

**PREDICTION OF TRANSIENT HEAD ON SEEPAGE PATH
USING BOUNDARY FITTED COORDINATE (BFC) SYSTEM**

A Thesis

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ABSTRACT

One of the main causes of levee failure is seepage through levee. This seepage can cause weakening in the levee structure, which in turn can cause sudden levee failure or overtopping of the levee due to levee depression. The purpose of this study is to determine the transient head on the seepage path due to changing water level during flood period. Transient seepage problems are very important as critical condition can occur during this time rather than during steady state condition. Seepage path through a levee can be predicted using the Boundary Fitted Coordinate (BFC) system method. The physical coordinate system is transformed into computational curvilinear coordinate system by using this method. One advantage of using this method is the grid generated can fit itself into the boundary of the system. The model developed is tested on a steady state rectangular dam for verification by comparing it with the analytical solutions. The model is then implemented on an unsteady state problem to determine the location of free surface and seepage face for several time steps until steady state is reached.

CHAPTER I

INTRODUCTION

Flooding has been one of the biggest and most continuous natural disasters in the world. In an effort to avert disasters in terms of loss of life and damage to properties caused by flooding, dams were built all over the world to control the flow of rivers. A dam is defined as a natural or manmade embankment, designed to prevent flooding of the landward side of the embankment, with a specific purpose to hold back water. Dams can also provide additional advantages apart from flood management, such as to provide water for irrigation, furnish hydroelectric power, and improve the navigability of waterways.

Dams that are built to protect land alongside rivers from frequent flooding caused by a rise in the water level of the river are called levees. These structures prevent flooding of the adjacent area from small floods and thus reduce the occurrence of flood. Levees normally contain flood waters when the river stage rises. When the river stage drops, levees do not hold back water. Thus, levees see intermittent use only during times of high river stages. However, when floods of large magnitude occur, such as floods with a recurrence interval of more than 100 years, there are situations in which levees can cause more damage as these structures increase the volume of water that is held in the channel and if levee failure occurs, the sudden release of water can increase the size of the flooded area. To avert this disaster, levee design, construction, maintenance and supervision holds a very important role. In order to avoid levee failure, it is important to analyze possible causes of failure. The most common causes are overtopping, erosion, seepage, and piping.

A mechanism that can cause levee failure is overtopping resulting from the river surface elevation rising higher than the levee. Another mechanism of levee failure is piping in which water seeping under or through a levee erodes fine particles of soil, opens a channeler slit, and the levee loses support so that the crown slumps below the water surface elevation, allowing overtopping to occur through the depression, leading to failure. Levees may fail because the floodwater erodes the side of the levee until the remaining part of the levee collapses from the pressure of the water. Extended periods of flooding can weaken levees by seepage water saturating the levee soils, allowing them to lose structural strength, so that the levee creeps, deforms, and may fail under the pressure of flood water. Levee failure can also be caused by erosion of the structure due to current or wave action. Current or wave action can cause the particles of earthen material to be eroded from the dam at an accelerated pace, causing a local sinking of the dam section. Continuation of the process will allow the floodwater to flow directly over the crest causing a levee failure.

Seepage through or under a levee may occur at a high enough rate to cause a boil, usually called a sand boil. Presence of sand boils can play a major role in levee failure. Seepage of floodwater through or under a levee is a normal process. However, when seepage occurs at a high rate, the seepage water can carry soil material with it. Seepage through a levee is relatively common, but when the seepage creates a drainage path and soil material is washed out through a boil on the landside of the structure, a potentially dangerous condition can be created. A boil is a condition under which enough pressure is produced to pipe water through or under the dam with sufficient velocity to carry earthen material to the landward side. Experience suggests that soil piping occurs when the hydraulic gradient, which is the ratio between the difference in hydraulic head at two points and the length along the boundary between the two points, is greater than

approximately one. When the hydraulic gradient equals one, the force on the soil, which is the seepage force, is equal to the unit weight of the water. When this gradient is greater than one, the seepage force becomes greater than the unit weight of the water and transports the soil along the direction of the force. Continuous piping can cause sufficient material to exit through the boil that a large void is created inside the levee, which can result in the weakening of the structure and eventually failure. Not all sand boils lead to levee failure. The flood stage that initiates formation of a sand boil may drop before significant erosion has occurred, or the flood stage that initiates sand boil formation may occur so infrequently that a levee does not fail.

Any seepage problems must consider three principal factors, which are soil media, type of flow, and boundary conditions. The soil media are important in the determination of seepage characteristics since different soil media will exhibit different behavior. Some of the most important characteristics that need to be determined from the soil media are transmissivity and coefficient of permeability. Soil media can be classified by these different characteristics. For example, if the coefficient of permeability is the same at all points in the flow region and it is independent of the direction of the flow, the soil is classified as homogeneous and isotropic. If on the other hand it is dependent on the direction of the flow, it is classified as heterogeneous and isotropic. For soil that is only independent on the direction of the flow, it is classified as homogeneous and anisotropic.

The type of flow can be classified as either steady state or transient flow. In a steady state type of flow, time is not considered as a variable and the position of the water table does not change. On the other hand, a transient problem requires time as a variable and so an initial condition needs to be described aside from boundary conditions and a time step needs to be determined to correctly illustrate the influence of time on the problem. In transient problems, it is

always important to choose the right time steps needed to solve the problems. Solutions obtained from too large time steps can be inaccurate or unstable, while solution from too small time steps is inefficient. Since time steps vary with each problem, it seems reasonable to base the time steps on the maximum movement of the water table. As a general guideline, it is reasonable to assume the initial time steps to be less than ten percent of the maximum movement of the water table. If a critical condition should occur during certain time steps, these time steps can be more refined to accurately describe the movement.

Boundary conditions are also needed to correctly describe the problem. In flow domains where all the boundaries are fixed and therefore known initially, the flow is said to be confined, but where one boundary is a free surface, the flow pattern is said to be unconfined. For this instance, where seepage is evaluated through a levee, the important boundary conditions that need to be defined are the upstream face, downstream face and free water surface. To accurately describe the problem, on the upstream and downstream faces, the pressure head is due to water pressure and varies with the height. The free water surface on the other hand has two conditions that need to be satisfied, the first of which is that atmospheric pressure is maintained on the boundary, while the second is that no flow crosses the free surface. Both of these conditions need to be satisfied simultaneously in order to accurately describe the location of the free surface.

Seepage analysis is an important tool which can be applied to predict seepage, and to investigate measures to prevent or reduce the magnitude of seepage flow. Losses due to flow through levees must be minimized and seepage flows that may cause piping must be controlled. Several seepage control measures can be implemented on levees to avoid failure. Control measures that can be implemented to control seepage to reduce risk of levee failure include installation of a pervious toe drain in the levee which will provide a ready exit for seepage

through the embankment, placement of a horizontal drainage layer during levee construction, and incorporation of an inclined drainage layer in levee design.

Control of seepage is not the only reason for analyzing seepage. Seepage analysis can provide information with which to evaluate other consequences such as excessive soil saturation, the magnitude of seepage forces and uplift pressure that can lead to levee failures. Seepage analysis can also be used in evaluating proposed future design alternatives and levee maintenance design. Seepage path prediction is important in order to assess the necessary steps to avoid levee failure. Determination of the seepage path is an important tool that can be used to determine the likelihood of damage to the levee due to piping and in the evaluation of piping design control measures.

During flood periods, the water level on the levee can change dramatically. Therefore, it is reasonable to assume that unsteady state flow will occur through the levee during part or all of the flood period. In time variant seepage problems for levees, time dependent features usually arise from the moving free surface. For example, as the river stage rises or falls, the location of the free surface at the upstream levee face changes. This movement of the free surface stores or yields far more water than quantity of water stored or released due to compressibility effects. In the description of free surface transient problems, not only boundary conditions must be known, but also the initial condition at the start of the event must be known. The boundary is then moving in time subject to certain conditions.

Whenever seepage paths are disturbed, the transition from the original flow line configuration to the new flow line configuration can take significant time. Changes in position of the free water surface are not instantaneous because of specific yields of the soil material combined with slow seepage of water through levee. Consequently, the movement of the free

surface tends to dominate flow pattern within the levee. In certain instances, critical condition can occur during this change and it is therefore important to investigate time variant seepage behavior. When head no longer changes with time, then a steady state is reached.

CHAPTER 2

LITERATURE REVIEW

2.1. Historical Review

Solutions for groundwater seepage problems have been developed since the pioneering work of Henry Darcy (1856). For example, many analytical solutions were developed and presented by Harr (1962) and Polubarinova-Kochina (1962). The groundwater seepage problem can be described by Darcy's law and continuity equations. The seepage equation is obtained by combining these two equations. Since the seepage equations are based on these two equations, the assumptions and limitations that apply to these equations also apply to the seepage equation. Darcy's law basically demonstrates a linear dependency between the hydraulic gradient and the discharge velocity. In the groundwater and seepage problems, where the flow is primarily of a single relatively incompressible fluid subject to small changes in temperature, it is more convenient to use Darcy's law with k , coefficient of permeability. Physically, all flow systems extend in three dimensions. However, in many problems the features of the groundwater motions are essentially planar, with the motion being substantially the same in parallel planes. Therefore these problems can be described with two dimensional flow only, and thereby reduce considerably the work necessary to obtain a solution. Although, analytical solutions are the most accurate methods in calculating seepage, the analytical method has several drawbacks. One major drawback of this method is that it is difficult to apply, as it is limited to groundwater flow with uniform hydraulic properties and simple boundary geometry. In some cases, analytical solutions cannot be obtained because of non-linear features such as variable permeability or moving boundaries.

Dupuit-Forchheimer (1930) introduced an approximation formula for calculating seepage through dams. The Dupuit theory of unconfined flow stems from two assumptions: for small inclinations of the line of seepage the streamlines can be taken as horizontal, therefore the equipotential lines approach the vertical and the hydraulic gradient was equal to the slope of the free surface and was invariant with depth. The validity of Dupuit's assumptions in a given flow situation is highly contingent on the steepness of the line of seepage. Dupuit's formula specifies a parabolic free surface, commonly referred to as Dupuit's parabola. In the derivation above no knowledge has been taken of the entrance or exit conditions of the line of seepage or of the development of a surface of seepage. Therefore in the absence of tail water, the line of seepage is seen to intersect the impervious base. Also, it should be noted that both the discharge quantity and the locus of the free surface are independent of the slopes of the dam. Although Dupuit's formula is easy to apply to different shapes of domain, the solution obtained may not be the best one. The most apparent drawback of this method is that the seepage face is ignored and considered non-existent. Since the late 1950s the introduction of digital computers has shifted most seepage analysis efforts to numerical methods.

Richardson (1911) introduced finite difference method for analyzing seepage problems. The method was applied to steady state problems prior to introduction of digital computers in the late 1950s. Finite differences were widely applied to unsteady state problems during the 1960s, but the method was limited in being able to describe complex boundary shapes. In this method, the domain is described as a collection of grids and the equation is evaluated at each point of the grid. This method greatly improved the solution of seepage problem as more complicated domains can now be described and the solution of the seepage problem can be obtained by means other than analytical solution which is often time consuming. Finite difference formula is

defined at each grid in the domain and the partial differential equation can be described as a set of finite difference formula. However, when the boundary is curved, the previously derived finite difference formulas for derivatives will have to be modified in the neighborhood of the boundary. For this approximation on the curved boundary, the accuracy for the approximation of the second derivative in the neighborhood of the boundary is not as good as the approximation at an ordinary mesh point. The situation becomes more difficult if the problem boundary conditions give information on the derivative of the function along a curved boundary. Errors caused by this method of solution must also be considered. The errors in approximate, numerical solutions arise due to three primary causes. The first and most important is the discretization error, which is due to the incomplete satisfaction of the governing equations and their boundary conditions. The second, the roundoff error, is due to the fact that only a finite amount of information may be stored at any stage of the calculation process. The third error is due to the approximations involved in the mathematical model to which the numerical solution is applied.

While finite difference formula greatly advances the science of seepage calculation, there are some limitations to this method, particularly in the solution of unsteady state seepage. Todsen (1971) has suggested that the finite difference method can become unstable as time increases. A stability criterion has to be defined to ensure a stable solution and the stability criterion can only be applied to a linearized differential equation. Another limitation of this method is that even if stability criterion for the linearized difference equation is satisfied, some kind of smoothing of the numerical solution was necessary to proceed in time. The success of the solution depends on the accuracy and on the stability of the difference schemes adopted. Moreover, the schemes of stability criterion and smoothing demand for their solutions more boundary and initial conditions than are needed for the corresponding differential equation problem. Therefore care must be

exercised in choosing these extra conditions so that the stability is not adversely affected. Unfortunately, there is no straightforward method to ascertain the criteria for computational stability of a nonlinear difference equation. If centered, three levels or two step formula is used, two initial conditions to find H at successive time levels will be needed. One is the physical condition, specified in the original problem, while the other may be called a computational condition, and is not given in the problem. It is needed because the difference equation contains more time levels than the minimum possible. Similar arguments hold for space derivatives, when grid points are substituted for time levels, and boundary conditions for initial conditions. It is realized that this procedure, which was a linear or a parabolic extrapolation, could have adverse effect upon the stability of the difference scheme. A linear extrapolation may cause instability of the two step centered difference approximation to the equation. Any tendency for instability developed from the extrapolation of the boundary values can be counteracted by the space smoothing introduced. For nonlinear finite difference equation a simple analysis of the stability conditions is not possible. Aliasing error will occur for linear equations with variable coefficients in space, and may cause the same type of instability. This type of computational instability is a major obstacle which must be overcome in order to prevent the numerical computations to become unstable after some time. Smoothing procedure was based on a Fourier analysis on the gridpoint data. Other smoothing procedure, such as adding diffusion type terms to the differential equation or incorporating smoothing in the finite difference scheme, may be used. More refined smoothing factor curves may be constructed by successive application of smoothing elements with different indices. For each time step new values of the height H were found. If smoothing is not done, wiggles can develop after some time steps which obviously had nothing to do with the solution of the physical flow problem.

Neumann and Witherspoon (1970) introduced Finite Element Method (FEM) in analyzing steady seepage with a free surface. This method improved on the finite difference method since the division of the domain can be controlled by different shapes and sizes of pieces that make up the domain. The basic principle of this method is to divide the domain of the problem into a number of subdomains and then construct the approximation in a piecewise manner over each subdomain. The trial functions used in the approximation process can also be defined in a piecewise manner. If the subdomains are relatively simple shapes and if the definition of the trial functions over these subdomains are repeatable, then the solution can be obtained easily. The piecewise definition of the trial or shape functions means that discontinuities in the approximating function or in its derivatives can occur. Different shapes and sizes can be used to describe the domain such as rectangular or triangular shapes. However, for complex problem domain, it is probably not possible to divide the domain accurately into subdomains. Also the solution of the governing equation is more complicated in this method compared to the simple method of finite difference. The finite difference process is in fact a particular case of the general finite element weighted residual methodology. With the use of the simple finite elements, refinement of the solution can be obtained by successive creation of finer meshes of elements. Higher order element shape functions can be generated in a straightforward manner for geometrically simple elements. Unfortunately, the simple shapes of the elements so far derived restrict severely their application in the analysis of practical problems, where often quite complex geometrical boundaries have to be modeled. Clearly, although techniques are available for solving linear problems, they are going to be difficult to apply in practice, and such techniques cannot be used for all nonlinear problems. Finite element method can also be used in solving unsteady state equations. Finite element will be used to represent the time domain which

is of infinite extent. Conditions at the end of the first element will be determined by use of the governing equations plus the initial conditions. This process is then repeated for subsequent elements, using the newly calculated information as the initial conditions for each element in turn. There are many reasons why an approach of this form is not normally adopted: with a large time domain, the size of the problems can become excessively large, the resulting equation system is nonsymmetric, and the geometrically simple nature of the time domain offers little incentive for the use of an irregular subdivision of space time element.

Another approach that was introduced was boundary element method by Liggett (1977). Boundary Element Method (BEM) is used to write equations for the location of discrete points on the free surface. These equations depend only on the boundary data and thus the free surface can be located without solving the complete problem. Also the resulting algebraic equations completely defined the location of the free surface and no iteration is necessary, except in the sense that the equations are nonlinear and thus an iterative solution is used such as in most cases of nonlinear simultaneously equations. The fundamental of this method is that the integration was divided into parts. The points are spaced so that four pairs of points appear when the boundary values undergo sudden changes. Not more than two such points should appear close together or the coefficient matrix of the equations will be singular. Since it uses boundary data only, it has the disadvantage of not being able to describe the solution everywhere in the domain of the problem.

Although there are many analytical and numerical solutions developed by numerous authors for steady state seepage, there is still a need for seepage path evaluation in a transient condition. This analysis can be important since critical condition can appear during a time when steady state condition has not been reached. Model studies of two dimensional moving boundary

problems was described by Bear (1972). These models emphasize the importance of boundary and initial condition definition of the problem. If the problem is one in which the dependent variables are also time dependent, the boundary conditions must be specified for all time t greater than zero. In addition, certain conditions, called initial conditions, must be satisfied at all points of the domain being considered at the particular instant of time at which the physical process begins. The problem is then referred to as an initial value problem. In general, any specification of boundary conditions for the second order partial differential equations considered here should include the geometric shape of the boundary and a statement of how the dependent variable vary on the boundary.

It is clear that a better numerical solution methodology with more flexibility is needed to describe transient seepage problems. The Boundary Fitted Coordinate (BFC) system method makes it possible to construct a grid system that can fit itself to irregular boundaries. Since the characteristics and properties of the equations transformed do not change, this method can be a powerful tool in solving seepage problems. According to Thompson (1977), the grid generation process begins by first defining the boundary geometry, then distributing points on the curves that form the edges of boundary sections. The next step is the generation of the corresponding type of grid on the boundary surface. Structured grid then can be generated as a solution to partial differential equations (PDE). Smoothness of the solution is obtained through the smoothing properties inherent in the elliptic system generation. The BFC system can be used for both steady and transient problems solution. The BFC system, which utilizing the fitting of the boundary can be utilized in solving transient problems and the smoothness of the solution can be obtained through the Laplace system generation.

2.2. Objectives and Scopes of Research

The objectives and scope of this research are as follows

1. Use Boundary Fitted Coordinate (BFC) system to represent the problem domain into a perfectly rectangular domain for more accurate and stable computations.
2. Implement different shapes of domain to ensure the grid generation process in the BFC method is working successfully.
3. Develop a model to predict the seepage path for a steady state condition and comparing the result to known analytical solutions.
4. Implement the model developed for a sudden drawdown in a rectangular earth dam to ensure the stability of the solution, which was a problem in regular finite difference scheme, can be maintained.
5. Implement the model developed to predict the seepage path during fall of river stages to simulate transient conditions which governs the seepage in the levee.

CHAPTER 3

MODEL DEVELOPMENT

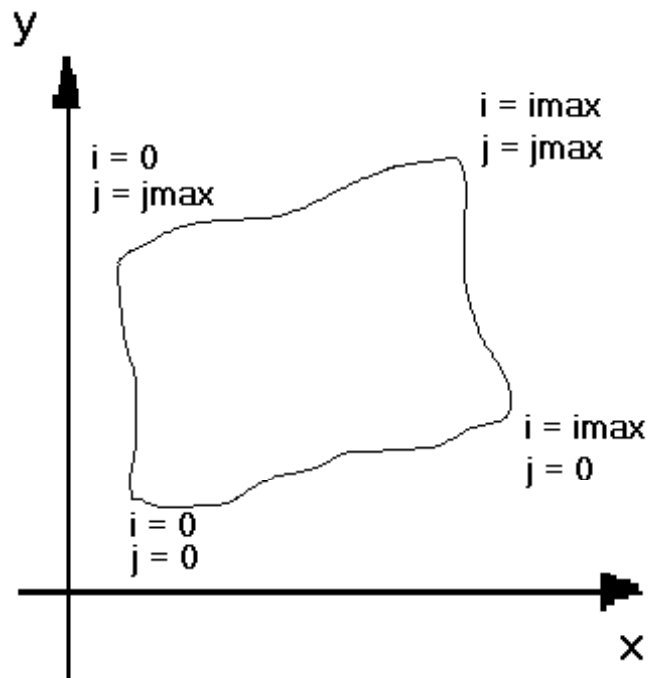
3.1. Grid Generation

As mentioned in the literature review, the Boundary Fitted Coordinate (BFC) system has certain advantages compared to the regular finite difference method because of its ability to conform to the boundaries of the system regardless of the shape. This feature increases the accuracy of the solution developed.

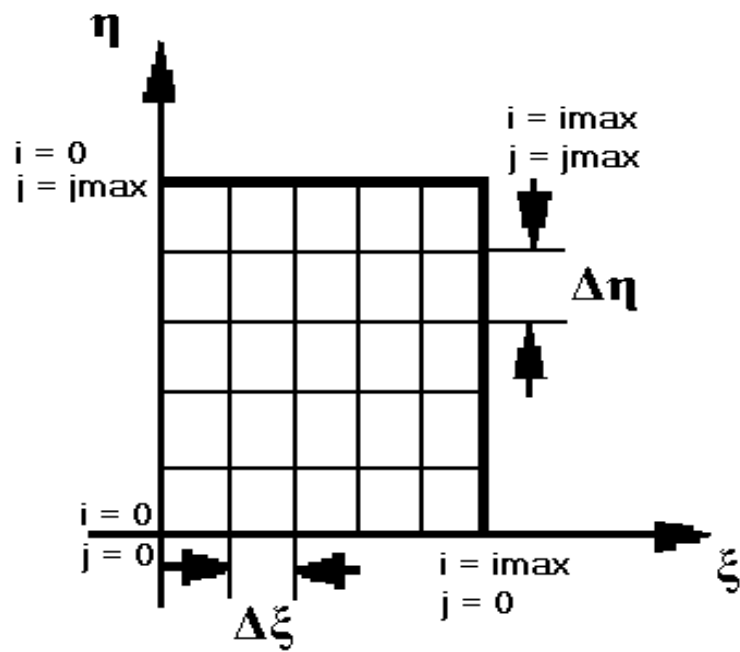
The most important factor of this method is that transformation is done so that the physical Cartesian coordinates (x,y) become the dependent variables and the curvilinear coordinates (ξ, η) becomes the independent variables. A generated grid is then defined as a set of points formed by the intersections of the lines of a boundary conforming curvilinear coordinate system. The principal feature of this system is that some coordinate line is coincident with each section of the boundary in the physical region.

The use of coordinate line intersections to define the grid points provides a means that allows all computation to be done on a fixed square grid after the governing partial differential equations are transformed as shown in Figure 1. The size of the intervals then determined the number of grid lines generated for a particular problem.

The number of grid lines generated or the spacing of the grid must be considered carefully. If the spacing is too large, it may effect the accuracy of the solution generated. If the spacing is too small, it will make the solution matrix to be really large and ineffective. Smaller spacing can be placed on regions where the gradient changes dramatically to make sure that the regions are represented accurately.



(a)



(b)

Figure 1. Transformation of problem domain from the (a) physical Cartesian coordinate system to (b) computational curvilinear coordinate system

The problem may be simplified for computation, however, by first transforming so that the physical Cartesian coordinates become the dependent variable with the curvilinear coordinate as the independent variables. Since a constant value of one curvilinear coordinate, with monotonic variation of the other, has been specified on each boundary segment, it follows that these boundary segments in the physical field will correspond to vertical or horizontal lines in the transformed field. Also, since the range of the variation of the curvilinear coordinate varying along a boundary segment has been made the same over opposing sides, it follows that the transformed field will be composed of rectangular blocks. The boundary value problem in the transformed field then involves generating the values of the physical Cartesian coordinates, $x(\xi, \eta)$ and $y(\xi, \eta)$ in the transformed field from the specified boundary values of $x(\xi, \eta)$ and $y(\xi, \eta)$ on the rectangular boundary of the transformed field, the boundary being formed of segments of constant ξ or η .

From Thompson (1977), the following transformation relationships are used in conversion between the Cartesian and curvilinear coordinate system.

$$\xi_x = \frac{y_\eta}{J} \quad (3.1)$$

$$\xi_y = -\frac{x_\eta}{J} \quad (3.2)$$

$$\eta_x = -\frac{y_\xi}{J} \quad (3.3)$$

$$\eta_y = \frac{x_\xi}{J} \quad (3.4)$$

where

$$J = x_\xi y_\eta - x_\eta y_\xi \quad (3.5)$$

In generating a grid, a set of points distributed over a calculation field for a numerical solution for a set of partial difference equations must be sufficiently dense that the numerical approximation is an accurate one, but it cannot be so dense that the solution is impractical to obtain. Generally, the grid spacing should be smooth and sufficiently refined to resolve changes in the gradients of the solution. There must also be a unique correspondence between the Cartesian and the curvilinear coordinate system. The mapping of the physical region into the transformed region must be one to one, so that every point in the physical field corresponds to one and only one point in the transformed field and vice versa.

In order to make use of a general boundary-conforming curvilinear coordinate system in the solution of partial differential equations, the equations must also be transformed to the curvilinear coordinates, where the independent variables are time and the Cartesian coordinates. The resulting equations are the same type as the original ones, but are more complicated because they contain more terms and variable coefficients. The domain, on the other hand, is greatly simplified since it is transformed to a fixed rectangular region regardless of its shape and movement in physical space.

Transformation from the Cartesian coordinate system (x,y) to the curvilinear coordinate system (ξ,η) is not only done on the governing equation, but it is also done on the boundary conditions and initial conditions which described the problem. This is done so that there is consistency in the coordinate system used in evaluating the problem. After the problem is evaluated and solved, the corresponding result is then transformed back into the original coordinate system which is the Cartesian coordinate system (x,y) .

As mentioned before, there is a need for a smoothing procedure to avoid instability. In BFC method, this smoothing is done by using the elliptic grid generator, more specifically Laplace grid generator which is known to have smoothing properties and thus the grids generated from this equation are the smoothest grid possible. Since smoothing procedure is already incorporated into the BFC method, finite differences can then be applied to solve the problem without experiencing the instability that can occur on the previous method.

In order to accurately describe the domain, at least four corner points must be chosen to represent the start and end of the boundaries in each direction. The rest of the boundaries can then be obtained from the equation which described the boundary formulation. After determination of boundary points, a grid generator is used to calculate the interior nodes of the solution domain. Two kinds of grid generator are used in generating a solution, which are the algebraic grid generator and the partial differential grid generator.

3.2. Algebraic Grid Generator

For initial grid generation, interior nodes of the problem area can be generated from direct interpolation from the boundaries using algebraic grid generation system. There are several types of algebraic generation system:

- **Unidirectional Interpolation**

The drawback of this interpolation method is that the interpolations of the interior nodes are only done in one direction at a time, which can increase the error in the resulting grid. Several examples of this type of interpolation are horizontal or vertical interpolation, Lagrange interpolation, and Hermite interpolation. Since the unidirectional interpolation technique doesn't give the best result in interpolating the interior nodes, it will not be used in this paper.

- Multi-directional interpolation

Multi-directional interpolation incorporates the boundary data in multi-dimensions simultaneously which increases the accuracy of the grid generated. Transfinite interpolation and tensor product interpolation are included in this category. The transfinite interpolation is commonly used to generate the interior nodes because it gives the best approximation compared to other methods of grid generator. This technique is usually done in sections of loops for each counters in each directions of the coordinate system. These calculations are then combined to calculate the interior grids, therefore combining the two directions of the coordinate system in one calculation. Transfinite interpolation for a two dimensional domain is shown by the following formula

$$\begin{aligned}
 RI1 &= \frac{i}{i_{\max}} \\
 RI2 &= \frac{i_{\max} - i}{i} \\
 x_1 &= RI1 * x(i_{\max}, j) + RI2 * x(0, j) \\
 y_1 &= RI1 * y(i_{\max}, j) + RI2 * y(0, j) \\
 RJ1 &= \frac{j}{j_{\max}} \\
 RJ2 &= \frac{j_{\max} - j}{j_{\max}} \\
 x_2 &= RJ1 * (x(i, j_{\max}) - x_1(i, j_{\max})) + RJ2 * (x(i, 0) - x_1(i, 0)) \\
 x_2 &= RJ1 * (x(i, j_{\max}) - x_1(i, j_{\max})) + RJ2 * (x(i, 0) - x_1(i, 0)) \\
 x(i, j) &= x_1(i, j) + x_2(i, j) \\
 y(i, j) &= y_1(i, j) + y_2(i, j)
 \end{aligned} \tag{3.6}$$

where i = counter in the ξ direction which can vary from 0 to i_{\max}

j = counter in the η direction which can vary from 0 to j_{\max}

3.3. Partial Differential Equation Grid Generator

There exists several kinds of partial differential grid generator: parabolic, hyperbolic, and elliptic. The use of these partial differential grid generators is determined by the type of coordinate points specification on the boundary of the physical domain. If only a portion of the boundary is specified, parabolic or hyperbolic grid generators may be used. Elliptic grid generator however requires that all coordinate points are specified on the entire boundaries of the problem. The parabolic and hyperbolic grid generator are generally faster than the elliptic grid generator, however, elliptic grid generator has the advantage of inherent smoothness. Since smoothness is a problem in the previous method developed using finite difference method, the elliptic grid generator will be used to solve the problem. The simplest form of the elliptic grid generator is the Laplace system.

As mentioned before, in order to guarantee the smoothness of the solution, Laplace equation is used to generate a grid system that will be used in solving the governing equations. From Thompson (1977), Thompson and Warsi (1982), Thompson *et al.* (1985), the following formulas are used, where ξ and η are the transformed curvilinear coordinate

$$\xi_{xx} + \xi_{yy} = 0 \quad (3.7)$$

$$\eta_{xx} + \eta_{yy} = 0 \quad (3.8)$$

Chain rule differentiation is used to represent Laplacian operator and can be represented by the following formula

$$\begin{aligned} \nabla^2 = & \left(\xi_x^2 + \xi_y^2 \right) \frac{\partial^2}{\partial \xi^2} + 2(\xi_x \eta_x + \xi_y \eta_x) \frac{\partial^2}{\partial \xi \partial \eta} + \left(\eta_x^2 + \eta_y^2 \right) \frac{\partial^2}{\partial \eta^2} \\ & + \left(\xi_{xx} + \xi_{yy} \right) \frac{\partial}{\partial \xi} + \left(\eta_{xx} + \eta_{yy} \right) \frac{\partial}{\partial \eta} \end{aligned} \quad (3.9)$$

Using equation (3.9), the grid-generating equations can then be derived for the Cartesian coordinate system which results in the following formulation

$$\begin{aligned} \nabla_x^2 = & (\xi_x^2 + \xi_y^2)x_{\xi\xi} + 2(\xi_x\eta_x + \xi_y\eta_y)x_{\xi\eta} + (\eta_x^2 + \eta_y^2)x_{\eta\eta} \\ & + (\xi_{xx} + \xi_{yy})x_\xi + (\eta_{xx} + \eta_{yy})x_\eta \end{aligned} \quad (3.10)$$

$$\begin{aligned} \nabla_y^2 = & (\xi_x^2 + \xi_y^2)y_{\xi\xi} + 2(\xi_x\eta_x + \xi_y\eta_y)y_{\xi\eta} + (\eta_x^2 + \eta_y^2)y_{\eta\eta} \\ & + (\xi_{xx} + \xi_{yy})y_\xi + (\eta_{xx} + \eta_{yy})y_\eta \end{aligned} \quad (3.11)$$

Following a procedure described by Thompson (1977) in utilizing the equations described above, equations (3.7) and (3.8) can be rearranged to result in the following formula

$$ax_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = 0 \quad (3.12)$$

$$ay_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = 0 \quad (3.13)$$

where

$$a = x_\eta^2 + y_\eta^2 \quad (3.14)$$

$$\beta = x_\xi x_\eta + y_\xi y_\eta \quad (3.15)$$

$$\gamma = x_\xi^2 + x_\eta^2 \quad (3.16)$$

Figure 2 and 3 show an application of the partial differential equation grid generators. In Figure 2, a circular domain is transformed into a rectangular domain in curvilinear coordinate system. In Figure 3, a more abstract shape is chosen to show that any domain shape can be transformed into a rectangular domain in curvilinear coordinate system. The domain chosen is Java island which is one of the main islands in Indonesia. The capital of Indonesia, Jakarta, is located on this island. As can be seen on Figure 3, the transformation of the domain is successfully done using elliptic partial differential grid generator.

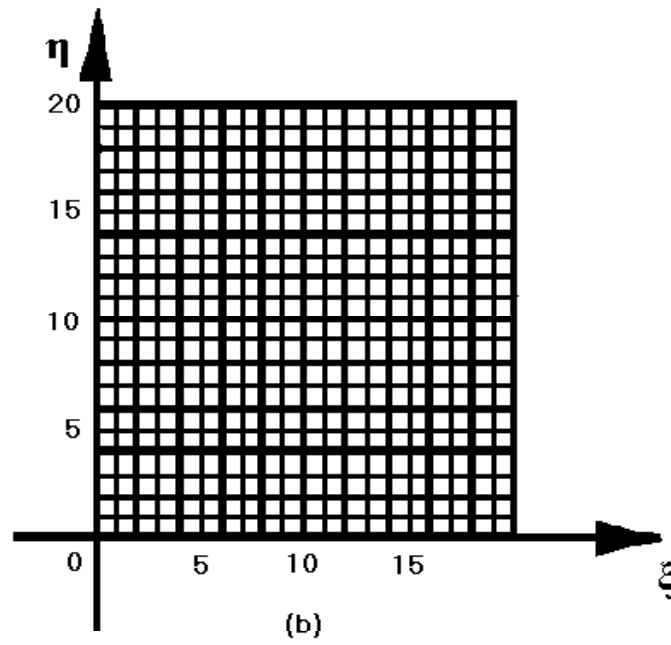
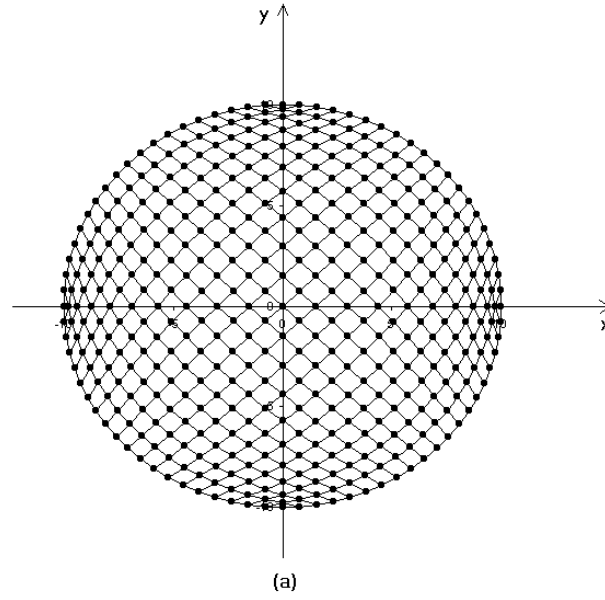
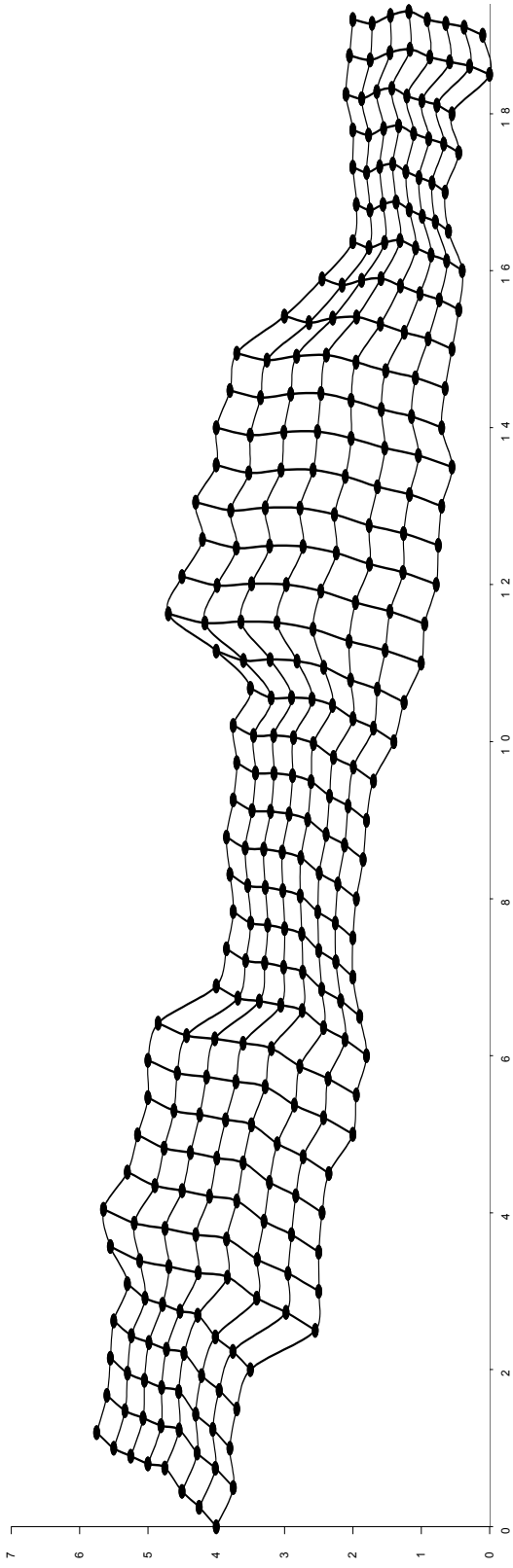
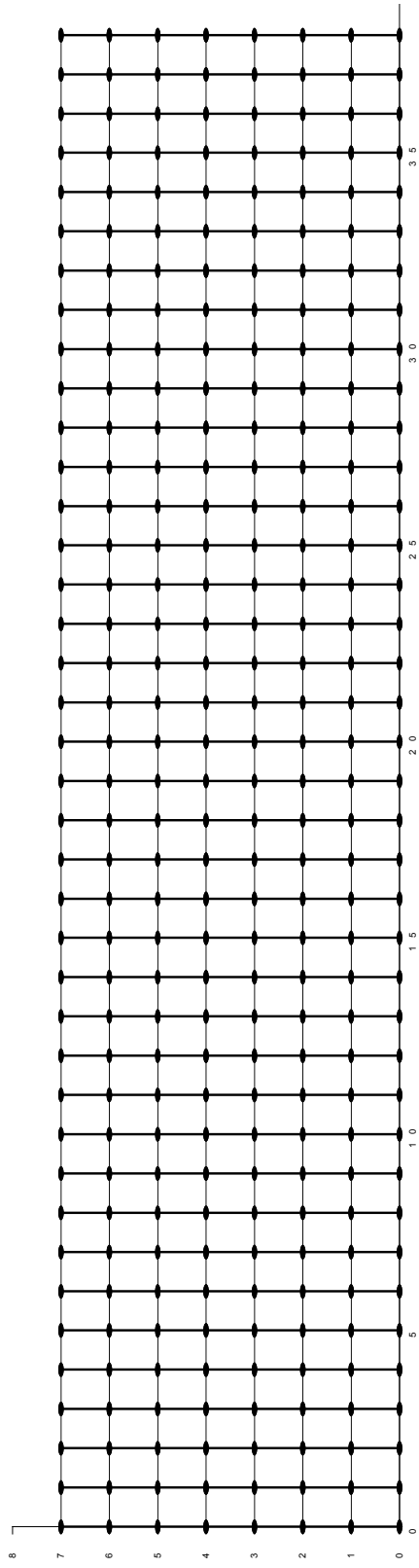


Figure 2. An example of transformation for a circular domain from (a) Cartesian coordinate to (b) curvilinear coordinate



(a)



(b)

Figure 3. An example of transformation for Java Island from (a) Cartesian coordinate to (b) curvilinear coordinate

3.4. Transformation of Governing Equations

For seepage through a levee shown in Figure 4, the governing equations can be described by the following equation which is basically the transient form of the Dupuit-Forchheimer equation (Jacobs, 1950)

$$\frac{\partial}{\partial x} \left(k_{xx} \frac{\partial h}{\partial x} + k_{xy} \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial y} \left(k_{yx} \frac{\partial h}{\partial x} + k_{yy} \frac{\partial h}{\partial y} \right) = S \frac{\partial h}{\partial t} \quad (3.17)$$

where h = head.

$k_{xx}, k_{xy}, k_{yx}, k_{yy}$ = permeability coefficient in corresponding directions.

S = specific storage coefficient.

t = time

As with any problems, boundary and initial conditions must be specified along with the governing equations for seepage problems. Boundary conditions and initial condition can be specified by the following equations

$$h(x,y,t) = H \quad \text{on free surface (F)} \quad (3.18)$$

$$\nabla h(x,y,t) \cdot n = 0 \quad \text{on free surface (F)} \quad (3.19)$$

$$h(x = 0, y, t) = H_1 \quad \text{on left side (L)} \quad (3.20)$$

$$h(x = w, y, t) = H_2 \quad \text{on right side (R)} \quad (3.21)$$

$$h(x,y,t) = H \quad \text{on seepage face (S)} \quad (3.22)$$

$$\nabla h(x,y,t) \cdot n = 0 \quad \text{on bottom side (B)} \quad (3.23)$$

$$h(x,y,t = 0) = H \quad (3.24)$$

where H = water elevation at the specified location

H_1 = water elevation on the left side of the domain

H_2 = water elevation on the right side of the domain

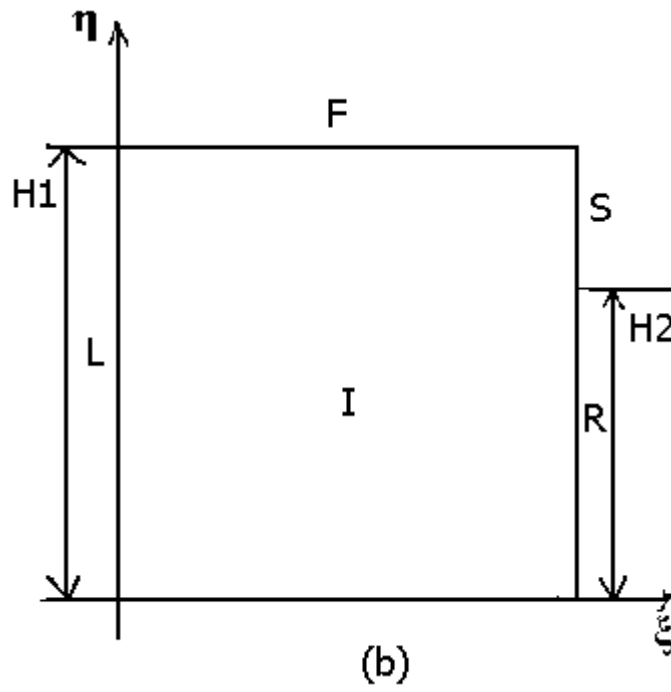
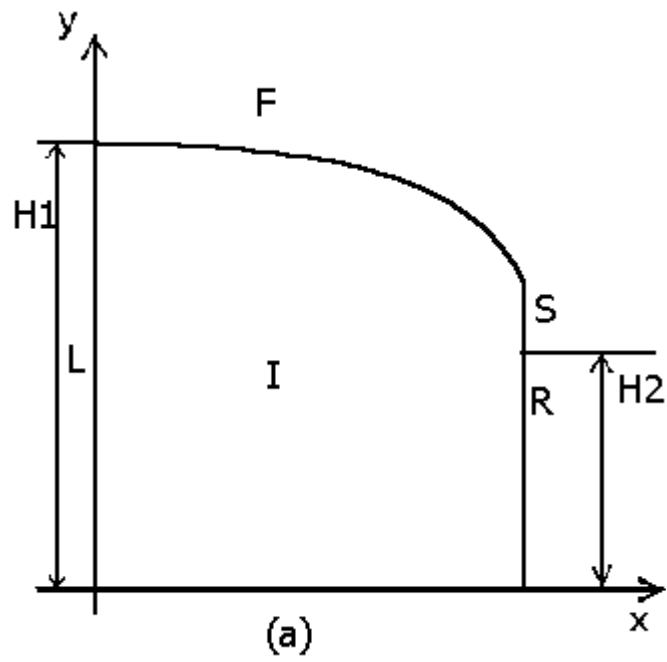


Figure 4. Cross section of a dam in (a) physical Cartesian coordinate and (b) transformed curvilinear coordinate

The groundwater flow equation in (3.17) is then transformed into curvilinear coordinates using the same procedure as described before

$$\begin{aligned}
S \frac{\partial h}{\partial t} = & \frac{\alpha' h_{\xi\xi} - 2\beta' h_{\xi\eta} + \gamma' h_{\eta\eta} + \delta h_{\xi} + \varepsilon h_{\eta}}{J^2} \\
& + \frac{\left[y_{\eta} (k_{xx})_{\xi} - y_{\xi} (k_{xx})_{\eta} \right] \cdot \left[y_{\eta} h_{\xi} - y_{\xi} h_{\eta} \right]}{J^2} \\
& + \frac{\left[x_{\eta} (k_{yy})_{\xi} - x_{\xi} (k_{yy})_{\eta} \right] \cdot \left[x_{\eta} h_{\xi} + x_{\xi} h_{\eta} \right]}{J^2} \\
& + \left[\begin{aligned}
& -x_{\eta} y_{\eta} h_{\xi\xi} + (x_{\xi} y_{\eta} + x_{\eta} y_{\xi}) h_{\xi\eta} - x_{\xi} y_{\xi} h_{\eta\eta} \\
& (x_{\xi} y_{\eta\eta} + x_{\eta\eta} y_{\xi}) h_{\xi} \\
& + \frac{2k_{xy}}{J^2} + \left(x_{\eta} y_{\eta} \frac{J_{\xi}}{J} - x_{\xi} y_{\eta} \frac{J_{\eta}}{J} - x_{\eta} y_{\xi} \frac{J_{\eta}}{J} \right) h_{\xi} \\
& (x_{\eta} y_{\xi\xi} + x_{\xi\xi} y_{\eta}) h_{\eta} \\
& + \left(x_{\xi} y_{\xi} \frac{J_{\eta}}{J} - x_{\xi} y_{\eta} \frac{J_{\xi}}{J} - x_{\eta} y_{\xi} \frac{J_{\xi}}{J} \right) h_{\eta}
\end{aligned} \right] \\
& + \frac{\left[y_{\eta} (k_{xy})_{\xi} - y_{\xi} (k_{xy})_{\eta} \right] \cdot \left[-x_{\eta} h_{\xi} + x_{\xi} h_{\eta} \right]}{J^2} \\
& + \frac{\left[x_{\eta} (k_{xy})_{\xi} - x_{\xi} (k_{xy})_{\eta} \right] \cdot \left[-y_{\eta} h_{\xi} + y_{\xi} h_{\eta} \right]}{J^2}
\end{aligned} \tag{3.25}$$

where

$$\alpha' = k_{xx} y_{\eta}^2 + k_{yy} x_{\eta}^2 \tag{3.26}$$

$$\beta' = k_{xx} y_{\xi} y_{\eta} + k_{yy} x_{\xi} x_{\eta} \tag{3.27}$$

$$\gamma' = k_{xx} y_{\xi}^2 + k_{yy} x_{\xi}^2 \tag{3.28}$$

$$\delta = k_{xx} \left[(y_{\eta} y_{\eta \zeta} - y_{\zeta} y_{\eta \eta}) - \frac{y_{\eta}^2 J_{\zeta} - y_{\zeta} y_{\eta} J_{\eta}}{J} \right] - k_{yy} \left[(x_{\eta} x_{\eta \zeta} - x_{\zeta} x_{\eta \eta}) - \frac{x_{\eta}^2 J_{\zeta} - x_{\zeta} x_{\eta} J_{\eta}}{J} \right] \quad (3.29)$$

$$\varepsilon = k_{xx} \left[(y_{\zeta} y_{\eta \zeta} - y_{\eta} y_{\zeta \zeta}) - \frac{y_{\eta}^2 J_{\eta} - y_{\zeta} y_{\eta} J_{\zeta}}{J} \right] - k_{yy} \left[(x_{\zeta} x_{\eta \zeta} - x_{\eta} x_{\zeta \zeta}) - \frac{x_{\zeta}^2 J_{\eta} - x_{\zeta} x_{\eta} J_{\zeta}}{J} \right] \quad (3.30)$$

$$J = x_{\zeta} y_{\eta} - x_{\eta} y_{\zeta} \quad (3.31)$$

To simplify the problem, permeability can be considered isotropic, then $k_{xx} = k_{yy} = k$ and $k_{xy} = k_{yx} = 0$ and equation (3.25) becomes

$$S \frac{\partial h}{\partial t} = \frac{k}{J^2} \cdot (\alpha h_{\zeta \zeta} - 2\beta h_{\zeta \eta} + \gamma h_{\eta \eta}) + \frac{[y_{\eta} k_{\zeta} - y_{\zeta} k_{\eta}] \cdot [y_{\eta} h_{\zeta} - y_{\zeta} h_{\eta}] + [-x_{\eta} k_{\zeta} + x_{\zeta} k_{\eta}] \cdot [-x_{\eta} h_{\zeta} + x_{\zeta} h_{\eta}]}{J^2} \quad (3.32)$$

Transformation of boundary conditions and initial condition into the curvilinear coordinate system was also done to ensure the uniformity of the system used and produces the following set of equations

$$h(\zeta, \eta, t) = H \quad \text{on free surface (F)} \quad (3.33)$$

$$\gamma h_{\eta} - \beta h_{\zeta} = 0 \quad \text{on free surface (F)} \quad (3.34)$$

$$h(\zeta = 0, \eta, t) = H_1 \quad \text{on left side (L)} \quad (3.35)$$

$$h(\zeta = w, \eta, t) = H_2 \quad \text{on right side (R)} \quad (3.36)$$

$$h(\zeta, \eta, t) = H \quad \text{on seepage face (S)} \quad (3.37)$$

$$\gamma h_{\eta} - \beta h_{\xi} = 0 \quad \text{on bottom side (B)} \quad (3.38)$$

$$h(\xi(\xi, \eta = 0)) = H \quad (3.39)$$

In order to obtain a solution for the problem, the first and second order partial derivatives are then represented by finite differences with respect to a point (i, j) , where i is a counter in the ξ direction and j is a counter in the η direction. These discretizations are done after the equations have been transformed. The following formulas represent the central difference formulations. The corresponding forward and backward finite differences are applied when point (i, j) being calculated is located on the edge of the domain.

$$x_{\xi} = \frac{x_{i+1,j} - x_{i-1,j}}{2} \quad (3.40)$$

$$y_{\xi} = \frac{y_{i+1,j} - y_{i-1,j}}{2} \quad (3.41)$$

$$x_{\eta} = \frac{x_{i,j+1} - x_{i,j-1}}{2} \quad (3.42)$$

$$y_{\eta} = \frac{y_{i,j+1} - y_{i,j-1}}{2} \quad (3.43)$$

$$x_{\xi\xi} = x_{i+1,j} - 2x_{i,j} + x_{i-1,j} \quad (3.44)$$

$$y_{\xi\xi} = y_{i+1,j} - 2y_{i,j} + y_{i-1,j} \quad (3.45)$$

$$x_{\eta\eta} = x_{i,j+1} - 2x_{i,j} + x_{i,j-1} \quad (3.46)$$

$$y_{\eta\eta} = y_{i,j+1} - 2y_{i,j} + y_{i,j-1} \quad (3.47)$$

$$x_{\xi\eta} = \frac{x_{i+1,j+1} - x_{i+1,j-1} - x_{i-1,j+1} + x_{i-1,j-1}}{4} \quad (3.48)$$

$$y_{\xi\eta} = \frac{y_{i+1,j+1} - y_{i+1,j-1} - y_{i-1,j+1} + y_{i-1,j-1}}{4} \quad (3.49)$$

where

$$\alpha = \frac{(x_{i,j+1} - x_{i,j-1})^2 + (y_{i,j+1} - y_{i,j-1})^2}{4} \quad (3.51)$$

$$\beta = \frac{(x_{i+1,j} - x_{i-1,j})(x_{i,j+1} - x_{i,j-1}) + (y_{i+1,j} - y_{i-1,j})(y_{i,j+1} - y_{i,j-1})}{4} \quad (3.52)$$

$$\gamma = \frac{(x_{i+1,j} - x_{i-1,j})^2 + (y_{i+1,j} - y_{i-1,j})^2}{4} \quad (3.53)$$

$$J = \frac{(x_{i+1,j} - x_{i-1,j})(y_{i,j+1} - y_{i,j-1}) - (x_{i,j+1} - x_{i,j-1})(y_{i+1,j} - y_{i-1,j})}{4} \quad (3.54)$$

CHAPTER 4

MODEL APPLICATION

4.1. Model Verification for Steady State Seepage on a Rectangular Earth Dam

The model developed in the previous section is tested on a rectangular earth dam with rock filled sides and compared with an analytical solution to ensure the model is working. Ignoring the rock filled sides, the domain of the problem becomes a rectangular earth dam. The water elevation on the left side is at 20 m and the water elevation on the right side is at 10 m. The width of the cross section of the dam is 20 m. Figure 5 shows the cross section of the real problem domain and the domain used in the computation. Transformation is done on the domain to curvilinear coordinate system. Figure 6 shows the physical domain and the transformed domain for this particular problem.

The general governing equation for a steady state seepage problem through an earth dam can be described by the following equation

$$\frac{\partial}{\partial x} \left(k_{xx} \frac{\partial h}{\partial x} + k_{xy} \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial y} \left(k_{yx} \frac{\partial h}{\partial x} + k_{yy} \frac{\partial h}{\partial y} \right) = 0 \quad (4.1)$$

For problem simplification, the domain is considered to be isotropic and homogeneous which results in the following equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (4.2)$$

Boundary conditions need to be specified on all sides of the domain to ensure proper problem solving. Two boundary conditions describe the free surface and these two conditions must be fulfilled simultaneously to correctly represent the free surface. The boundary conditions for the problem domain described in Figure 6 are as follows

$$h(x,y) = H \quad \text{on free surface (F)} \quad (4.3)$$

$$\nabla h(x,y) \cdot \hat{n} = 0 \quad \text{on free surface (F)} \quad (4.4)$$

$$h(x = 0, y) = 20 \quad \text{on left side (L)} \quad (4.5)$$

$$h(x = w, y) = 10 \quad \text{on right side (R)} \quad (4.6)$$

$$h(x,y) = H \quad \text{on seepage face (S)} \quad (4.7)$$

$$\nabla h(x,y) \cdot \hat{n} = 0 \quad \text{on bottom side (B)} \quad (4.8)$$

Transformation of the governing equation is then done to ensure compatibility with the computational domain as described in the previous section. The transformed equation assuming homogeneous and isotropic condition on the domain is as follows

$$\alpha h_{\zeta\zeta} + 2\beta h_{\zeta\eta} + \gamma h_{\eta\eta} = 0 \quad (4.9)$$

The boundary conditions are also transformed into curvilinear coordinates so that they will be compatible to the governing equations

$$h(\zeta, \eta) = H \quad \text{on free surface (F)} \quad (4.10)$$

$$\gamma h_{\eta\eta} - \beta h_{\zeta\zeta} = 0 \quad \text{on free surface (F)} \quad (4.11)$$

$$h(\zeta = 0, \eta) = 20 \quad \text{on left side (L)} \quad (4.12)$$

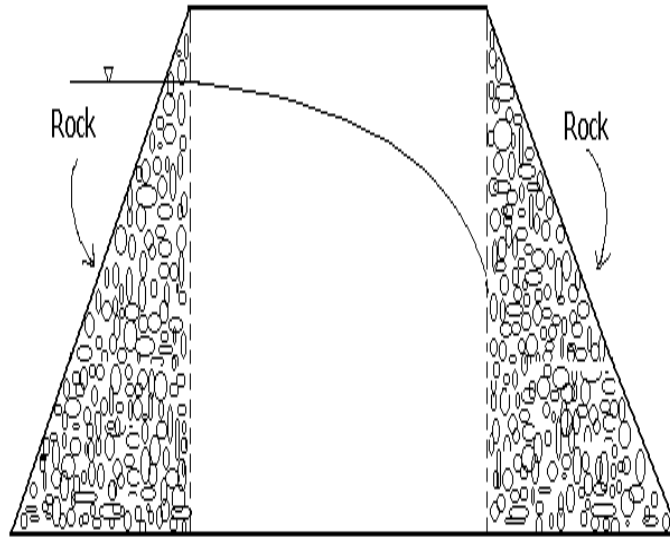
$$h(\zeta = w, \eta) = 10 \quad \text{on right side (R)} \quad (4.13)$$

$$h(\zeta, \eta) = H \quad \text{on seepage face (S)} \quad (4.14)$$

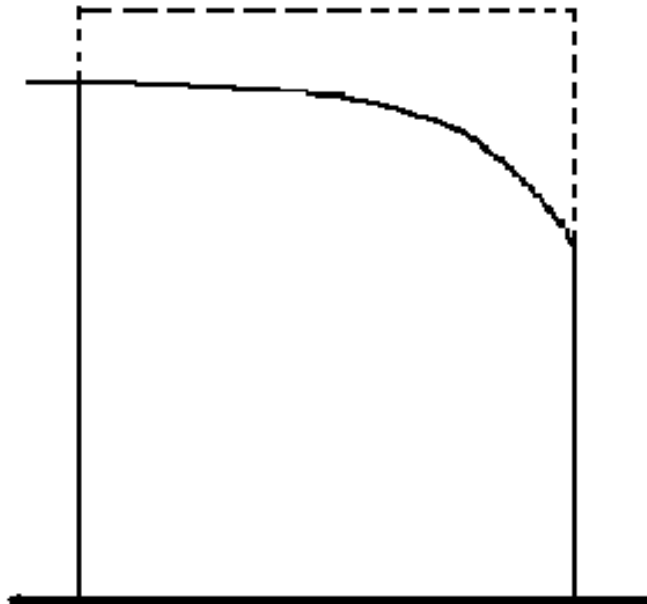
$$\gamma h_{\eta\eta} - \beta h_{\zeta\zeta} = 0 \quad \text{on bottom side (B)} \quad (4.15)$$

After all the transformations are done, the following steps are followed in solving the problem

1. Guess the initial locations of free surface and seepage face, and with the other prescribed boundary conditions, the boundaries of the domain are established.



(a)



(b)

Figure 5. (a) Cross section of an earth dam with rock filled sides and (b) domain problem representation

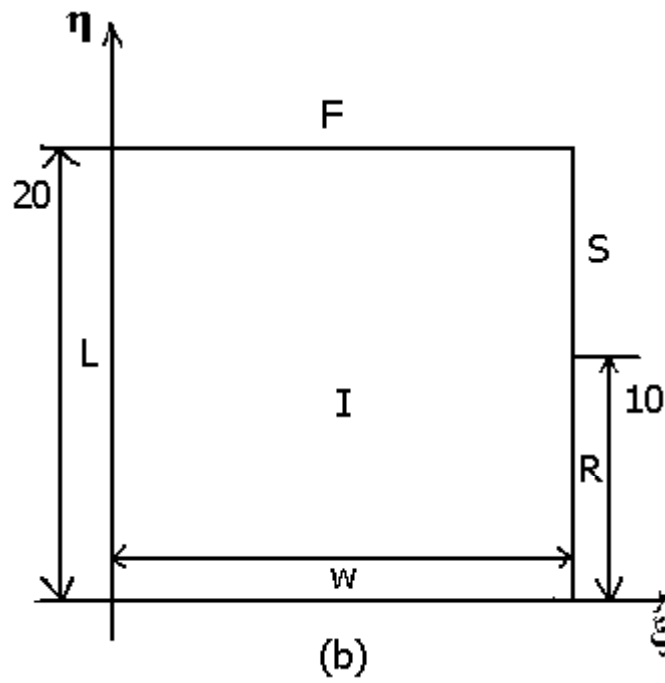
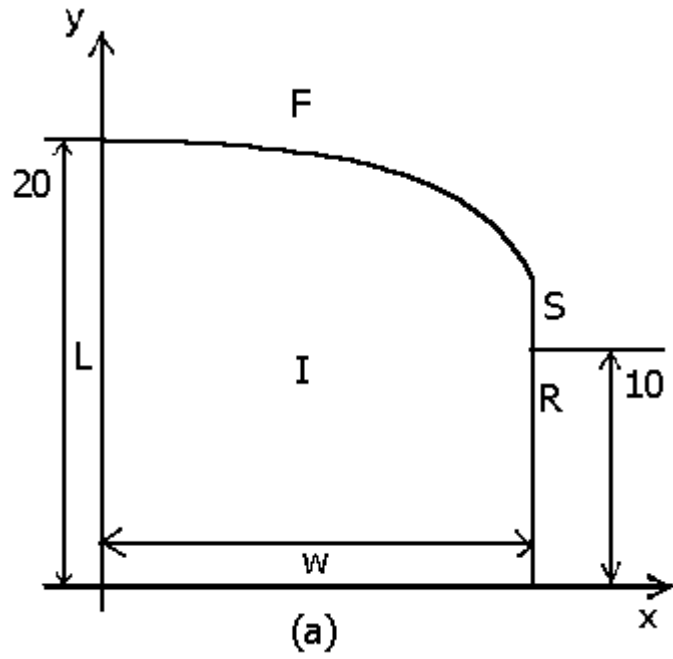


Figure 6. Cross section of a dam in (a) physical Cartesian coordinate and (b) transformed curvilinear coordinate for the prescribed boundary conditions

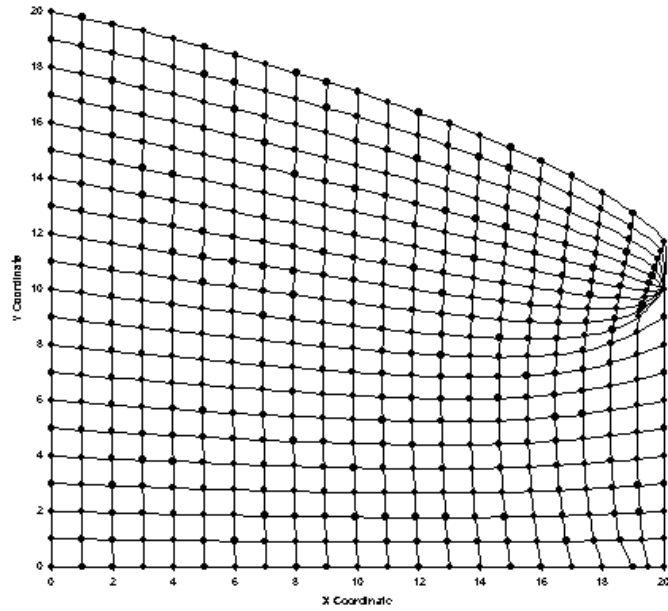
2. Generate the initial grid using transfinite interpolation. The coordinates of the boundary fitted grid are then calculated using successive over relaxation method.
3. Discretize the governing equation and use it to solve the problem at each iteration.
4. Implement corresponding boundary conditions after each iteration.
5. Compare the computed hydraulic head on the free surface and the elevation on the free surface. If the difference between them is greater than a set tolerance number, then iteration has to be repeated. If the difference is smaller than the tolerance, then the solution reaches the final iteration, and the solution obtained is the final steady solution, where head no longer changes with time.

The results of this problem can be seen on Figures 7 and 8. The final elevation of the seepage face after the final iteration is found to be 10.888 m. The seepage face elevation obtained from analytical solution is described by Uginchus (1966). The equation used for the analytical solution is as follows

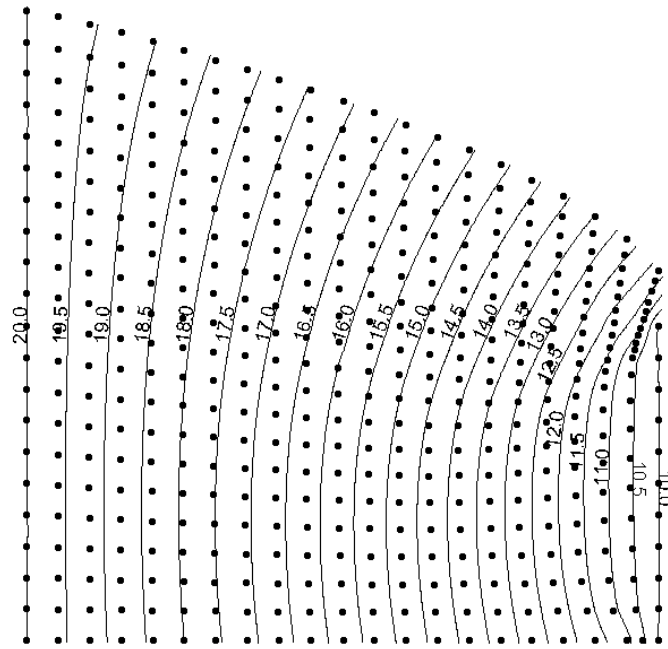
$$\frac{q}{kH} = bh_o \quad (4.16)$$

$$\frac{q}{kH} = f\left(\frac{L}{H}\right) \quad (4.17)$$

where b is a coefficient which is a function of ratio of width of the dam and the water elevation. Coefficient b varies inversely with the ratio, where it is the strongest for steep slopes. Values of b are tabulated in the appendix of the book. After coefficient b is determined, the value of q/kH is determined based on the value of coefficient b . The elevation of the seepage face then can be determined from the previous equation. For this particular problem, the elevation of the seepage face is found to be 10.8 m. This shows that BFC method can adequately described the solution for the problem.



(a)



(b)

Figure 7. (a) Grid generated and (b) the corresponding head distribution contour after iteration number 50

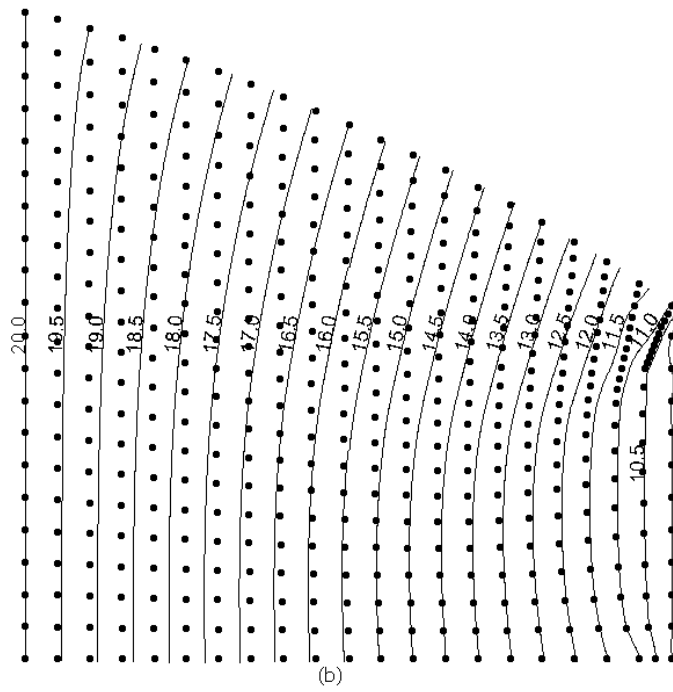
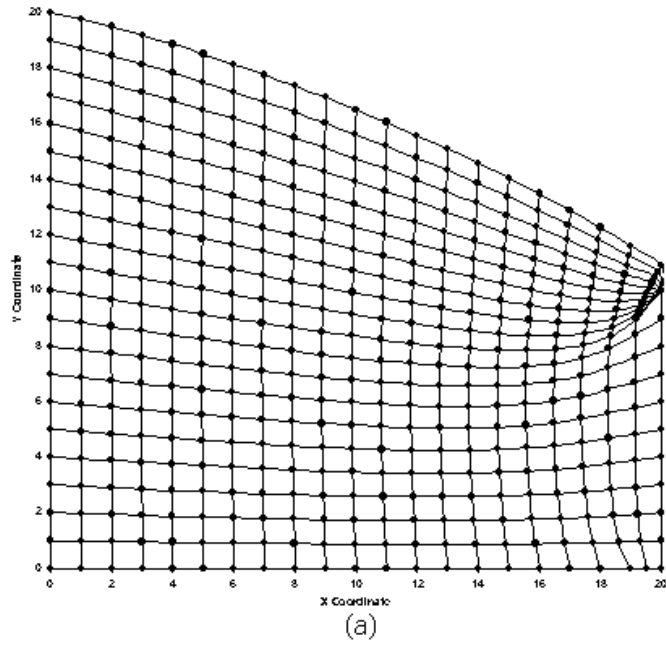


Figure 8. (a) Grid generated and (b) the corresponding head distribution contour after final iteration

4.2. Model Verification for Heat Distribution on a Circular Domain

The BFC method is tested on a circular domain for a heat distribution problem. A circle with radius 1 m is the domain used in this model. The governing equation for the cylindrical coordinate system is shown in the following equation, which is essentially a Laplace equation.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = 0 \quad (4.18)$$

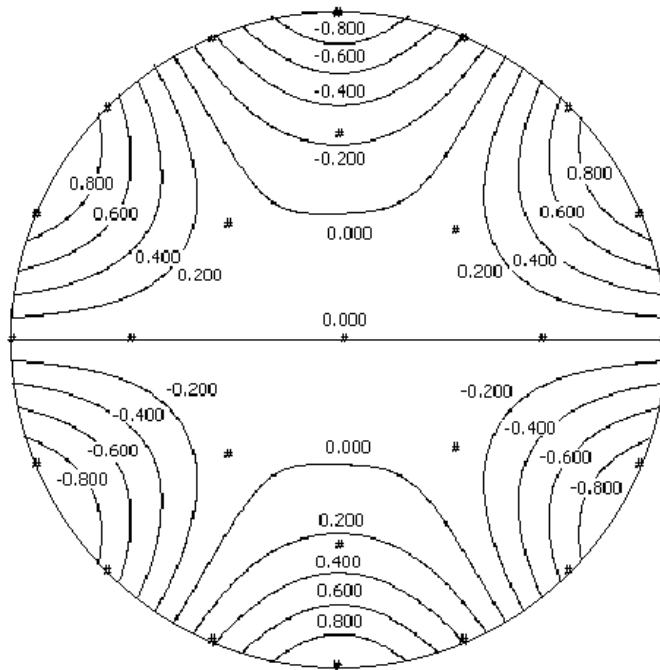
The boundary condition at the outer boundary is

$$T(1, \theta) = \sin(3\theta) \quad (4.19)$$

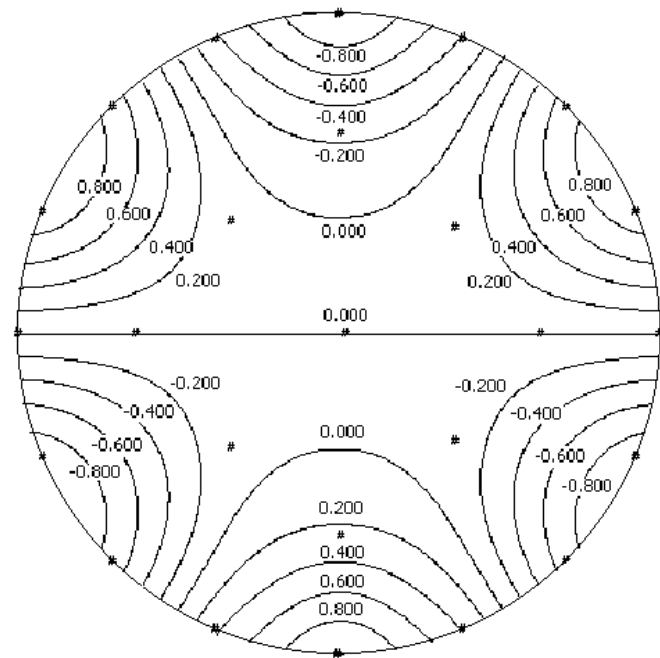
The general analytical solution for this problem type is provided by Ozisik (1968) and after application of boundary condition yields the following equation

$$T(r, \theta) = r^3 \sin(3\theta) \quad (4.20)$$

Figure 9 shows the contour of the heat distribution in the domain for analytical solution and BFC method. The values of the contour for both solutions appear to be similar with only a slight difference in the geometry near the center the circle. To better show the comparison between the two solutions, these two graphs are imposed onto each other. The comparison of these two solution techniques is shown in Figure 10. As can be seen, the BFC method described the solution quite well as the error produced by the difference between the analytical solution and the BFC method is relatively small. This shows that BFC method can be used satisfactory in cylindrical coordinate as well as in the Cartesian coordinate. For this particular problem, the analytical solution can be simply obtained as the boundary condition is simple. However since the BFC method can be used to described the solution quite accurately, for more complex problems for which analytical solution is not readily available, the BFC method can be used to solve these problems.



(a)



(b)

Figure 9. (a) The contour of heat distribution using BFC method and (b) the contour of heat distribution using analytical method.

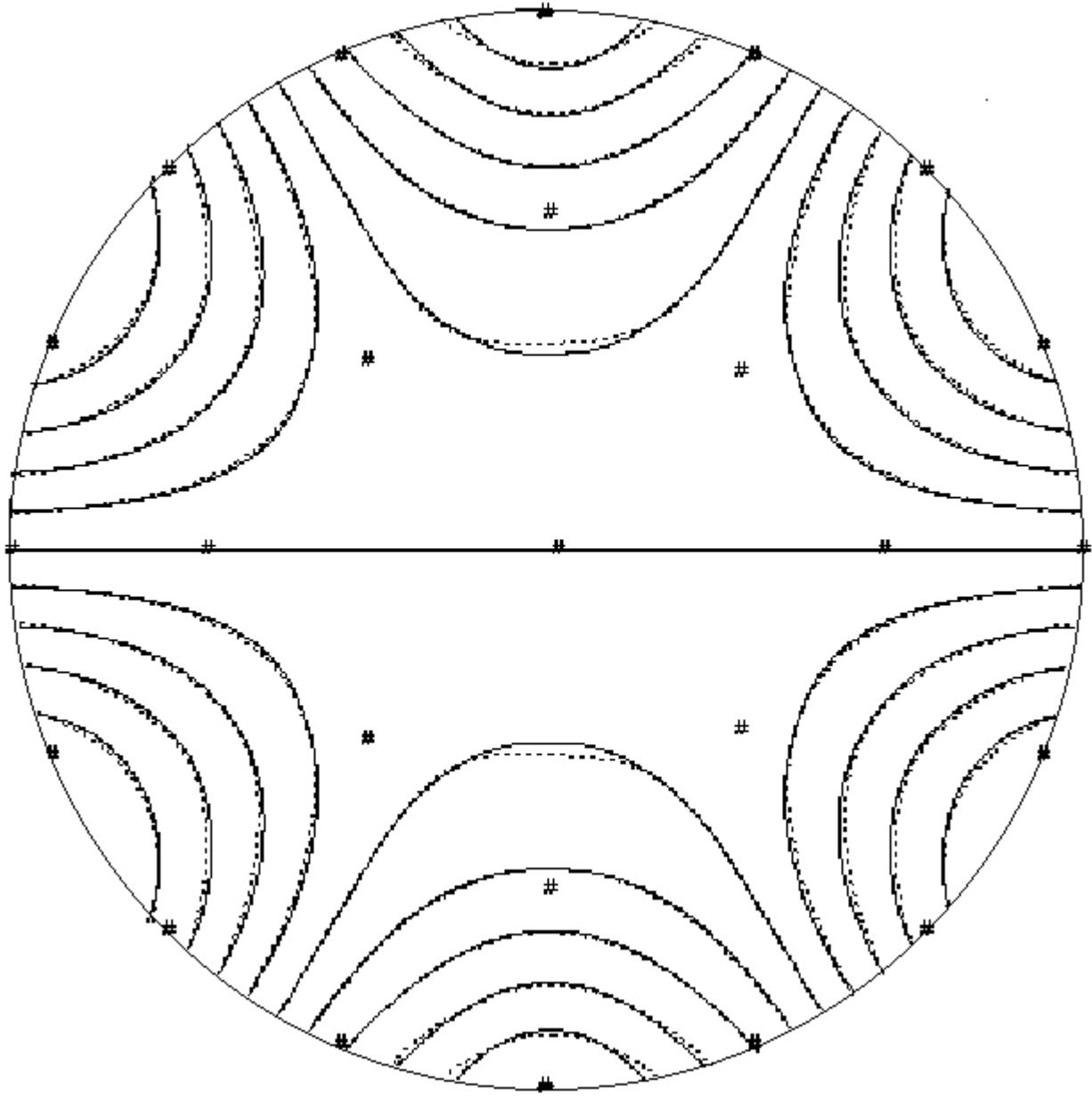


Figure 10. Comparison of contour of heat distribution for analytical solution and BFC method where the solid line is the analytical solution and the dotted lines is the BFC solution.

4.3. Model Application for Transient Seepage on a Rectangular Earth Dam

The model developed in the previous section is applied to a rectangular earth dam shown in Figure 11 where a sudden drawdown condition occurs. A rectangular earth dam with head water elevation of 20 m and a cross section width of 20 m is used in this example. Tail water elevation previously at 20 m is suddenly dropped to 6 m at time $t = 0$ and kept at that position. Since the seepage is moving slower than this drop of water table, the seepage will not reach a steady state instantaneously. Therefore, this problem can be considered as an unsteady state problem. The seepage of this unsteady state problem is then solved using BFC method.

The general governing equation for an unsteady state seepage problem through an earth dam can be described by the following equation

$$\frac{\partial}{\partial x} \left(k_{xx} \frac{\partial h}{\partial x} + k_{xy} \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial y} \left(k_{yx} \frac{\partial h}{\partial x} + k_{yy} \frac{\partial h}{\partial y} \right) = S \frac{\partial h}{\partial t} \quad (4.21)$$

For problem simplification, the domain is considered to be isotropic and homogeneous which results in the following equation

$$k \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) = S \frac{\partial h}{\partial t} \quad (4.22)$$

Boundary conditions need to be specified on all sides of the domain to ensure proper problem solving. Two boundary conditions describe the free surface and these two conditions must be fulfilled simultaneously to correctly represent the free surface. The boundary conditions for the problem domain are described as follows

$$h(x,y,t) = H \quad \text{on free surface (F)} \quad (4.23)$$

$$\nabla h(x,y,t) \cdot \hat{n} = 0 \quad \text{on free surface (F)} \quad (4.24)$$

$$h(x = 0, y, t) = 20 \quad \text{on left side (L)} \quad (4.25)$$

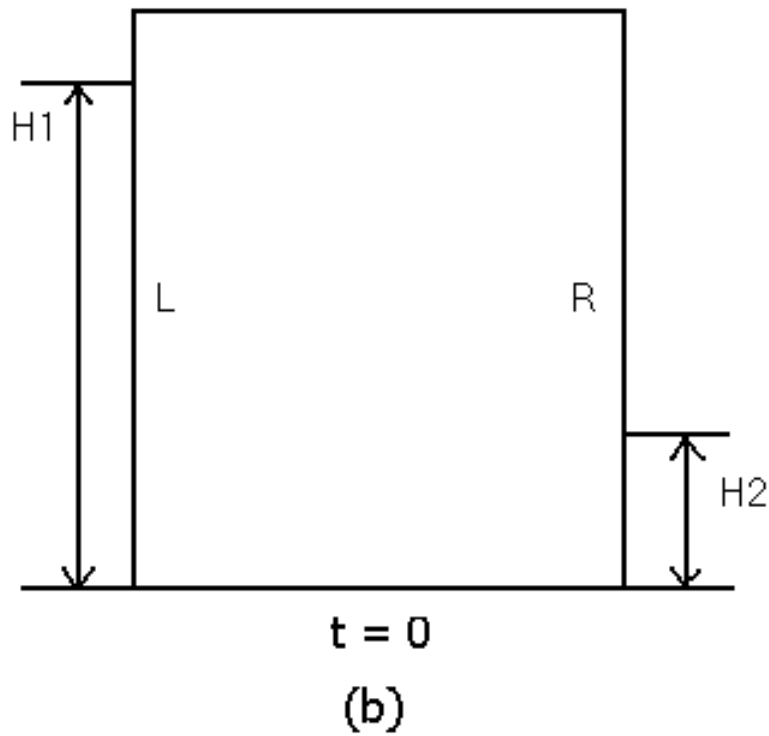
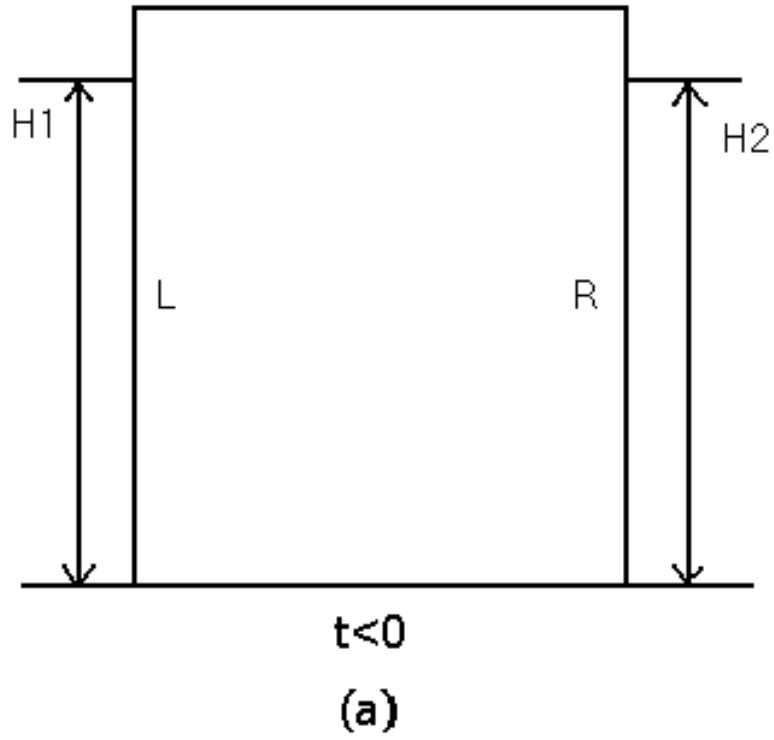


Figure 11. Domain with falling water table at (a) before the start of the simulation, $t < 0$ and (b) at the start of the simulation, $t = 0$

$$h(x = w, y, t) = 6 \quad \text{on right side (R)} \quad (4.26)$$

$$h(x, y, t) = H \quad \text{on seepage face (S)} \quad (4.27)$$

$$\nabla h(x, y, t) \cdot \hat{n} = 0 \quad \text{on bottom side (B)} \quad (4.28)$$

The previous and initial conditions are described by the following equations

$$h(x = w, y, t < 0) = 20 \quad (4.29)$$

$$h(x = w, y, t = 0) = 6 \quad (4.30)$$

Transformation of the governing equation is then done to ensure compatibility with the computational domain as described in the previous section. The transformed equation assuming homogeneous and isotropic condition on the domain is as follows

$$\alpha h_{\zeta\zeta} + 2\beta h_{\zeta\eta} + \gamma h_{\eta\eta} = S \frac{\partial h}{\partial t} \quad (4.31)$$

The boundary conditions are also transformed into curvilinear coordinates so that they will be compatible to the governing equations

$$h(\zeta, \eta, t) = H \quad \text{on free surface (F)} \quad (4.32)$$

$$\gamma h_{\eta} - \beta h_{\zeta} = 0 \quad \text{on free surface (F)} \quad (4.33)$$

$$h(\zeta = 0, \eta, t) = 20 \quad \text{on left side (L)} \quad (4.34)$$

$$h(\zeta = w, \eta, t) = 6 \quad \text{on right side (R)} \quad (4.35)$$

$$h(\zeta, \eta, t) = H \quad \text{on seepage face (S)} \quad (4.36)$$

$$\gamma h_{\eta} - \beta h_{\zeta} = 0 \quad \text{on bottom side (B)} \quad (4.37)$$

Todd (1959) suggested that for each time step of the solution, the seepage reach a steady state solution. Equation (4.31) can then be simplified to a steady state equation

$$\alpha h_{\zeta\zeta} + 2\beta h_{\zeta\eta} + \gamma h_{\eta\eta} = 0 \quad (4.38)$$

The formula for calculation of new head position after each time step is described by the following equation

$$\delta H = \frac{\delta t}{S_y} (v_y + v_x \tan a) \quad (4.39)$$

where

$\tan a$ = the head difference between two nodes adjacent to the nodes being calculated at the moment divided by the length between the two nodes.

S_y = specific yield.

Applying Darcy's equation into equation (4.39) yields the following equation

$$\delta H = -\frac{\delta t}{S_y} \left(k_y \frac{\partial h}{\partial y} + k_x \frac{\partial h}{\partial x} \tan a \right) \quad (4.40)$$

Combining equation (4.40) with

$$\frac{\partial h}{\partial x} = \left(\frac{\partial h}{\partial y} - 1 \right) \tan a \quad (4.41)$$

results in the following equation

$$\delta H = \frac{-\delta t}{S_y} \left[k_y \frac{\partial h}{\partial y} - k_x \left(1 - \frac{\partial h}{\partial y} \right) \tan^2 a \right] \quad (4.42)$$

If the problem is assumed to be isotropic and homogeneous, equation (4.42) simplified into

$$\Delta H = \frac{-k\Delta t}{S_y w} \left[\frac{\partial h}{\partial y} - \left(1 - \frac{\partial h}{\partial y} \right) \tan^2 a \right] \quad (4.43)$$

Equation (4.43) is then transformed into curvilinear coordinate system

$$\Delta H = \frac{-k\Delta t}{S_y w} \left[\left(\frac{\gamma h_\eta - \beta h_\zeta}{J\sqrt{\gamma}} \right) - \left(1 - \frac{\gamma h_\eta - \beta h_\zeta}{J\sqrt{\gamma}} \right) \tan^2 a \right] \quad (4.44)$$

Equation (4.44) is used to calculate initial head drop for each time iteration. After initial coordinate of the new free surface is calculated, the boundary conditions are calculated again until steady state is achieved. For this problem, the values $k = 0.12$ m/d and $S_y = 0.15$ are used.

For initial description where the water level is at 20 m, a solution is not done as the hydraulic head will be the same throughout the domain. Time steps are needed for the next solution procedures. Determination of scale of the time steps is important to ensure accurate representation for unsteady state seepage. A general rule suggested by Todd (1959) is that the average time steps are determined by the maximum movement of the water table. The time steps on average should be about ten percent of the maximum movement to ensure accurate solution. In this instance, the first two time steps are chosen to be smaller than ten percent of the maximum movement, while the ensuing time steps are proportionally larger. As suggested by Herbert (1965), the rate of fall of the water table on a vertical line becomes less as the slope of the water table increases. Using these rules, the time steps for this particular problem are taken as follows

Time step 1: $0.04 S_y w/k$

Time step 2: $0.08 S_y w/k$

Time step 3: $0.14 S_y w/k$

Time step 4: $0.23 S_y w/k$

Time step 5: $0.35 S_y w/k$

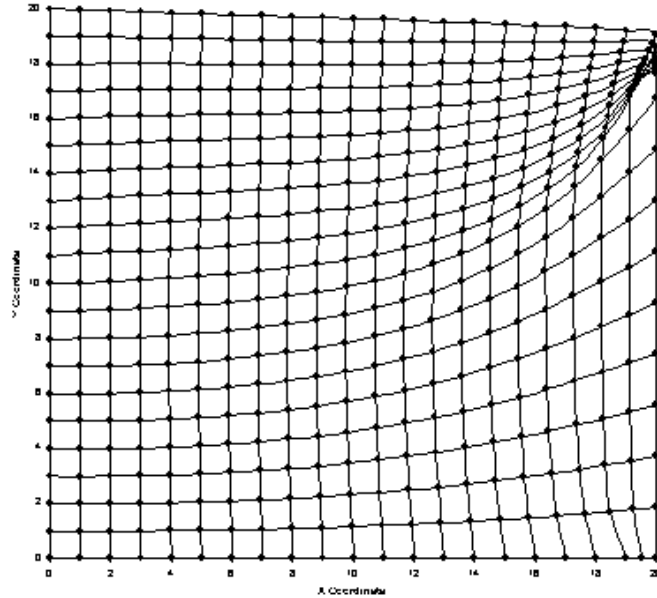
Time step 6: $0.60 S_y w/k$

The following steps are used in solving unsteady state seepage

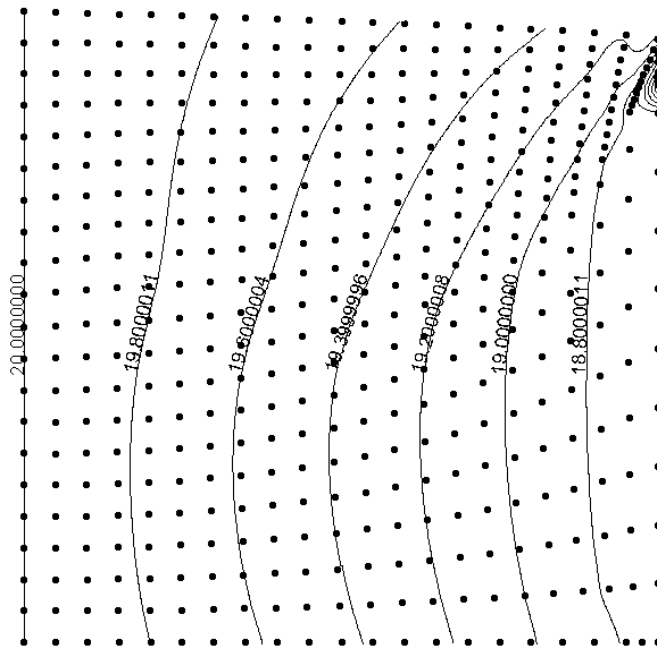
1. Guess the initial locations of free surface and seepage face, and with the other prescribed boundary conditions, the boundaries of the domain are established.

2. Generate the initial grid using transfinite interpolation. The coordinates of the boundary fitted grid are then calculated using successive over relaxation method.
3. Discretize the governing equation and use it to solve the problem at each iteration.
4. Implement boundary conditions after each iteration.
5. Compare the computed hydraulic head on the free surface and the elevation on the free surface. If the difference between them is greater than a set tolerance number, then iteration has to be repeated. If the difference is smaller than the tolerance, then the solution reaches the final iteration, and the solution obtained is the final steady solution.
6. The next elevation is calculated using the equation specified.
7. These steps are repeated until the water table reaches a steady state.

Figure 12 – 18 show the grid generation and the head distribution for successive time steps. The grid points are more concentrated on the seepage face near the free surface since this area needs the most attention in the problem. Distribution of grid points can be adjusted if other areas are thought to need more attention. As can be seen from the figures, the solution is stable and this illustrates further the advantage of BFC method compared to the finite difference solution. The solution is also smooth as a result of Laplace grid generator. The steady state is reached when height and hydraulic head no longer changes at each iteration. At the first time step, the free surface has dropped slightly and the seepage face elevation is found to be at 18.9 m. The subsequent time steps are then done according to the rules suggested before and the free surface drop become more steep as time progresses. For each time steps, iterations are done until the solution reaches steady state where the hydraulic head in the domain no longer changes at each iteration. At final steady state, the solution no longer changes with time as the water elevation is kept at that level. The seepage face elevation at the final steady state is found to be at 8.4 m.

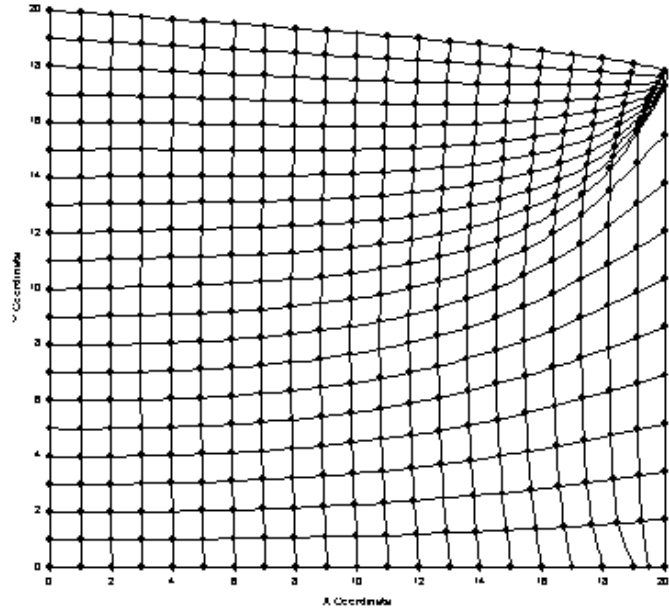


(a)

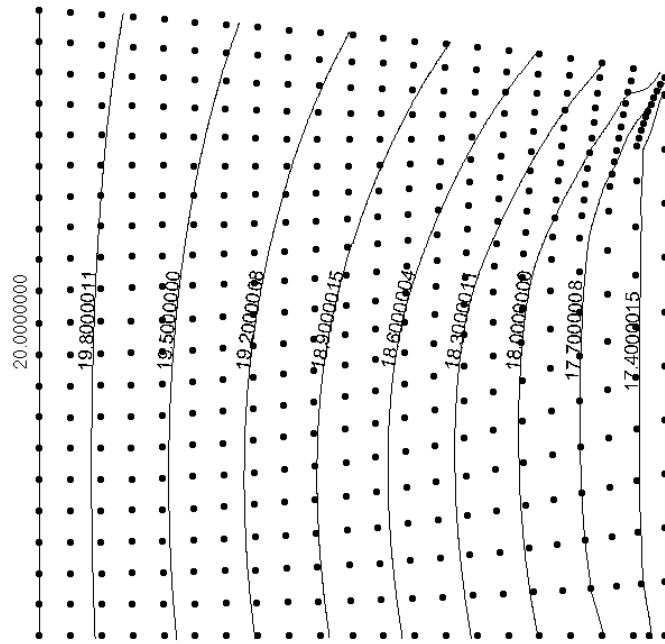


(b)

Figure 12. (a) Grid distribution and (b) corresponding head distribution contour for time step 1

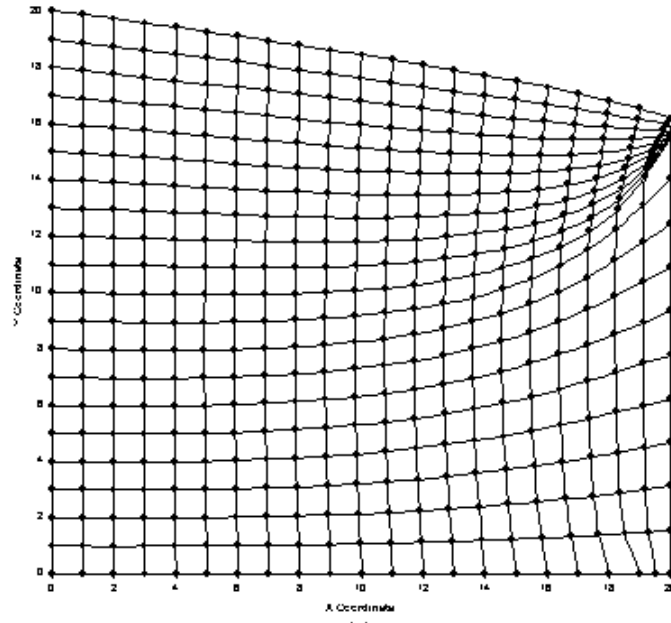


(a)

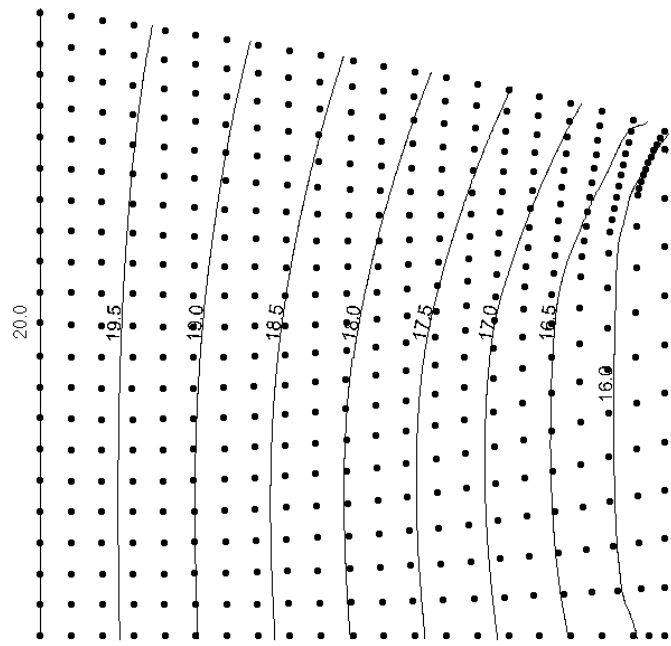


(b)

Figure 13. (a) Grid generated and (b) corresponding head distribution contour for time step 2

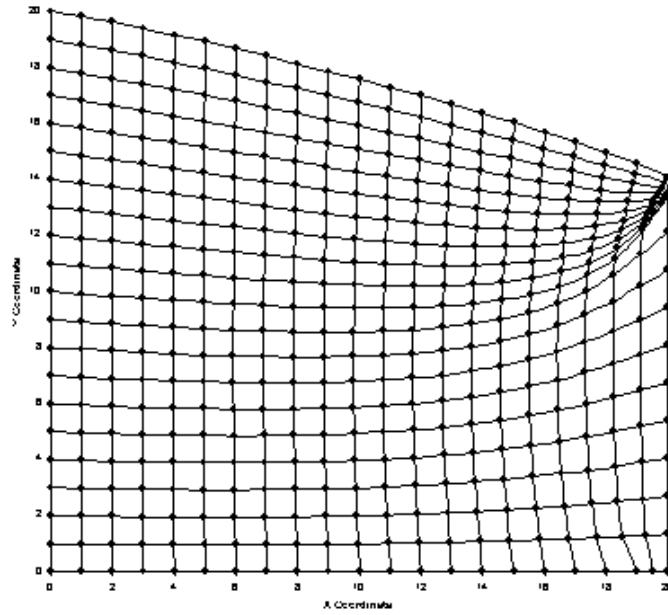


(a)

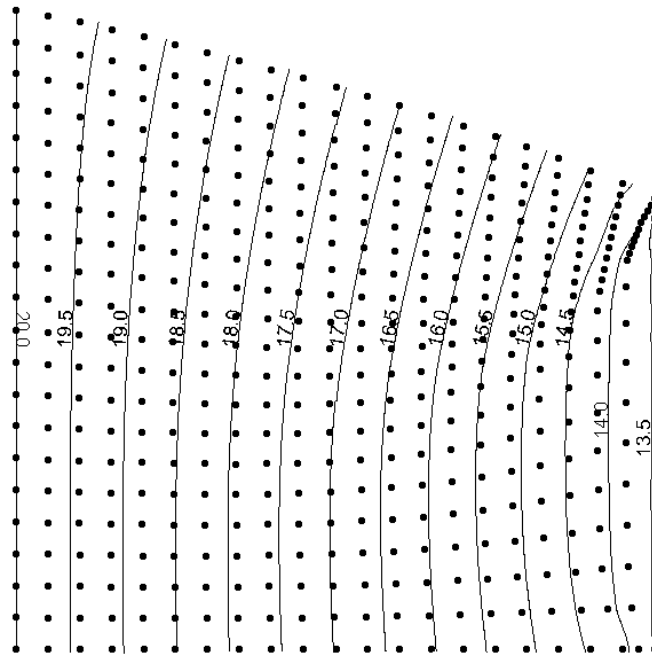


(b)

Figure 14. (a) Grid generated and (b) corresponding head distribution contour for time step 3.

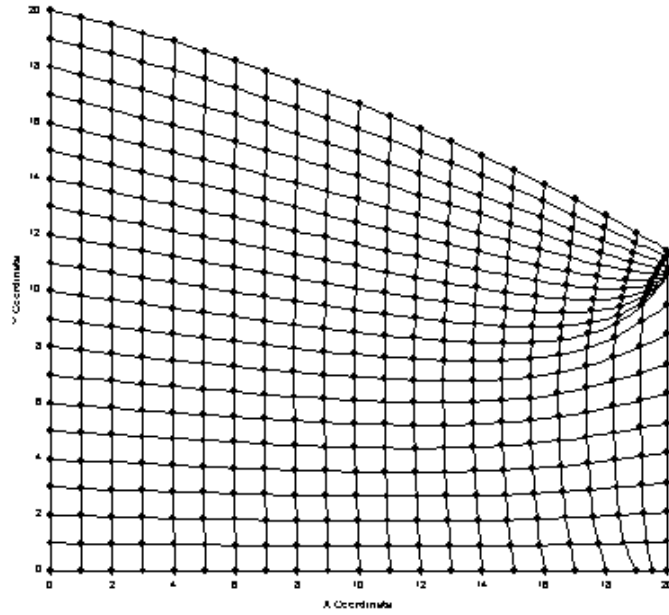


(a)

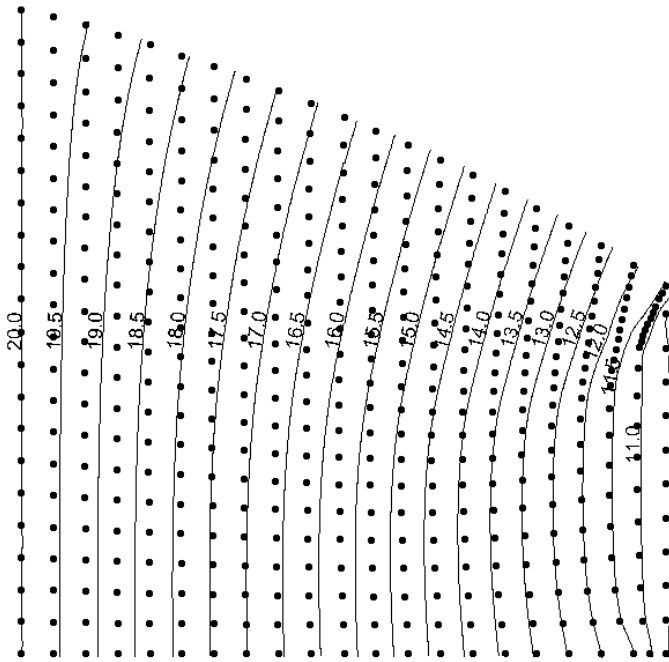


(b)

Figure 15. (a) Grid generated and (b) corresponding head distribution contour for time step 4



(a)



(b)

Figure 16. (a) Grid generated and (b) corresponding head distribution contour for time step 5

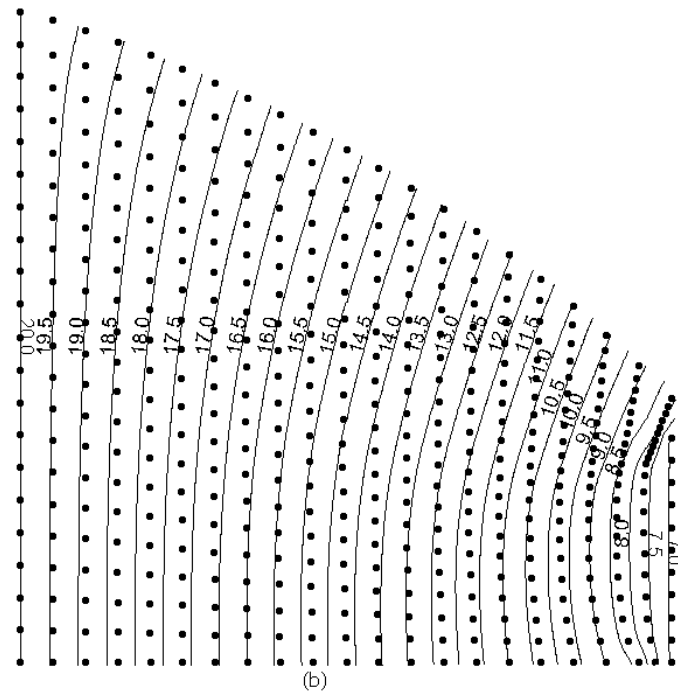
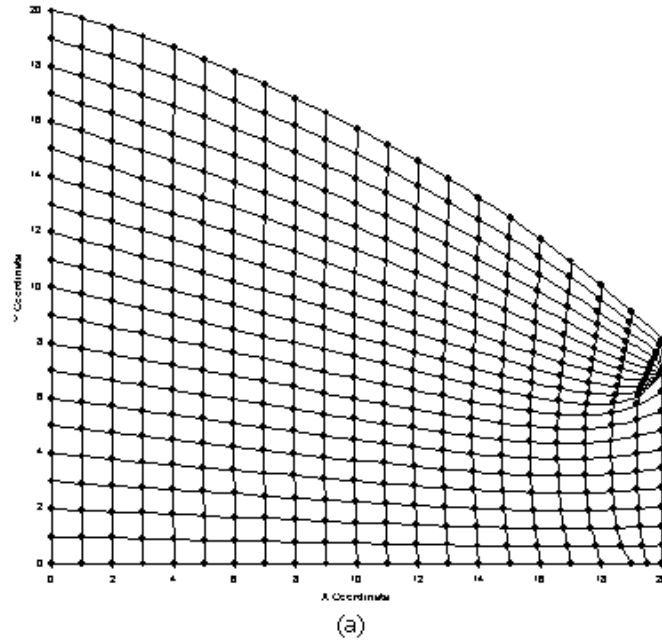
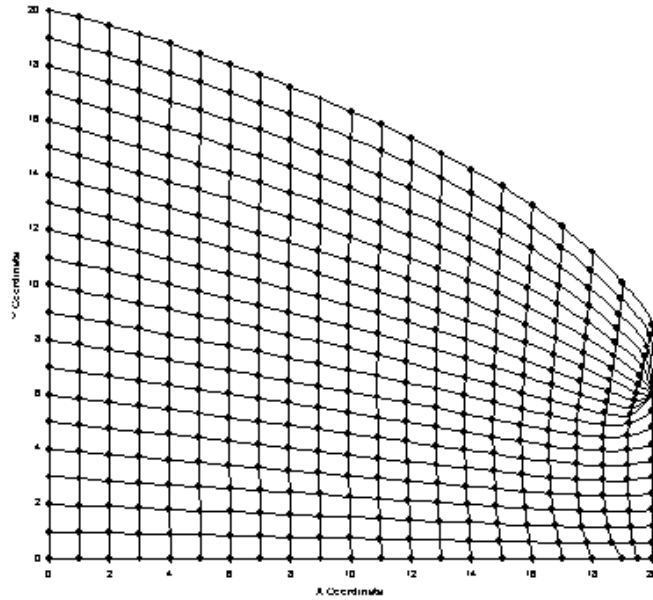
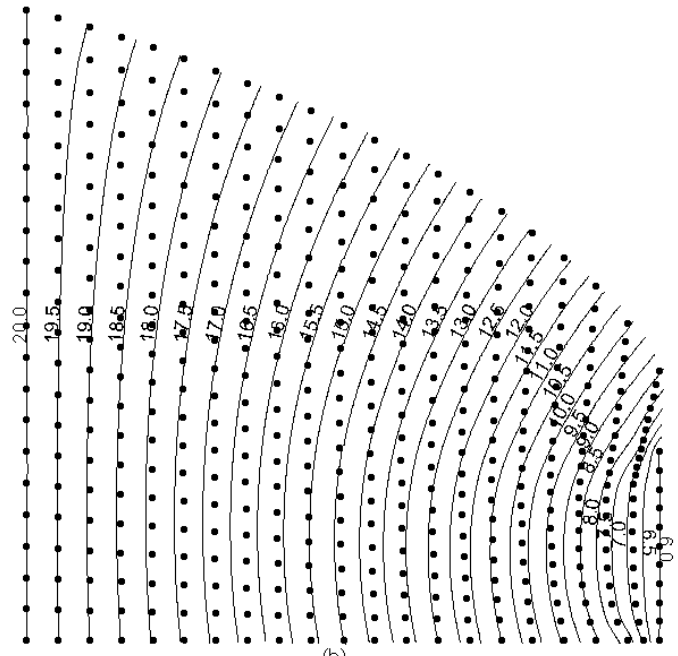


Figure 17. (a) Grid generated and (b) corresponding head distribution contour for time step 6



(a)



(b)

Figure 18. (a) Grid generated and (b) corresponding head distribution contour for final steady state

4.4. Model Application for Transient Seepage on a Levee

The model developed on the previous chapter is applied on a levee to show the versatility of the model in handling different kinds of domain. A levee with different slopes on the riverside and the landside is used as the domain of this problem. The slope on the landside of the levee is steeper than the slope on the riverside of the levee. The height of the levee is 8 m and the width of the crest is 20 m. After transformation by equations described in the previous chapter, the domain becomes rectangular in the computational coordinate system. A cross section of the levee in the physical domain and the computational domain is shown in Figure 19. To simulate unsteady state condition, the water level on the riverside of the levee is suddenly dropped to 2 m elevation at time $t = 0$ from its previous position at 5 m. The water elevation is kept at 2 m until the solution reaches a steady state. There is no tailwater on the landside of the levee. These conditions are illustrated in Figure 20.

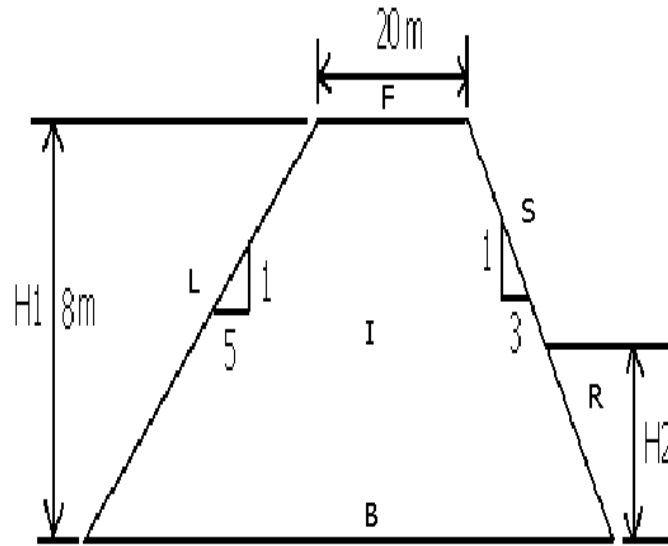
The general governing equation for an unsteady state seepage problem through the levee can be described by the following equation

$$\frac{\partial}{\partial x} \left(k_{xx} \frac{\partial h}{\partial x} + k_{xy} \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial y} \left(k_{yx} \frac{\partial h}{\partial x} + k_{yy} \frac{\partial h}{\partial y} \right) = S \frac{\partial h}{\partial t} \quad (4.45)$$

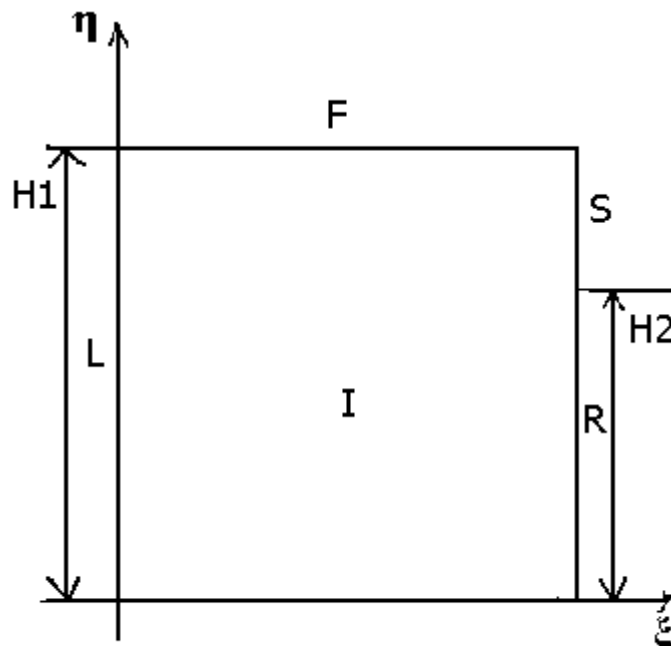
For problem simplification, the domain is considered to be isotropic and homogeneous which results in the following equation

$$k \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) = S \frac{\partial h}{\partial t} \quad (4.46)$$

Boundary conditions need to be specified on all sides of the domain to ensure proper problem solving. Two boundary conditions describe the free surface and these two conditions must be fulfilled simultaneously to correctly represent the free surface. The boundary conditions for the problem domain are described as follows



(a)



(b)

Figure 19. Domain with falling water table at (a) start of the simulation, $t = 0$ and (b) end of simulation, $t = \infty$

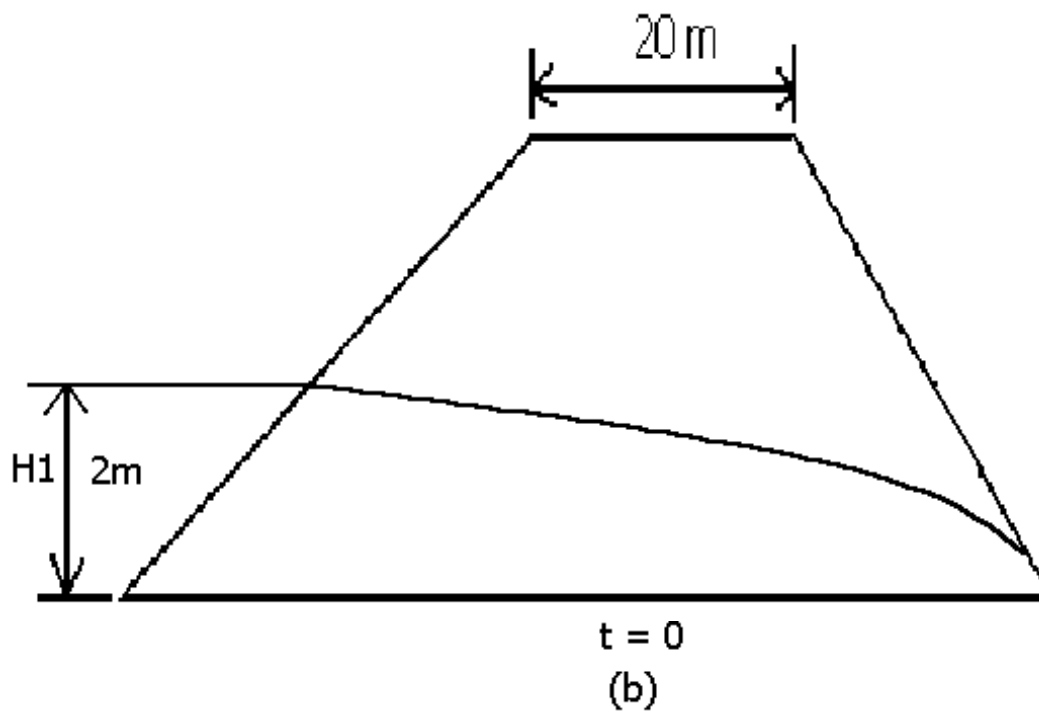
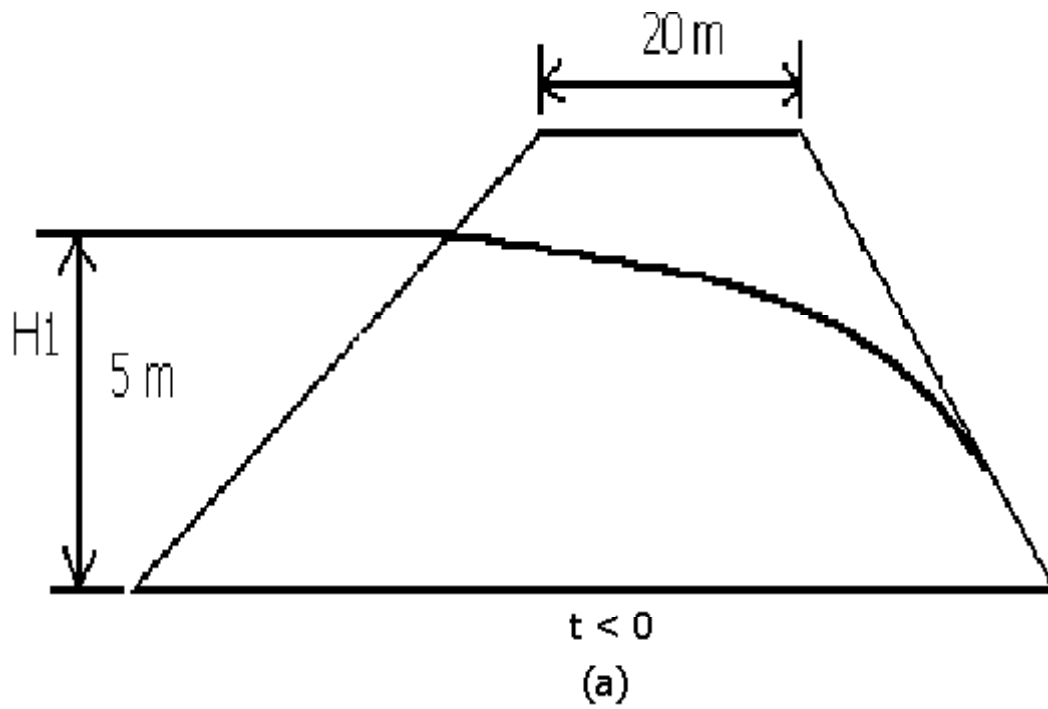


Figure 20. Levee with falling water table at (a) before the start of the simulation, $t < 0$ and (b) at the start of the simulation, $t = 0$

$$h(x,y,t) = H \quad \text{on free surface (F)} \quad (4.47)$$

$$\nabla h(x,y,t) \cdot \hat{n} = 0 \quad \text{on free surface (F)} \quad (4.48)$$

$$h(x = 0, y, t) = 2 \quad \text{on left side (L)} \quad (4.49)$$

$$h(x = w, y, t) = 0 \quad \text{on right side (R)} \quad (4.50)$$

$$h(x, y, t) = H \quad \text{on seepage face (S)} \quad (4.51)$$

$$\nabla h(x, y, t) \cdot \hat{n} = 0 \quad \text{on bottom side (B)} \quad (4.52)$$

The initial and final conditions are described by the following equations

$$h(x = w, y, t < 0) = 5 \quad (4.53)$$

$$h(x = w, y, t = 0) = 2 \quad (4.54)$$

Transformation of the governing equation is then done to ensure compatibility with the computational domain as described in the previous section. The transformed equation assuming homogeneous and isotropic condition on the domain is as follows

$$\alpha h_{\zeta\zeta} + 2\beta h_{\zeta\eta} + \gamma h_{\eta\eta} = S \frac{\partial h}{\partial t} \quad (4.55)$$

The boundary conditions are also transformed into curvilinear coordinates so that they will be compatible to the governing equations

$$h(\zeta, \eta, t) = H \quad \text{on free surface (F)} \quad (4.56)$$

$$\gamma h_{\eta} - \beta h_{\zeta} = 0 \quad \text{on free surface (F)} \quad (4.57)$$

$$h(\zeta = 0, \eta, t) = 2 \quad \text{on left side (L)} \quad (4.58)$$

$$h(\zeta = w, \eta, t) = 0 \quad \text{on right side (R)} \quad (4.59)$$

$$h(\zeta, \eta, t) = H \quad \text{on seepage face (S)} \quad (4.60)$$

$$\gamma h_{\eta} - \beta h_{\zeta} = 0 \quad \text{on bottom side (B)} \quad (4.61)$$

Todd (1959) suggested that for each time step of the solution, the seepage reach a steady state solution. Equation (4.55) can then be simplified to a steady state equation

$$ah_{\xi\xi} + 2\beta h_{\xi\eta} + \gamma h_{\eta\eta} = 0 \quad (4.62)$$

Using the method described previously, the formula for calculation of new head position after each time step is described by the following equation

$$\Delta H = \frac{-k\Delta t}{S_y w} \left[\frac{\partial h}{\partial y} - \left(1 - \frac{\partial h}{\partial y} \right) \tan^2 a \right] \quad (4.63)$$

Equation (4.63) is then transformed into curvilinear coordinate system

$$\Delta H = \frac{-k\Delta t}{S_y w} \left[\left(\frac{\gamma h_{\eta} - \beta h_{\xi}}{J\sqrt{\gamma}} \right) - \left(1 - \frac{\gamma h_{\eta} - \beta h_{\xi}}{J\sqrt{\gamma}} \right) \tan^2 a \right] \quad (4.64)$$

Equation (4.64) is used to calculate initial head drop for each time iteration. After initial coordinate of the new free surface is calculated, the boundary conditions are calculated again until steady state is achieved. For this problem, the values $k = 0.12$ m/d and $S_y = 0.15$ are used.

The time steps for this particular problem are taken as follows

Time step 1: 0

Time step 2: $0.04 S_y w/k$

Time step 3: $0.08 S_y w/k$

Time step 4: $0.15 S_y w/k$

Time step 5: $0.26 S_y w/k$

Time step 6: $0.38 S_y w/k$

Time step 7: $0.54 S_y w/k$

Time step 8: $0.72 S_y w/k$

Time step 9: final steady state

The following steps are used in solving unsteady state seepage

1. Guess the initial locations of free surface and seepage face, and with the other prescribed boundary conditions, the boundaries of the domain are established.
2. Generate the initial grid using transfinite interpolation. The coordinates of the boundary fitted grid are then calculated using successive over relaxation method.
3. Discretize the governing equation and use it to solve the problem at each iteration.
4. Implement boundary conditions after each iteration.
5. Compare the computed hydraulic head on the free surface and the elevation on the free surface. If the difference between them is greater than a set tolerance number, then iteration has to be repeated. If the difference is smaller than the tolerance, then the solution reaches the final iteration, and the solution obtained is the final steady solution.
6. The next elevation is calculated using the equation specified.
7. These steps are repeated until the water table reaches a steady state.

Figure 21 – 29 show the grid generation and the head distribution for successive time steps. For each time steps, the solutions shown are solution at steady state, where height and hydraulic head no longer changes with each iteration. The elevation of the seepage face from the start of the simulation decreases until it reaches the end of the simulation. This is to be expected as the ratio of the width of the levee and the water elevation increases as the water elevation decreases. This in turn produces less steep slopes and therefore decreases the amount of the seepage face. In Figure 21, the elevation of the seepage face it found to be at 1.55 m, while in Figure 29 at the end of the simulation, the elevation of the seepage face is found to be at 0.41 m.

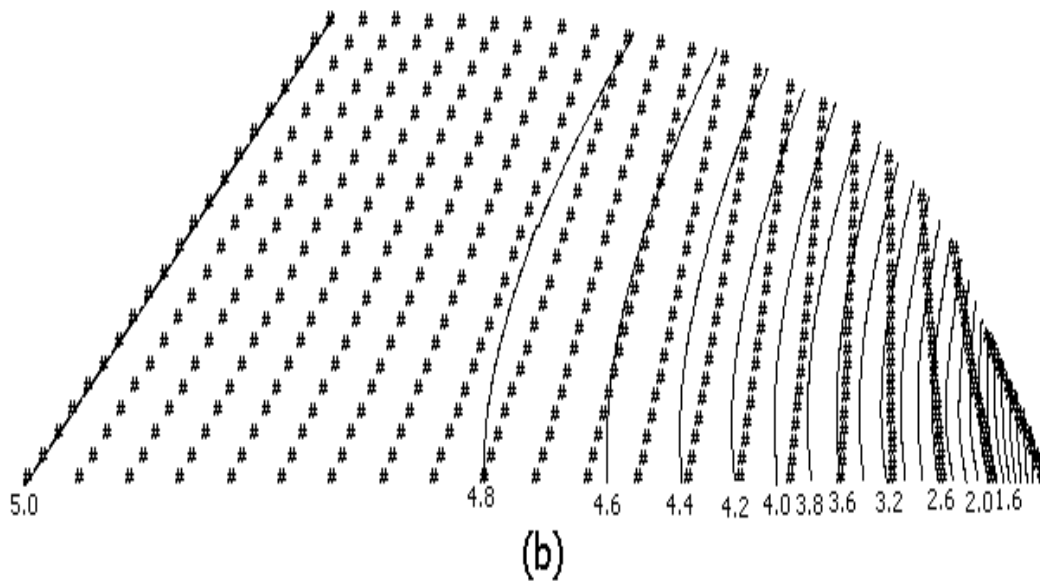
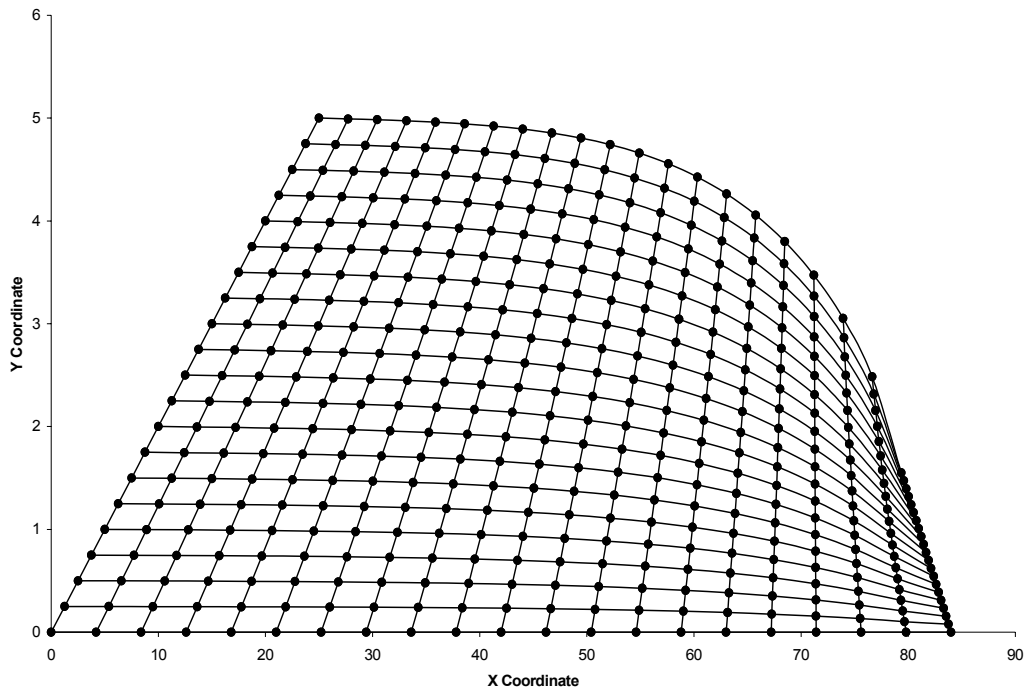


Figure 21. (a) Grid distribution and (b) corresponding head distribution contour for time step 1

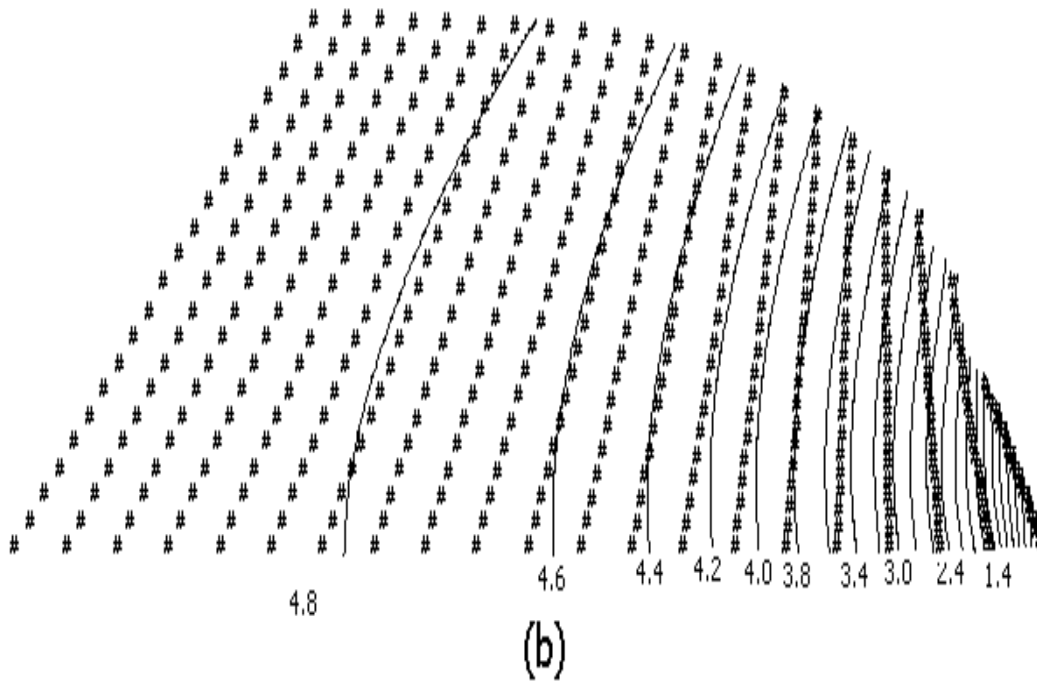
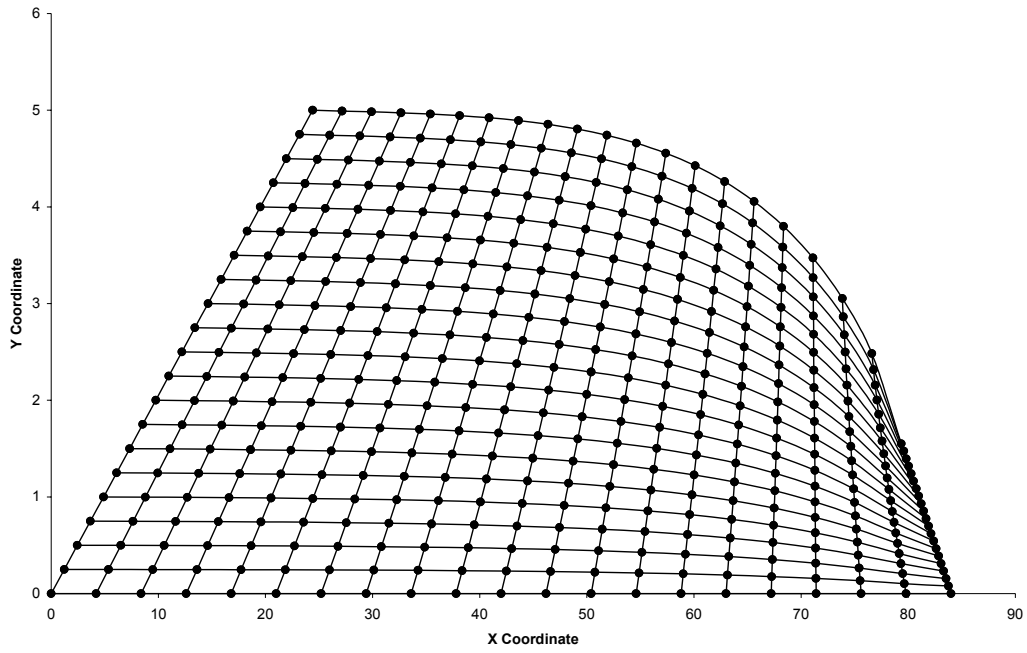


Figure 22. (a) Grid generated and (b) corresponding head distribution contour for time step 2

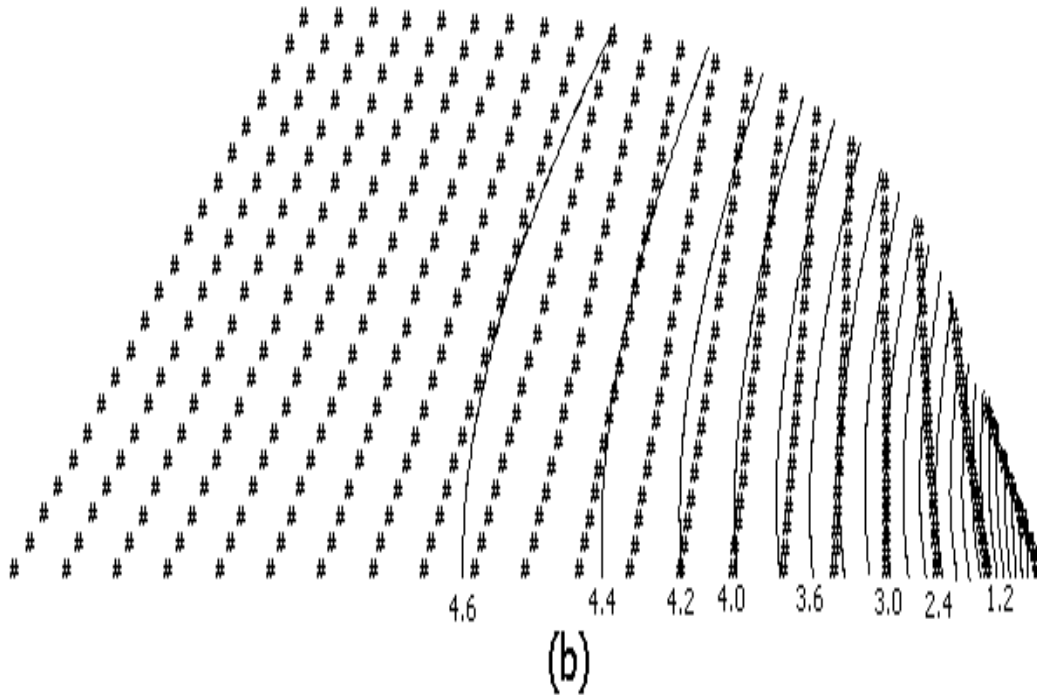
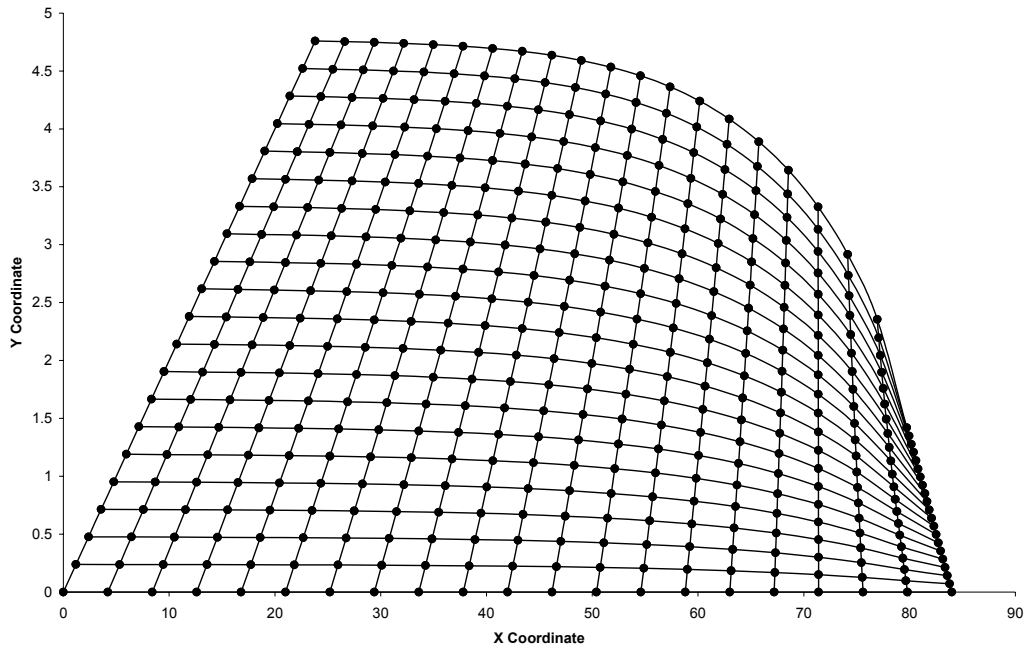


Figure 23. (a) Grid generated and (b) corresponding head distribution contour for time step 3.

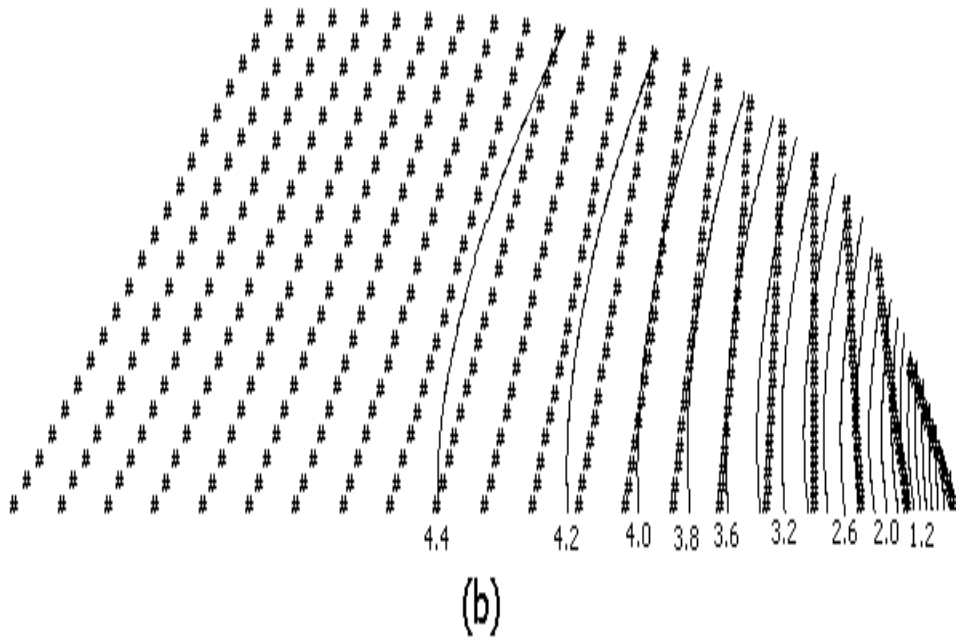
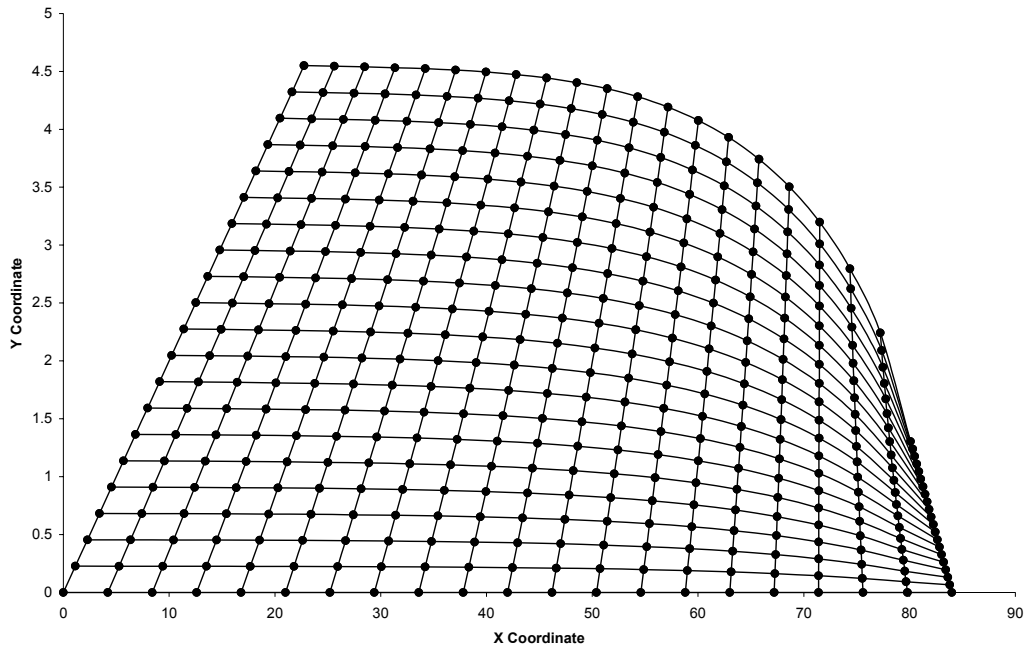


Figure 24. (a) Grid generated and (b) corresponding head distribution contour for time step 4

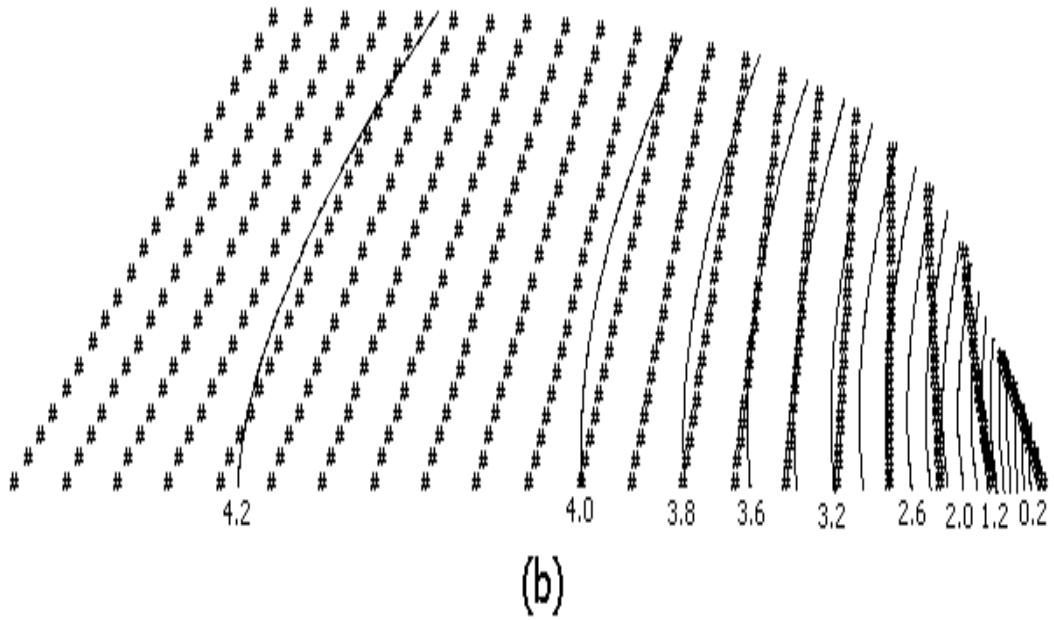
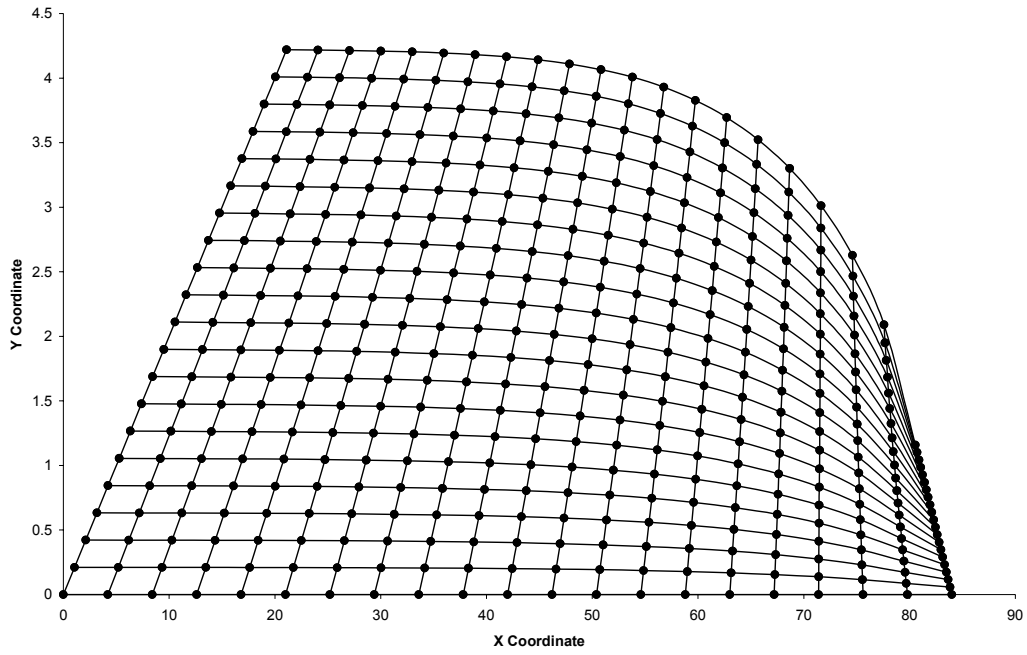
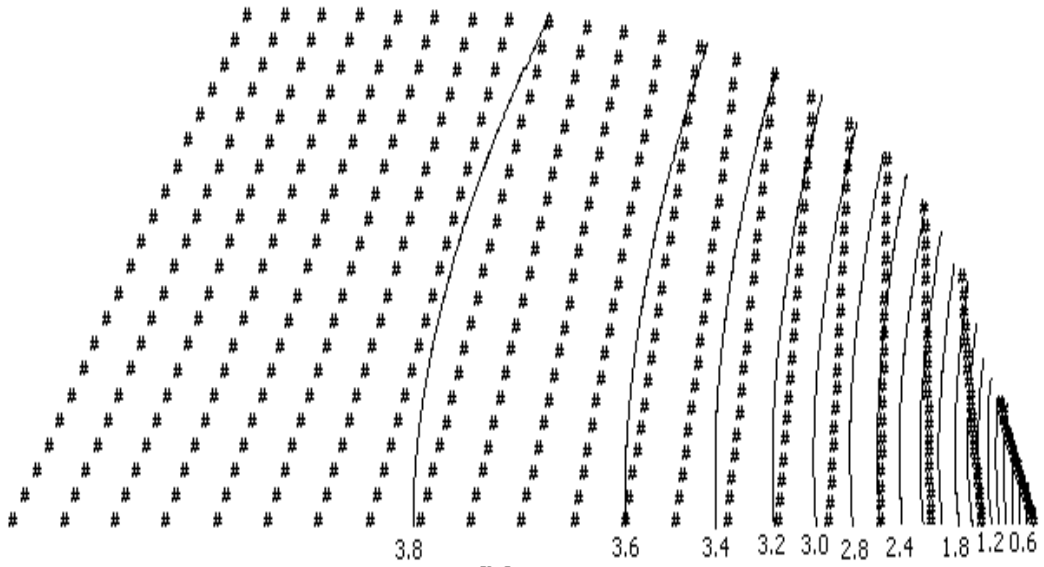
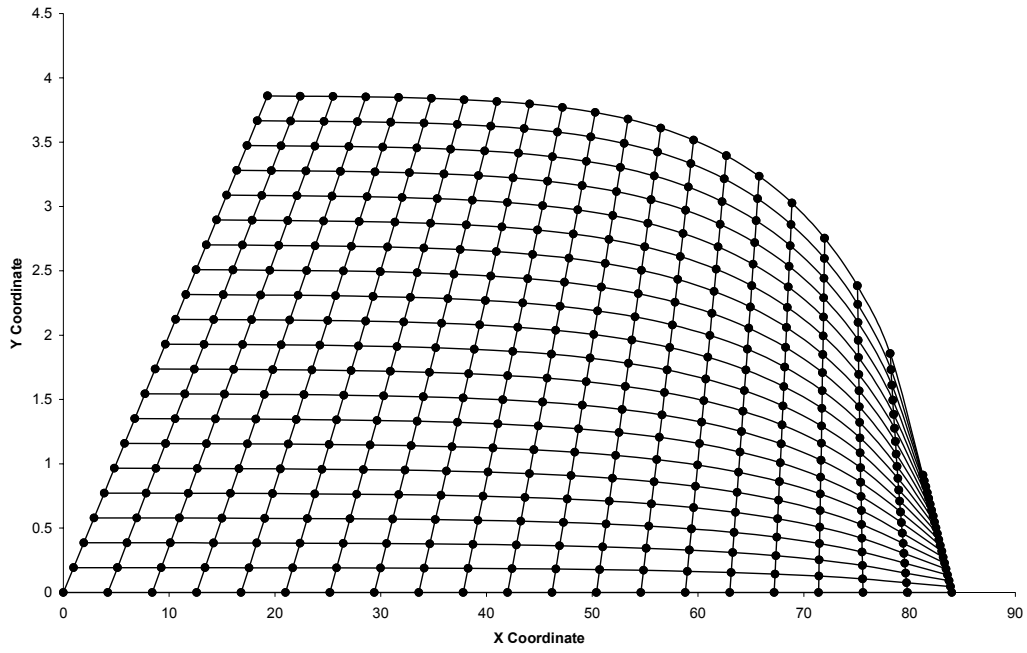
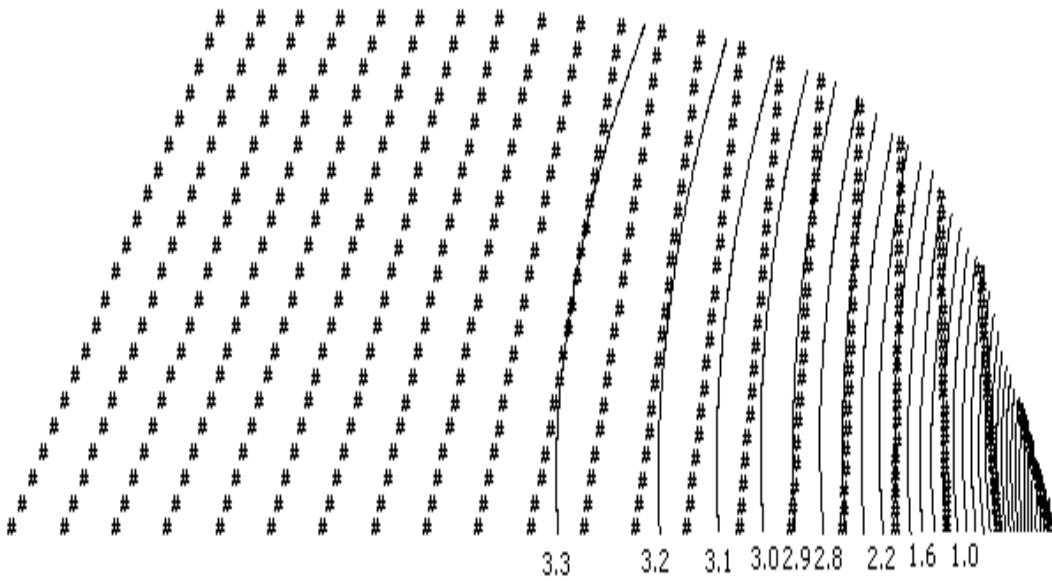
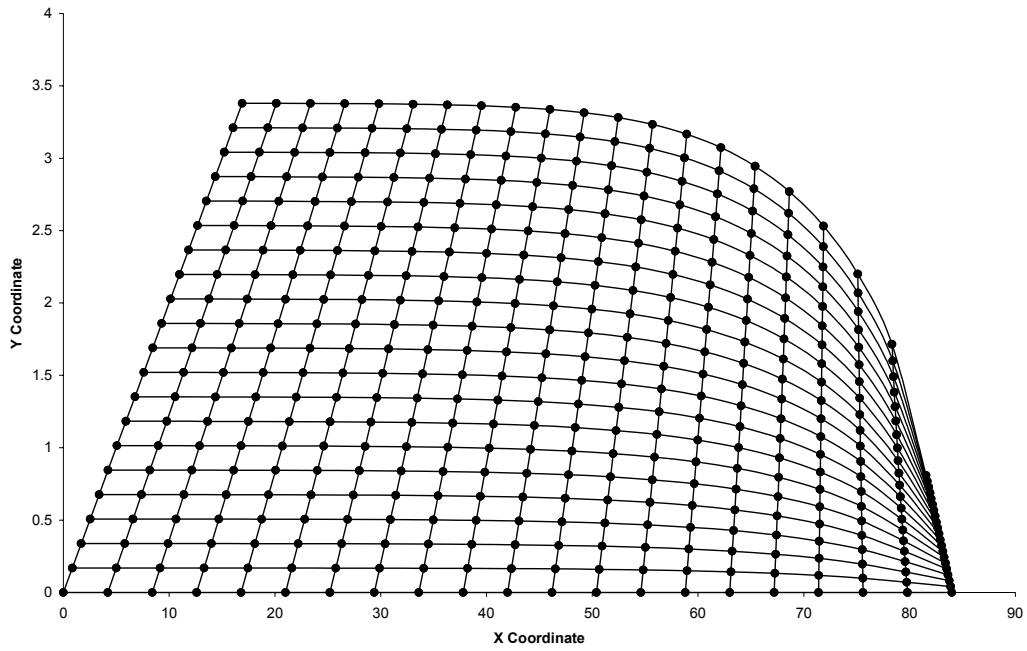


Figure 25. (a) Grid generated and (b) corresponding head distribution contour for time step 5



(b)

Figure 26. (a) Grid generated and (b) corresponding head distribution contour for time step 6



(b)

Figure 27. (a) Grid generated and (b) corresponding head distribution contour for time step 7

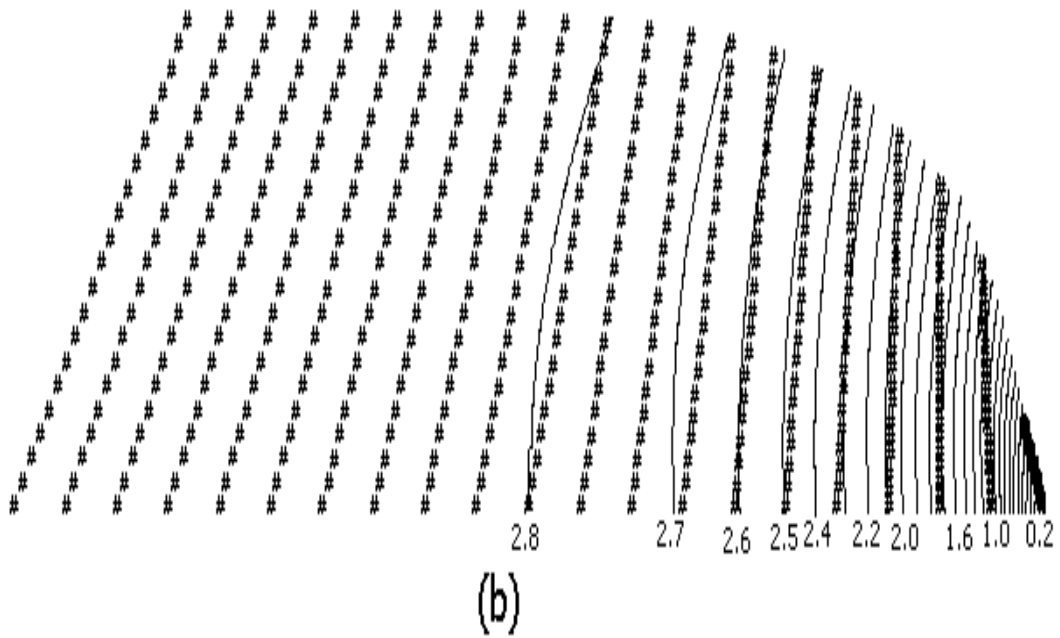
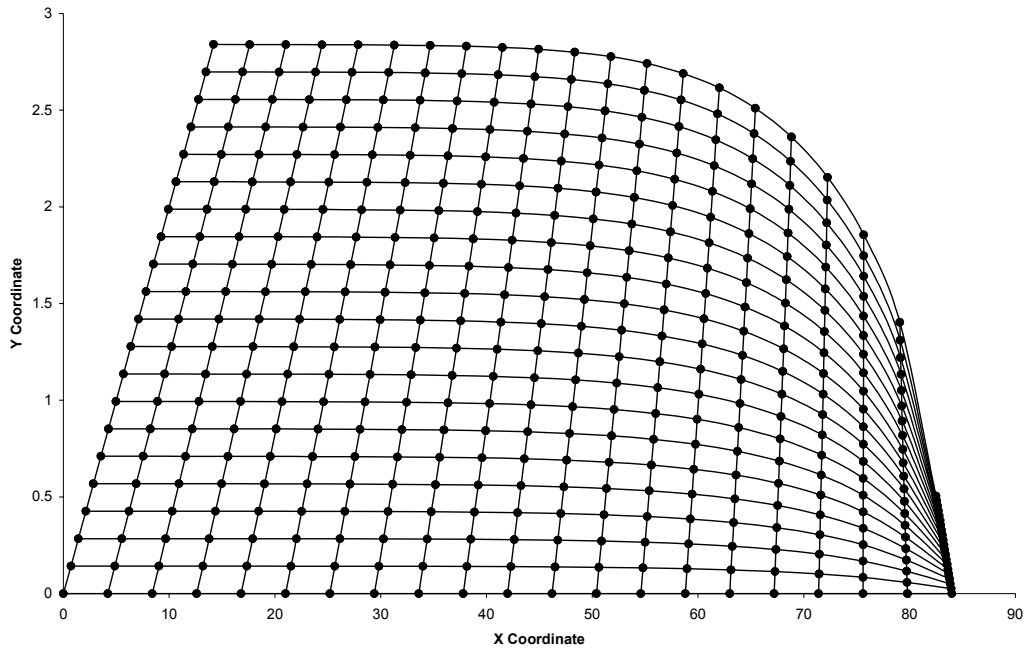


Figure 28. (a) Grid generated and (b) corresponding head distribution contour for time step 8

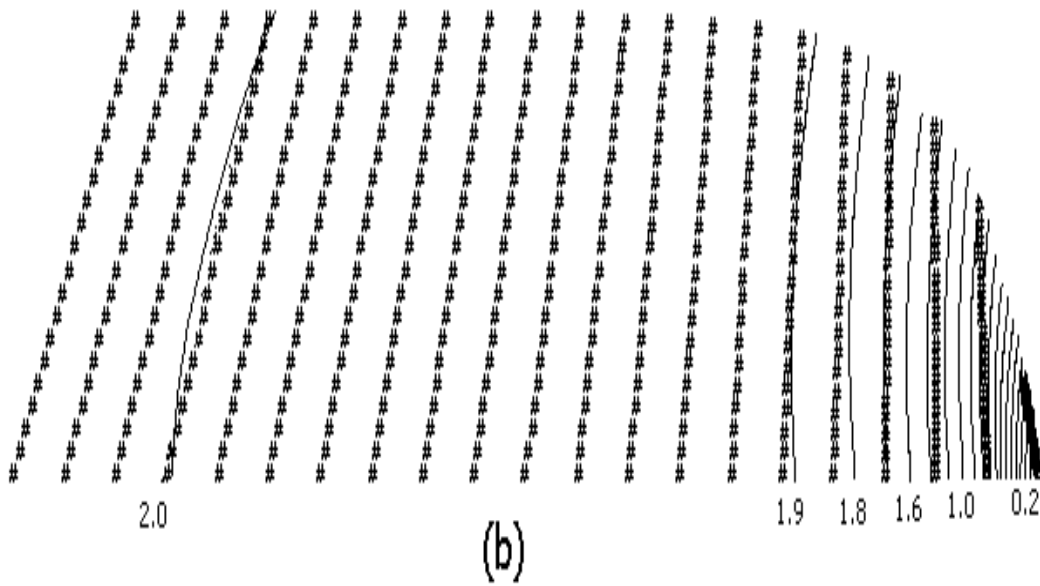
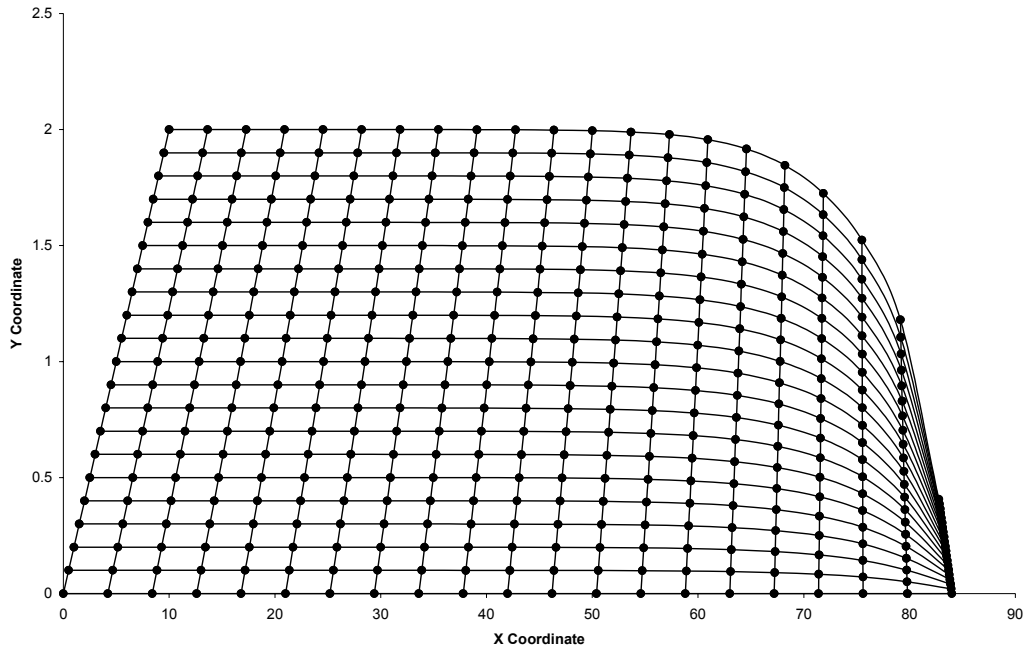


Figure 29. (a) Grid generated and (b) corresponding head distribution contour for final steady state.

CHAPTER 5

SUMMARY AND CONCLUSIONS

The following summary and conclusions can be derived from model development and model application described in previous sections:

1. Boundary Fitted Coordinate (BFC) System can be used to transform an irregularly shaped domain into a perfectly rectangular computational domain. This is shown in this paper from the transformation of irregular domain of Java Island to a perfectly rectangular domain in the curvilinear coordinate system.
2. Transformation into curvilinear coordinate system also needs to be done on the governing equations and boundary conditions. These transformation relations are provided for the seepage through a levee in the previous chapter. Since the transformation of these equations does not change the basic of the equation, these equations still retain their original properties and limitations.
3. The problem of unsteady seepage can be considered a steady state seepage at each time steps. The problems are iterated until a steady solution is obtained, where height and hydraulic head no longer changes with each iteration after all the boundary conditions are satisfied.
4. As shown in model application for unsteady seepage through a rectangular earth dam and a levee, solutions can be obtained while stability is maintained throughout the simulation. This is a definite advantage from the regular finite difference method.
5. The free surface of the seepage can be represented in smooth form using the BFC method and easier to use than previous methods used such as finite difference or finite element method.

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APPENDIX: DERIVATION OF EQUATIONS

The partial differential equation grid generator utilizes the Laplace system which guarantees the smoothest grid generated and is represented by the following equations

$$\xi_{xx} + \xi_{yy} = 0 \quad (\text{A1})$$

$$\eta_{xx} + \eta_{yy} = 0 \quad (\text{A2})$$

Chain rule differentiation is used to calculate the Laplacian representation

$$\begin{aligned} \nabla^2 = & \left(\xi_x^2 + \xi_y^2 \right) \frac{\partial^2}{\partial \xi^2} + 2(\xi_x \eta_x + \xi_y \eta_y) \frac{\partial^2}{\partial \xi \partial \eta} + \left(\eta_x^2 + \eta_y^2 \right) \frac{\partial^2}{\partial \eta^2} \\ & + (\xi_{xx} + \xi_{yy}) \frac{\partial}{\partial \xi} + (\eta_{xx} + \eta_{yy}) \frac{\partial}{\partial \eta} \end{aligned} \quad (\text{A3})$$

Applying the Laplacian on the x and y coordinates yields the following equations

$$\begin{aligned} \nabla_x^2 = & \left(\xi_x^2 + \xi_y^2 \right) x_{\xi\xi} + 2(\xi_x \eta_x + \xi_y \eta_y) x_{\xi\eta} + \left(\eta_x^2 + \eta_y^2 \right) x_{\eta\eta} \\ & + (\xi_{xx} + \xi_{yy}) x_{\xi} + (\eta_{xx} + \eta_{yy}) x_{\eta} \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} \nabla_y^2 = & \left(\xi_x^2 + \xi_y^2 \right) y_{\xi\xi} + 2(\xi_x \eta_x + \xi_y \eta_y) y_{\xi\eta} + \left(\eta_x^2 + \eta_y^2 \right) y_{\eta\eta} \\ & + (\xi_{xx} + \xi_{yy}) y_{\xi} + (\eta_{xx} + \eta_{yy}) y_{\eta} \end{aligned} \quad (\text{A5})$$

Substitute the following relationship into the Laplace operator from equation (A4) and (A5)

$$\xi_x = \frac{y\eta}{J} \quad (\text{A6})$$

$$\xi_y = -\frac{x\eta}{J} \quad (\text{A7})$$

$$\eta_x = -\frac{y\xi}{J} \quad (\text{A8})$$

$$\eta_y = \frac{x\xi}{J} \quad (\text{A9})$$

After the substitution the equations become

$$\nabla_x^2 = (x_\eta^2 + y_\eta^2)x_{\xi\xi} - 2(x_\xi x_\eta + y_\xi y_\eta)x_{\xi\eta} + (x_\xi^2 + y_\xi^2)x_{\eta\eta} \quad (\text{A10})$$

$$\nabla_y^2 = (x_\eta^2 + y_\eta^2)y_{\xi\xi} - 2(x_\xi x_\eta + y_\xi y_\eta)y_{\xi\eta} + (x_\xi^2 + y_\xi^2)y_{\eta\eta} \quad (\text{A11})$$

Simplifying the equations

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = 0 \quad (\text{A12})$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = 0 \quad (\text{A13})$$

where

$$\alpha = x_\eta^2 + y_\eta^2 \quad (\text{A14})$$

$$\beta = x_\xi x_\eta + y_\xi y_\eta \quad (\text{A15})$$

$$\gamma = x_\xi^2 + y_\xi^2 \quad (\text{A16})$$

Derivation of governing equation

$$\frac{\partial}{\partial x} \left(k_{xx} \frac{\partial h}{\partial x} + k_{xy} \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial y} \left(k_{xy} \frac{\partial h}{\partial x} + k_{yy} \frac{\partial h}{\partial y} \right) = S \frac{\partial h}{\partial t} \quad (\text{A17})$$

The above equation can be expanded to become the following equation

$$\frac{\partial}{\partial x} \left(k_{xx} \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial x} \left(k_{xy} \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial y} \left(k_{xy} \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_{yy} \frac{\partial h}{\partial y} \right) = S \frac{\partial h}{\partial t} \quad (\text{A18})$$

Using product rule:

$$\frac{d}{dx} [f(x).g(x)] = f(x).g'(x) + g(x).f'(x) \quad (\text{A19})$$

Equation (A18) will become

$$\begin{aligned}
S \frac{\partial h}{\partial t} = & k_{xx} \frac{\partial^2 h}{\partial x^2} + (k_{xx})_x \frac{\partial h}{\partial x} + k_{xy} \frac{\partial^2 h}{\partial x \partial y} + (k_{xy})_x \frac{\partial h}{\partial y} \\
& + k_{yx} \frac{\partial^2 h}{\partial y \partial x} + (k_{yx})_y \frac{\partial h}{\partial y} + k_{yy} \frac{\partial^2 h}{\partial y^2} + (k_{yy})_y \frac{\partial h}{\partial y}
\end{aligned} \tag{A20}$$

In a more compact form, equation (A20) can be written as

$$\begin{aligned}
S \frac{\partial h}{\partial t} = & k_{xx} h_{xx} + (k_{xx})_x h_x + k_{xy} h_{xy} + (k_{xy})_x h_y \\
& + k_{yx} h_{yx} + (k_{yx})_y h_y + k_{yy} h_{yy} + (k_{yy})_y h_y
\end{aligned} \tag{A21}$$

The general, second order, quasi-linear partial differential equation can be described by the following formula

$$A f_{xx} + B f_{xy} + C f_{yy} + D f_x + E f_y = g \tag{A22}$$

This general formula can be transformed into curvilinear coordinate system and yield the following equation

$$A f_{\xi\xi} + B f_{\xi\eta} + C f_{\eta\eta} + D f_{\xi} + E f_{\eta} = g \tag{A23}$$

Comparing with transformed boundary value problems:

$$(\alpha f_{\xi})_{\xi} + (\beta f_{\xi})_{\eta} + (\beta f_{\eta})_{\xi} + (\gamma f_{\eta})_{\eta} = g \tag{A24}$$

Expanding the equation

$$\alpha f_{\xi\xi} + \alpha_{\xi} f_{\xi} + \beta f_{\xi\eta} + \beta_{\eta} f_{\xi} + \beta f_{\eta\xi} + \beta_{\xi} f_{\eta} + \gamma f_{\eta\eta} + \gamma_{\eta} f_{\eta} = g \tag{A25}$$

Using

$$A = \frac{1}{J} \left(\alpha y_{\eta}^2 - 2\beta x_{\eta} y_{\eta} + \gamma x_{\eta}^2 \right) \tag{A26}$$

$$B = -\frac{1}{J} \left(\alpha y_{\xi} y_{\eta} - \beta (x_{\xi} y_{\eta} + x_{\eta} y_{\xi}) + \gamma x_{\xi} x_{\eta} \right) \tag{A27}$$

$$C = \frac{1}{J} \left(\alpha y_{\xi}^2 - 2\beta x_{\xi} y_{\xi} + \gamma x_{\xi}^2 \right) \tag{A28}$$

Then

$$\alpha' = \frac{1}{J} (k_{xx} y_{\eta}^2 - 2k_{xy} x_{\eta} y_{\eta} + k_{yy} x_{\eta}^2) \quad (\text{A29})$$

$$\beta' = -\frac{1}{J} (k_{xx} y_{\xi} y_{\eta} - k_{xy} (x_{\xi} y_{\eta} + x_{\eta} y_{\xi}) + k_{yy} x_{\xi} x_{\eta}) \quad (\text{A30})$$

$$\gamma' = \frac{1}{J} (k_{xx} y_{\xi}^2 - 2k_{xy} x_{\xi} y_{\xi} + k_{yy} x_{\xi}^2) \quad (\text{A31})$$

Also

$$\alpha_{\xi} = \left[\frac{1}{J} (k_{xx} y_{\eta}^2 - 2k_{xy} x_{\eta} y_{\eta} + k_{yy} x_{\eta}^2) \right]_{\xi} \quad (\text{A32})$$

Using product rule as before:

$$\alpha_{\xi} = \frac{1}{J} \left[(k_{xx} y_{\eta}^2 - 2k_{xy} x_{\eta} y_{\eta} + k_{yy} x_{\eta}^2) \right]_{\xi} + \left(\frac{1}{J} \right)_{\xi} (k_{xx} y_{\eta}^2 - 2k_{xy} x_{\eta} y_{\eta} + k_{yy} x_{\eta}^2) \quad (\text{A33})$$

Equation becomes:

$$\alpha_{\xi} = \frac{1}{J} \left[\begin{aligned} & k_{xx} (y_{\eta}^2)_{\xi} + (k_{xx})_{\xi} (y_{\eta}^2) - 2k_{xy} (x_{\eta} y_{\eta})_{\xi} \\ & - 2(k_{xy})_{\xi} (x_{\eta} y_{\eta}) + k_{yy} (x_{\eta}^2)_{\xi} + (k_{yy})_{\xi} x_{\eta}^2 \end{aligned} \right] + \frac{J_{\xi}}{J^2} (k_{xx} y_{\eta}^2 - 2k_{xy} x_{\eta} y_{\eta} + k_{yy} x_{\eta}^2) \quad (\text{A34})$$

$$\alpha_{\xi} = \frac{1}{J} \left[\begin{aligned} & 2k_{xx} y_{\eta} y_{\eta \xi} + (k_{xx})_{\xi} (y_{\eta}^2) - 2k_{xy} (x_{\eta} y_{\eta \xi} + x_{\eta \xi} y_{\eta}) \\ & - 2(k_{xy})_{\xi} (x_{\eta} y_{\eta}) + 2k_{yy} x_{\eta} x_{\eta \xi} + (k_{yy})_{\xi} x_{\eta}^2 \end{aligned} \right] + \frac{J_{\xi}}{J^2} (k_{xx} y_{\eta}^2 - 2k_{xy} x_{\eta} y_{\eta} + k_{yy} x_{\eta}^2) \quad (\text{A35})$$

For β_{ξ} :

$$\beta_{\xi} = \left[-\frac{1}{J} (k_{xx} y_{\xi} y_{\eta} - k_{xy} (x_{\xi} y_{\eta} + x_{\eta} y_{\xi}) + k_{yy} x_{\xi} x_{\eta}) \right]_{\xi} \quad (\text{A36})$$

$$\beta_\xi = -\frac{1}{J} \left[(k_{xx} y_\xi y_\eta - k_{xy} (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} x_\xi x_\eta) \right]_\xi - \left(\frac{1}{J} \right)_\xi (k_{xx} y_\xi y_\eta - k_{xy} (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} x_\xi x_\eta) \quad (\text{A37})$$

$$\beta_\xi = -\frac{1}{J} \left[k_{xx} (y_\xi y_\eta)_\xi + (k_{xx})_\xi y_\xi y_\eta - k_{xy} (x_\xi y_\eta + x_\eta y_\xi)_\xi - (k_{xy})_\xi (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} (x_\xi x_\eta)_\xi + (k_{yy})_\xi x_\xi x_\eta \right] - \frac{J_\xi}{J^2} (k_{xx} y_\xi y_\eta - k_{xy} (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} x_\xi x_\eta) \quad (\text{A38})$$

$$\beta_\xi = -\frac{1}{J} \left[k_{xx} (y_\xi y_\eta)_\xi + (k_{xx})_\xi y_\xi y_\eta + (k_{yy})_\xi x_\xi x_\eta - k_{xy} (x_\xi y_\eta)_\xi + x_{\xi\xi} y_\eta + x_\eta y_{\xi\xi} + x_\eta y_\xi y_\xi \right] - (k_{xy})_\xi (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} (x_\xi x_\eta)_\xi + x_{\xi\xi} x_\eta - \frac{J_\xi}{J^2} (k_{xx} y_\xi y_\eta - k_{xy} (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} x_\xi x_\eta) \quad (\text{A39})$$

For β_η :

$$\beta_\eta = \left[-\frac{1}{J} (k_{xx} y_\xi y_\eta - k_{xy} (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} x_\xi x_\eta) \right]_\eta \quad (\text{A40})$$

$$\beta_\eta = -\frac{1}{J} \left[(k_{xx} y_\xi y_\eta - k_{xy} (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} x_\xi x_\eta) \right]_\eta - \left(\frac{1}{J} \right)_\eta (k_{xx} y_\xi y_\eta - k_{xy} (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} x_\xi x_\eta) \quad (\text{A41})$$

$$\beta_\eta = -\frac{1}{J} \left[k_{xx} (y_\xi y_\eta)_\eta + (k_{xx})_\eta y_\xi y_\eta - k_{xy} (x_\xi y_\eta + x_\eta y_\xi)_\eta - (k_{xy})_\eta (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} (x_\xi x_\eta)_\eta + (k_{yy})_\eta x_\xi x_\eta \right] - \frac{J_\eta}{J^2} (k_{xx} y_\xi y_\eta - k_{xy} (x_\xi y_\eta + x_\eta y_\xi) + k_{yy} x_\xi x_\eta) \quad (\text{A42})$$

$$\beta_\eta = -\frac{1}{J} \left[\begin{aligned} &k_{xx}(y_\xi y_{\eta\eta} + y_{\xi\eta} y_\eta) + (k_{xx})_\eta y_\xi y_\eta \\ &-k_{xy}(x_\xi y_{\eta\eta} + x_{\xi\eta} y_\eta + x_\eta y_{\xi\eta} + x_{\eta\eta} y_\xi) \\ &- (k_{xy})_\eta (x_\xi y_\eta + x_\eta y_\xi) + k_{yy}(x_\xi x_{\eta\eta} + x_{\xi\eta} x_\eta) + (k_{yy})_\eta x_\xi x_\eta \\ &- \frac{J_\eta}{J^2} (k_{xx} y_\xi y_\eta - k_{xy}(x_\xi y_\eta + x_\eta y_\xi) + k_{yy} x_\xi x_\eta) \end{aligned} \right] \quad (\text{A43})$$

For γ :

$$\gamma_\eta = \left[\frac{1}{J} (k_{xx} y_\xi^2 - 2k_{xy} x_\xi y_\xi + k_{yy} x_\xi^2) \right]_\eta \quad (\text{A44})$$

$$\gamma_\eta = \frac{1}{J} \left[(k_{xx} y_\xi^2 - 2k_{xy} x_\xi y_\xi + k_{yy} x_\xi^2) \right]_\eta + \left(\frac{1}{J} \right) (k_{xx} y_\xi^2 - 2k_{xy} x_\xi y_\xi + k_{yy} x_\xi^2) \quad (\text{A45})$$

$$\gamma_\eta = \frac{1}{J} \left[\begin{aligned} &k_{xx}(y_\xi^2)_\eta + (k_{xx})_\eta (y_\xi^2) - 2k_{xy}(x_\xi y_\xi)_\eta \\ &- 2(k_{xy})_\eta (x_\xi y_\xi) + k_{yy}(x_\xi^2)_\eta + (k_{yy})_\eta x_\xi^2 \end{aligned} \right] \\ + \frac{J_\eta}{J^2} (k_{xx} y_\eta^2 - 2k_{xy} x_\eta y_\eta + k_{yy} x_\eta^2) \quad (\text{A46})$$

$$\gamma_\eta = \frac{1}{J} \left[\begin{aligned} &2k_{xx} y_\xi y_{\xi\eta} + (k_{xx})_\eta (y_\xi^2) - 2k_{xy}(x_\xi y_{\xi\eta} + x_{\xi\eta} y_\xi) \\ &- 2(k_{xy})_\eta (x_\xi y_\xi) + 2k_{yy} x_\xi x_{\xi\eta} + (k_{yy})_\eta x_\xi^2 \end{aligned} \right] \\ + \frac{J_\eta}{J^2} (k_{xx} y_\xi^2 - 2k_{xy} x_\xi y_\xi + k_{yy} x_\xi^2) \quad (\text{A47})$$

Compare the equations with the basic form:

$$\alpha h_{\xi\xi} + \alpha_\xi h_\xi + \beta h_{\xi\eta} + \beta_\eta h_\xi + \beta h_{\eta\xi} + \beta_\xi h_\eta + \gamma h_{\eta\eta} + \gamma_\eta h_\eta = JS \frac{\partial h}{\partial t} \quad (\text{A48})$$

Divide equation by J

$$\frac{\alpha}{J} h_{\xi\xi} + \frac{\alpha_\xi}{J} h_\xi + \frac{\beta}{J} h_{\xi\eta} + \frac{\beta_\eta}{J} h_\xi + \frac{\beta}{J} h_{\eta\xi} + \frac{\beta_\xi}{J} h_\eta + \frac{\gamma}{J} h_{\eta\eta} + \frac{\gamma_\eta}{J} h_\eta = S \frac{\partial h}{\partial t} \quad (\text{A49})$$

Substituting equations:

$$\begin{aligned}
S \frac{\partial h}{\partial t} = & \left[\frac{k_{xx}y_\eta^2 - 2k_{xy}x_\eta y_\eta + k_{yy}x_\eta^2}{J^2} \right] h_{\xi\xi} + \left[\frac{k_{xx}y_\xi^2 - 2k_{xy}x_\xi y_\xi + k_{yy}x_\xi^2}{J^2} \right] h_{\eta\eta} \\
& - 2 \left[\frac{k_{xx}y_\xi y_\eta - k_{xy}(x_\xi y_\eta + x_\eta y_\xi) + k_{yy}x_\xi x_\eta}{J^2} \right] h_{\xi\eta} \\
& + \left[\frac{1}{J^2} \left[2k_{xx}y_\eta y_{\eta\xi} + (k_{xx})_\xi (y_\eta^2) - 2k_{xy}(x_\eta y_{\eta\xi} + x_{\eta\xi} y_\eta) \right] \right. \\
& \left. - 2(k_{xy})_\xi (x_\eta y_\eta) + 2k_{yy}x_\eta x_{\eta\xi} + (k_{yy})_\xi x_\eta^2 \right. \\
& \left. + \frac{J_\xi}{J^3} (k_{xx}y_\eta^2 - 2k_{xy}x_\eta y_\eta + k_{yy}x_\eta^2) \right] h_\xi \\
& + \left[-\frac{1}{J^2} \left[k_{xx}(y_\xi y_{\eta\xi} + y_{\xi\xi} y_\eta) + (k_{xx})_\xi y_\xi y_\eta - k_{xy}(x_\xi y_{\eta\xi} + x_{\xi\xi} y_\eta) \right] \right. \\
& \left. + k_{xy}(x_\eta y_{\xi\xi} + x_{\eta\xi} y_\xi) - (k_{xy})_\xi (x_\xi y_\eta + x_\eta y_\xi) \right. \\
& \left. + k_{yy}(x_\xi x_{\eta\xi} + x_{\xi\xi} x_\eta) + (k_{yy})_\xi x_\xi x_\eta \right] h_\eta \\
& - \frac{J_\xi}{J^2} (k_{xx}y_\xi y_\eta - k_{xy}(x_\xi y_\eta + x_\eta y_\xi) + k_{yy}x_\xi x_\eta) \\
& + \left[-\frac{1}{J^2} \left[k_{xx}(y_\xi y_{\eta\eta} + y_{\xi\eta} y_\eta) + (k_{xx})_\eta y_\xi y_\eta - k_{xy}(x_\xi y_{\eta\eta} + x_{\xi\eta} y_\eta) \right] \right. \\
& \left. - k_{xy}(x_\eta y_{\xi\eta} + x_{\eta\eta} y_\xi) - (k_{xy})_\eta (x_\xi y_\eta + x_\eta y_\xi) \right. \\
& \left. + k_{yy}(x_\xi x_{\eta\eta} + x_{\xi\eta} x_\eta) + (k_{yy})_\eta x_\xi x_\eta \right] h_\xi \\
& - \frac{J_\eta}{J^3} (k_{xx}y_\xi y_\eta - k_{xy}(x_\xi y_\eta + x_\eta y_\xi) + k_{yy}x_\xi x_\eta) \\
& + \left[\frac{1}{J^2} \left[2k_{xx}y_\xi y_{\xi\eta} + (k_{xx})_\eta (y_\xi^2) - 2k_{xy}(x_\xi y_{\xi\eta} + x_{\xi\eta} y_\xi) \right] \right. \\
& \left. - 2(k_{xy})_\eta (x_\xi y_\xi) + 2k_{yy}x_\xi x_{\xi\eta} + (k_{yy})_\eta x_\xi^2 \right. \\
& \left. + \frac{J_\eta}{J^3} (k_{xx}y_\xi^2 - 2k_{xy}x_\xi y_\xi + k_{yy}x_\xi^2) \right] h_\eta
\end{aligned} \tag{A50}$$

$$\begin{aligned}
S \frac{\partial h}{\partial t} = & \left[\frac{k_{xx}y_\eta^2 + k_{yy}x_\eta^2}{J^2} \right] h_{\xi\xi} + \left[\frac{k_{xx}y_\xi^2 + k_{yy}x_\xi^2}{J^2} \right] h_{\eta\eta} \\
& - 2 \left[\frac{k_{xx}y_\xi y_\eta + k_{yy}x_\xi x_\eta}{J^2} \right] h_{\xi\eta} \\
& + \frac{k_{xx}}{J^2} \left[2y_\eta y_{\eta\xi} - y_\eta^2 \frac{J_\xi}{J} - y_\xi y_{\eta\eta} - y_{\eta\xi} y_\eta + y_\xi y_\eta \frac{J_\eta}{J} \right] h_\xi \\
& + \frac{k_{xx}}{J^2} \left[2y_\xi y_{\xi\eta} - y_\xi^2 \frac{J_\eta}{J} - y_\xi y_{\eta\xi} - y_{\xi\xi} y_\eta + y_\xi y_\eta \frac{J_\xi}{J} \right] h_\eta \\
& - \frac{k_{yy}}{J^2} \left[2x_\eta x_{\eta\xi} - x_\eta^2 \frac{J_\xi}{J} - x_\xi x_{\eta\eta} - x_{\eta\xi} x_\eta + x_\xi x_\eta \frac{J_\eta}{J} \right] h_\xi \\
& - \frac{k_{yy}}{J^2} \left[2x_\xi x_{\xi\eta} - x_\xi^2 \frac{J_\eta}{J} - x_\xi x_{\eta\xi} - x_{\xi\xi} x_\eta + x_\xi x_\eta \frac{J_\xi}{J} \right] h_\eta \\
& + \frac{2k_{xy}}{J^2} \left[-x_\eta y_\eta h_{\xi\xi} + (x_\xi y_\eta + x_\eta y_\xi) h_{\xi\eta} - x_\xi y_\xi h_{\eta\eta} \right] \\
& + \frac{2k_{xy}}{J^2} \left[\begin{aligned} & -x_\eta y_{\eta\xi} - x_{\eta\xi} y_\eta + x_\eta y_\eta \frac{J_\xi}{J} + x_\xi y_{\eta\eta} + x_\eta y_{\eta\xi} \\ & + x_{\eta\xi} y_\eta + x_{\eta\eta} y_\xi - x_\xi y_\eta \frac{J_\eta}{J} - x_\eta y_\xi \frac{J_\eta}{J} \end{aligned} \right] h_\xi \\
& + \frac{2k_{xy}}{J^2} \left[\begin{aligned} & -x_\xi y_{\eta\xi} - x_{\eta\xi} y_\xi + x_\xi y_\xi \frac{J_\eta}{J} + x_\xi y_{\eta\xi} + x_\eta y_{\xi\xi} \\ & + x_{\xi\xi} y_\eta + x_{\eta\xi} y_\xi - x_\xi y_\eta \frac{J_\xi}{J} - x_\eta y_\xi \frac{J_\xi}{J} \end{aligned} \right] h_\eta \\
& + \frac{(k_{xx})_\xi}{J^2} \left[y_\eta^2 h_\xi - y_\xi y_\eta h_\eta \right] + \frac{(k_{xx})_\eta}{J^2} \left[y_\xi^2 h_\eta - y_\xi y_\eta h_\xi \right] \\
& + \frac{(k_{yy})_\xi}{J^2} \left[x_\eta^2 h_\xi - x_\xi x_\eta h_\eta \right] + \frac{(k_{yy})_\eta}{J^2} \left[x_\xi^2 h_\eta - x_\xi x_\eta h_\xi \right] \\
& + \frac{(k_{xy})_\xi}{J^2} \left[x_\xi y_\eta h_\eta - 2x_\eta y_\eta h_\xi + x_\eta y_\xi h_\eta \right] \\
& + \frac{(k_{xy})_\eta}{J^2} \left[x_\xi y_\eta h_\xi - 2x_\xi y_\xi h_\eta + x_\eta y_\xi h_\xi \right]
\end{aligned} \tag{A51}$$

$$\begin{aligned}
S \frac{\partial h}{\partial t} = & \left[\frac{\alpha' h_{\xi\xi\xi} - 2\beta' h_{\xi\xi\eta} + \gamma' h_{\eta\eta\eta} + \delta h_{\xi} + \varepsilon h_{\eta}}{J^2} \right] \\
& + \frac{\left[y_{\eta} (k_{xx})_{\xi} - y_{\xi} (k_{xx})_{\eta} \right] \cdot \left[y_{\eta} h_{\xi} - y_{\xi} h_{\eta} \right]}{J^2} \\
& + \frac{\left[x_{\eta} (k_{yy})_{\xi} - x_{\xi} (k_{yy})_{\eta} \right] \cdot \left[x_{\eta} h_{\xi} + x_{\xi} h_{\eta} \right]}{J^2} \\
& + \frac{2k_{xy}}{J^2} \left[\begin{aligned} & -x_{\eta} y_{\eta} h_{\xi\xi\xi} + (x_{\xi} y_{\eta} + x_{\eta} y_{\xi}) h_{\xi\xi\eta} - x_{\xi} y_{\xi} h_{\eta\eta\eta} \\ & (x_{\xi} y_{\eta\eta} + x_{\eta\eta} y_{\xi}) h_{\xi} + \left(x_{\eta} y_{\eta} \frac{J_{\xi}}{J} - x_{\xi} y_{\eta} \frac{J_{\eta}}{J} - x_{\eta} y_{\xi} \frac{J_{\eta}}{J} \right) h_{\xi} \\ & (x_{\eta} y_{\xi\xi} + x_{\xi\xi} y_{\eta}) h_{\eta} + \left(x_{\xi} y_{\xi} \frac{J_{\eta}}{J} - x_{\xi} y_{\eta} \frac{J_{\xi}}{J} - x_{\eta} y_{\xi} \frac{J_{\xi}}{J} \right) h_{\eta} \end{aligned} \right] \\
& + \frac{\left[y_{\eta} (k_{xy})_{\xi} - y_{\xi} (k_{xy})_{\eta} \right] \cdot \left[-x_{\eta} h_{\xi} + x_{\xi} h_{\eta} \right]}{J^2} \\
& + \frac{\left[x_{\eta} (k_{xy})_{\xi} - x_{\xi} (k_{xy})_{\eta} \right] \cdot \left[-y_{\eta} h_{\xi} + y_{\xi} h_{\eta} \right]}{J^2}
\end{aligned} \tag{A52}$$

where:

$$\alpha' = k_{xx} y_{\eta}^2 + k_{yy} x_{\eta}^2 \tag{A53}$$

$$\beta' = k_{xx} y_{\xi} y_{\eta} + k_{yy} x_{\xi} x_{\eta} \tag{A54}$$

$$\gamma' = k_{xx} y_{\xi}^2 + k_{yy} x_{\xi}^2 \tag{A55}$$

$$\begin{aligned}
\delta = & k_{xx} \left[(y_{\eta} y_{\eta\xi} - y_{\xi} y_{\eta\eta}) - \frac{y_{\eta}^2 J_{\xi} - y_{\xi} y_{\eta} J_{\eta}}{J} \right] \\
& - k_{yy} \left[(x_{\eta} x_{\eta\xi} - x_{\xi} x_{\eta\eta}) - \frac{x_{\eta}^2 J_{\xi} - x_{\xi} x_{\eta} J_{\eta}}{J} \right]
\end{aligned} \tag{A56}$$

$$\varepsilon = k_{xx} \left[(y_{\xi} y_{\eta \xi} - y_{\eta} y_{\xi \xi}) - \frac{y_{\eta}^2 J_{\eta} - y_{\xi} y_{\eta} J_{\xi}}{J} \right] - k_{yy} \left[(x_{\xi} x_{\eta \xi} - x_{\eta} x_{\xi \xi}) - \frac{x_{\xi}^2 J_{\eta} - x_{\xi} x_{\eta} J_{\xi}}{J} \right] \quad (\text{A57})$$

A unit vector normal to the line of constant ξ and η can each be described by the following equations

$$n^{(\xi)} = \frac{\nabla_{\xi}^z}{|\nabla_{\xi}^z|} = \frac{(y_{\eta} i - x_{\eta} j)}{\sqrt{\alpha}} \quad (\text{A58})$$

$$n^{(\eta)} = \frac{\nabla_{\eta}}{|\nabla_{\eta}|} = \frac{(-y_{\xi} i + x_{\xi} j)}{\sqrt{\gamma}} \quad (\text{A59})$$

Normal directional derivatives for no flow boundary conditions can then be described by the following equations:

$$\left. \frac{\partial h}{\partial n} \right|_{\xi = \text{const}} = \nabla h \cdot \frac{\nabla_{\xi}^z}{|\nabla_{\xi}^z|} = \frac{1}{J\sqrt{\alpha}} (\alpha h_{\xi} - \beta h_{\eta}) \quad (\text{A60})$$

$$\left. \frac{\partial h}{\partial n} \right|_{\eta = \text{const}} = \nabla h \cdot \frac{\nabla_{\eta}}{|\nabla_{\eta}|} = \frac{1}{J\sqrt{\gamma}} (\gamma h_{\eta} - \beta h_{\xi}) \quad (\text{A61})$$

VITA

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