulsive Systems*, (in progress).

[35] P.R. Wolenski and Y. Zhuang, Proximal analysis and the minimal time function, *SIAM J. Control Optim.* **36** (1998), 1048–1072, no. 3.

Vita

Stanislav Žabić was born on March 31, 1973, in Vukovar, Croatia. He finished his undergraduate studies at University of Belgrade in 1998 in Belgrade, Serbia. In August 1999 he came to Louisiana to pursue graduate studies in mathematics. In 2002 he earned a Master of Science degree in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2005. This thesis was defended on final exam in April 2005.