A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mathematics

by

Lee Windsperger

B.S., Saint Louis University, 2005
M.S., Louisiana State University, 2007
August 2012
Acknowledgments

Thank you, Frank. Thank you, family and friends. I love you, Jayci.
# Table of Contents

Acknowledgments ................................................................. ii

Abstract ........................................................................ iv

1 The Asymptotic Laplace Transform ........................................ 1
   1.1 The Classical Laplace Transform ........................................ 1
   1.2 The Asymptotic Laplace Transform ..................................... 15
   1.3 Generalized Functions ...................................................... 41
   1.4 Advanced Type Delay-Differential Equations ....................... 55

2 Numerical Inversion and Approximation ................................. 63
   2.1 Rational Padé Approximations of the Exponential ............... 65
   2.2 Hille-Phillips Functional Calculus ...................................... 77
   2.3 Approximations of Semigroups without Scaling and Squaring, Part I 80
   2.4 The Constant in the Brenner and Thomée Theorem .............. 83
   2.5 Approximations of Semigroups without Scaling and Squaring, Part II 91
   2.6 Inversion of Asymptotic Laplace Transforms of Generalized Functions 97

References ............................................................................ 108

Vita .................................................................................. 111
Abstract

This dissertation refines and further develops numerical methods for the inversion of the classical Laplace transform and explores the effectiveness of these methods when applied (a) to an asymptotic generalization of the Laplace transform for generalized functions and (b) to the numerical approximation of solutions of ill-posed evolution equations (e.g. backwards in time problems).

Chapter 1 of the dissertation reviews some of the key features of asymptotic Laplace transform theory and its application to evolution equations. Although some of the statements and results contain slight modifications and improvements, the material presented in Chapter 1 is known to the experts in the field. The main contributions of this work is in Chapter 2 where an attempt is made to help clarify and determine the size of the constant in the celebrated Hersh-Kato and Brenner-Thomée approximation theorem of semigroup theory. In particular, by improving an earlier estimate, we are able to show that matrix semigroups $e^{tA}$ can be approximated “without scaling and squaring” in terms of the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ of the generating matrix $A$ (see Theorem 2.3.2). Also, our estimate of the Brenner-Thomée constant given in Section 2.4 improves earlier estimates given by Neubrander, Özer, and Sandmaier in [28]. The techniques used in Section 2.4 open the door to Theorem 2.5.1, a first attempt to lift the matrix result (Theorem 2.3.2) to the general semigroup setting. Finally, in (2.31) we present a new approach on how to approximate the continuous representatives $f = k * u$ of a generalized function $u$ in terms of its Laplace transform $\hat{u}$. 
1. The Classical Laplace Transform

The purpose of this section is to illustrate that classical Laplace transform methods require significant assumptions that occasionally severely limit the applicability of the method to linear evolutionary problems. To see this, let $X$ be a (complex) Banach space and $f \in L^1_{\text{loc}}([0, \infty), X)$. For the classical Laplace transform

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda s} f(s) \, ds := \lim_{t \to \infty} \int_0^t e^{-\lambda s} f(s) \, ds$$

to exist, the following conditions must hold (see also Sections 1.4 and 1.5 of [2]).

**Proposition 1.1.1.** For a (complex) Banach space $X$ and $f \in L^1_{\text{loc}}([0, \infty), X)$, the following statements are equivalent.

(i) The Laplace transform $\hat{f}(\lambda)$ converges for some $\lambda \in \mathbb{C}$.

(ii) There exists an abscissa of convergence $\omega \in \mathbb{R}$ such that $\hat{f}(\lambda)$ converges for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > \omega$ and does not converge for $\text{Re}(\lambda) < \omega$.

(iii) There exists $M, \bar{\omega} > 0$ such that $\|F(t)\| \leq Me^{\bar{\omega}t}$, where $F(t) := \int_0^t f(s) \, ds$.

Moreover, if any of the above statements holds, then $\omega \leq \bar{\omega}$, $\lambda \to \hat{f}(\lambda)$ is analytic for $\text{Re}(\lambda) > \omega$, and

(a) $\|\hat{f}(\lambda)\| \leq \frac{M|\lambda|}{\text{Re}(\lambda)-\bar{\omega}}$ for $\text{Re}(\lambda) > \bar{\omega}$.

(b) $|\hat{f}(\lambda)| \leq cM$ for all $\lambda > \frac{c}{(c-1)} \bar{\omega}$ with $c > 1$.

(c) $|\hat{f}(\bar{\omega} + re^{i\alpha})| \leq \frac{cM}{\cos(\alpha)}$ for $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, and $r \geq \frac{\bar{\omega}}{(c-1)}$ with $c > 1$.
(d) \( \hat{f}(a + ib) = O(|b|) \) as \( |b| \to \infty \) for all \( a > \hat{\omega} \).

(e) \( \hat{f}(a + ib) = o(|b|) \) as \( |b| \to \infty \) uniformly in \( \omega + \delta \leq a \leq \infty \) for any \( \delta > 0 \).

Proof. To show \( (i) \Rightarrow (ii) \), suppose \( \hat{f}(\lambda_0) \) exists for \( \lambda_0 \in \mathbb{C} \). Define \( G_0(t) := \int_0^t e^{-\lambda_0 s} f(s) \, ds \). Since \( \hat{f}(\lambda_0) \) exists, \( G_0(t) \) is bounded. Integration by parts implies that
\[
\int_0^t e^{-\lambda s} f(s) \, ds = \int_0^t e^{-(\lambda-\lambda_0)s} e^{-\lambda_0 s} f(s) \, ds = e^{-(\lambda-\lambda_0)t} G_0(t) + (\lambda - \lambda_0) \int_0^t e^{-(\lambda-\lambda_0)s} G_0(s) \, ds.
\]
Therefore,
\[
\hat{f}(\lambda) = (\lambda - \lambda_0) \int_0^\infty e^{-(\lambda-\lambda_0)s} G_0(s) \, ds
\]
exists for all \( \text{Re}(\lambda) > \text{Re}(\lambda_0) \). By defining \( \omega := \inf \{ \text{Re}(\lambda_0), \hat{f}(\lambda_0) \text{ converges} \} \) one obtains \( (ii) \).

For \( (ii) \Rightarrow (iii) \), let \( \lambda_0 > 0 \) be such that \( \hat{f}(\lambda_0) \) exists. Again define \( G_0(t) := \int_0^t e^{-\lambda_0 s} f(s) \, ds \) and let \( C := \sup_{t \geq 0} \|G_0(t)\| \). Then
\[
F(t) := \int_0^t f(s) \, ds = \int_0^t e^{\lambda_0 s} e^{-\lambda_0 s} f(s) \, ds = e^{\lambda_0 t} G_0(t) - \lambda_0 \int_0^t e^{\lambda_0 s} G_0(s) \, ds
\]
implies that
\[
\|F(t)\| \leq C e^{\lambda_0 t} + C(e^{\lambda_0 t} - 1) \leq 2C e^{\lambda_0 t}.
\]
The implication \( (iii) \Rightarrow (i) \) follows from \( \|F(T)\| \leq M e^{\hat{\omega} T} \) and the fact that
\[
\hat{f}(\lambda) = \lim_{T \to \infty} \int_0^T e^{-\lambda t} f(t) \, dt = \lim_{T \to \infty} \left[ e^{-\lambda T} F(T) + \lambda \int_0^T e^{-\lambda t} F(t) \, dt \right] = \lambda \int_0^\infty e^{-\lambda t} F(t) \, dt
\]
exists for \( \text{Re}(\lambda) > \hat{\omega} \).

Moreover, if any of the above statements \( (i), (ii), (iii) \) holds, then \( (a) \) follows since
\[
\|\hat{f}(\lambda)\| = \left\| \int_0^\infty e^{-\lambda t} f(t) \, dt \right\| = \left\| \lambda \int_0^\infty e^{-\lambda t} F(t) \, dt \right\| \leq \frac{M |\lambda|}{\text{Re}(\lambda) - \hat{\omega}}.
\]
where \( M, \tilde{\omega} > 0 \) are given by (iii). Statements (b), (c), and (d) follow directly by plugging in the specific choices of \( \lambda \) into (a). To show that \( \lambda \to \hat{f}(\lambda) \) is analytic, define \( q_k : \mathbb{C} \to X \) with \( k \in N_0 := N \cup \{ 0 \} \) by

\[
q_k(\lambda) := \int_0^k e^{-\lambda t} f(t) \, dt.
\]

As \( N \to \infty \), \( \sum_{n=0}^{N} \frac{(-\lambda)^n}{n!} \to e^{-\lambda t} \) uniformly for \( \lambda \) in bounded subsets of \( \mathbb{C} \). Because of the uniform convergence,

\[
q_k(\lambda) := \int_0^k e^{-\lambda t} f(t) \, dt = \int_0^k \lim_{N \to \infty} \sum_{n=0}^{N} \frac{(-\lambda)^n}{n!} f(t) \, dt = \lim_{N \to \infty} \int_0^k \sum_{n=0}^{N} \frac{\lambda^n}{n!} (-t)^n f(t) \, dt = \lim_{N \to \infty} P_N(\lambda),
\]

where \( P_N(\lambda) = \sum_{0}^{N} a_n \lambda^n \) and

\[
a_n = \frac{(-1)^n}{n!} \int_0^k t^n f(t) \, dt.
\]

Since \( P_N(\lambda) \) converges uniformly to \( q_k(\lambda) \), by the Weierstrass convergence theorem (see A.5 of [2]), the functions \( q_k \) are entire, and

\[
q_k^{(j)}(\lambda) = \int_0^k e^{-\lambda t} (-t)^j f(t) \, dt
\]

for all \( j \in N_0 \). As above, let \( \lambda_0 > \omega \) such that \( \hat{f}(\lambda_0) \) exists and define \( G_0(t) := \int_0^t e^{-\lambda_0 s} f(s) \, ds \). Then \( G_0 \) is bounded, and it follows that \( q_k \) converges to \( \hat{f} \) uniformly on compact subsets of \( \{ \lambda : \text{Re}(\lambda) > \omega \} \) as \( k \to \infty \) since

\[
\hat{f}(\lambda) - q_k(\lambda) = \int_k^\infty e^{-(\lambda-\lambda_0)s} e^{-\lambda_0 s} f(s) \, ds = -e^{-(\lambda-\lambda_0)k} G_0(k) + (\lambda - \lambda_0) \int_k^\infty e^{-(\lambda-\lambda_0)s} G_0(s) \, ds.
\]

By the Weierstrass convergence theorem, \( \hat{f} \) is analytic, and \( q_k^{(j)}(\lambda) \to \hat{f}^{(j)}(\lambda) \) as \( k \to \infty \) for \( \text{Re}(\lambda) > \lambda_0 > \omega \).
To prove statement (e) on the growth of Laplace transforms on vertical strips, the argument from Chapter 2 Section 13 of [38] will be followed. To begin, let \( \varepsilon, \delta > 0 \). The following will show that there exists a \( b_0 > 0 \) such that

\[
|\hat{f}(a + ib)| \leq \varepsilon |b|
\]

for all \( \omega + \delta \leq a < \infty \) and \( |b| \geq b_0 \). Choose \( \omega < a_1 < \omega + \delta \). Then \( a - a_1 > \omega + \delta - a_1 > 0 \) and there exists \( C > 0 \) such that \( G_0(t) := \int_0^t e^{a_1 s} f(s) \, ds \) is bounded by \( C \). Moreover, it follows from

\[
\frac{|\hat{f}(a+ib)|}{|b|} \leq \frac{1}{|b|} \left| \int_0^T e^{-(a+ib)t} f(t) \, dt \right| + \frac{1}{|b|} \left| \int_T^\infty e^{-(a-a_1+ib)t} G_0(t) \, dt \right| \\
\leq \frac{1}{|b|} \left| \int_0^T e^{-(\omega+\delta)t} |f(t)| \, dt \right| + \frac{1}{|b|} \left| e^{-(a-a_1+ib)T} G_0(T) \right| \\
+(a - a_1 + ib) \int_T^\infty e^{-(a-a_1+ib)t} G(t) \, dt \\
\leq \frac{1}{|b|} \left| \int_0^T e^{-(\omega+\delta)t} |f(t)| \, dt \right| + \frac{1}{|b|} \left( C e^{-(a-a_1)T} + \sqrt{\frac{(a-a_1)^2 + b^2}{a-a_1}} e^{-(a-a_1)T} \right) \\
\leq \frac{1}{|b|} \left| \int_0^T e^{-(\omega+\delta)t} |f(t)| \, dt \right| + \left( C e^{-(a-a_1)T} \right) \left( \frac{1}{|b|} + \sqrt{\frac{1}{b^2} + \frac{1}{(a-a_1)^2}} \right) \\
\leq \frac{1}{|b_0|} \left| \int_0^T e^{-(\omega+\delta)t} |f(t)| \, dt \right| + \left( C e^{-(\omega+\delta-a_1)T} \right) \left( \frac{1}{|b_0|} + \sqrt{\frac{1}{b_0^2} + \frac{1}{(\omega+\delta-a_1)^2}} \right).
\]

Now choose \( T \) large enough such that the second term is less than \( \frac{\varepsilon}{2} \) for all \( |b| \geq b_0 > 0 \). Then choose \( b_0 \) large enough such that the first term is less than \( \frac{\varepsilon}{2} \) for all \( |b| \geq b_0 \).

As Proposition 1.1.1 shows, the application of classical Laplace transform techniques to linear differential, delay, integral, or evolution equations requires

(a) restrictive assumptions on the domain \( \Omega \) and growth of an analytic function \( r : \mathbb{C} \supset \Omega \to X \) to have a Laplace transform representation \( r = \hat{f} \) for some \( f \in L^1_{\text{loc}}([0, \infty), X) \), and

(b) restrictive integrability and growth assumptions on a function \( f : \mathbb{R}_+ \to X \) to be Laplace transformable.
The following examples illustrate why the above restrictions severely limit the applicability of the classical Laplace transform to certain linear evolutionary problems so as to motivate the need for an extension of the classical transform which removes all such limitations.

Remark 1.1.2. Since multiplication semigroups will play a prominent role in this dissertation for constructing examples and counterexamples, first recall some basic examples and general characteristics of this important class of semigroups.

For the multiplication semigroups \( T(t)f(x) : t \mapsto e^{ta(x)}f(x) \) and their generators \( Af(x) : x \mapsto a(x)f(x) \) on \( X = C_0([0, \infty), \mathbb{C}) \) where \( a : \mathbb{R}^+ \to \mathbb{C} \) is continuous, the following statements hold:

(a) The spectrum \( \sigma(A) := \{ \lambda \in \mathbb{C} : \lambda \notin \rho(A) \} \) coincides with the closure of the range of the function \( a(\cdot) \) in \( \mathbb{C} \). Clearly,

\[
R(\lambda, A)f = (\lambda I - A)^{-1}f : x \mapsto \frac{1}{\lambda - a(x)}f(x)
\]

is a well-defined bounded linear operator from \( X \to X \) if and only if \( \lambda \) is not in the closure of the range \( \{a(x), x \geq 0\} \) of \( a(\cdot) \). Furthermore, in this case, \( \|R(\lambda, A)\| = \frac{1}{d(\lambda, \sigma(A))} \), where \( d(\lambda, \sigma(A)) \) denotes the distance of \( \lambda \) to the spectrum \( \sigma(A) := \{ \lambda : \lambda I - A \text{ not invertible} \} = \{a(x) : x \geq 0\} \).

(b) \( A \) is a bounded linear operator if and only if \( \sigma(A) \) is bounded if and only if the range of \( a \) is bounded. Moreover, if \( A \) is bounded, then \( t \to T(t) \) is an entire function from \( \mathbb{C} \) into the space \( L(X) \) of all bounded linear operators on \( X \). Clearly, the operator \( (Af)(x) = a(x)f(x) \) is defined for all \( f \in X \) if and only if \( |a(x)| \leq M \) for all \( x \geq 0 \). In this case, \( \|A\| = \|a(x)\|_\infty = \sup_{x \geq 0} |a(x)| \) and \( e^{zA}f := \sum \frac{z^n}{n!}A^nf \) is an entire function on \( \mathbb{C} \), where

\[
e^{zA}f(x) = \left( \sum \frac{z^n}{n!}a^n(x) \right)f(x) = e^{z_{a(x)}}f(x)
\]
for all $x \geq 0$ and $\|e^{xA}\| = \sup_{x \geq 0} e^{Re(za(x))}$.

(c) $A$ is the generator of a strongly continuous semigroup if and only if there exists $\omega \in \mathbb{R}$ such that $s(A) := \sup_{\lambda \in \sigma(A)} Re(\lambda) = \sup_{x \geq 0} Re(a(x)) \leq \omega$. If $A$ is the generator of a strongly continuous semigroup, then by Proposition 5.5 of Chapter 1 in [13], there exist constants $M > 1$ and $\omega \geq 0$ such that

$$
\|T(t)f\| = \sup_{x \geq 0} |e^{ta(x)}f(x)| = \sup_{x \geq 0} e^{tRe(a(x))}|f(x)| \leq Me^{\omega t}\|f\|
$$

for all $f \in X$. This implies that $Re(a(x)) \leq \omega$ for all $x \geq 0$. Conversely, if there exists $\omega \geq 0$ such that $s(A) := \sup_{\lambda \in \sigma(A)} Re(\lambda) = \sup_{x \geq 0} Re(a(x)) \leq \omega$, for $f \in X$ and given $\varepsilon > 0$, choose a $N > 0$ such that $|f(x)| \leq \frac{\varepsilon}{2(e^\omega + 1)}$ for all $x \geq N$. Then, for $0 \leq t \leq 1$,

$$
\|T(t)f - f\| = \sup_{x \geq 0} |e^{ta(x)} - 1||f(x)|
\leq \sup_{x \in [0,N]} |e^{ta(x)} - 1|\|f\|_{\infty} + \sup_{x \geq N}(e^\omega + 1)|f(x)|
\leq (\sup_{x \in [0,N]} |a(x)|) e^{\omega t}\|f\|_{\infty} + \frac{\varepsilon}{2}
\leq t(\sup_{x \in [0,N]} |a(x)|) e^{\omega t}\|f\|_{\infty} + \frac{\varepsilon}{2}.
$$

This shows that $T(t)f \to f$ as $t \to 0$ for all $f \in X$. Since a semigroup is strongly continuous on $[0,\infty)$ if it is continuous at $t = 0$ (see Proposition 5.3 of Chapter 1 in [13]), $t \to T(t)f$ is continuous on $[0,\infty)$ for all $f \in X$.

(d) If $a(x) = -x$ ($x \geq 0$), then the semigroup is strongly continuous on $[0,\infty)$ and $z \to T(z)$ is an analytic function from $H_0 := \{z \in \mathbb{C} : Re(z) > 0\}$ into $\mathcal{L}(X)$. By (c), since $\sup_{x \geq 0} Re(a(x)) \leq 0$, $t \to T(t)$ is strongly continuous on $[0,\infty)$. The map $z \to T(z)$ where $T(z)f : x \to e^{-xz}f(x)$ is analytic for $z \in \mathbb{C}$ with $Re(z) > 0$. To see this, let $H_0 = \{z : Re(z) > 0\}$. By Proposition A.3 in [2] the map $\Pi : H_0 \to \mathcal{L}(X)$ given by $z \to T(z)$ is analytic if and only if the maps $T_f : z \to T(z)f$ are analytic for all $f \in X = C_0([0,\infty), \mathbb{C})$. Since
the linear functions $\delta_x \in X^*$ defined by $\delta_x(f) := f(x)$ are a norming subset of $X^*$ (i.e. sup$_{\delta_x} \langle f, \delta_x \rangle$ defines the norm on $X$), it follows that $T_f : z \to T(x)f$ is analytic if and only if $T_{f,x} : z \to T(z)f(x) = e^{-zx}f(x)$ is analytic for all $f \in X$ and all $x \geq 0$. Thus, $z \to T(z)$ is analytic on $H_0$. In particular, $t \to T(t)$ is continuous for $t \in (0, \infty)$.

However, the map $t \to T(t)$ is not continuous in 0. To see this, consider functions $f_n \in X$ with $\|f_n\| = 1$ and $f_n(x) = 1$ for $x \in [0, n]$. Then

$$\|T(t) - I\| = \sup_{\|f\| \leq 1} \|T(t)f - f\| \\
\geq \sup_{f_n} \sup_{x \geq 0} |e^{-tx}f_n(x) - f_n(x)| = 1$$

for all $t > 0$.

(e) If $a(x) = -x + ix$ ($x \geq 0$), then the semigroup is strongly continuous on $[0, \infty)$ and $z \to T(z)$ is an analytic function from $H_{\pi/4} := \{z \in \mathbb{C} : |\arg(z)| < \pi/4 \}$ into $\mathcal{L}(X)$. The proof of this statement follows the arguments in part (d) by observing that $\text{Re}(a(x)z) < 0$ for all $z \in H_{\pi/4}$.

(f) If $a(x) = -x + ix^2$ ($x \geq 0$), then the semigroup is strongly continuous on $[0, \infty)$, $t \to T(t)$ is continuous on $(0, \infty)$, but $t \to T(t)$ is not analytic in any open sector containing $(0, \infty)$. However, $t \to T(t)f$ is infinitely often differentiable on $(0, \infty)$ for all $f \in X$. Because of the presence of the $ix^2$ term in $a(x)$, $\text{Re}(a(x)z) < 0$ only for real-valued $z$. Thus, there does not exist any open sector containing $(0, \infty)$ where the semigroup is well-defined and analytic.

By (c), since sup$_{x \geq 0} \text{Re}(a(x)) \leq 0$, $t \to T(t)$ is strongly continuous on $[0, \infty)$. To see that $t \to T(t)$ is norm-continuous on $(0, \infty)$, let $t_0 > 0$ and given any $\varepsilon > 0$, choose $N > 0$ sufficiently large such that $e^{-tx} \leq \frac{\varepsilon}{4}$ for all $x \geq N$. 

7
Then with $h \leq \frac{\varepsilon}{2M_a}$ where $M_a = \sup_{x \in [0,N]} |a(x)|$, $\|T(t_0 + h) - T(t_0)\| = 
\sup_{\|f\| \leq 1} \sup_{x \geq 0} |e^{-(t_0+h)x} e^{i(t_0+h)x^2} - e^{-t_0x} e^{it_0x^2}| \|f\| \leq \sup_{x \geq 0} e^{-t_0x} |e^{-hx} e^{ihx^2} - 1| 
\leq \sup_{x \in [0,N]} |e^{ha(x)} - 1| + \sup_{x \in [N,\infty)} 2e^{-t_0x} 
\leq \sup_{x \in [0,N]} |a(x) \int_0^h e^{sa(x)} ds| + \frac{\varepsilon}{2} \leq M_a h + \frac{\varepsilon}{2} \leq \varepsilon.

To see that $t \to T(t)$ is not only norm-continuous but also infinitely often differentiable on $(0, \infty)$, let $t > 0$. Observe first that

$$A^n T(t) f := x \to a(x)^n e^{ta(x)} f(x) \in C_0[0, \infty)$$

and that $A^n T(t) \in \mathcal{L}(X)$ for all $n \in \mathbb{N}$. It follows that

$$\|A^n \left( \frac{T(t+h) - T(t)}{h} \right) - A^{n+1} T(t)\| = \sup_{\|f\| \leq 1} \|a(\cdot)^n (\frac{e^{ha(x)} - e^{ta(x)}}{h}) f(\cdot) - a(\cdot)^{n+1} e^{ta(\cdot)} f(\cdot)\|$$

$$= \sup_{\|f\| \leq 1} \|a(\cdot)^n (\frac{e^{ha(x)} - e^{ta(x)}}{h}) f(\cdot) - a(\cdot)^{n+1} e^{ta(\cdot)} f(\cdot)\|$$

$$\leq \sup_{x \in [0,N]} \left| \frac{e^{ha(x)} - e^{ta(x)}}{h} - a(x) \right| a(x)^n + \sup_{x \in [N,\infty)} \left| a(x)^n e^{ta(x)} \left( \frac{1}{h} \int_0^h a(x) e^{sr(x)} dr - a(x) \right) \right|$$

$$\leq \sup_{x \in [0,N]} \left| \frac{e^{ha(x)} - e^{ta(x)}}{h} - a(x) \right| a(x)^n + \sup_{x \in [N,\infty)} \left| a(x)^n e^{ta(x)} \right| 2|a(x)|.$$

Now choose $N > 0$ such that $\sup_{x \in [N,\infty)} 2|a(x)^n e^{ta(x)}| \leq \frac{\varepsilon}{2}$ and set $M_{n,a,N} := \sup_{x \in [0,N]} \left| a(x)^n \right|$. Then

$$\|A^n \left( \frac{T(t+h) - T(t)}{h} \right) - A^{n+1} T(t)\| \leq \sup_{x \in [0,N]} \left| \frac{e^{ha(x)} - e^{ta(x)}}{h} - a(x) \right| \cdot M_{n,a,N} + \frac{\varepsilon}{2}.$$

Now consider the map $\phi : z \to e^{za(\cdot)}$ as a map from $\mathbb{C}$ into $C[0, N]$. Then $\phi$ is entire since the maps $t \to \langle \delta_x, \phi \rangle$ are entire for $x \in [0, N]$ and $\phi'(\zeta) = a(\cdot)e^{za(\cdot)}$. This implies that

$$\left\| \frac{e^{ha(x)} - e^{ta(x)}}{h} - a(x) \right\|_{C[0,N]} = \sup_{x \in [0,N]} \left| \frac{e^{ha(x)} - e^{ta(x)}}{h} - a(x) \right| \to 0$$
as $h \to 0$. 
(g) If \(a(x) = -x + ie^x (x \geq 0)\), then the semigroup is strongly continuous on \([0, \infty)\), \(t \rightarrow T(t)\) is continuous on \((0, \infty)\) and \(t \rightarrow T(t)f\) is \(n\)-times continuously differentiable on \((n, \infty)\) for all \(f \in X\). Similar arguments as in part (f) show that the semigroup is strongly continuous on \([0, \infty)\), \(t \rightarrow T(t)\) is norm-continuous on \((0, \infty)\), and that \(t \rightarrow T(t)f\) is \(n\)-times continuously differentiable on \((n, \infty)\) with derivatives

\[
\left(\frac{d^n}{dt^n}T(t)\right) f : x \mapsto (-x + ie^x)^n e^{-t(x-ie^x)} f(x).
\]

(h) If \(a(x) = ix (x \geq 0)\), then the semigroup is strongly continuous on \([0, \infty)\), \(t \rightarrow T(t)\) is not measurable on \((0, \infty)\), and \(t \rightarrow T(t)f\) is, in general, nowhere differentiable. By part (c), since \(\sup_{x \geq 0} \text{Re}(a(x)) = 0\), \(t \rightarrow T(t)\) is strongly continuous on \([0, \infty)\). To verify that \(t \rightarrow T(t)\) is not measurable on \((0, \infty)\), let \(0 < s < t\). Now choose \(f \in C_0[0, \infty)\) with \(f(x) = 1\) for \(0 \leq x \leq \frac{2\pi}{t-s}\). Then

\[
\|T(t) - T(s)\| = 2 \text{ since } \|T(t) - T(s)\| \leq \|T(t)\| + \|T(s)\| \leq 2 \text{ and }
\]

\[
\|T(t) - T(s)\| \geq \|T(t)f - T(s)f\| = \sup_{x \geq 0} |e^{i(t-s)x} - 1| |f(x)| \geq 2.
\]

Since \(\|T(t)-T(s)\| = 2\) for \(t \neq s\), one can construct disjoint open balls around every value of \(T(t)\) with \(t > 0\). Since there are uncountably many values of \(t > 0\), it is not possible for \(t \rightarrow T(t)\) to be approximated by a sequence of simple functions. Therefore, \(t \rightarrow T(t)\) is not measurable. Furthermore, note that

\[
\frac{d}{dt}(t \rightarrow T(t)f(x)) = a(x)e^{ta(x)}f(x) = ixe^{i\pi}f(x).
\]

Since \(x \rightarrow ixe^{i\pi}f(x) \notin C_0[0, \infty)\) for certain \(f\), in general, the continuous function \(t \rightarrow T(t)f\) is nowhere differentiable.

In the above examples, each of the operators share the common characteristic that their spectrums are contained in a left half plane; i.e., \(s(A) = \sup_{x \geq 0} \text{Re}(a(x)) \leq 9\).
As a result, all the multiplication operators studied so far generated strongly continuous semigroups (also called $C_0$-semigroups). The investigation now turns to examples of operators $A$ that do not generate strongly continuous semigroups.

**Example 1.1.3.** Let $-A$ be the generator of a $C_0$-semigroup that is not a $C_0$-group; that is, since $-A$ generates a $C_0$-semigroup, $A$ does not generate a $C_0$-semigroup. For such an operator, the abstract Cauchy problem

$$w'(t) = -A w(t), \ w(0) = x$$

is well-posed; i.e., the semigroup $T(t)$ ($t \geq 0$) generated by $-A$ provides unique classical solutions $w(t) = T(t)x$ for all $x \in D(A)$ ($x \in X$). Consider the backwards-in-time problem

$$(ACP_T) \quad w'(t) = -A w(t), \ w(T) = f \text{ for } t \in [0,T],$$

where $f$ is an observed state of the system at time $T > 0$, and the intention is to find the history $w(t)$ for $0 \leq t < T$. Then, by letting $u(t) := w(T - t)$, $(ACP_T)$ is equivalent to

$$(ACP) \quad u'(t) = A u(t), \ u(0) = f \text{ for } t \in [0,T].$$

To illustrate with a particular example, suppose $X = C_0([0,\infty), \mathbb{C})$ and

$$Af(x) = a(x)f(x), \ a(x) = x + i e^{x^2},$$

with maximal domain $D(A) \subset X$. Then $-A$ generates the semigroup

$$T(t)f(x) = e^{-tx}e^{-ite^{x^2}}f(x)$$

which is strongly continuous for $t \geq 0$ and norm-continuous for $t > 0$. To see this, by part (c) of Remark 1.1.2, since $\sup_{x \geq 0} Re(a(x)) \leq 0$, $t \to T(t)$ is strongly
continuous on $[0, \infty)$. To show that $t \to T(t)$ is norm-continuous on $(0, \infty)$, given any $\varepsilon > 0$, choose $N > 0$ sufficiently large such that for $t_0 > 0$, $e^{-t_0 x} \leq \frac{\varepsilon}{4}$ for all $x \geq N$. Then with $h \leq \frac{\varepsilon}{2M_a}$ where $M_a = \sup_{x \in [0,N]} |a(x)|$, $\|T(t_0 + h) - T(t_0)\| = \\
\sup\{\|f\| \leq 1 \sup_{x \geq 0} |e^{-(t_0 + h)x}e^{-i(t_0 + h)e^{x^2}} - e^{-t_0 x}e^{-it_0 e^{x^2}}| \|f\| \} \\
\leq \sup_{x \geq 0} e^{-t_0 x}|e^{-hx}e^{-ihx^2} - 1| \\
\leq \sup_{x \in [0,N]} |e^{ha(x)} - 1| + \sup_{x \in [N,\infty)} 2e^{-t_0 x} \\
\leq \sup_{x \in [0,N]} |a(x) \int_0^h e^{sa(x)} ds| + \frac{\varepsilon}{2} \leq M_a h + \frac{\varepsilon}{2} \leq \varepsilon.

Note that $x \to \frac{d}{dt} [T(t)f(x)] = a(x)e^{ta(x)}f(x) = (-x - ie^{x^2})e^{-t(x+ie^{x^2})}f(x)$ is not in $C_0[0,\infty)$ for all $f \in X$. This shows that $t \to T(t)$ is continuous on $(0, \infty)$ but nowhere differentiable since there are functions $f \in X$ for which $f \to T(t)f$ is nowhere differentiable.

However, $A$ does not generate a semigroup since

$$T(t)f(x) = e^{tx}e^{ite^{x^2}}f(x)$$

is not defined for all $f \in X$. Furthermore, if $f$ does not have compact support, $(ACP)$ may not have a global solution that is Laplace transformable in the classical sense. Moreover, the resolvent

$$\lambda \to R(\lambda, A)f : x \to \frac{1}{\lambda - x - ie^{x^2}}f(x)$$

cannot be the Laplace transform of a function in the classical sense since its domain (the resolvent set)

$$\{\lambda \in \mathbb{C} : \lambda \neq x + ie^{x^2}, x \geq 0\}$$

is not contained in any right half-plane (see Proposition 1.1.1 (ii)).

**Example 1.1.4.** On $X = C_0[0, \infty) \times C_0[0, \infty)$ consider the multiplication operator

$$A(f, g)(x) = \begin{pmatrix} ix & ix \\ 0 & ix \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$$
with maximal domain $D(A)$. Then $\sigma(A) = i\mathbb{R}^+$ and

$$R(\lambda, A)(f, g)(x) = \left( \begin{array}{cc} \lambda - ix & -ix \\ 0 & \lambda - ix \end{array} \right)^{-1} \left( \begin{array}{c} f(x) \\ g(x) \end{array} \right) = \left( \begin{array}{cc} 1 & \frac{ix}{(\lambda - ix)^2} \\ 0 & \frac{1}{\lambda - ix} \end{array} \right) \left( \begin{array}{c} f(x) \\ g(x) \end{array} \right).$$

Since $|\lambda - ix| \geq \lambda$ and $|\lambda - ix| \geq |ix|$ for all $\lambda > 0$ and $x \geq 0$, it follows that $\|R(\lambda, A)\| \leq \frac{2}{\lambda}$ for all $\lambda > 0$. However, if $\lambda = a + ib$ with $a > 0$, then

$$\|R(a + ib, A)(0, g)\| = \sup_{x \geq 0} \left| \frac{ix}{(\lambda - ix)^2} g(x), \frac{1}{\lambda - ix} g(x) \right| \geq \sup_{x \geq 0} \left| \frac{ix}{(\lambda - ix)^2} g(x) \right|.$$

Since

$$\sup_{x \geq 0} \left| \frac{ix}{(\lambda - ix)^2} \right| = \sup_{x \geq 0} \frac{x}{a^2 + (b - x)^2} = \frac{\sqrt{a^2 + b^2} + b}{2a^2}, \quad (1.1)$$

there exists a $g \in C_0([0, \infty))$ with $\|g\|_{\infty} = 1$ such that

$$\|R(a + ib, A)(0, g)\| \geq \frac{\sqrt{a^2 + b^2} + b}{2a^2}.$$

Then, as $b \to \infty$, $R(a + ib, A)(0, g) \neq o(|b|)$. By Proposition (1.1.1) condition $(e)$, $R(\lambda, A)(0, g)$ cannot be a classical Laplace transform. Since (1.1) shows that

$$\|R(a + ib, A)(0, g)\| = O|b|,$$

the condition $(e)$ of Proposition (1.1.1) is a rather sharp growth estimate.

If $g \in C_0[0, \infty)$ is such that $x \to xg(x) \notin C_0[0, \infty)$, then

$$t \to T(t)(f, g) : x \to \left( \begin{array}{cc} e^{ixt} & ixe^{ixt} \\ 0 & e^{ixt} \end{array} \right) \left( \begin{array}{c} f(x) \\ g(x) \end{array} \right)$$

is not a function from $[0, \infty) \to X$. However, the once integrated semigroup

$$S(t)(f, g) := t \to \int_0^t T(s)(f, g) \, ds$$
becomes a norm-Lipschitz continuous family of bounded linear operators. In fact, for
\[ S(t)(f, g)(x) = \begin{pmatrix} \int_0^t e^{ixs} \, ds & \int_0^t ise^{ixs} \, ds \\ 0 & \int_0^t e^{ixs} \, ds \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \]
it follows that
\[ \|S(t) - S(t_0)\| \leq 2|t - t_0|. \]
Hence, \( t \to S(t) \) is a norm Lipschitz continuous function. Since
\[ R(\lambda, A) = \lambda \int_0^\infty e^{-\lambda t} S(t) \, dt, \]
\( \lambda \to R(\lambda, A) \) is an example of a function that has a \( \lambda \)-multiplied Laplace representation, but cannot be a Laplace transform itself. Also \( t \to S(t) \) is a norm Lipschitz continuous function that is nowhere differentiable with respect to \( t \) since its formal derivative, \( t \to T(t) \) is not a function from \([0, \infty) \to \mathcal{L}(X)\).

**Example 1.1.5.** The linear PDE
\[
\begin{align*}
    u_t(t, x) &= x^2 u_x(t, x) \quad t \geq 0 \\
    u(t, 0) &= 0 \quad t \geq 0 \\
    u(0, x) &= x \quad x \in [0, 1]
\end{align*}
\]
has the unique solution \( u(t, x) = \frac{x^2}{1-tx} \). Because of the singularity at \( t = \frac{1}{x} \), \( u(t, x) \) is not in \( L^1 \) and therefore not Laplace transformable. Thus, this PDE has a solution that cannot be found with classical Laplace transform methods.

**Example 1.1.6.** The functions \( t \to e^{n^2} \) for \( n \geq 1 \) are not Laplace transformable in the classical sense, whereas the functions \( f(t) = e^t e^{st} \cos(e^{st}) \) and
\[
g(t) = \begin{cases} 
    e^{n^2} & : t \in [n, n + (\frac{1}{2})n e^{-n^2}] \\
    0 & : \text{else}
\end{cases}
\]
are transformable for \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) > 0 \) since the antiderivatives

\[
F(t) := \int_0^t f(t) \, dt = \sin(e^t) - \sin(e) < 2
\]

and \( G(t) := \int_0^t g(t) \, dt < \int_0^\infty g(t) \, dt = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = 2 \) are bounded (see Proposition 1.1.1 (iii)).

The fact that functions like \( t \to e^{t^2} \) are not classically Laplace transformable makes the application of the classical transform to inhomogeneous linear ODE’s somewhat cumbersome since equations like

\[
f'(t) + f(t) = (2t + 1)e^{t^2} \quad f(0) = 1
\]

(with solution \( f(t) = e^{t^2} \)) cannot be solved directly with the Laplace transform method but require an additional step via the variation of parameters formula.

By using asymptotic versions of the Laplace transform, introduced first by J.C. Vignaux [37] and further developed by Boris Bäumer [4], [5] and Günter Lumer and Frank Neubrander [24], [25], one can remove all growth restrictions on the functions \( f \) and obtain a Laplace transform based operational calculus that is applicable to all Banach space valued functions (classical functions, distributions, hyperfunctions) defined on an interval \([a, b)\) or on a half-line \([a, \infty)\) (and, in particular, to all of the problems described above).

The power of the classical Laplace transform lies in its operational properties and its injectivity. Among the operational properties, most significant is the property that the Laplace transform maps convolution \( (f * g)(t) := \int_a^b f(t - s)g(s) \, ds \) to multiplication; i.e.,

\[
\hat{f * g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda).
\]

Because of this property, the Laplace transform converts linear integral and differential equations into algebraic equations which are often, at least theoretically,
easier to solve. In order to retain its applicability, any desired extension of the Laplace transform must maintain all operational properties and the injectivity of the classical Laplace transform while being applicable to functions that are either not exponentially bounded or that exist only on finite intervals.

1.2 The Asymptotic Laplace Transform

A natural extension of the Laplace transform is the finite Laplace transform

\[ \hat{f}_T(\lambda) := \int_0^T e^{-\lambda t} f(t) \, dt \]

which is essential for the definition of the asymptotic Laplace transform to be discussed later in this section. The finite Laplace transform is applicable to functions in \( L^1_{\text{loc}}([0, \infty), X) \) regardless of their growth at infinity. Unfortunately, the finite Laplace transform does not retain the essential property of mapping convolution onto multiplication. Or does it? Let \( f \in L^1([0, T], X) \) and \( g \in L^1([0, T], \mathbb{C}) \), then \( f \ast g \in L^1([0, T], X) \) (see [33], Theorem 7.14) and

\[
\hat{f}_T(\lambda) \hat{g}_T(\lambda) = \int_0^T e^{-\lambda t} f(t) \, dt \int_0^T e^{-\lambda s} g(s) \, ds = \int_0^T \int_0^T e^{-\lambda(t+s)} f(t) g(s) \, dt \, ds \\
= \int_0^T \int_s^T e^{-\lambda t} f(t-s) g(s) \, dt \, ds \\
= \int_0^T \int_0^T e^{-\lambda t} f(t-s) g(s) \, ds \, dt + \int_T^{2T} \int_{t-T}^T e^{-\lambda t} f(t-s) g(s) \, ds \, dt \\
= \int_0^T e^{-\lambda t} (f \ast g)(t) \, dt + \int_T^{2T} e^{-\lambda t} \int_{t-T}^T f(t-s) g(s) \, ds \, dt \\
= \hat{(f \ast g)}_T(\lambda) + e^{-\lambda T} \int_0^T e^{-\lambda t} \int_t^T f(t+s) g(s) \, ds \, dt.
\]

Hence the finite Laplace transform retains the essential property of mapping convolution onto multiplication up to a remainder term

\[ a(\lambda) := e^{-\lambda T} \int_0^T e^{-\lambda t} \int_t^T f(t + T - s) g(s) \, ds \, dt \]
satisfying \( \|a(\lambda)\| \leq M e^{-\lambda T} \) (\( \lambda > 0 \)) or

\[
\limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln \|a(\lambda)\| \leq -T. \tag{1.2}
\]

To have the multiplicative property restored, one needs to introduce an equivalence relation under which the remainder term is equivalent to the zero element; i.e., one defines

\[ a \approx_T 0 \text{ if (1.2) holds}. \]

Observe that (1.2) holds if and only if for all \( \varepsilon > 0 \) there exists a \( \lambda_0 \) and a \( M_\varepsilon > 0 \) such that

\[
\|a(\lambda)\| \leq M_\varepsilon e^{\lambda(-T+\varepsilon)} \tag{1.3}
\]

for all \( \lambda > \lambda_0 \). Furthermore, it is essential to ensure that the injectivity of the Laplace transform remains valid. If the injectivity holds, then the asymptotic extension of the finite Laplace transform retains all essential operational properties of the Laplace transform while being applicable to all \( L^1 \)-functions on finite intervals.

In working toward such an equivalence relation, it is necessary to justify why functions of exponential decay \( T \), like the above remainder term \( a(\lambda) \), can be treated as zero-elements when using the finite or classical Laplace transform. The Phragmén-Doetsch Inversion Formula is the key to unlocking the needed justification.

**Theorem 1.2.1.** (Phragmén-Doetsch Inversion Formula) Let \( f \in L^1_{\text{loc}}([0, \infty), X) \) and \( F : t \to \int_0^t f(s) \, ds \) for all \( t \geq 0 \). If \( \hat{f}(\lambda) = \int_0^\infty e^{-\lambda s} f(s) \, ds \) converges absolutely for some \( \lambda \geq 0 \), then

\[
F(t) = \lim_{k \to \infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{jk t} \hat{f}(jk)
\]

for all \( t \geq 0 \).

In order to be able to state the following lemma that will be used in the proof of the Phragmén-Doetsch Inversion Formula, the following definition is needed.
Definition 1.2.2. A postsectorial region $\Sigma$ in $\mathbb{C}$ is defined to be an open subset of the right-half plane such that for all $0 < \phi_0 < \pi/2$ there exists $r_0 > 0$ such that $\lambda = re^{i\phi} \in \Sigma$ for all $r > r_0$ and $|\phi| < \phi_0$. An analytic function $u : \mathbb{C} \supset \Sigma \to X$ is of minimal exponential type on a postsector $\Sigma$ if for every sector

$$\Sigma_\phi = \{ \lambda : |\arg(\lambda)| \leq \phi < \frac{\pi}{2}, \lambda \neq 0 \}$$

and all $\varepsilon > 0$ there exists $M > 0$ such that

$$\|u(\lambda)\| \leq Me^{\varepsilon|\lambda|}$$

in $\Sigma \cap \Sigma_\phi$. Denote by $O_\Sigma(X)$ the set of $X$-valued analytic functions of minimal exponential type on a given postsector $\Sigma$. Also denote the set of analytic functions that are of minimal exponential type on some postsector $\Sigma$ by $O(\Sigma, X) := \cup_\Sigma O_\Sigma(X)$.

Lemma 1.2.3. Let $f \in L_{loc}^1([0, \infty), X)$ such that $\hat{f}(\lambda)$ exists for some $\lambda \in \mathbb{C}$. Then

(a) $\hat{f}$ is of minimal exponential type on the half-plane $\text{Re}(\lambda) > \bar{\omega} + \varepsilon$ for any $\varepsilon > 0$, where $\bar{\omega}$ is as in Proposition 1.1.1, and

(b) $a(\lambda) := \int_T^\infty e^{-\lambda t}f(t)\,dt$ is of minimal exponential type on the half-plane $\text{Re}(\lambda) > \bar{\omega} + \varepsilon$ and $a(\lambda) \approx_T 0$ for any $T > 0$.

Proof. Statement (a) follows directly from Proposition 1.1.1. By (a) of the proposition, there exist $M, \bar{\omega} > 0$ such that $\|\hat{f}(\lambda)\| \leq M\frac{|\lambda|}{\text{Re}(\lambda) - \bar{\omega}}$ for $\text{Re}(\lambda) > \bar{\omega}$. Thus $\hat{f}$ is of minimal exponential type on the half-plane $\{ \lambda : \text{Re}(\lambda) > \bar{\omega} + \varepsilon \}$. To verify (b), note that $a(\lambda) = \int_0^\infty e^{-\lambda t}f(t)(\chi_{(T,\infty)}(t)\,dt$. Therefore, $a$ is of minimal exponential type on the half-plane $\{ \lambda : \text{Re}(\lambda) > \bar{\omega} + \varepsilon \}$. Since $\|F(t)\| \leq Me^{\varepsilon t}$ where $F(t) := \int_0^t f(s)\,ds$, it follows that for $T \geq 0$,

$$\int_T^\infty e^{-\lambda t}f(t)\,dt = -e^{-\lambda T}F(T) + \lambda \int_T^\infty e^{-\lambda t}F(t)\,dt.$$
Furthermore, there exists a constant $C_T > 0$ such that 
\[
\| \int_T^\infty e^{-\lambda t} f(t) \, dt \| = \| -e^{-\lambda T} F(T) + \lambda \int_T^\infty e^{-\lambda t} F(t) \, dt \|
\leq e^{-\lambda T} \| F(T) \| + \frac{M|\lambda|}{(\lambda - \bar{\omega})} e^{-(\lambda - \bar{\omega})T}
\leq (M + \frac{M|\lambda|}{(\lambda - \bar{\omega})}) e^{-(\lambda - \bar{\omega})T} \leq C_T e^{-(\lambda - 2\bar{\omega})T}
\]
for all $\lambda > 2\bar{\omega}$. Thus $\int_T^\infty e^{-\lambda t} f(t) \, dt \approx_T 0$. 

The following proof of Theorem 1.2.1 follows the argument outlined in [11], Kap. 8, Satz 1.

Proof. (Phragmén-Doetsch Inversion Formula) Let $\lambda_0 > 0$ be such that $\hat{f}(\lambda_0)$ exists. For $k > \omega$, $\hat{f}(jk)$ exists and, by (a) of Proposition 1.1.1, there exists $\tilde{M} > 0$ such that $|\hat{f}(jk)| \leq \tilde{M}$ as $k \to \infty$. Thus for $k \in \mathbb{N}$ with $k > \omega$, the infinite sum
\[
y := \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j!} e^{jkt} \hat{f}(jk) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j!} e^{jkt} \int_0^\infty e^{-jks} f(s) \, ds
\]
converges uniformly on compacts. Moreover, by the dominated convergence theorem,
\[
z_T := \int_0^T (1 - e^{-e^{k(t-s)}}) f(s) \, ds = \int_0^T \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{jkt} f(s) \, ds
\leq \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{jkt} \int_0^T e^{-jks} f(s) \, ds.
\]
By Lemma 1.2.3, for all $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that
\[
\left\| \int_T^\infty e^{-\lambda s} f(s) \, ds \right\| \leq M_\varepsilon e^{-\lambda (T-\varepsilon)}
\]
for all $\lambda > \omega + \varepsilon$. This implies
\[
\|z_T - y\| = \left\| \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{jkt} \int_T^\infty e^{-jks} f(s) \, ds \right\|
\leq M_\varepsilon \sum_{j=1}^{\infty} \frac{1}{j!} e^{jkt} e^{-jk(T-\varepsilon)} = M_\varepsilon \left[ 1 - e^{k(t-T+\varepsilon)} \right] \to 0
\]
as $T \to \infty$ for all $t \geq 0$. This means that
\[
y_k := y = \int_0^\infty (1 - e^{-e^{k(t-s)}}) f(s) \, ds.
\]
Then, for \(0 < \delta < t\) and \(F(t) := \int_0^t f(s) \, ds\),
\[
\|y_k - F(t)\| \leq \left\| \int_0^{t-\delta} e^{-k(t-s)} f(s) \, ds \right\| + \left\| \int_{t-\delta}^t e^{-k(t-s)} f(s) \, ds \right\| + \left\| \int_t^{t+\delta} (1 - e^{-k(t-s)}) f(s) \, ds \right\| + \left\| \int_{t+\delta}^\infty (1 - e^{-k(t-s)}) f(s) \, ds \right\|.  
\] (1.4)

Observe that \(0 < e^{-k(t-s)} < 1\) and \(0 < 1 - e^{-k(t-s)} < 1\). Thus, for the second and third terms of (1.4), given any \(\varepsilon > 0\), one can choose a \(\delta\) small enough that
\[
\left\| \int_{t-\delta}^t e^{-k(t-s)} f(s) \, ds \right\| \leq \int_{t-\delta}^t |f(s)| \, ds < \frac{\varepsilon}{4}
\]

and
\[
\left\| \int_{t}^{t+\delta} (1 - e^{-k(t-s)}) f(s) \, ds \right\| \leq \int_{t}^{t+\delta} |f(s)| \, ds < \frac{\varepsilon}{4}
\]
independent of \(k\). This follows from the fact that \(t \to \int_0^t |f(t)| \, ds\) is continuous. Now fix this \(\delta\). To continue, notice that for \(0 \leq s \leq t - \delta\), \(0 < e^{-k(t-s)} \leq e^{-k\delta}\).

This means that for sufficiently large \(k\), say \(k > k_1\),
\[
\left\| \int_0^{t-\delta} e^{-k(t-s)} f(s) \, ds \right\| \leq e^{-k\delta} \int_0^{t-\delta} |f(s)| \, ds < \frac{\varepsilon}{4}.
\]

Now if \(z > 0\), by the Mean Value Theorem, \(\frac{e^{-z} - 1}{z} = -e^{-cz}\) for some \(0 < c < 1\). This implies that \(1 - e^{-z} = ze^{-cz} < z\). Therefore \(0 < 1 - e^{-k(t-s)} < e^{k(t-s)}\), and thus
\[
\left\| \int_{t+\delta}^\infty (1 - e^{-k(t-s)}) f(s) \, ds \right\| \leq e^{kt} \int_{t+\delta}^\infty e^{-ks} |f(s)| \, ds.
\]

Since \(\hat{f}(\lambda)\) converges absolutely for some \(\lambda > 0\), by Lemma 1.2.3 there exists \(C > 0\) such that \(\left\| \int_{t+\delta}^\infty e^{-ks} |f(s)| \, ds \right\| \leq Ce^{-k(t+\delta')}\) for sufficiently large \(k\) and some \(0 < \delta' < \delta\). This implies
\[
\left\| \int_{t+\delta}^\infty (1 - e^{-k(t-s)}) f(s) \, ds \right\| \leq Ce^{-k\delta'}.
\]

It follows that, for sufficiently large \(k\), say \(k > k_2\),
\[
\left\| \int_{t+\delta}^\infty (1 - e^{-k(t-s)}) f(s) \, ds \right\| < \frac{\varepsilon}{4}.
\]
Thus, from (1.4), for \( t > 0 \), for any \( \varepsilon > 0 \) with \( \delta > 0 \) and for all \( k > \max\{k_1, k_2\} \) as described above,

\[
\left\| \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{jkt} \hat{f}(jk) - F(t) \right\| < \varepsilon.
\]

For \( t = 0 \), it follows that (using Lemma 1.2.3 again)

\[
\| y_k - F(0) \| = \| y \| = \left\| \int_0^{\infty} \left( 1 - e^{-k(t-s)} \right) f(s) \, ds \right\|
\leq \int_0^{\delta} |f(s)| \, ds + \int_\delta^{\infty} |1 - e^{-k(t-s)}| |f(s)| \, ds
\leq \int_0^{\delta} |f(s)| \, ds + \int_\delta^{\infty} e^{-k(t-s)} |f(s)| \, ds
\leq \int_0^{\delta} |f(s)| \, ds + M_3 e^{-k\delta'}
\]

for \( 0 < \delta' < \delta \). Now choose \( \delta > 0 \) small enough such that the first term is less than \( \frac{\varepsilon}{2} \) for all \( k \) and then pick \( k_0 \) large enough such that the second term is less than \( \frac{\varepsilon}{2} \) for all \( k \geq k_0 \). Thus, Theorem 1.2.1 holds for all \( t \geq 0 \).

A more restrictive version of the Phragmén-Doetsch Inversion Formula can be found in Section 2.3 of [2]. There it is assumed that \( f \in L^\infty([0, \infty), X) \) or \( F(t) := \int_0^t f(s) \, ds \in Lip_0([0, \infty), X) \). In Example 1.1.6, the function

\[
g(t) = \begin{cases} 
  e^{n^2} & : t \in \left[n, n + \left(\frac{1}{2}\right)n e^{-n^2}\right] \\
  0 & : \text{else}
\end{cases}
\]

has been shown to be Laplace transformable. Note that \( t \to g(t) \) is not a function in \( L^\infty([0, \infty), X) \), and therefore \( \hat{g}(\lambda) \) could not be inverted using the version of the Phragmén-Doetsch Inversion Formula found in [2]. However, since \( g \) is positive, it follows that \( \int_0^{\infty} e^{-\lambda t} |g(t)| \, dt = \int_0^{\infty} e^{-\lambda t} g(t) \, dt = \hat{g}(\lambda) \) exists. This means \( \hat{g}(\lambda) \) converges absolutely. Thus the Laplace transform of \( g \) can be inverted using the version of the Phragmén-Doetsch Inversion Formula of Theorem 1.2.1.

Although the numerical usefulness of the Phragmén-Doetsch Inversion Formula remains questionable, the theoretical power of the theorem is evident by Corollary
1.2.5 below. Another theoretical advantage of the formula is that one only needs to know the values of the function $\lambda \rightarrow \hat{f}(\lambda)$ to be inverted on equidistance sequences. For example, this attribute implies that both the sine and cosine functions cannot be Laplace transform representations of some function $f$ since both sine and cosine are zero on an equidistance sequence of function values.

A significant extension of the Phragmén-Doetsch Inversion Formula due to Boris Bäumer is the Phragmén-Mikusiński Inversion (see [5] or [26]). For this inversion, the functions $f$ are assumed to be $C_0([0,T],X)$, which is a slightly more restrictive class of functions when compared to the $L_{loc}^1([0,\infty),X)$ used in Theorem 1.2.1. However, for the Phragmén-Mikusiński Inversion, one only needs to evaluate the function $\lambda \rightarrow \hat{f}(\lambda)$ to be inverted on Müntz sequences. A sequence $(\beta_n) \subset \mathbb{R}^+$ is defined to be a Müntz sequence, if for all $n \in \mathbb{N}$, $\beta_{n+1} - \beta_n \geq 1$, and $\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty$.

This allows for more flexibility on the values of $\lambda \rightarrow \hat{f}(\lambda)$ that are needed for inversion, and more importantly, it allows for the following fundamental theorem concerning the support of a $L_{loc}^1$-function in terms of its Laplace transforms.

**Theorem 1.2.4.** (Support Theorem) Let $0 \leq T$ and let $f \in L_{loc}^1([0,\infty),X)$ be Laplace transformable. Then the following are equivalent:

(i) Every Müntz sequence $(\beta_n)_{n \in \mathbb{N}}$ satisfies
\[ \limsup_{n \to \infty} \frac{1}{\beta_n} \ln \| \hat{f}(\beta_n) \| = -T. \]

(ii) For every Müntz sequence $(\beta_n)_{n \in \mathbb{N}}$ there exists a Müntz subsequence $(\beta_{n_k})_{k \in \mathbb{N}}$ satisfying
\[ \lim_{k \to \infty} \frac{1}{\beta_{n_k}} \ln \| \hat{f}(\beta_{n_k}) \| = -T. \]

(iii) There exists a Müntz sequence $(\beta_n)_{n \in \mathbb{N}}$ satisfying
\[ \limsup_{n \to \infty} \frac{1}{\beta_n} \ln \| \hat{f}(\beta_n) \| = -T. \]

(iv) $f(t) = 0$ almost everywhere on $[0,T]$ and $T \in \text{supp}(f)$. 

21
$$\limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln \| \hat{f}(\lambda) \| = -T.$$ 

For this investigation of the asymptotic Laplace transform, the most important theoretical consequence of the Phragmén-Doetsch Inversion Formula is the following corollary.

**Corollary 1.2.5.** Let $f \in L^1_{\text{loc}}([0, \infty), X)$, $F : t \to \int_0^t f(s) \, ds$ for all $t \geq 0$, and $a : (0, \infty) \to X$ be a function of exponential decay $T$. If $\hat{f}(\lambda) = \int_{-\infty}^{\lambda} e^{-\lambda t} f(t) \, dt$ converges absolutely for some $\lambda \geq 0$, then

$$F(t) = \lim_{k \to \infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{jkt} [\hat{f}(jk) + a(jk)];$$

i.e. the Phragmén-Doetsch inversion of the Laplace transform is invariant under perturbations $a(\cdot)$ with $a \approx_T 0$.

**Proof.** By definition of exponential decay $T$, for all $T_0 < T$ there exists $\lambda_0 > \omega$ such that $\|a(\lambda)\| \leq e^{-T_0 \lambda}$ for all $\lambda \geq \lambda_0$. Hence for $jk \geq \lambda_0$,

$$\left\| \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{jkt} a(jk) \right\| \leq \sum_{j=1}^{\infty} \frac{1}{j!} e^{-T_0 jk} = e^{-(T_0 - t)k} = e^{-(T_0 - t)k} - 1.$$

In particular, $\| \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{jkt} a(jk) \| \to 0$ as $k \to \infty$ for all $0 \leq t < T_0 < T$. \hfill $\Box$

The invariance of the Phragmén-Doetsch Inversion Formula under perturbations of exponential decay $T$ justifies the use of the equivalence relation $\approx_T$ in Laplace transform theory. With the introduction of this equivalence relation, one is now able to modify the definition of the finite Laplace transform. In order to motivate the desired modification, the following observations are needed. Let $f \in L^1([0, \infty), X)$ and suppose $\hat{f}(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t) \, dt$ converges for some $\lambda \in \mathbb{C}$. Then

$$\hat{f}(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t) \, dt = \int_0^T e^{-\lambda t} f(t) \, dt + \int_T^{\infty} e^{-\lambda t} f(t) \, dt. \quad (1.5)$$
Lemma 1.2.3, Corollary 1.2.5, and the equality (1.5) show that a function on \([0,T]\) is uniquely determined by its finite Laplace transform plus any perturbation that is \(T\)-asymptotically equal to zero. This property leads to the following definition.

**Definition 1.2.6.** Let \(f \in L^1([0,T],X)\) and \(0 < T < \infty\), then the \(T\)-asymptotic Laplace transform of \(f\) is defined as the set of all analytic functions \(r\) defined on some post-sectorial region \(\Sigma\) with values in \(X\) which are of minimal exponential type, i.e.
\[
\{f\}_T := \hat{f}_T + O_T,
\]
where \(O_T = \{a \in O(\Sigma,X) : a \approx_T 0\}\). Moreover, for \(f \in L^1_{loc}([0,\infty),X)\), the asymptotic Laplace transform \(\{f\}\) is given by the intersection of the sets \(\{f\}_T\), i.e.,
\[
\{f\} := \cap_{T>0} \{f\}_T = \{f\}_\infty.
\]
The following result, due to Lumer and Neubrander based on the early, groundbreaking work of J.C. Vignaux [37], shows that \(\{f\} \neq \emptyset\) for all functions \(f \in L^1_{loc}([0,\infty),X)\) and that all operational properties of the classical Laplace transform remain valid.

**Theorem 1.2.7.** Let \(f,g \in L^1([0,T),X)\) or \(f,g \in L^1_{loc}([0,\infty),X)\). Then, for all \(0 < T \leq \infty\) (where \(\{f\}_\infty := \{f\} = \cap_{T>0} \{f\}_T\)),

(a) \(\emptyset \neq \{f\} \subset \{f\}_T\),

(b) if \(\hat{f}\) exists, then \(\hat{f} \in \{f\}_T\),

(c) \(\{f\}_T \cap \{g\}_T \neq \emptyset\) if and only if \(f = g\) a.e. on \([0,T]\),

(d) \(\{f\}'_T = \lambda \{f\}_T - f(0)\),

(e) if \(g \in L^1([0,T),\mathbb{C})\) or \(g \in L^1_{loc}([0,\infty),\mathbb{C})\), then \(\{f * g\}_T = \{f\}_T \cdot \{g\}_T\), and
\( (f) \) \( \{ f \}^T_T(\lambda) = \{-tf\}^T_T. \)

\( (g) \) Let \( A \) be a closed linear operator on \( X \) and \( v, w \in L_{loc}^1([0, \infty), X) \).

(i) If \( r \in \{ v \}^T_T \), and \( Ar \in \{ w \}^T_T \), then \( Av = w \) on \([0, T]\).

(ii) If \( (1 \ast v)(t) \in D(A) \) and \( A(1 \ast v)(\cdot) \in L_{loc}^1([0, \infty), X) \), then

\[
\{ A(1 \ast v) \}^T_T = \frac{A}{\lambda} \{ v \}^T_T.
\]

Proof. For complete proofs of the above statements see either [24] or [26]. In order to be able to answer in detail questions like: “What is the asymptotic Laplace transform of \( t \rightarrow e^{t^2} \)” we recall the proof of (a) from [26]. Because \( \{ f \} := \cap_{T > 0} \{ f \}^T_T \), it is clear that \( \{ f \} \subset \{ f \}^T_T \). Let \( f \in L_{loc}^1([0, \infty), X) \). Then for \( 0 < T < \infty \), since the Laplace transform is analytic for all \( \lambda \) where \( \hat{f}(\lambda) \) exists (see Proposition 1.1.1), the finite Laplace transform \( \hat{f}_T(\lambda) = \int_0^\infty e^{-\lambda t} f(t) \cdot \chi_{[0,T]}(t) \, dt \) is an entire function. Furthermore the finite Laplace transform is of minimal exponential type by Lemma 1.2.3 (even bounded on \( Re(\lambda) > 0 \)). Thus,

\[
\{ f \}^T_T = \hat{f}_T + O_T \not\in \emptyset,
\]

where \( O_T = \{ a \in O(\Sigma, X) : a \approx_T 0 \} \). Now assume \( 0 < T' < T < \infty \), \( F(t) := \int_0^t f(s) \, ds \) for \( t \geq 0 \), and

\[
h(\lambda) := \int_{T'}^T e^{-\lambda t} f(t) \, dt = e^{-\lambda T'} F(T') - e^{-\lambda T} F(T) + \lambda \int_T^{T'} e^{-\lambda t} F(t) \, dt.
\]

Now, \( F \) is bounded on \([T', T]\) and thus \( h \approx_{T'} 0 \). It follows that

\[
\{ f \}^T_T = \hat{f}_{T'} + h + O_T \subset \hat{f}_{T'} + h + O_{T'} = \hat{f}_{T'} + O_{T'} = \{ f \}^T_{T'}.
\]

This means that

\[
\{ f \} = \cap_{T > 0} \{ f \}^T_T \subset \{ f \}^T_T \subset \{ f \}^T_{T'} \text{ if } 0 < T' < T < \infty.
\]
In order to show that the asymptotic Laplace transform \( \{f\} \) of \( f \) is well defined, (i.e., \( \{f\} \neq \emptyset \) for all \( f \in L^1_{loc}((0, \infty), X) \)), define

\[
r(\lambda) := \lambda \int_0^\infty e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{\sigma(\lambda)}})F(t) \, dt,
\]

(1.6)

where \( G(t) := \max\{\|F(t)\|, 1\} \), and \( \lambda \to d(\lambda) \) is an analytic function later to be determined such that \( r \in \{f\}_T \) for all \( T > 0 \). Let \( z \in \mathbb{C} \) with \( \text{Re}(z) \geq 0 \), then

\[|1 - e^{-z}| = |z| \int_0^1 e^{-zt} \, dt| \leq |z|.
\]

This means that

\[
\|e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{\sigma(\lambda)}})F(t)\| \leq e^{-\text{Re}(\lambda)t}|d(\lambda)|
\]

(1.7)

for all \( t \geq 0 \) and \( \lambda \in \Omega := \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > 0 \text{ and } \text{Re}(d(\lambda)) \geq 0\} \). With the assumptions that \( \Omega \) contains a post-sectorial region and that \( d(\lambda) \) is of minimal exponential type then

\[
\|r(\lambda)\| \leq \frac{1}{\text{Re}(\lambda)}|\lambda d(\lambda)|
\]

gives that \( r \in O(\Sigma, X) \). Now consider \( a(\lambda) = r(\lambda) - \int_0^T e^{-\lambda t} f(t) \, dt \). By definition \( r(\lambda) \in \{f\} \) if and only if \( a \approx_{\lambda T} 0 \) for all \( T > 0 \). Note that

\[
a(\lambda) = r(\lambda) - \int_0^T e^{-\lambda t} f(t) \, dt
\]

\[
= r(\lambda) - e^{-\lambda T} F(T) - \lambda \int_0^T e^{-\lambda t} F(t) \, dt
\]

\[
= \lambda \int_0^\infty e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{\sigma(\lambda)}})F(t) \, dt - \lambda \int_0^T e^{-\lambda t} e^{-\frac{d(\lambda)}{\sigma(\lambda)}} F(t) \, dt - e^{-\lambda T} F(T)
\]

\[
:= a_1(\lambda) - a_2(\lambda) - a_3(\lambda).
\]

Therefore it is necessary to show that \( a_1(\lambda), a_2(\lambda), \) and \( a_3(\lambda) \) are all \( \approx_{\lambda T} 0 \) for all \( T > 0 \). First consider \( a_3(\lambda) \). Clearly \( a_3(\lambda) \approx_{\lambda T} 0 \) for all \( T > 0 \). Next examine \( a_2(\lambda) = -\lambda \int_0^T e^{-\lambda t} e^{-\frac{d(\lambda)}{\sigma(\lambda)}} F(t) \, dt \). If \( \lambda \in \Sigma \cap \mathbb{R}^+ \) then \( \|a_2(\lambda)\| \leq MT e^{-\frac{\text{Re}(d(\lambda))}{M}} \), where

\[M = \sup_{0 \leq t \leq T} \|G(t)\|, \text{ and this means that } \limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln \|a_2(\lambda)\| = \limsup_{\lambda \to \infty} \frac{1}{\lambda} \frac{\text{Re}(d(\lambda))}{M}.
\]

Therefore, to make sure that \( a_2(\lambda) \approx_{\lambda T} 0 \) for all \( T > 0 \), it must be assumed that

\[
\frac{1}{\lambda} \text{Re}(d(\lambda)) \to \infty \text{ as } \lambda \to \infty.
\]

Finally, to show \( a_1(\lambda) = \lambda \int_T^\infty e^{-\lambda t}(1 - e^{-\frac{d(\lambda)}{\sigma(\lambda)}})F(t) \, dt \)
is of exponential decay $T$ for all $T > 0$, observe that it follows from (1.7) that 
\[ \|a_1(\lambda)\| \leq |d(\lambda)| e^{-\lambda T} \] for all $\lambda \in \Sigma \cap \mathbb{R}_+$. This means if $\limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln(|d(\lambda)|) \leq 0$, then $a_1(\lambda) \approx_T 0$ for all $T > 0$. In order to ensure $a(\lambda) \approx_T 0$ for all $T > 0$, it is necessary to have the damping function $d(\lambda)$ be an analytic function on a post-sectorial subregion of $\Sigma$ of the open right half-plane. Furthermore, $\Re(d(\lambda)) \geq 0$ for all $\lambda \in \Sigma$, $\frac{1}{\lambda} \Re(d(\lambda)) \to \infty$ as $\lambda \to \infty$, and $\limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln(|d(\lambda)|) \leq 0$.

An example of a function that holds these desired properties is

\[ d(\lambda) = \lambda \ln(\lambda). \]

Notice that $\ln(\lambda) := \ln|\lambda| + i \arg(\lambda)$ with $-\pi < \arg(\lambda) \leq \pi$ is of minimal exponential type in any postsector in the region $\{\lambda : |\lambda| > 1\} \cap \{\lambda : \Re(\lambda) > 0\}$.

To see this, observe that $|\ln(\lambda)| < |\ln(|\lambda|) + i \arg(\lambda)| = \sqrt{\ln^2|\lambda| + \arg^2(\lambda)}$ for $|\lambda| \geq 1$. Then for all $\varepsilon > 0$ there exists a $M > 0$ such that $|\ln(\lambda)| \leq Me^{\varepsilon|\lambda|}$. Since $\lambda \to \ln(\lambda)$ is of minimal exponential type, $\lambda \to \lambda \ln(\lambda)$ must also be of minimal exponential type. Let $\lambda = re^{i\theta}$, then $\Re(d(\lambda)) = r \ln r \cos \theta - r \theta \sin \theta$.

Thus if $\Sigma = \{\lambda \in \mathbb{C} : \Re(\lambda) > 1\} \cap \{\lambda = re^{i\theta} : r > e^{\theta \tan(\theta)}, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\}$, then $\Re(d(\lambda)) \geq 0$ for all $\lambda \in \Sigma$ and $\frac{1}{\lambda} \Re(d(\lambda)) \to \infty$ as $\lambda \to \infty$. Also because $|d(\lambda)| = \sqrt{r^2(\ln r)^2 + r^2 \theta^2}$, then $\limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln(|d(\lambda)|) = 0$. Thus $d(\lambda) = \lambda \ln \lambda$ satisfies all of the desired properties.

Since property ($g$) is essential for many applications of the asymptotic Laplace transform, the proof of Theorem 1.2.7 ($g$) from [24] is given below. For the proof, the following remark from [24] is needed.

**Remark 1.2.8.** If $f \in L^1_{loc}([0, \infty), X)$, then $F := 1 * f = \int_0^t f(s) \, ds$ is bounded on $[0, T]$, and for every $y \in \{f\}_T$, Theorem 1.2.7 gives that $\frac{1}{\lambda} y \in \{1\}_T \{f\}_T \subset \{F\}_T$. 

26
Thus, there exists $a \approx_T 0$ such that
\[
\frac{1}{\lambda} y(\lambda) = \int_0^T e^{-\lambda t} F(t) \, dt + a(\lambda) = \int_0^\infty e^{-\lambda t} dG(t) + a(\lambda)
\]
for all $\lambda > 0$, where $G$ is piecewise defined by
\[
G(t) := \begin{cases} 
(1 * F)(t) = \int_0^t (t - s) f(s) \, ds & : 0 \leq t \leq T \\
(1 * F)(T) & : t > T
\end{cases}
\]
By Lemma 1.2.3, it follows from the Phragmén-Doetsch inversion that
\[
G(t) = \int_0^t (t - s) f(s) \, ds = \lim_{k \to \infty} \sum_{j=1}^\infty \frac{(-1)^j e^{jk t} y(jk)}{j!} \]
for all $f \in L^1_{\text{loc}}([0, \infty), X)$, $y \in \{f\}_T$, and $0 \leq t < T \leq \infty$.

**Proof.** (Statement (g) of Theorem 1.2.7) Let $A$ be a closed linear operator on $X$ and $v, w \in L^1_{\text{loc}}([0, \infty), X)$. For $(i)$, if $r \in \{v\}_T$, $r(\lambda) \in D(A)$ for all $\lambda$ in a post-sectorial region $S$, and $Ar \in \{w\}_T$, then by 1.8, it follows that
\[
\int_0^t (t - s) v(s) \, ds = \lim_{k \to \infty} \sum_{j=1}^\infty \frac{(-1)^j e^{jk t} y(jk)}{j!} \]
and
\[
\int_0^t (t - s) w(s) \, ds = \lim_{k \to \infty} \sum_{j=1}^\infty \frac{(-1)^j e^{jk t} y(jk) A r(jk)}{j!} \]
for all $t \in [0, T)$. Since $A$ is closed, $\int_0^t (t - s) v(s) \, ds \in D(A)$ and $A \int_0^t (t - s) v(s) \, ds = \int_0^t (t - s) w(s) \, ds$ for all $t \in [0, T)$. Then, by using the closedness of $A$ again, $v(t) \in D(A)$ and $Av(t) = w(t)$ for almost all $t \in [0, T)$.

To see $(ii)$, note that with $(1 * v)(t) \in D(A)$, $A(1 * v)(\cdot) \in L^1_{\text{loc}}([0, \infty), X)$ and the closedness of $A$, one has that
\[
\{A(1 * v)\}_T = \int_0^T e^{-\lambda t} A \int_0^t v(s) \, ds \, dt + \{0\}_T
\]
\[
= A \int_0^T e^{-\lambda t} \int_0^t v(s) \, ds \, dt + \{0\}_T
\]
\[
= -\frac{A}{\lambda} \int_0^T v(s) \, ds + \frac{A}{\lambda} v_T + \{0\}_T.
\]
Then because $e^{-\lambda T} \left[ \frac{-A}{\lambda} \right] \int_0^T v(s) \, ds \approx_T 0$, it follows that $\{A(1 * v)\}_T = \frac{A}{\lambda} \{v\}_T$. \qed
Remark 1.2.9. All of the properties of Theorem 1.2.7 except \( (f) \) hold for a less stringent and more easily applicable definition of the asymptotic Laplace transform. Namely, let \( f \in L^1([0,T],X) \), then an alternative definition of the \( T \)-finite Laplace transform is
\[
\{f\}^*_T := \hat{f}_T + O^*_T,
\]
where \( O^*_T \) is the equivalence class of all functions of exponential decay \( T \) without any additional regularity assumptions on the zero elements \( a \in O^*_T \) (i.e., the functions \( a \in O^*_T \) do not have to be analytic functions of minimal exponential type). Similarly, for \( f \in L^1_{loc}([0,\infty),X) \), an alternative definition of the asymptotic Laplace transform is given by
\[
\{f\}^* := \cap_{T>0} \{f\}^*_T.
\]
Again, \( \{f\}^* \neq \emptyset \) and all operational properties of the classical Laplace transform remain valid except property \( (f) \) of Theorem 1.2.7 (see [24] or [26]). Since the equivalence classes \( \{f\}^* \) and \( \{f\}^*_T \) are much larger than \( \{f\} \) and \( \{f\}_T \), it is in general much easier to find \( r \in \{f\}^* \) than \( r \in \{f\} \) with \( r \) given as in (1.6). The following example from [25] illustrates the need for the condition of minimal exponential type in order for property \( (f) \) to hold.

Example 1.2.10. The following property of Laplace transform theory, \( \hat{f}'(\lambda) = -tf(t) \) does not extend to the definition of the asymptotic Laplace transform given in (1.9). Let \( X = \mathbb{C} \), \( 1 < \alpha < 2 \), and \( a(\lambda) := e^{-\lambda^\alpha \sin(e^{\lambda^2})} \). Clearly \( a \approx_T 0 \) for any \( T > 0 \). Thus, \( a \in O^*_T \) for all \( T \). But
\[
\limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln |a'(\lambda)| = \limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln | -\alpha \lambda^{\alpha-1} e^{-\lambda^\alpha \sin(e^{\lambda^2})} + 2\lambda e^{\lambda^2} e^{-\lambda^\alpha \cos(e^{\lambda^2})} | = +\infty.
\]
Therefore, \( a' \not\in O^*_T = \{-t0\}^*_T \). Which means that \( \{(0)\}^*_T \) is not contained in \( \{-t0\}^*_T \), and hence, in general \( \{(f)\}^*_T \neq \{-tf\}^*_T \).
The fact that the elements in the class of \( \{0\}_T \) are assumed to be of minimal exponential type is alone strong enough to ensure the validity of the operational property \( \{f\}_T = \{-tf\}_T \) for the T-finite Laplace transform (as well as the asymptotic Laplace transform). Consider again the function \( \lambda \to a(\lambda) := e^{-\lambda^\alpha \sin(e^{\lambda^2})} \). On any postsector \( \lambda \to a(\lambda) \) is not of minimal exponential type. For \( \lambda = re^{i\theta} \) it follows that

\[
a(\lambda) = e^{-r^\alpha e^{i\alpha\theta}} \left[ \frac{e^{ir^2(\cos(2\theta)+i\sin(2\theta))} - e^{-ir^2(\cos(2\theta)+i\sin(2\theta))}}{2i} \right]
\]

Using the equality \(|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2Re(z_1 \cdot z_2)\), it follows that

\[
|a(\lambda)| = e^{-r^\alpha \cos(\alpha\theta)} \left[ \sqrt{\left(\frac{e^{-r^2 \sin(2\theta)}}{2} + \left(\frac{e^{r^2 \sin(2\theta)}}{2}\right)^2 - 2 \cos^2(2r^2 \cos(2\theta))} \right].
\]

Then for \( \omega > 0 \),

\[
\frac{|a(\lambda)|}{e^{\omega r}} = e^{-r^\alpha \cos(\alpha\theta) - \omega r} \left[ \sqrt{\left(\frac{\left(\frac{e^{-r^2 \sin(2\theta)}}{2} + \left(\frac{e^{r^2 \sin(2\theta)}}{2}\right)^2 - 2 \cos^2(2r^2 \cos(2\theta))} \right)} \right]
\]

Assuming \( \pi > \theta > 0 \) it follows that

\[
\lim_{r \to \infty} \frac{|a(\lambda)|}{e^{\omega r}} = \infty.
\]

This shows that \( a(\cdot) \) is not of minimal exponential type on any postsector and thus \( a \notin O_T \).

**Example 1.2.11.** The asymptotic Laplace transform is a convenient tool to show the existence and uniqueness of solutions of inhomogeneous systems of linear ordinary differential equation. To illustrate this, consider the problem

\[
\begin{align*}
u'(t) &= Au(t) + f(t), \quad u(0) = x
\end{align*}
\]
for $A \in M_{n \times n}(\mathbb{C})$ and $f \in L^1_{loc}([0, \infty), \mathbb{C}^n)$. Then the existence and uniqueness for the solutions of (1.10) follows from

$$u' = Au + f, \ u(0) = x$$

$$\iff \{u'\} = \{Au + f\}, \ u(0) = x$$

$$\iff \lambda \{u\} - x = A\{u\} + \{f\}$$

$$\iff (\lambda - A)\{u\} = x + \{f\}$$

$$\iff \{u\} = R(\lambda, A)x + R(\lambda, A)\{f\}$$

\[\ast \iff \ast\]

$$\iff \{u\} = \{e^{tA}x\} + \{e^{tA}\{f\}\}$$

$$\iff \{u\} = \{e^{tA}x\} + \{e^{tA} \ast f\}$$

$$\iff \{u\} = \{e^{tA}x + \int_0^t e^{(t-s)A} f(s) \, ds\}$$

$$\iff u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s) \, ds$$

for all $t \in [0, \infty)$, where the equivalence $\ast \iff \ast$ holds since

$$\{e^{tA}x\} = R(\lambda, A)x + \{0\}$$

for $\text{Re}(\lambda) > \|A\|$ and $\{0\} \cdot \{f\} = \{0\}$. This proof holds not only for all $L^1_{loc}$-functions $f$ (like $t \to e^{t^2}$) but also for generalized functions $f$ (to be defined in Section 1.3). Equally important to this extension is that the proof is constructive when one applies the numerical approximation results of Chapter 2 to approximate $u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s) \, ds$.

As observed in ([24] and [7]), the asymptotic Laplace transform is also useful to characterize the existence and, in particular, uniqueness of solutions of the abstract Cauchy problem

\[(ACP) \quad u'(t) = Au(t), \ u(0) = x\]

for $t \in [0, T]$, where $A$ is a closed linear operator with domain and range in a Banach space $X$. Instead of $(ACP)$, it is often convenient to study the integral
equation

\((ACP_0)\) \hspace{1cm} u(t) - x = A \int_0^t u(s) \, ds,

where \(t \in [0, T]\) (or \(t \in [0, \infty)\) if \(T = \infty\)). A function \(u \in C([0, T], X)\) with \(\int_0^t u(s) \, ds \in D(A)\) for all \(t \in [0, T]\) that satisfies \((ACP_0)\) is called a mild solution of \((ACP)\). Clearly, by the closedness of \(A\), if \(u\) is a mild solution that is continuously differentiable then \(u(t) \in D(A)\) and \(u'(t) = Au(t)\), \(u(0) = x\) for all \(t \in [0, T]\), i.e., differentiable solutions of \((ACP)_0\) are solutions of \((ACP)\).

**Corollary 1.2.12. (Existence)** Let \(A\) be a closed linear operator on a Banach space \(X\). For \(0 < T \leq \infty\) and with \(u \in C([0, T], X)\), the following are equivalent:

(i) \(u\) solves \((ACP_0)\) on \([0, T]\).

(ii) There exists \(y \in \{u\}_T\) such that \((\lambda I - A)y(\lambda) - x \approx_T 0\).

(iii) There exists \(y \in \{u\}_T\) and \(p \in X\) such that \((\lambda I - A)y(\lambda) = x + e^{-\lambda T}p\).

**Proof.** To show (i) implies (iii), suppose that \(u\) solves \((ACP_0)\) on \([0, T]\). This means that \(u(t) - x = A \int_0^t u(s) \, ds\). This implies that \((1 * u)(t) \in D(A)\) and that \(t \to A(1 * u)(t) \in L^1([0, T], X)\). Hence, by the closedness of \(A\), it follows that \(\int_0^T e^{-\lambda T} \int_0^t u(s) \, ds \in D(A)\) and that

\[
A \int_0^T e^{-\lambda T} \int_0^t u(s) \, ds = \int_0^T e^{-\lambda T} A \int_0^t u(s) \, ds.
\]

Using this property, one now has that

\[
\int_0^T e^{-\lambda T} u(t) \, dt = \int_0^T e^{-\lambda T} x \, dt + \int_0^T e^{-\lambda T} A \int_0^t u(s) \, ds
\]

\[
= \int_0^T e^{-\lambda T} x \, dt + A \int_0^T e^{-\lambda T} \int_0^t u(s) \, ds
\]

\[
= \int_0^T e^{-\lambda T} x \, dt - \frac{A}{\lambda} e^{-\lambda T} \int_0^T u(s) \, ds + \frac{A}{\lambda} \int_0^T e^{-\lambda T} u(s) \, ds.
\]

31
Now using the fact that \( u \) solves \((ACP_0)\),
\[
(\lambda I - A)\dot{u}_T(\lambda) = \lambda \left[ x \int_0^T e^{-\lambda t} \, dt - \frac{1}{\lambda} e^{-\lambda T} (u(T) - x) \right] = x - e^{-\lambda T} u(T).
\]

Since \( \dot{u}_T \in \{ u \}_T \), \((iii)\) holds.

To see that \((iii)\) implies \((ii)\), suppose there exists \( y \in \{ u \}_T \) and \( p \in X \) such that
\[
(\lambda I - A)y(\lambda) = x + e^{-\lambda T} p.
\]
Then \( \lambda \to e^{-\lambda T} p \) is a function of exponential decay \( T \).

Therefore \( (\lambda I - A)y(\lambda) \approx_T x \) which implies that \((ii)\) holds.

To see that \((ii)\) implies \((i)\), suppose there exists \( y \in \{ u \}_T \) such that \((\lambda I - A)y(\lambda) - x \approx_T 0\). Note that
\[
\frac{1}{\lambda} y(\lambda) \in \frac{1}{\lambda} \{ u \}_T \subset \{ 1 \ast u \}_T.
\]

Then it follows that
\[
(\lambda I - A)y(\lambda) - x \approx_T 0 \quad \iff \quad A\left(\frac{1}{\lambda} y(\lambda)\right) - y(\lambda) + \frac{1}{\lambda} x \approx_T 0
\]
and this means that
\[
A\left(\frac{1}{\lambda} y(\lambda)\right) \in y(\lambda) - \frac{1}{\lambda} x + \{0\}_T \subset \{ u \}_T - \{1\}_T x + \{0\}_T \subset \{ u - x \}_T.
\]

Then by property \((g)\) part \((i)\) of Theorem 1.2.7, since \( \frac{1}{\lambda} y(\lambda) \in \{ 1 \ast u \}_T \) and \( A\left(\frac{1}{\lambda} y(\lambda)\right) \in \{ u - x \}_T \), it follows that \( A(1 \ast u) = u - x \) and thus \( u(t) - x = A \int_0^t u(s) \, ds \). Therefore, \( u \) solves \((ACP_0)\) on \([0, T]\).

\( \square \)

**Corollary 1.2.13.** (Local Uniqueness Theorem) Let \( A \) be a closed linear operator on a Banach space \( X \) and \( 0 < T < \infty \). The following statements are equivalent:

(i) All solutions of \((ACP)_0\) are unique on \([0, T]\).

(ii) If \( y \in \{ u \}_T \) for some continuous function \( u \) and if \( (\lambda I - A)y(\lambda) \approx_T 0 \), then \( y \approx_T 0 \).
(iii) If \( y \in \{u\}_{T} \) for some continuous function \( u \) and if \( (\lambda I - A)y(\lambda) = e^{-\lambda T}p \) for some \( p \in X \), then \( y \approx_T 0 \).

Proof. All solutions of \((ACP)_0\) are unique on \([0, T]\) if and only if the only solution of \( u(t) = A \int_{0}^{t} u(s) \, ds \) is the zero-solution. Without restricting generality, it will be shown that the only continuous solution of \( u(t) = A \int_{0}^{t} u(s) \, ds \) is the zero solution. To see that (i) implies (ii), suppose that all solutions of \((ACP_0)\) are unique, i.e. the only continuous solution of \( u(t) = A \int_{0}^{t} u(s) \, ds \) is the zero solution. Let \( y \in \{u\}_{T} \) for some continuous function \( u \) such that \( (\lambda I - A)y(\lambda) \approx_T 0 \). Then, by Corollary 1.2.12, \( u \) solves \( u(t) = A \int_{0}^{t} u(s) \, ds \) on \([0, T]\). Because all solutions of \((ACP_0)\) are unique, then \( u(t) \) is the zero solution which implies \( y(\lambda) \approx_T 0 \).

To show that (ii) implies, (iii), suppose \( y \in \{u\}_{T} \) for some continuous function \( u \) and that \( (\lambda I - A)y(\lambda) = e^{-\lambda T}p \) for some \( p \in X \). Then \( \lambda \rightarrow e^{-\lambda T}p \) is a function of that is of exponential decay \( T \). Therefore \( (\lambda I - A)y(\lambda) \approx_T 0 \) and by (ii), \( y(\lambda) \approx_T 0 \).

Finally, to see that (iii) implies (i), let \( u \) be a continuous function such that \( u(t) = A \int_{0}^{t} u(s) \, ds \) for all \( t \in [0, T] \). Let \( y(\lambda) := \int_{0}^{T} e^{-\lambda s}u(s) \, ds \). Then

\[
\frac{1}{\lambda} y(\lambda) = \frac{1}{\lambda} e^{-\lambda T} \int_{0}^{T} u(s) \, ds + \int_{0}^{T} e^{-\lambda s} \int_{0}^{s} u(r) \, dr \, ds.
\]

Now, since \( \int_{0}^{t} u(s) \, ds \in D(A) \) and \( \int_{0}^{T} e^{-\lambda s} \int_{0}^{s} u(r) \, dr \, ds \in D(A) \) and

\[
A \int_{0}^{T} e^{-\lambda s} \int_{0}^{s} u(r) \, dr \, ds = \int_{0}^{T} e^{-\lambda s} A \int_{0}^{s} u(r) \, dr \, ds = \int_{0}^{T} e^{-\lambda s} u(s) \, ds.
\]

It follows that \( \frac{1}{\lambda} y(\lambda) \in D(A) \) and

\[
\frac{A}{\lambda} y(\lambda) = e^{-\lambda T} A \int_{0}^{T} u(s) \, ds + \int_{0}^{T} e^{-\lambda s} A \int_{0}^{s} u(r) \, dr \, ds
= e^{-\lambda T} u(T) + y(\lambda)
\]

or \( (\lambda I - A)y(\lambda) = e^{-\lambda T}(-u(T)) \). Then by (iii), \( y \approx_T 0 \). This means that \( u(t) \) must be zero for all \( t \in [0, T] \). □
Corollary 1.2.14. (Global Uniqueness Theorem) Let $A$ be a closed linear operator on a Banach space $X$. The following statements are equivalent:

(i) $(ACP_0)$ has the uniqueness property on $[0, \infty)$.

(ii) If $y \in \{u\}$ for some continuous function $u : [0, \infty) \to X$ and if $(\lambda I - A)y(\lambda) \approx 0$, then $y \approx 0$.

Proof. As in the previous corollary, all solutions of $(ACP_0)$ are unique on $[0, T]$ if and only if the only solution of $u(t) = A \int_0^t u(s) \, ds$ is the zero-solution. Without restricting generality, it will be shown that the only continuous solution of $u(t) = A \int_0^t u(s) \, ds$ is the zero solution.

To see that $(i)$ implies $(ii)$, suppose that all solutions of $(ACP_0)$ are unique, i.e. the only continuous solution of $u(t) = A \int_0^t u(s) \, ds$ on $[0, \infty)$ is the zero solution. Let $y \in \{u\}$ for some continuous function $u$ such that $(\lambda I - A)y(\lambda) \approx 0$. Then, by Corollary 1.2.12, $u$ solves $u(t) = A \int_0^t u(s) \, ds$ on $[0, \infty)$. Because all solutions of $(ACP_0)$ are unique, then $u$ is the zero solution which implies $y(\lambda) \approx 0$.

To verify $(ii)$ implies $(i)$, let $u$ be a continuous function such that $u(t) = A \int_0^t u(s) \, ds$ for all $t \in [0, \infty)$. Let $y \in \{u\}$. Then $\frac{1}{\lambda} y(\lambda) \in \frac{1}{\lambda} \{u\} \subset \{1 \ast u\}$. By Theorem 1.2.7 (g), $\frac{1}{\lambda} y(\lambda) \in A\{1 \ast u\} = \{u\}$. Thus $y(\lambda) - \frac{A}{\lambda} y(\lambda) \approx 0$ or $(\lambda I - A)y(\lambda) \approx 0$. Thus, $y \approx 0$ and therefore $u = 0$ on $[0, \infty)$. 

Corollary 1.2.15. (Lyubich’s Uniqueness Theorem) Let $(\omega, \infty) \subset \rho(A)$ where $\rho(A)$ is the resolvent set of $A$. If

$$\limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln \|R(\lambda, A)\| = 0,$$

then $A$ has the uniqueness property.

Proof. Let $T > 0$, if $\rho(A) \neq 0$, then $A$ is closed. If $\lambda \to y(\lambda) \in D(A)$ and satisfies $(\lambda I - A)y(\lambda) \approx_T 0$ (i.e. $(\lambda I - A)y(\lambda) = a(\lambda)$ where $a(\lambda)$ is of exponential decay
then \( y(\lambda) = R(\lambda, A)a(\lambda) \). This means that

\[
\limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln \|y(\lambda)\| \leq \limsup_{\lambda \to \infty} \left[ \frac{1}{\lambda} \ln \|R(\lambda, A)\| + \frac{1}{\lambda} \ln \|a_\lambda\| \right] \leq -T.
\]

Therefore \( y(\lambda) \approx_T 0 \) and thus \( A \) has the uniqueness property.

For the following corollary, recall that an operator \( A \) is called \textit{dissipative} if

\[
\|(\lambda I - A)y\| \geq \lambda\|y\|
\]

for all \( \lambda \to y(\lambda) \in D(A) \) and \( \lambda > 0 \).

**Corollary 1.2.16.** Let \( A \) be a closed linear operator. If \( A \) is dissipative, then \( A \) has the uniqueness property.

**Proof.** Suppose \( y(\lambda) \in D(A) \) and \( (\lambda I - A)y(\lambda) \approx_T 0 \), i.e. \( (\lambda I - A)y(\lambda) = a(\lambda) \) where \( a(\lambda) \) is of exponential decay \( T \). Because \( A \) is dissipative,

\[
\lambda\|y(\lambda)\| \leq \|(\lambda I - A)y(\lambda)\| = \|a(\lambda)\|.
\]

It follows that \( y(\lambda) \) is of exponential decay \( T \) and thus \( A \) has the uniqueness property.

**Example 1.2.17.** The linear PDE

\[
\begin{align*}
  u_t(t, x) &= u_x(t, x) & x &\in [0, 1] \\
  u(0, x) &= f(x) & t &\geq 0 \\
  f &\in C[0, 1]
\end{align*}
\]

which corresponds to the Cauchy problem

\[
u'(t) = Au(t) \quad u(0) = f \]

for where \( A = \frac{d}{dx} \) for \( f \in X = C[0, 1] \) has the whole complex plane as its pointspectrum \( p\sigma(A) \). To verify this, note that \( Af = f' = \lambda f \) is satisfied by all exponential
functions $x \to e^{\lambda x}$, and thus $p\sigma(A) = \mathbb{C}$. Since the pointspectrum of $A$ is the whole complex plane, the property that $\| (\lambda I - A) y \| \geq \lambda \| y \|$ for all $\lambda > 0$ cannot hold. Thus $A$ is non-dissipative.

Moreover, the most notable characteristic of the linear PDE for this investigation is that the uniqueness property does not hold for $A$. To see this let $\tilde{f} \in C^1[0, \infty)$ be an arbitrary continuously differentiable extension of $f \in C^1[0, 1]$. Then $u_t(t, x) := \tilde{f}(t + x)$ solves the linear PDE since

$$u_t(t, x) = u_x(t, x) \text{ for all } t > 0, x \in [0, 1]$$

and $u(0, x) = \tilde{f}(x) = f(x)$ for all $x \in [0, 1]$. Since there are infinitely many ways to construct the continuous, everywhere differentiable extension of $f$, the uniqueness property does not hold for $A$.

However, as the next example shows, there are closed operators $A$ where the pointspectrum of $A$ is the whole complex plane that possess the uniqueness property. In order to show this, the following extension of Lemma 4.1.1 from [31] is needed.

**Lemma 1.2.18.** Let $u \in C([0, T], X)$. If there exist $c, M > 0$ such that

$$\left| \int_0^T e^{\lambda^2 s} u(s) \, ds \right| \leq Me^{\lambda c}$$

for $\lambda > \omega_0$, then $u(t) \equiv 0$ on $[0, T]$.

**Proof.** Let $x^* \in X^*$ and set $\varphi(t) = \langle x^*, u(t) \rangle = x^*(u(t)) = u(t)(x^*)$. Then $\varphi$ is continuous on $[0, T]$ and

$$\left| \int_0^T e^{\lambda^2 s} \varphi(t) \, dt \right| = \left| \langle x^*, \int_0^T e^{\lambda^2 s} u(t) \, dt \rangle \right| \leq \|x^*\| \cdot Me^{\lambda c} = M_1 e^{\lambda c} \quad (1.11)$$

for all $\lambda > \omega_0$. The proof will show that (1.11) implies that $\varphi(t) \equiv 0$ on $[0, T]$ and since $x^* \in X^*$ was arbitrary, it follows that $u(t) \equiv 0$ on $[0, T]$. 

36
Consider the series
\[
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{k^2\lambda^2} = 1 - e^{-e^{2\lambda^2}}.
\]
This series converges uniformly in \( \tau \) on bounded intervals. Therefore, for \( t \in [0, T) \),
\[
\left| \int_0^T \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{k(t-s)\lambda^2} \varphi(s) \, ds \right| = \left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{k(t)\lambda^2} \int_0^T e^{ks\lambda^2} \varphi(s) \, ds \right|.
\]
Furthermore
\[
\left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{k(t)\lambda^2} \int_0^T e^{ks\lambda^2} \varphi(s) \, ds \right| \leq \sum_{k=1}^{\infty} \frac{1}{k!} e^{k(t)\lambda^2} \left| \int_0^T e^{ks\lambda^2} \varphi(s) \, ds \right|
\]
for \( \sqrt{k} \cdot \lambda > \omega_0 \). Since \( \sqrt{k} < k \), it follows that
\[
\left| \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{k(t)\lambda^2} \int_0^T e^{ks\lambda^2} \varphi(s) \, ds \right| \leq M_1 \left( 1 - e^{e^{(t)\lambda^2+\lambda^2}} \right).
\]
Since \( \lim_{\lambda \to \infty} \left( 1 - e^{e^{(t)\lambda^2+\lambda^2}} \right) = 0 \) for \( t \in [0, T) \), then
\[
\left| \int_0^T \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{k(t-s)\lambda^2} \varphi(s) \, ds \right| \to 0 \quad (1.12)
\]
as \( \lambda \to \infty \).

Again for \( t \in [0, T) \), observe that
\[
\int_0^T \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{k(t-s)\lambda^2} \varphi(s) \, ds = \int_0^T \left( 1 - e^{-e^{(t-s)\lambda^2}} \right) \varphi(s) \, ds.
\]
Using Lebesgue’s dominated convergence theorem,
\[
\lim_{\lambda \to \infty} \int_0^T \left( 1 - e^{-e^{(t-s)\lambda^2}} \right) \varphi(s) \, ds = \int_0^T \lim_{\lambda \to \infty} \left( 1 - e^{-e^{(t-s)\lambda^2}} \right) \varphi(s) \, ds = \int_T^t \varphi(s) \, ds,
\]
because
\[
\lim_{\lambda \to \infty} \left( 1 - e^{-e^{(t-T+s)\lambda^2}} \right) = \begin{cases} 
1 & \text{for } t - T + s > 0 \Rightarrow s > T - t \\
0 & \text{for } t - T + s < 0 \Rightarrow s < T - t
\end{cases}
\]
Combining this with (1.12) gives that for every \(0 \leq t < T\), then \(\int_{T-t}^{T} \varphi(s) \, ds = 0\). This implies that \(\varphi(t) \equiv 0\) on \([0, T]\).

\textbf{Example 1.2.19.} Consider the observation problem for the one-dimensional heat equation \(w_s(s, r) = w_{rr}(s, r)\) for \(r \in \mathbb{R}, \ s \in [0, 1]\) with observation values \(w(s, 0) = x_0(s), \ w_s(s, 0) = x_1(s)\) where \(x_0, x_1 \in C([0, 1])\). Notice that one cannot include any boundary conditions such as \(x(0) = 0\) or \(x(1) = 0\) on \(C[0, 1]\). In this model, the heat and flux of the rod are only observed at one point. The goal is to estimate the heat distribution \(w(s_0, r)\) in the interior of the rod, say for \(r \in [0, r_0]\) for some \(0 < r_0 \leq \infty\) and for some \(s_0 \in [0, 1]\). Setting \(u(r) = (w(\cdot, r), w_r(\cdot, r))\) gives the Cauchy problem
\[
u'(r) = Au(r), \quad u(0) = (x_0, x_1), \quad r \in [0, r_0]
\]
on \(X := C[0, 1] \times C[0, 1]\) where
\[
A := \begin{pmatrix}
0 & I \\
\frac{d}{ds} & 0
\end{pmatrix}.
\]
To see that the pointspectrum of \(A\) is the whole complex plane, recall that if \(\lambda\) is in the pointspectrum if and only if there exists an \((f, g) \neq (0, 0)\) in \(X \times X\) such that \((\lambda I - A)(f, g) = 0\). This will hold if and only if
\[
\lambda f - g = 0 \iff g = \lambda f \iff \lambda^2 = Df \iff f(s) = e^{\lambda^2 s}
\]
and thus the pointspectrum is equal to the whole complex plane since one cannot include any boundary conditions in the domain of \(\frac{d}{ds}\) or in the space \(C[0, 1] \times C[0, 1]\).
Because of this property, one can not expect global solutions for all smooth initial data \(x_0, x_1\), and the solutions will not necessarily depend continuously on the initial data \(x_0, x_1\). Since the pointspectrum of \(A\) is the whole complex plane, the property that \(\|(\lambda I - A)y\| \geq \lambda \|y\|\) for all \(\lambda > 0\) cannot hold. Thus \(A\) is non-dissipative.

Now let \(u\) be a continuous solution of (1.13). In order to go back to standard notation, we change the problem

\[
u'(r) = Au(r) \quad u(0) = (x_0, x_1), \quad r \in [0, r_0]
\]

for \(x_i = x_i(s) \in C[0, 1]\) by replacing \(r\) with \(t\), \(r_0\) with \(T\), and \(s\) by \(x\) into

\[
u'(t) = Au(t) \quad u(0) = (x_0, x_1), \quad t \in [0, T]
\]

for \(x_i = x_i(x) \in C[0, 1]\), and

\[
A := \begin{pmatrix}
0 & I \\
\frac{d}{dx} & 0
\end{pmatrix}.
\]

Then

\[
e^{-\lambda T}u(T) + \lambda \int_0^T e^{-\lambda t}u(t) \, dt = \int_0^T e^{-\lambda t}u'(t) \, dt = \int_0^T e^{-\lambda t}Au(t) \, dt = A\int_0^T e^{-\lambda t}u(t) \, dt
\]

and therefore \((\lambda - A)\hat{v}(\lambda) = -e^{-\lambda T}u(T)\) for

\[
v(t) := \begin{cases}
u(t) : 0 \leq t \leq T \\
0 : t \geq T
\end{cases}.
\]

Now let \(u(T) = u(T, \cdot) = (h(\cdot), g(\cdot))\) and \(\hat{v}(\lambda) = (\hat{v}_1(\lambda, \cdot), \hat{v}_2(\lambda, \cdot)) = (\hat{v}_1(\cdot), \hat{v}_2(\cdot))\).

Then

\[
\begin{cases}
\lambda \hat{v}_1(x) - \hat{v}_2(x) = -e^{-\lambda T}h(x) \\
-\hat{v}_1'(x) + \lambda \hat{v}_2(x) = -e^{-\lambda T}g(x)
\end{cases}.
\]
This implies that $\hat{v}_2(x) = \lambda \hat{v}_1(x) + e^{-\lambda T} h(x)$ and

$$-\hat{v}_1'(x) - \lambda^2 \hat{v}_1(x) = -e^{-\lambda T} g(x) + \lambda e^{-\lambda T} h(x)$$

$$\iff e^{-\lambda^2 x} \hat{v}_1'(x) - \lambda^2 e^{-\lambda^2 x} \hat{v}_1(x) = e^{-\lambda T} e^{-\lambda^2 x} g(x) + \lambda e^{-\lambda T} e^{-\lambda^2 x} h(x)$$

$$\iff e^{-\lambda^2 x} \hat{v}_1(x) = \int_0^x e^{-\lambda T} e^{-\lambda^2 r} (g(r) + \lambda h(r)) \, dr + c(\lambda)$$

where $c(\lambda)$ is an arbitrary real function independent of $x$. In particular

$$c(\lambda) = e^{-\lambda^2 x} \hat{v}_1(\lambda, x) - \int_0^x e^{-\lambda T} e^{-\lambda^2 r} (g(r) + \lambda h(r)) \, dr.$$ 

It follows with $x = 0$ that $c(\lambda) = \hat{v}_1(\lambda, 0)$, hence $c(\lambda)$ is a Laplace transform. Also note that with $x = 1$ that

$$c(\lambda) = e^{-\lambda^2} \hat{v}_1(\lambda, 1) - \int_0^1 e^{-\lambda T} e^{-\lambda^2 r} (g(r) + \lambda h(r)) \, dr$$

and this illustrates that $c(\lambda)$ is of exponential decay $T$. Because $c(\lambda)$ is a Laplace transform and is of exponential decay, by Corollary 1.2.5, the inverse Laplace transform of $c(\lambda)$ is 0 on $[0, T]$ and thus $v_1(t, 0) = 0$ on $[0, T]$. This means that for sufficiently large $\lambda$, $\hat{v}_1(\lambda, 0) \equiv 0$ which implies $c(\lambda) = 0$. Thus,

$$\hat{v}_1(x) = \int_0^x e^{-\lambda T} e^{\lambda^2 (x-r)} (g(r) + \lambda h(r)) \, dr$$

$$= \int_0^x e^{-\lambda T} e^{\lambda^2 s} (g(x-s) + \lambda h(x-s)) \, ds.$$ 

Now, since $\hat{v}_1(x)$ is a Laplace transform, by Proposition 1.1.1

$$|\hat{v}_1(x)| = \left| \int_0^x e^{-\lambda T} e^{\lambda^2 s} (g(x-s) + \lambda h(x-s)) \, ds \right| \leq M$$

for all $x \in [0, 1]$ and for all sufficiently large $\lambda$. By Lemma 1.2.18, $g \equiv 0$ and $\lambda h \equiv 0$. Hence $g = h \equiv 0$, and thus $\hat{v}(\lambda) = (\hat{v}_1(\lambda, \cdot), \hat{v}_2(\lambda, \cdot)) = (0, 0)$. It follows that $u(t) \equiv 0$, and therefore, $A$ has the uniqueness property.
1.3 Generalized Functions

Since the asymptotic Laplace transform will be used to treat “ill-posed” problems, it is necessary to extend the definition of the asymptotic and $T$-asymptotic Laplace transform to generalized functions. There are several different approaches to defining generalized functions. Usually, generalized functions are defined as functionals on a space of test functions. An alternative approach, used below, builds on the Titchmarsh-Foias theorem (see [6], [4], [26] for details and proof).

**Theorem 1.3.1.** (Titchmarsh-Foias) Let $k \in L^1_{\text{loc}}([0, \infty), X)$, and $a > 0$ and consider the convolution operator $T : f \rightarrow k \ast f$, where

$$(k \ast f)(t) := \int_0^t k(t - s)f(s)\,ds,$$

as an operator from $C([0, a], X)$ into $C_0([0, a], X)$. The following are equivalent:

(i) $T$ has dense range,

(ii) $T$ is injective, and

(iii) $0 \in \text{supp}[k]$; i.e., $k \not\equiv 0$ on $[0, \varepsilon)$ for all $0 < \varepsilon < 1$.

Furthermore, if $0 \in \text{supp}[k]$, then $\|f\|_T := \|Tf\|$ defines a norm on $C([0, a], X)$ and $T$ extends to an isometric isomorphism between the completion of $C([0, a], X)$ under that norm and $C_0([0, a], X)$.

Hence for such $k$, by Theorem 1.3.1, generalized functions $u : [0, a] \rightarrow X$ can be defined as elements of the (Banach space) completion $C^{[k]}([0, a], X)$ of the vector space $C([0, a], X)$ with norm $\|f\|_T := \|k \ast f\|$. This means that generalized functions $u$ with support in a compact interval $[0, a]$ are regarded as equivalence classes $[f_n]$ of continuous functions $f_n \in C([0, a], X)$ such that $k \ast f_n \rightarrow g \in C_0([0, a], X)$. The convolution $k \ast u$ is then defined to be $g$. 

41
To obtain generalized functions with support in \([0, \infty)\) one takes \(T\) to be an injective operator on \(C([0, \infty), X)\) with dense range in the Fréchet space \(C_0([0, \infty), X)\). Then, by Titchmarsh-Foias, \(\|f\|_{T,n} := \sup_{t \in [0,n]} \|Tf\|\) defines a family of seminorms on \(C([0, \infty), X)\), \(T\) will extend to an isomorphism between the Fréchet completion \(C^{[k]}([0, \infty), X)\) of \(C([0, \infty), X)\) with respect to the family of seminorms on \(C([0, \infty), X)\), and generalized functions \(u\) are defined as elements of the Fréchet completion \(C^{[k]}([0, \infty), X)\).

To simplify the presentation, a function \(k\) will be defined as admissible if either \(k \in L^1[0, T]\) if \(0 < T < \infty\) or if \(k \in L^1_{loc}[0, \infty)\) such that \(\hat{k}(\lambda)\) exists for some \(\lambda \in \mathbb{C}\). For such \(k\), statement \((iii)\) of Theorem 1.3.1 is equivalent to \(\limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln \|\hat{k}(\lambda)\| = 0\) (see \((v)\) of Theorem 1.2.4). Now let \(u \in C^{[k]}([0, T], X)\) for some admissible convolution kernel \(k\), then \(m := k * u \in C_0([0, T], X)\). This allows one to define the 
T-asymptotic Laplace transform \(\{u\}\) of a generalized function \(u\) as

\[
\{u\}_T := \frac{\{m\}_T}{k_T}.
\]

Similarly, for a generalized function \(u\) defined as elements of the Fréchet completion \(C^{[k]}([0, \infty), X)\), then \(m := k * u \in C_0([0, \infty), X)\). The asymptotic Laplace transform \(\{u\}\) of \(u\) is then defined as

\[
\{u\} := \frac{\{m\}}{k}.
\]

(see III.2 in [4]). In particular, the asymptotic Laplace transform \(\{u\}\) of a generalized function \(u\) is an equivalence class of meromorphic functions defined in a post-sectoral region.

Now, a \(k\)-generalized function \(u \in \mathcal{C}^k\) is said to be Laplace transformable (in the classical sense) if the continuous function \(m := k * u\) is Laplace transformable and the Laplace transform of \(u\) is then defined as

\[
\hat{u}(\lambda) := \frac{\hat{m}(\lambda)}{k(\lambda)}.
\]
The following examples illustrate how to find a specific $k$ for a particular, irregular multiplication semigroup and thus outlines how to apply asymptotic Laplace transform methods to this problem. In the first two examples, it will be shown that integrating (or integrating $n$-times) is enough to regularize the solutions. This is equivalent to looking at the $k$-generalized solutions where $k = \frac{1}{\lambda}$ for integrating one time and $k = \left(\frac{1}{\lambda}\right)^n$ for integrating $n$-times. In the third example, integration is not enough to regularize the solutions.

**Example 1.3.2.** Let $X = C_0([0, \infty), \mathbb{C})$ and

$$ Af(x) = a(x)f(x), \quad a(x) = x + ie^{x^2}, $$

with maximal domain $D(A) \subset X$. As shown in Example 1.1.3, $-A$ generates the $C_0$-semigroup

$$ W(t)f(x) = e^{-ta(x)}f(x), $$

and $A$ does not generate a semigroup because

$$ T(t)f(x) = e^{ta(x)}f(x) = e^{tx}e^{ite^{x^2}}f(x) $$

is not defined for all $f \in X$. Moreover, the presence of the amplifying factor $e^{tx}$ and the oscillating factor $e^{ite^{x^2}}$ makes every solution $u(t, x) = T(t)f(x) = e^{tx}e^{ite^{x^2}}f(x)$ of the corresponding Cauchy problem

\[
\begin{align*}
  u_t(t, x) &= a(x)u(t, x) \\
  u(0, x) &= f(x) \\
  u(t, \infty) &= 0
\end{align*}
\]

physically meaningless if the observed data $f(x)$ contains some small error. To illustrate, observe the following: let $\omega(x) = \frac{1}{1000}$ be the error on the observed initial data $f(x)$. Then the error $r(t, x)$ on the solution is given as

$$ r(t, x) = e^{tx}e^{ite^{x^2}}\omega(x). $$

43
Since $|r(t, x)| = \frac{1}{10^{10}}e^{tx}$, the solution is immensely effected by the small, initial error in measurement even for moderately large values of $t$ and $x$ (see Figures 1.1(a), 1.2(a), 1.3(a) below). However, the once integrated semigroup

$$S(t)f(x) = \int_0^t T(s)f(x) \, ds = \frac{1}{x + ie^{x^2}}[e^{t(x+ie^{x^2})} - 1]f(x)$$

becomes a family of bounded linear operators. This follows from the fact that now the factor of $e^{x^2}$ in the denominator will cancel the growth of $e^{tx}$ in the numerator. Although the solutions themselves do not depend continuously on the initial data, the time averages of the solutions do. That is,

$$S_\varepsilon(t)f(x) = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} T(s)f(x) \, ds = \frac{1}{\varepsilon} \frac{1}{x + ie^{x^2}}[e^{(t+\varepsilon)(x+ie^{x^2})} - e^{t(x+ie^{x^2})}]f(x)$$

depends continuously on the data $f$. Also note that

$$\left| \sup_{x \geq 0} \frac{1}{\varepsilon} \frac{1}{x + ie^{x^2}}[e^{(t+\varepsilon)(x+ie^{x^2})} - e^{t(x+ie^{x^2})}] \right| \leq \frac{2}{\varepsilon} e^{\frac{(t+\varepsilon)^2}{4}},$$

thus $\|S_\varepsilon(t)f(x)\| = \frac{2}{\varepsilon} e^{\frac{(t+\varepsilon)^2}{4}} \|f(x)\|$. 

![Graph](image1.png) ![Graph](image2.png)

**FIGURE 1.1.** Let $x = 20$, $t \in [1.9, 2]$, and $\varepsilon = 0.1$. 

44
FIGURE 1.2. Let $x = 10$ and $t \in [5, 5.1]$ and $\varepsilon = 0.1$.

![Figure 1.2](image1.png)

(a) $\text{Re}(r(t, x))$

(b) $\text{Re}(S_\varepsilon(t)\omega(x))$

FIGURE 1.3. Let $x = 1$, $t \in [50, 55]$ and $\varepsilon = 0.1$.

![Figure 1.3](image2.png)

(a) $\text{Re}(r(t, x))$ and $-\text{Re}(r(t, x))$

(b) $\text{Re}(S_\varepsilon(t)\omega(x))$ and $-\text{Re}(S_\varepsilon(t)\omega(x))$

Furthermore, if one investigates time averages of the time average solutions, that is

$$\frac{1}{\varepsilon} \int_t^{t+\varepsilon} S_\varepsilon(s)f(x)\, ds = \frac{1}{\varepsilon^2} \frac{1}{(x + ie^{2\varepsilon^2})^2} \left( e^{\varepsilon(x+ie^{2\varepsilon^2})} - 1 \right)^2 e^{t(x+ie^{2\varepsilon^2})} \|f(x)\|,$$

the solutions will gain more continuous time dependence, hence become more regular. In addition, these solutions will grow slightly slower for large $t$ as evident by

$$\left\| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} S_\varepsilon(s)f(x)\, ds \right\| \leq \frac{4}{\varepsilon^2} e^{\frac{(t+2\varepsilon)^2}{8}} \|f(x)\|.$$
To summarize, the once integrated semigroup $S(t)$ defined by the bounded linear extension of the operator $f \mapsto \int_0^t T(s)f(x)\,ds$ to all of $X$ is strongly continuous on $[0, \infty)$, norm-Lipschitz continuous, and nowhere differentiable on $(0, \infty)$.

**Example 1.3.3.** For the multiplication semigroup $T(t)f(x) : t \mapsto e^{ta(x)}f(x)$ and the generator $Af(x) : x \mapsto a(x)f(x)$ on $X = C_0([0, \infty),\mathbb{C})$, where $a(x) = x + ie^x$ ($x \geq 0$), the operators $T(t)$ are no longer in $\mathcal{L}(X)$. However, the $n$-times integrated semigroup $S_n(t)$ defined by the bounded linear extension of the operator $f \mapsto \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} T(s)f\,ds$ ($0 \leq t \leq n$) to all of $X$ is continuous a function from $[0,n)$ into $X$ for all $f \in X$. Moreover, if $k(t) := \mathcal{L}^{-1}(e^{-bt})$ where $\mathcal{L}^{-1}(f)$ is the inverse Laplace transform of $f$ and $b > 0$ and $0 < \delta < 1$, then the $k$-convoluted semigroup $S_k(t)$ defined by the bounded linear extension of $f \mapsto \int_0^t k(t-s)T(s)f\,ds$ is strongly continuous in $[0, \infty)$.

**Example 1.3.4.** Consider $X = C_0([0, \infty),\mathbb{C})$ and

$$Af(x) = a(x)f(x), \quad a(x) = x + 2ix^2,$$

with maximal domain $D(A) \subset X$. Then $A$ generates the irregular semigroup of unbounded linear operators

$$T(t)f : x \mapsto e^{t(x+2ix^2)}f(x).$$

The integrated semigroup

$$S(t)f : x \mapsto \frac{1}{x + 2ix^2} [e^{t(x+2ix^2)} - 1]f(x)$$

is still unbounded since the $x^2$-term in the denominator fails to offset the $e^{tx}$ in the numerator. So, the question arrises, how does one find a regularized $k$ that makes

\[(ACP_k) \quad v(t) = A \int_0^t v(s)ds + (1 * k)(t)f\]
a well-posed problem on \([0, T]\)? If there is a \(v \in C_0([0, T], X)\) that solves \((ACP_K)\), then it follows from Theorem 1.2.7 that

\[
\{v\}_T = \left\{ A(1 \ast v) \right\}_T + \{1 \ast k\}_T f = \frac{A}{\lambda} \{v\}_T + \frac{\dot{k}(\lambda)}{\lambda} f \quad \text{or} \quad \{v\}_T = \dot{k}(\lambda)(\lambda I - A)^{-1} f + \{0\}_T.
\]

In other words, \(v\) is the inverse asymptotic Laplace transform of \(\dot{k}(\lambda)(\lambda I - A)^{-1} f\).

To construct \(v\), we must find \(\dot{k}(\lambda)\) such that:

(a) the integral

\[
v(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \dot{k}(\lambda) R(\lambda, A) f \, d\lambda
\]

is well-defined for \(t \in [0, T]\), where

\[
\Gamma = \Gamma_- \cup \Gamma_1 \cup \Gamma_+, \quad \Gamma_\pm = \{ r \pm ir^2, 1 \leq r < \infty \}, \quad \Gamma_1 = \{ 1 + ir, -1 \leq r \leq 1 \},
\]

(b) \(v \in C_0([0, T], X)\),

(c) \(\int_0^T e^{-\lambda t} v(t) \, dt = \hat{v}_T(\lambda) \in \dot{k}(\lambda)(\lambda I - A)^{-1} f + r(\lambda) \) with \(r(\lambda) \approx_T 0\).

To see how \(\dot{k}\) has to be chosen so that the integral in (a) exists, consider the integrand

\[
I(\lambda) = e^{\lambda t} \dot{k}(\lambda) R(\lambda, A) f \, d\lambda
\]

for \(\lambda \in \Gamma_\pm\). Since \(\|R(\lambda, A)f\| = \sup_{x \geq 0} \left| \frac{1}{\lambda - (x + 2ix^2)} f(x) \right| \leq C\|f\|\), for \(\lambda \in \Gamma_\pm\) it follows that

\[
\|I(\lambda)\| \leq e^{Re(\lambda)t} |\hat{k}(\lambda)| C\|f\| \cdot |d\lambda| = C e^{rt} |\hat{k}(r + ir^2)| \|f\| |1 + 2ir| \, dr
\]

for all \(r > 1\). In order for the integral in (a) to exist for all \(t \in [0, T]\), \(\dot{k}(\lambda)\) must be chosen so that

\[
|\hat{k}(r + ir^2)| \leq \tilde{C} e^{-r^2}
\]
for some $\tilde{T} > T$, because then
\[
\| I(\lambda) \| \leq Ce^{rT} |\hat{k}(r + ir^2)| \| f \| |1 + 2ir| dr \\
\leq Ce^{rT} \tilde{C} e^{-r\tilde{T}} |1 + 2ir| dr \\
\leq Me^{rT} e^{-r\tilde{T}'} dr
\]
for some $T < \tilde{T}' < \tilde{T}$. Now let $\hat{k}(\lambda) = e^{-\sqrt{\lambda}}$ where $C = \frac{\tilde{T}}{\cos(\frac{\pi}{8})}$. Then
\[
|\hat{k}(\lambda)| = |e^{-C(\sqrt{\lambda})(\cos(\frac{\text{Arg}(\lambda)}{2}) + i\sin(\frac{\text{Arg}(\lambda)}{2}))}| = e^{-C(\sqrt{\lambda}) \cos(\frac{\pi}{8})}
\leq e^{-C(\sqrt{\lambda}) \cos(\frac{\pi}{8})} = e^{-C\sqrt{\lambda}}
\]
and thus
\[
|\hat{k}(r + ir^2)| \leq e^{-\tilde{T} \sqrt{r^2 + r^4}} \leq e^{-\tilde{T} \sqrt{r^4}} = e^{-\tilde{T}r}.
\]
Now with this choice of $\hat{k}(\lambda)$, the integrand in (a) exists and one can show that (b) and (c) are also satisfied (see Theorem 1.2.7). Since the corresponding function $k$ is given by
\[
k(t) = Ce^{-\frac{c^2}{2\sqrt{\pi}t}} \text{ where } C = \frac{\tilde{T}}{\cos(\frac{\pi}{8})},
\]
it follows from the above that the $k$-convoluted semigroup $S(t) f = v(t) = (k * T)(t) f :$
\[
x \rightarrow \int_0^t k(t - s)e^{s(x + ix^2)} ds f(x) \in C_0([0, \infty))
\]
and $\| S(t) f \| \leq M_t \| f \|$ for all $t \in [0, T)$.

\[\square\]

In general, finding an appropriate regularizing convolution kernel $k$ function for a given problem is difficult. However, the next result and its corollary offer universal guidelines for the construction of such a function. To state the next result, which coincides with a result of Lumer-Neubrander (see [24]) but also contains additional growth estimates for the inverse transforms, the following framework is needed:
consider analytic functions $u : \Omega \to X$ with $\sup_{\lambda \in \Omega} \|\lambda u(\lambda)\| < \infty$, where the domain $\Omega$ contains a post-sectorial region

$$\Omega_\Psi := \{ \lambda : \text{Re}(\lambda) \geq \beta > 0, |\text{Im}(\lambda)| \leq \Psi(\text{Re}(\lambda)) \},$$

where $\Psi$ is a positive, strictly increasing $C^1$-function with $\Psi(r) \to \infty$ as $r \to \infty$ and $\sup_{r \geq \beta} \frac{\Psi'(r)}{r \Psi(r)} < \infty$ for some $\alpha \geq 0$, i.e., $\Psi$ is dominated by some exponential function $t \to e^{cr^\alpha}$.

**Theorem 1.3.5.** *(Lumer-Neuhrander)* Let $\Omega_\Psi \subset \Omega$, $u : \Omega \to X$ analytic, and $\sup_{\lambda \in \Omega} \|\lambda u(\lambda)\| < \infty$.

(a) If there exists $c > 0$ and an analytic function $k : \Omega_\Psi \to \mathbb{C}$ such that

$$|k(\lambda)| \leq e^{-c\Psi^{-1}(|\lambda|)} \text{ for all } \lambda \in \Omega_\Psi,$$

then there exists $m \in C_0([0, c), X)$ such that $ku \in \{ m \}_T$ for all $0 < T < c$.

(b) If there exists $c > 0$, $d > 1$, and an analytic function $k : \Omega_\Psi \to \mathbb{C}$ such that

$$|k(\lambda)| \leq e^{-c(\Psi^{-1}(|\lambda|))^d} \text{ for all } \lambda \in \Omega_\Psi$$

then there exists $m \in C_0([0, \infty), X)$ with $\|m(t)\| \leq M e^{c\omega e^{\frac{1}{\omega \Psi^{-1}(|\lambda|)}}}$ for all $\omega > \frac{d(\frac{1}{c})^{\frac{1}{d-1}}}{C_0}$ such that $ku \in \{ m \}$.

**Proof.** (a) Let $\Gamma$ be the oriented boundary of the region $\Omega_\Psi$; i.e., $\Gamma = \Gamma_- \cup \Gamma_\beta \cup \Gamma_+$, where $\Gamma_{\pm} := \{ r \pm i\Psi(r) ; \beta \leq r < \infty \}$, and $\Gamma_\beta := \{ \beta + ir; \Psi(\beta) \leq r \leq \Psi(\beta) \}$. Additionally, let $M$, $C_0$, $C_1 > 0$ satisfy $\|\lambda u(\lambda)\| \leq M$ and $\left| \frac{1+\Psi'(r)}{r \Psi(r)} \right| \leq C_0 + C_1 r^\alpha$ for all $\lambda \in \Omega_\Psi$ and all $r \geq \beta$. Since $\Psi$ is an increasing function, then $\Psi^{-1}$ must also be increasing. Thus, $\Psi^{-1}(|r \pm i\Psi(r)|) \geq \Psi(|\Psi(\beta)|) = r$ for all $r \geq \beta$. Then, it follows that for $\lambda = r \pm i\Psi(r) \in \Gamma_{\pm}$,

$$\|e^{\lambda t}k(\lambda)u(\lambda) d\lambda\| \leq Me^{rt}e^{-c\Psi^{-1}(|r \pm i\Psi(r)|)} \left| \frac{1+\Psi'(r)}{r \Psi(r)} \right| dr \leq M(C_0 + C_1 r^\alpha) e^{-r(c-t)} dr.$$
The above inequality implies that

\[ m(t) := \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} k(\lambda) u(\lambda) \, d\lambda \]

is well-defined and continuous for \(0 \leq t < c\). Next it will be shown that for all \(0 < T < c\) and \(\lambda\) in the interior of \(\Omega_\Psi\) that

\[ \int_0^T e^{-\lambda t} m(t) \, dt = k(\lambda) u(\lambda) - a_T(\lambda) \]

where \(a_T(\lambda) := \frac{e^{-\lambda T}}{2\pi i} \int_{\Gamma} \frac{1}{\mu - \lambda} e^{\mu T} k(\mu) u(\mu) \, d\mu\). To start the argument, let \(\Gamma(n) := \Gamma \cap \{\Re(\lambda) \leq n\}\) and \(\Pi_n := \{n + ir : -\Psi(n) \leq r \leq \Psi(n)\}\). Then by Fubini’s theorem, for \(\lambda \in \Omega_\Psi^0\),

\[
\int_0^T e^{-\lambda t} m(t) \, dt = \int_0^T e^{-\lambda t} \frac{1}{2\pi i} \int_{\Gamma} e^{\mu t} k(\mu) u(\mu) \, d\mu \, dt
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} \int_0^T e^{-\lambda t} e^{\mu t} k(\mu) u(\mu) \, dt \, d\mu
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\mu - \lambda} \left[ e^{(\mu - \lambda)T} - 1 \right] k(\mu) u(\mu) \, d\mu.
\]

\[
= \lim_{n \to \infty} \frac{1}{2\pi i} \left[ \int_{\Gamma(n)} \frac{1}{\mu - \lambda} e^{(\mu - \lambda)T} k(\mu) u(\mu) \, d\mu \right]
\]

\[
- \lim_{n \to \infty} \frac{1}{2\pi i} \left[ \int_{\Pi(n)} \frac{1}{\mu - \lambda} k(\mu) u(\mu) \, d\mu \right].
\]

This means that

\[
\int_0^T e^{-\lambda t} m(t) \, dt = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\Gamma(n)} \frac{1}{\mu - \lambda} e^{(\mu - \lambda)T} k(\mu) u(\mu) \, d\mu
\]

\[ + \lim_{n \to \infty} \frac{1}{2\pi i} \left( - \int_{\Gamma(n)} + \int_{\Pi(n)} - \int_{\Pi_n} \right) \frac{1}{\mu - \lambda} k(\mu) u(\mu) \, d\mu. \tag{1.14} \]

For \(n > \Re(\lambda)\), the Cauchy residue theorem gives that

\[
\frac{1}{2\pi i} \left( - \int_{\Gamma(n)} + \int_{\Pi(n)} \right) \frac{1}{\mu - \lambda} k(\mu) u(\mu) \, d\mu = k(\lambda) u(\lambda).
\]

Now observe that \(\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\Pi_n} \frac{1}{\mu - \lambda} k(\mu) u(\mu) \, d\mu = 0\) since

\[
\left\| \int_{\Pi_n} \frac{1}{\mu - \lambda} k(\mu) u(\mu) \, d\mu \right\| \leq \int_{-\Psi(n)}^{\Psi(n)} \frac{1}{|n + ir - \lambda|} e^{-\Psi^{-1}((n+ir))} \frac{M}{|n + ir|} \, dr
\]

\[
\leq M e^{-\Psi^{-1}(n)} \int_{-\infty}^{\infty} \frac{1}{|n + ir|^2} \to 0 \text{ as } n \to \infty.
\]
Hence, from (1.14), it follows that

\[
\int_0^T e^{-\lambda t} m(t) \, dt = k(\lambda)u(\lambda) + \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu - \lambda} e^{(\mu - \lambda)T} k(\mu)u(\mu) \, d\mu
\]

\[
= k(\lambda)u(\lambda) + \frac{e^{-\lambda T}}{2\pi i} \int_\Gamma \frac{1}{\mu - \lambda} e^{\mu T} k(\mu)u(\mu) \, d\mu
\]

\[
= k(\lambda)u(\lambda) + a_T(\lambda).
\]

In order to prove that \( a_T \approx_T 0 \), it is sufficient to show that \( \int_\Gamma \frac{1}{\mu - \lambda} e^{\mu T} k(\mu)u(\mu) \, d\mu \) is bounded for \( \lambda \to \infty \). On the path \( \Gamma_\beta \) as \( \lambda \to \infty \), \( \int_{\Gamma_\beta} \frac{1}{\mu - \lambda} e^{\mu T} k(\mu)u(\mu) \, d\mu \to 0 \).

Furthermore, on the paths \( \Gamma_\pm \),

\[
\| \int_{\Gamma_\pm} \frac{1}{\mu - \lambda} e^{\mu T} k(\mu)u(\mu) \, d\mu \| \leq \int_\beta^\infty \frac{M}{\text{dist}(\lambda, T \pm \gamma)} e^{\mu T} e^{-c\Psi^{-1}(|r + i\Psi(r)|)} \left| \frac{1 + i\Psi'(r)}{r \pm \Psi(r)} \right| \, dr
\]

\[
\leq \frac{M}{\text{dist}(\lambda, T \pm \gamma)} \int_\beta^\infty (C_0 + C_1 r^\alpha) e^{-r(c-T)} \, dr.
\]

Hence \( a_T \approx 0 \) and \( k(\lambda)u(\lambda) \in \{ m \}_T \) for all \( 0 < T < c \).

Before continuing to the proof of (b), \( \| m(t) \| \) will be estimated for \( 0 < t < c \). First, define \( \Gamma^* = \Gamma^*_+ \cup \Pi_\gamma \cup \Gamma^*_- \), where

\[
\Gamma^*_\pm := \{ r \pm i\Psi(r); \gamma \leq r < \infty \}, \text{ and } \Pi_\gamma := \{ \gamma + \Psi(\gamma)e^{ir}; -\frac{\pi}{2} \leq r \leq \frac{\pi}{2} \}.
\]

By Cauchy’s theorem, \( m(t) = \frac{1}{2\pi i} \int_{\Gamma^*} e^{\lambda t} k(\lambda)u(\lambda) \, d\lambda \). Then since

\[
\| \frac{1}{2\pi i} \int_{\Pi_\gamma} e^{\lambda t} k(\lambda)u(\lambda) \, d\lambda \| \leq \frac{M}{2\pi} \int_0^{\infty} \left| e^{\gamma t + i\Psi(\gamma)} \right| e^{-c\Psi^{-1}(\gamma + \Psi(\gamma))} \left| \frac{\Psi'(\gamma)}{\gamma + \Psi(\gamma)} \right| \, d\gamma
\]

\[
\leq \frac{M}{2\pi} \int_0^{\infty} e^{\gamma t + i\Psi(\gamma)\cos(r)} e^{-c\Psi^{-1}(\Psi(\gamma))} \, dr
\]

and since \( \| e^{\lambda t} k(\lambda)u(\lambda) \, d\lambda \| \leq M(C_0 + C_1 r^\alpha)e^{-r(c-t)} \, dr \) along \( \Gamma_\pm \), it follows that

\[
\| \frac{1}{2\pi i} \int_{\Gamma_\pm} e^{\lambda t} k(\lambda)u(\lambda) \, d\lambda \| \leq \frac{M}{2\pi} \int_0^{\infty} (C_0 + C_1 r^\alpha)e^{-r(c-t)} \, dr
\]

\[
\leq \frac{MC_0}{2\pi} e^{-\gamma(c-t)} \left( \frac{1}{c-t} + \frac{MC_1}{2\pi} \int_0^\infty r^\alpha e^{-r(c-t)} \, dr \right).
\]

Hence,

\[
\| m(t) \| \leq \frac{M}{\pi} e^{-\gamma(c-t)} \left( \int_0^{\frac{\pi}{2}} e^{i\Psi(\gamma)\cos(r)} \, dr + \frac{C_0}{c-t} \right) + \frac{MC_1}{\pi} \int_0^{\infty} r^\alpha e^{-r(c-t)} \, dr.
\]
Moreover, choosing \( \gamma \) such that \( \Psi(\gamma) = \frac{\Psi(\beta)c}{t} \) (and thus \( \gamma := \Psi^{-1}(\frac{\Psi(\beta)c}{t}) \)), then

\[
\|m(t)\| \leq \frac{M}{\pi} e^{-\Psi^{-1}(\frac{\Psi(\beta)c}{t})(c-t)} \left( \int_{\beta}^{\gamma} e^{c\Psi(\beta)\cos(r)} dr + \frac{C_0}{c-t} \right) + \frac{MC_1}{\pi} \int_{\gamma}^{\infty} r^\alpha e^{-r(c-t)} dr
\]

for \( 0 < t < c \). Also note that \( m(0) = 0 \).

(b) Let \( \Gamma, M, C_0, \) and \( C_1 \) be defined as in (a). Then, when \( \lambda = r \pm i\Psi(r) \in \Gamma_\pm \),

\[
\|e^{\lambda t}k(\lambda)u(\lambda) d\lambda\| \leq Me^{\lambda t}e^{-c(\Psi^{-1}(|r \pm i\Psi(r)|))d} \left| \frac{1 \pm i\Psi'(r)}{\Psi(\gamma)} \right| dr
\]

\[
\leq M(C_0 + C_1 r^\alpha)e^{rT - c^\alpha} dr.
\]

This implies that

\[
m(t) := \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}k(\lambda)u(\lambda) d\lambda
\]

is well-defined and continuous for \( t > 0 \). The methods of part (a) can be used again to show that

\[
\int_{0}^{T} e^{-\lambda t}m(t) dt = k(\lambda)u(\lambda) + \frac{e^{-\lambda T}}{2\pi i} \int_{\Gamma} \frac{1}{\mu - \lambda} e^{\mu t}k(\mu)u(\mu) d\mu := k(\lambda)u(\lambda) + a_T(\lambda).
\]

On the path \( \Gamma_\beta \) as \( \lambda \rightarrow \infty \), \( \int_{\Gamma_\beta} \frac{1}{\mu - \lambda} e^{\mu t}k(\mu)u(\mu) d\mu \rightarrow 0 \). Furthermore, on the paths \( \Gamma_\pm \),

\[
\| \int_{\Gamma_\pm} \frac{1}{\mu - \lambda} e^{\mu t}k(\mu)u(\mu) d\mu \| \leq \int_{\beta}^{\infty} \frac{M}{\text{dist}(\lambda, \Gamma_\pm)} e^{rT} e^{-c(\Psi^{-1}(|r \pm i\Psi(r)|))d} \left| \frac{1 \pm i\Psi'(r)}{\Psi(\gamma)} \right| dr
\]

\[
\leq \frac{M}{\text{dist}(\lambda, \Gamma_\pm)} \int_{\beta}^{\infty} (C_0 + C_1 r^\alpha)e^{-rT - c^\alpha} dr.
\]

Thus \( \int_{\Gamma} \frac{1}{\mu - \lambda} e^{\mu t}k(\mu)u(\mu) d\mu \) is bounded which means \( a_T(\lambda) \approx_T 0 \). This implies \( ku \in \{m\}_T \) for all \( T > 0 \) and thus, \( ku \in \{m\} \).

In order to estimate \( \|m(t)\| \) for \( 0 < t \leq 1 \), let \( t \) be fixed, \( \gamma \geq \beta \) be arbitrary, and use the path \( \Gamma^* \) as defined in (a). Again by Cauchy's theorem, \( m(t) = \frac{1}{2\pi i} \int_{\Gamma^*} e^{\lambda t}k(\lambda)u(\lambda) d\lambda \). Now for \( \lambda \in \Pi_\gamma \),

\[
\| \frac{1}{2\pi i} \int_{\Pi_\gamma} e^{\lambda t}k(\lambda)u(\lambda) d\lambda \| \leq \frac{M}{2\pi} \int_{\gamma}^{\beta} \left| e^{\gamma t + i\Psi(\gamma)} e^{t\gamma} e^{-c(\Psi^{-1}(|\gamma + i\Psi(\gamma)e^{ir}|))d} \frac{\Psi(\gamma)}{|\gamma + i\Psi(\gamma)e^{ir}|} \right| dr
\]

\[
\leq \frac{M}{2\pi} \int_{\gamma}^{\beta} e^{\gamma t + t\Psi(\gamma)\cos(r)} e^{-c(\Psi^{-1}(\Psi(\gamma)))d} dr
\]

\[
\leq \frac{M}{\pi} e^{\gamma t - c^\alpha} \int_{0}^{\gamma} e^{t\Psi(\gamma)\cos(r)} dr.
\]
Then, since \( \|e^{\lambda t}(\omega)u(\omega)\| \leq M(C_0 + C_1r^\alpha)e^{rt-cr^d} \) along the paths \( \Gamma_\pm \),
\[
\left\| \frac{1}{2\pi i} \int_{\Gamma_\pm} e^{\lambda t}(\omega)u(\omega) \, d\lambda \right\| \leq M \int_0^\infty (C_0 + C_1r^\alpha)e^{rt-cr^d} \, dr
\]
Thus, for \( t \geq 0 \)
\[
\|m(\lambda)\| \leq \frac{M}{\pi} e^{\gamma t-cr^d} \int_0^\infty e^{\Psi(\gamma)\cos(r)} \, dr + \frac{M}{\pi} \int_0^\infty (C_0 + C_1r^\alpha)e^{rt-cr^d} \, dr.
\]
If \( 0 < t < 1 \), by choosing \( \gamma \) such that \( \Psi(\gamma) = \frac{\Psi(\beta)}{t} \) and thus \( \gamma := \Psi^{-1}(\frac{\Psi(\beta)}{t}) \), it follows that
\[
\|m(t)\| \leq M \int_0^\infty e^{\gamma t-cr^d} \int_0^\infty e^{\Psi(\gamma)\cos(r)} \, dr + M \int_0^\infty (C_0 + C_1r^\alpha)e^{rt-cr^d} \, dr.
\]
Again, note that \( m(0) = 0 \).
In order to estimate \( \|m(t)\| \) for large \( t > 0 \), \( \alpha \geq 0 \), and \( d > 1 \), it will be shown that
\[
\int_0^\infty r^\alpha e^{rt-cr^d} \, dr \leq M_\omega c_\alpha c_\varepsilon e^{\omega t}\frac{d}{d}\frac{d}{d}
\]
for all \( \omega > \frac{d-1}{d}(\frac{1}{\alpha d})^{\frac{1}{d}} \). To illustrate this, first note that there exists a constant \( c_\alpha \) such that \( r^\alpha \leq c_\alpha e^{sr} \). Then
\[
\int_0^\infty r^\alpha e^{rt-cr^d} \, dr \leq c_\alpha \int_0^\infty e^{r(t+\varepsilon)-cr^d} \, dr.
\]
Now let \( g(r) := e^{r(t+\varepsilon)-cr^d} \), it follows that \( g(r) \) reaches a maximum at
\[
r = \left( \frac{1}{d} \right)^{\frac{1}{d-1}} \left( \frac{(t+\varepsilon)}{c} \right)^{\frac{1}{d-1}}.
\]
Let \( M \) be that maximum value where \( M := e^{\frac{d-1}{2\alpha d}(\frac{1}{d})^{\frac{1}{d-1}}(t+\varepsilon)^\frac{d}{d}} \). Also, if \( r \geq \left( \frac{2(t+\varepsilon)}{c} \right)^{\frac{1}{d-1}} \) then
\[
r(t+\varepsilon) - cr^d \leq -r(t+\varepsilon).
\]
This implies that
\[
c_\alpha \int_0^\infty e^{r(t+\varepsilon)-cr^d} \, dr \leq c_\alpha \left[ \int_0^{\left( \frac{2(t+\varepsilon)}{c} \right)^{\frac{1}{d-1}}} e^{r(t+\varepsilon)-cr^d} \, dr + \int_0^\infty \left( \frac{2(t+\varepsilon)}{c} \right)^{\frac{1}{d-1}} e^{-r(t+\varepsilon)} \, dr \right]
\leq c_\alpha \left[ \left( \frac{2(t+\varepsilon)}{c} \right)^{\frac{1}{d-1}} M + \frac{1}{(t+\varepsilon)e} - \left( \frac{2(t+\varepsilon)}{c} \right)^{\frac{1}{d-1}} (t+\varepsilon) \right]
\leq c_\alpha M_\omega e^{\omega(t+\varepsilon)}\frac{d}{d} = M_\omega c_\alpha c_\varepsilon e^{\omega t}\frac{d}{d}
\]
53
for some \( M_\omega > 1 \) and all \( \omega > (\frac{d-1}{d})(\frac{1}{c_\omega})^{\frac{1}{d-1}}. \)

\[ \square \]

**Corollary 1.3.6.** Let \( \Omega_\Psi \subset \Omega \), \( u : \Omega \to X \) analytic, and \( \sup_{\lambda \in \Omega} \| \lambda u(\lambda) \| < \infty. \)

(a) Let \( \Psi(x) = e^{ax\gamma} \ (a > 0, \ \gamma > 1) \). Then, for all \( b > 0 \), there exists \( m \in C_0([0, \infty), X) \) such that \( \frac{1}{X} u(\lambda) \in \{m\}. \)

(b) Let \( \Psi(x) = e^{ax} \ (a > 0) \). Then

\[ (i) \text{ for all } b > 0 \text{ there exists } m \in C_0([0, ab), X) \text{ such that } \frac{1}{X} u(\lambda) \in \{m\}_T (0 < T < ab), \]

\[ (ii) \text{ for all } b > 0, 0 < \delta < 1 \text{ there exists } m \in C_0([0, \infty); X) \text{ such that } e^{-\frac{\delta}{a} \lambda^d} u(\lambda) \in \{m\}, \text{ and } d := \cos(\frac{\pi}{2}). \]

(c) Let \( \Psi(x) = x^{1/a} \ (0 < a < 1). \) Then

\[ (i) \text{ for all } b > 0 \text{ there exists } m \in C_0([0, b \cos(\frac{\alpha\pi}{2})), X) \text{ such that } e^{-b^{\frac{\alpha}{a}}} u(\lambda) \in \{m\}_T (0 < T < b \cos(\frac{\alpha\pi}{2})), \]

\[ (ii) \text{ for all } b > 0, a < \gamma < 1 \text{ there exists } m \in C_0([0, \infty), X) \text{ such that } e^{-b^{\gamma}} u(\lambda) \in \{m\}. \]

**Proof.** To see (a), notice that because \( \Psi^{-1}(x) = (\frac{1}{\alpha} \ln(x))^{1/\gamma} \). Then \( e^{-ab(\psi^{-1}(|\lambda|))^{\gamma}} = \frac{1}{|X|^\gamma}. \) By choosing \( k(\lambda) = \frac{1}{X^\gamma}, \) statement (a) follows from Theorem 1.3.5 (b).

For (b), notice \( \Psi^{-1}(x) = \frac{1}{\alpha} \ln(x), \) and thus \( e^{-ab\psi^{-1}(|\lambda|)} = \frac{1}{|X|^\gamma}. \) If one defines \( k(\lambda) := \frac{1}{X^\gamma}, \) then statement (i) follows from Theorem 1.3.5 (a). In order to show (ii), first note that if \( \text{Re}(z) > 0 \) with \( \text{Arg}(z) = \alpha \) and \( 0 < \delta < 1, \) then

\[ |e^{-z^d}| = e^{-|z|^d \cos(\delta \alpha)} \leq e^{-|z|^d \cos(\delta \frac{\pi}{2})} := e^{-d|z|^d} \]
where \( d := \cos(\delta \pi/2) \). Now by defining \( k(\lambda) := e^{-b/2} \lambda^\delta \), it follows that there exists a constant \( c > 0 \) such that

\[
|k(\lambda)| \leq e^{-b|\lambda|^\delta} \leq e^{-c(\Psi^{-1}(|\lambda|))^2} = e^{-c\Psi^{-1}(|\lambda|)}
\]

for all \( \lambda \in \Omega\Psi \). Then statement (ii) follows from Theorem 1.3.5 (b).

For (c), choose \( k(\lambda) = e^{-b\lambda} \), then

\[
|k(\lambda)| \leq e^{-b\cos(\delta \pi/2)|\lambda|} = e^{-b\cos(\delta \pi/2)\Psi^{-1}(|\lambda|)}
\]

for \( \text{Re}(\lambda) > 0 \). Thus statement (i) follows from Theorem 1.3.5 (a). Statement (ii) follows from Theorem 1.3.5 (b) by defining \( k(\lambda) := e^{-b\gamma} \) since

\[
|k(\lambda)| \leq e^{-b\cos(\gamma \pi/2)|\lambda|} = e^{-b\cos(\gamma \pi/2)\Psi^{-1}(|\lambda|)^{\gamma/a}}
\]

for \( \text{Re}(\lambda) > 0 \).

\( \square \)

In the following section, Corollary 1.3.6 will be used to prove the existence of solutions for a particular class of delay-differential equations which are not classically Laplace transformable.

### 1.4 Advanced Type Delay-Differential Equations

This investigation concerns first order delay-differential equations

\[
\text{(DDE)} \quad a_0 u'(t) + a_1 u'(t - \omega) + b_0 u(t) + b_1 u(t - \omega) = f(t)
\]

with \( u(t) = g(t) \) for \( t \in [-\omega, 0] \), where the coefficients \( a_0, a_1, b_0, \) and \( b_1 \) will be real valued constants and \( \omega \) is a positive real number. The function \( f \) represents a given forcing term and \( g \) is a given initial value function [8]. First order DDE are
prototypes of systems in which the future behavior depends not only on its present state, but also on its past. Their application lies within systems that have a built in memory effect, such as chemical equilibria, financial interest, or population growth models.

First order linear DDE are well understood if they are of retarded type (i.e., $a_0 \neq 0, a_1 = 0$), but are more challenging if they are of advanced type (i.e., $a_0 = 0, a_1 \neq 0$). Equations of retarded type are well-posed in the sense that there are unique classical solutions that depend continuously on the data. Advanced equations can be considered as solving a specific retarded equation “backwards in time” and are generally ill-posed in the sense that the solutions no longer depend continuously on the data.

**Remark 1.4.1.** Given the DDE of retarded type,

$$a_0 u(t) + b_0 u(t) + b_1 u(t - \omega) = f(t)$$

with $u(t) = g(t)$ for $t \in [T - \omega, T]$ two substitutions, $w(t) := u(T - t)$ and $v(t) := w(t + \omega)$ yields the DDE of advanced type

$$a_0 v'(t - \omega) - b_0 v(t - \omega) - b_1 v(t) = \tilde{f}(t)$$

where $\tilde{f}(t) := -f(T - t)$ and $v(t) = \tilde{g}(t) := g(T - t - \omega)$ for $-\omega \leq t \leq 0$. Hence, solving a DDE of retarded type backwards in time is equivalent to solving a DDE of advanced type. Equations of retarded type and, more generally, equations with a built in time delay that depend not only the present state of the system but also the past state are extremely useful in modeling numerous systems occurring in biology, medicine, chemistry, engineering, and public health (see [8], [23] and [10]). Thus some importance is placed on the study of equations of advanced type when they are seen as retarded type equations backwards in time.
Laplace transform techniques are one approach to solving DDE’s. Let \( u(t) = g(t) \) for \( t \in [-\omega, 0] \). Then the Laplace transform \( \hat{u} \) of the solution \( u \) of DDE is given by
\[
\hat{u}(\lambda) = \frac{a_0 g(0) + a_1 g(-\omega) - [b_1 + a_1 \lambda] e^{-\lambda \omega} \int_{-\omega}^{0} e^{-\lambda s} g(s) \, ds + \hat{f}(\lambda)}{b_0 + a_0 \lambda + [b_1 + a_1 \lambda] e^{-\lambda \omega}}.
\]

Now
\[
\int_{-\omega}^{0} e^{-\lambda s} g(s) \, ds = \int_{0}^{\omega} e^{-\lambda (s-\omega)} g(s-\omega) \, ds = e^{\lambda \omega} \int_{0}^{\infty} e^{-\lambda s} g(\omega)(s) \, ds = e^{\lambda \omega} \tilde{g}_\omega(\lambda)
\]
where
\[
g_\omega(s) := \begin{cases} 
g(s-\omega) & 0 \leq s \leq \omega \\
0 & s > \omega \end{cases}.
\]

Let \( h(\lambda) \) be the characteristic function of DDE; i.e.,
\[
h(\lambda) = b_0 + \lambda a_0 + (b_1 + \lambda a_1) e^{-\lambda \omega}.
\]

The Laplace transform of DDE can now be represented by
\[
\hat{u}(\lambda) = \frac{a_0 g(0) + a_1 g(-\omega)}{h(\lambda)} - \frac{1}{h(\lambda)} b_1 \tilde{g}_\omega(\lambda) - \frac{1}{h(\lambda)} a_1 \lambda \tilde{g}_\omega(\lambda) + \frac{1}{h(\lambda)} \hat{f}(\lambda), \quad (1.15)
\]
and thus
\[
\begin{align*}
u(t) & = [a_0 g(0) + a_1 g(-\omega)] m(t) - b_1 (m * g_\omega)(t) - a_1 (m * g_\omega')(t) + (m * f)(t) \\
& = [a_0 g(0) + a_1 g(-\omega)] m(t) - b_1 \int_{0}^{t} m(t-s) g_\omega(s) \, ds \\
& \quad - a_1 \int_{0}^{t} m(t-s) g_\omega'(s) \, ds + \int_{0}^{t} m(t-s) f(s) \, ds
\end{align*}
\]
is a formal solution of DDE, where \( m \) is the inverse Laplace transform of the function \( \frac{1}{h(\lambda)} \). Since the Laplace transform of Laplace-transformable function has all its singularities in a left half-plane (see Proposition 1.1.1), for equations of retarded type, \( \frac{1}{h(\lambda)} \) is, in fact, a Laplace transform of a classical function \( m \). Hence, the unique solution of DDE is given by
\[
u(t) = [a_0 g(0) + a_1 g(-\omega)] m(t) - b_1 (m * g_\omega)(t) + a_1 (m * g_\omega')(t) + (m * f)(t) \quad (1.16)
\]
assuming \( g_\omega(s) \) is differentiable. However, for equations of advanced type the singularities of the analytic function \( \frac{1}{h(\lambda)} \) have arbitrarily large real parts.

**Proposition 1.4.2.** Given an advanced type DDE

\[
 u'(t - \omega) + b_0 u(t) + b_1 u(t - \omega) = f(t)
\]

with \( u(t) = g(t) \) for \( t \in [-\omega, 0] \) and its corresponding characteristic equation

\[
 h(\lambda) = b_0 + (\lambda + b_1)e^{-\lambda \omega},
\]

it follows that \( h(\lambda) \) has infinitely many zeros which lie on the curve

\[
 y^2 = b_0^2 e^{2\omega x} - (x + b_1)^2
\]

for \( \lambda = x + iy \).

**Proof.** Let \( \lambda = x + iy \). Now \( h(\lambda) = 0 \) implies that \( b_0 e^{\lambda \omega} + \lambda + b_1 = 0 \) which is equivalent to

\[
 b_0 e^{\omega x} \cos(\omega y) + ib_0 e^{\omega x} \sin(\omega y) + x + iy + b_1 = 0.
\]

This holds if

\[
 b_0 e^{\omega x} \cos(\omega y) = -(x + b_1) \quad (1.17)
\]

\[
 b_0 e^{\omega x} \sin(\omega y) = -y.
\]

Then the zeros for \( h(\lambda) \) lie on the curve

\[
 y^2 = b_0^2 e^{2\omega x} - (x + b_1)^2.
\]

To show that there are an infinite number of singularities, it follows from (1.17) that \( \tan(\omega y) = \frac{\omega y}{x + b_1} \) and thus \( x = \frac{y}{\tan(\omega y)} - b_1 \). Then, by substitution into the second equation of (1.17), the zeros of \( h(\lambda) \) occur whenever

\[
 b_0 e^{-b_1 \omega} \frac{\omega y}{\tan(\omega y)} \sin(\omega y) = -y.
\]
It is clear from the inspection of the graphs \( e^{\omega y} \tan(\omega y) \sin(\omega y) \) and \(-y\) that there will be an infinite number of intersections and hence an infinite number of zeros of \( h \), because

\[
\lim_{y \to (k\pi/\omega)^-} e^{\omega y} \tan(\omega y) \sin y = -\infty
\]

for all \( k \) where \( k \) is an odd positive integer. \( \square \)

Proposition 1.4.2 shows that the characteristic equation of an advanced type DDE has infinitely many zeros which grow exponentially in the right half-plane. These zeros correspond to singularities of the Laplace transform of the advanced type DDE, and thus, by Proposition 1.1.1, \( \frac{1}{h(\lambda)} \) can not be the Laplace transform of a function in the classical sense. This means that the formal solution given by (1.16) needs further justification for advanced type DDE’s.

Before applying the methods of Theorem 1.3.5 to prove the existence of solutions for advanced type DDE’s, the following remark will allow for a simplification in the representation of the advanced type DDE.

**Remark 1.4.3.** Let \( w(t) \) be a solution of

\[
w'(t - \omega) = cw(t).
\]

If \( c = -b_0 e^{-b_1 \omega} \) and \( u(t) := e^{-b_1 t} w(t) \) then

\[
u'(t - \omega) = -b_1 e^{-b_1(t-\omega)} w(t - \omega) + e^{-b_1(t-\omega)} w'(t - \omega)
\]

\[
= -b_1 u(t - \omega) - e^{-b_1(t-\omega)} b_0 e^{-b_1 \omega} w(t)
\]

\[
= -b_1 u(t - \omega) - b_0 e^{-b_1 t} w(t)
\]

\[
= -b_1 u(t - \omega) - b_0 u(t).
\]

This means \( u(t) := e^{-b_1 t} w(t) \) solves \( u'(t - \omega) = -b_0 u(t) - b_1 u(t - \omega) \). Moreover, the coefficient \( b_1 \) only affects the growth of the solutions \( u(t) \) and does not affect
regularity or existence of solutions. Instead of showing the existence of advanced type DDE for \( u'(t - \omega) + b_0 u(t) + b_1 u(t - \omega) = 0 \), one only needs to consider \( u'(t - \omega) + \tilde{b}_0 u(t) = 0 \) where \( \tilde{b}_0 = b_0 e^{-b_1 \omega} \) with the corresponding characteristic function \( h(\lambda) = \tilde{b}_0 + \lambda e^{-\lambda \omega} \).

Next, the asymptotic Laplace transform methods outlined in Theorem 1.3.5 and Corollary 1.3.6 will be followed to prove the existence of the family of functions whose asymptotic Laplace transform is \( u(\lambda) = \frac{1}{h(\lambda)} = \frac{1}{\tilde{b}_0 + \lambda e^{-\lambda \omega}} \).

Following the framework from Theorem 1.3.5, let \( \Gamma \) be the oriented boundary of the region \( \Omega_{\Psi} \); i.e., \( \Gamma = \Gamma_\Psi \cup \Gamma_\beta \cup \Gamma_\alpha \), where \( \Gamma_\pm := \{ r \pm i\Psi(r); \beta \leq r < \infty \} \), and \( \Gamma_\beta := \{ \beta + ir; -\Psi(\beta) \leq r \leq \Psi(\beta) \} \). The path \( \Psi(r) \) must be chosen so that \( \Gamma \) avoids the zeros of the characteristic function \( h(\lambda) = \tilde{b}_0 + \lambda e^{-\lambda \omega} \) (the singularities of \( u(\lambda) \)) which lie on the curve \( y = \pm \sqrt{b_0^2 e^{2\omega x} - x^2} \) (see Proposition 1.4.2). With this in mind, define \( \Psi(r) := ce^{\omega r} \) where \( c = \frac{3|\tilde{b}_0|}{5} \). Then

\[
\sqrt{b_0^2 e^{2\omega r} - r^2} \geq \frac{3|\tilde{b}_0|}{5} e^{\omega r} \Rightarrow \frac{4}{5}|\tilde{b}_0| \geq re^{-\omega r}.
\]

There exists an \( r_0 \) such that the above inequalities hold true for all \( r > r_0 \), since the function \( re^{-\omega r} \) has an absolute maximum of \( \frac{1}{\omega e^{\omega r}} \) at \( r = \frac{1}{\omega} \). With \( \beta > r_0 \), \( \Gamma \) will avoid the singularities of \( u(\lambda) \). Furthermore, with \( \Psi(r) := ce^{\omega r} \), \( \Psi(r) \to \infty \) as \( r \to \infty \) and \( \sup_{r \geq \beta} \frac{\Psi'(r)}{r^{\alpha} \Psi(r)} < \infty \) for \( \alpha \geq 0 \). The following lemmas are needed to show that the conditions of Theorem 1.3.5 are satisfied.

**Lemma 1.4.4.** If \( \Psi(r) := ce^{\omega r} \) where \( c = \frac{3|\tilde{b}_0|}{5} \), then \( \left| \frac{1 + i\Psi'(r)}{r + i\Psi(r)} \right| \leq C_0 \) where

\[
C_0 := \frac{1}{\sqrt{\beta^2 + c^2 e^{2\omega \beta}}} + \frac{\omega}{\sqrt{\beta^2 e^{-2\omega \beta} + 1}}.
\]

**Proof.**

\[
\left| \frac{1 + i\Psi'(r)}{r + i\Psi(r)} \right| \leq \frac{1}{|r + i\Psi(r)|} + \left| \frac{\Psi'(r)}{r + i\Psi(r)} \right| = \frac{1}{\sqrt{\beta^2 + c^2 e^{2\omega \beta}}} + \frac{\omega}{\sqrt{\beta^2 e^{-2\omega \beta} + 1}}.
\]
Let $\Omega^0_\Psi$ be defined as the interior of the region bounded by $\Omega_\Psi$.

**Lemma 1.4.5.** For $u(\lambda) = \frac{1}{h(\lambda)} = \frac{1}{b_0 + \lambda e^{-\omega}}$, $\sup_{\lambda \in \Omega^0_\Psi} \|u(\lambda)\| < \infty$.

**Proof.** To prove that $u(\lambda)$ is bounded in the domain $\Omega^0_\Psi$, it will be shown that $u(\lambda)$ is bounded along the oriented boundary of the region $\Omega_\Psi$ as described above. To show $u(\lambda)$ is bounded along $\Gamma_\pm := \{r \pm i\Psi(r); \beta \leq r < \infty\}$ with $\Psi(r) := ce^{\omega r}$ and $c = \frac{3|b_0|}{5}$, it is necessary that for $\lambda \in \Gamma_\pm$, $\|\tilde{b}_0 + \lambda e^{-\omega}\| \geq \varepsilon$. Then

$$\|\tilde{b}_0 + \lambda e^{-\omega}\| \geq |\tilde{b}_0| - |\lambda e^{-\omega}|$$

and $|\tilde{b}_0| - |\lambda e^{-\omega}| > 0$ whenever $|\tilde{b}_0|^2 > |\lambda|^2|e^{-\omega}|^2$ or $|\tilde{b}_0|^2 > [r^2 + c^2e^{2\omega r}]e^{-2\omega r}$ or $\frac{4}{5}|\tilde{b}_0| > re^{-r\omega}$. Recall that $r_0$ was chosen so the above inequalities hold. Thus $u(\lambda)$ is bounded on $\Gamma_\pm$.

To show that $u(\lambda)$ is bounded on $\Gamma_\beta := \{\beta + ir; -\Psi(\beta) \leq r \leq \Psi(\beta)\}$ note that for $\lambda \in \Gamma_\beta$,

$$\|q(\lambda)\| = \frac{|e^{(\beta+ir)\omega}|}{|\tilde{b}_0 e^{(\beta+ir)\omega} + \beta + ir|} = \frac{e^{\beta\omega}}{(\tilde{b}_0 e^{\beta\omega} \cos(r) + \beta)^2 + (e^{\beta\sin(r)} + r)^2} \leq \frac{e^{\beta\omega}}{\beta^2}.$$ 

Since $u(\lambda)$ is bounded along $\Gamma$, the maximum modulus principle implies that $u(\lambda)$ is bounded in $\Omega^0_\Psi$.

A slight increase of $\beta > r_0$ allows us to define $\Gamma^*$ as the oriented boundary of the region $\Omega^*_\Psi$ which is $\Omega_\Psi$ shifted to the right to lie inside $\Omega^0_\Psi$, i.e., $\Gamma^* = \Gamma_\pm \cup \Gamma^*_\beta \cup \Gamma_+$, where $\Gamma_\pm$ is defined as above and $\Gamma^*_\beta := \{\beta^* + ir; -\Psi(\beta^*) \leq r \leq \Psi(\beta^*)\}$ with $r_0 < \beta < \beta^*$. With $\Omega^*_\Psi$, one is now ready to apply the results of Theorem 1.3.5 and Corollary 1.3.6 to prove the existence of advanced type DDE’s.

**Theorem 1.4.6.** (Existence for Advanced Type DDE) Let $\Omega^*_\Psi \subset \Omega^0_\Psi$, $u : \Omega^*_\Psi \to X$ analytic, where $u(\lambda) = \frac{1}{h(\lambda)} = \frac{1}{b_0 + \lambda e^{-\omega}}$ and $\sup_{\lambda \in \Omega} \|u(\lambda)\| < \infty$.  


(a) For \( n > 0 \) and with an analytic function \( k : \Omega_\Psi^* \to \mathbb{C} \) where \( \hat{k}(\lambda) := \frac{c^n}{\lambda^{n+1}} \)

for all \( \lambda \in \Omega_\Psi^* \) and \( c = \frac{3|\lambda|}{5} \), then there exists \( m \in C_0([0, \omega n), X) \) such that

\[ ku \in \{m\}_T \text{ for all } 0 < T < \omega n. \]

(b) For \( n > 0 \) and \( \{\delta : 0 < \delta < 1\} \), choose an analytic function \( k : \Omega_\Psi^* \to \mathbb{C} \) such that \( \hat{k}(\lambda) := \frac{1}{\lambda} e^{-n/2|\lambda^\delta} \) for all \( \lambda \in \Omega_\Psi^* \) where \( d := \cos(\frac{\pi}{2}) \), then there exists \( m \in C_0([0, \infty), X) \) such that \( ku \in \{m\} \).

Proof. For (a), observe that \( \Psi^{-1}(r) := \frac{1}{\omega} \ln |\frac{\hat{r}}{c}| \). Also note that

\[
|\hat{k}(\lambda)| = \left| \frac{c^n}{\lambda^{n+1}} \right| = \frac{|c|^n}{|\lambda|^{n+1}} = \frac{1}{|\lambda|} \left( \frac{|c|}{|\lambda|} \right)^n = \left( \frac{1}{|\lambda|} \right) e^{-\omega n \Psi^{-1}(|\lambda|)}.
\]

To finish the proof of (a), follow the arguments of Theorem 1.3.5 and Corollary 1.3.6 with only a slight adjustment to account for the fact that \( \sup_{\lambda \in \Omega_\Psi} \|\lambda u(\lambda)\| \) is not bounded and only \( \sup_{\lambda \in \Omega_\Psi} \|u(\lambda)\| \leq \infty \). This problem is made up for by the extra \( \frac{1}{|\lambda|} \) which is gained from the estimate of \( |\hat{k}(\lambda)| \) with the given choice of \( k(\lambda) \).

To prove (b), observe that by defining \( \hat{k}(\lambda) := \frac{1}{\lambda} e^{-\frac{1}{2}|\lambda^\delta} \), it follows that there exists a constant \( l > 0 \) such that

\[
|\hat{k}(\lambda)| \leq \frac{1}{|\lambda|} e^{-k|\lambda|^\delta} \leq \frac{1}{|\lambda|} e^{-l(\frac{1}{2} \ln(|\lambda|^\delta))^2} = \frac{1}{|\lambda|} e^{-l(\Psi^{-1}(|\lambda|))^2}
\]

for all \( \lambda \in \Omega_\Psi^* \). Now, follow the same slightly adjusted arguments of Theorem 1.3.5 and Corollary 1.3.6 as outlined in the proof of (a).
Chapter 2
Numerical Inversion and Approximation

By recognizing the following fundamental connection between semigroups and Laplace transform theory, one can obtain new and efficient inversion procedures for the (classical) Laplace transform (see [18], [20]). In order to introduce the new inversion methods, notice first that on $C_b([0, \infty), X)$ (continuous and bounded functions from $\mathbb{R}^+ \to X$), the (left) shifts

$$T(t)f(\cdot) := f(t + \cdot)$$

define a (bi-continuous) contraction semigroup generated by the derivative operator $Af = f'$. The key to the new inversion methods is the following connection between the shift semigroup on spaces of continuous functions $f$ and the Laplace transform. If $A$ is the generator of the shift semigroup $T(t)$, then

$$T(t)f(0) = f(t) \ (t \geq 0),$$

$$R(\lambda, A)f(0) = \int_0^\infty e^{-\lambda t}T(t)f(0)\,dt = \int_0^\infty e^{-\lambda t}f(t)\,dt = \hat{f}(\lambda),$$

for $\lambda > 0$. That is, studying the shift semigroup on $C_b([0, \infty), X)$ evaluated at $x = 0$ is equivalent to studying Laplace transforms on $C_b([0, \infty), X)$. In particular, since

$$R(\lambda, A)^{n+1}f(0) = \frac{(-1)^n}{n!}R^{(n)}(\lambda, A)f(0) = \frac{(-1)^n}{n!} \int_0^\infty e^{-\lambda t}(-t)^nT(t)f(0)\,dt$$

$$= \frac{(-1)^n}{n!} \int_0^\infty e^{-\lambda t}(-t)^nf(t)\,dt = \frac{(-1)^n}{n!}\hat{f}^{(n)}(\lambda),$$

every approximation result for operator semigroups in terms of powers of the resolvent of the generator will yield an inversion formula for the Laplace transform. These observations are summarized in the following proposition which formalizes
the connection between the approximation of semigroups and the inversion of the Laplace transform.

**Proposition 2.0.7.** *(Transference Principle)* Let $X$ be a Banach space, $f \in C_b([0, \infty), X)$ with its Laplace transform $\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) \, dt$ existing for $\text{Re}\lambda > 0$, and let $A$ be the generator of a bounded, strongly continuous semigroup $T(t)$ on a Banach space $Z$ with resolvent $R(\lambda, A)$ for $\text{Re}\lambda > 0$. Then the following three problems are equivalent.

(i) Compute $f(t)$ in terms of $\hat{f}(\lambda)$.

(ii) Compute $T(t)z$ in terms of $R(\lambda, A)z$ for all $z \in Z$.

(iii) Let $A = \frac{d}{ds}$ on $F = C_b([0, \infty), X)$. Compute the shift semigroup $T(t)f = f(t + \cdot)$ in terms of $R(\lambda, A)f(\cdot)$ for all $f \in F$.

**Proof.** To see that (i) implies (ii) recall the fundamental fact that the resolvent of an operator $A$ is the Laplace transform of the operator semigroup $T(t)$ (see Proposition I.3 of [27] or Theorem 3.1.7 of [2]). In particular, $t \rightarrow T(t)z \in C_b([0, \infty), Z)$ and $R(\lambda, A)z = \int_0^\infty e^{-\lambda t} T(t)z \, dt$ for all $z \in Z$ and $\text{Re}(\lambda) > 0$ (since $\|T(t)\| \leq M$ for all $t \geq 0$ and some $M \geq 1$).

Now (ii) implies (iii) because, as stated above, the shift semigroup with generator $A = \frac{d}{ds}$ on $F = C_b([0, \infty), X)$ is bi-continuous. It has been shown that those approximation results for strongly continuous semigroups used in this dissertation can be extended to hold for bi-continuous semigroups also (see [18]).

Finally, (iii) implies (i) because, as shown above, studying the shift semigroup on $C_b([0, \infty), X)$ evaluated at $x = 0$ is equivalent to studying Laplace transforms on $C_b([0, \infty), X)$. \qed
2.1 Rational Padé Approximations of the Exponential

To introduce the rational approximation schemes discussed later in this chapter, first consider the one dimensional case \( X = \mathbb{C} \). If \( r : \Omega \subset \mathbb{C} \to \mathbb{C} \) is a continuously differentiable function with \( r(0) = r'(0) = 1 \), then by L’Hôpital’s rule,
\[
\lim_{n \to \infty} \ln r \left( \frac{ta}{n} \right)^n = \lim_{n \to \infty} \frac{\ln(r(ta)/n)}{1/n} = \lim_{n \to \infty} \frac{r'(ta)/n}{r(ta)/n} = ta
\]
and therefore \( e^{ta} = \lim_{n \to \infty} r(ta/n)^n \) for all \( t \in \mathbb{R} \). Furthermore, if \( r \) is analytic with \( r(z) = \sum_{n=0}^{\infty} a_n z^n \) where \( a_n = \frac{r^{(n)}(0)}{n!} \), then with the assumptions on \( r(0) \) and \( r'(0) \), it follows that the first two coefficients of the Taylor expansion of \( e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \) equal the first two coefficients of the expansion of \( r \). This means that
\[
|r(z) - e^z| = \left| \left( a_2 - \frac{1}{2} \right) z^2 + \left( a_3 - \frac{1}{3!} \right) z^3 + \ldots \right| \leq C \cdot |z|^2
\]
for \( |z| \) sufficiently small. In contrast to previous work on the subject by Hersh-Kato [17], Brenner-Thomée [9], Kovács [21] and [22], Jara [19], and Özer [29], the following sections are mostly concerned with the size of the constant \( C \) above (and in (2.3) below). Notice that by using the binomial formula, one is able to see that
\[
|r(ta/n)^n - e^{ta}| = \left| r(ta/n)^n - \left( e^{ta/n} \right)^n \right|
\]
\[
= \left| r(ta/n) - e^{ta/n} \right| \cdot \left| \sum_{j=0}^{n-1} r(ta/n) \frac{n-1-j}{j} e^{ta/n} \right|
\]
\[
\leq \left| r(ta/n) - e^{ta/n} \right| \cdot \left| \sum_{j=0}^{n-1} r(ta/n) \frac{n-1-j}{j} e^{ta/n} \right|
\]
\[
= \left| (I) \right| \cdot \left| (II) \right| \cdot \left| (III) \right|.
\]

Suppose that \( Re(a) \leq 0 \) and that \( r \) satisfies the condition
\[
|r(z)| \leq 1 \text{ for all } Re(z) \leq 0,
\]

(2.2)
then, since $\Re(a) \leq 0$, the term $(II)$ is bounded by $n$. Functions for which (2.2) holds shall be called $\mathcal{A}$-stable. Observe that the term $(III)$ is bounded by 1 since $\Re(a) \leq 0$. To estimate $(I)$, another definition will be introduced. Clearly, the estimate of $(I)$ will depend on the quality of approximation of $e^z$ by the function $r(z)$ in the following sense. A function $r : \Omega \subset \mathbb{C} \to \mathbb{C}$ is said to be an approximation of the exponential of order $m \in \mathbb{N}$ if the first $m + 1$ coefficients $a_0, \ldots, a_m$ of the Taylor expansion of $r$ at $z = 0$ match the first $m + 1$ coefficients of $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. If $r$ is an approximation of the exponential of order $m$, then the function $z \to \frac{r(z) - e^z}{z^{m+1}}$ is analytic in a neighborhood of 0 and there exists a $C > 0$ such that

$$|r(z) - e^z| \leq C|z|^{m+1} \quad (2.3)$$

for all $|z|$ sufficiently small. If it is assumed that $r$ is an approximation of the exponential of order $m$, then $(I) \leq C\|\frac{t}{n}\|^{m+1}$. Thus, from (2.1), if $r$ is an $\mathcal{A}$-stable approximation of the exponential of order $m$, then

$$\left| r \left( \frac{t}{n} \right)^n - e^{t} \right| \leq C \left| \frac{t}{n} \right|^{m+1} \cdot n = C \frac{1}{n^m} t^{m+1} |a^{m+1}|. \quad (2.4)$$

The investigation will now focus on the characteristics and properties of rational functions which are $\mathcal{A}$-stable approximation of the exponential of order $m$ and, in particular, on the size of the corresponding constant $C$ in (2.3) and (2.4).

One can assume without loss of generality that for $r = \frac{P}{Q}$, that $P$ and $Q$ are relatively prime and $Q(0) = 1$. The following lemma and its proof are taken from [32] and [35] where it is attributed to an exercise from a course of Christian Lubich in 2005 at the University of Tübingen.

**Lemma 2.1.1.** Let $m \in \mathbb{N}$ and consider $r = \frac{P}{Q}$ where $P(z) = \sum_{i=0}^{m} a_i z^i$ and $Q(z) = \sum_{i=0}^{m} b_i z^i$. The following statements are equivalent:

(i) The function $r$ is a rational approximation of the exponential of order $m$. 

...
(ii) The coefficients of $P$ and $Q$ are related by $m + 1$ linear equations

$$a_i = \sum_{j=0}^{i} \frac{1}{(i-j)!} b_j$$

for $0 \leq i \leq m$.

(iii) There is a unique polynomial $M$ with $\deg(M) \leq m$ such that $M^{(m-i)}(1) = a_i$ and $M^{(m-i)}(0) = b_i$ for all $0 \leq i \leq m$.

Proof. To show that (i) holds if and only if (ii) holds, first assume that $r$ is an approximation of the exponential of order $m$. This means that there exists a constant $C > 0$ such that

$$|r(z) - e^z| \leq C|z|^{m+1}$$

for all $|z|$ sufficiently small. Now $Q$ is continuous and thus bounded in a neighborhood of 0. This implies that $|\frac{1}{z^{m+1}} \left(P(z) - Q(z) - e^z\right)|$ is bounded in a neighborhood of 0 if and only if $|\frac{1}{z^{m+1}} (P(z) - Q(z)e^z)|$ is also bounded in a neighborhood of 0. The last statement is equivalent to asserting that the first $m + 1$ coefficients of the Taylor expansion of $P(z) - Q(z)e^z$ vanish. Because

$$Q(z)e^z = \sum_{i=0}^{m} b_i z^i \sum_{j=0}^{\infty} \frac{z^j}{j!} = \sum_{i=0}^{\infty} c_i z^i$$

with $c_i = \sum_{j=0}^{i} b_j \frac{1}{(i-j)!}$, then the first $m + 1$ coefficients of the Taylor expansion of $P(z) - Q(z)e^z$ vanish if and only if

$$a_i = c_i = \sum_{j=0}^{i} b_j \frac{1}{(i-j)!}$$

for $0 \leq i \leq m$.

To verify that (ii) holds if and only if (iii) holds, suppose that $b_0, \ldots, b_m \in \mathbb{R}$ such that $M(z) := \sum_{j=0}^{m} \frac{b_j}{(m-j)!} z^{m-j}$ is the unique polynomial with $\deg(M) \leq m$ such that $b_i = M^{(m-i)}(0)$ for all $0 \leq i \leq m$. The goal will be to show that the condition
$$a_i = M^{(m-i)}(1)$$ is equivalent to (ii). Now by changing the order of summation, 
$$M(z) = \sum_{j=0}^{m} \frac{b_{m-j}}{j!} z^j.$$ Then

$$M^{(i)}(z) = \sum_{j=i}^{m} \frac{b_{m-j}}{(j-i)!} z^{j-i} = \sum_{j=0}^{m-i} \frac{b_{m-i-j}}{j!} z^j$$

for $$0 \leq i \leq m$$. By changing indices and then reversing the order of summation, it follows that

$$M^{(m-i)}(z) = \sum_{j=0}^{m-(m-i)} \frac{b_{m-(m-i)-j}}{j!} z^j = \sum_{j=0}^{i} \frac{b_{j-i}}{j!} z^j = \sum_{j=0}^{i} \frac{(b_j)}{(i-j)!} z^{i-j}.$$ 

Thus $$M^{(m-i)}(1) = \sum_{j=0}^{i} \frac{b_j}{(i-j)!}$$ for all $$0 \leq i \leq m$$. This means that $$M^{(m-i)}(1) = a_i$$ if and only if (ii) holds.

The next result is originally from Henri Padé’s 1892 dissertation [30]. The proof is again from Christian Lubich’s course as documented in [32] and [35].

**Theorem 2.1.2. (Padé Approximation I)** Let $$r = \frac{P}{Q}$$ be a rational approximation of the exponential of order $$m$$ with $$Q(0) = 1$$. Then $$m \leq p + q$$, where $$p = \deg(P)$$ and $$q = \deg(Q)$$ and $$P$$ and $$Q$$ are relatively prime.

**Proof.** If $$p \geq m$$ or $$q \geq m$$, then $$m \leq p + q$$. But this cannot hold since $$r = \frac{P}{Q}$$ is of order $$m$$ with $$P$$ and $$Q$$ relatively prime. Therefore, let $$p \leq m$$, $$q \leq m$$, and suppose that

$$r(z) = \frac{P(z)}{Q(z)} = \sum_{i=0}^{m} \frac{a_i z^i}{\sum_{i=0}^{m} b_i z^i}.$$ 

Then by Lemma 2.1.1, there is a unique polynomial $$M$$ with $$\deg(M) \leq m$$ such that $$M^{(m-i)}(1) = a_i$$ and $$M^{(m-i)}(0) = b_i$$ for all $$0 \leq i \leq m$$. Because $$p = \deg(P)$$, then it must follow that $$a_i = M^{(m-i)}(1) = 0$$ for all $$p < i \leq m$$. This means that $$M^{(i)}(1) = 0$$ for all $$0 \leq i \leq m - p - 1$$. Hence, 1 is a root of $$M$$ of multiplicity $$m - p$$. Similarly, with $$q = \deg(Q)$$, it follows that $$b_i = M^{(m-i)}(0) = 0$$ for all $$q < i \leq m$$. 

68
This implies that 0 is a root of $M$ of multiplicity $m - q$. Thus, $z^{m-q}(z - 1)^{m-p}$ must divide $M$. Therefore, $m - q + m - p \leq \text{deg}(M) \leq m$ and $m \leq p + q$. \hfill \Box

If $r = \frac{P}{Q}$ with $Q(0) = 1$ is a rational approximation of the exponential of maximal order $m = p + q$, then $r$ is said to be a Padé Approximation of order $m$. The poles (zeros of $Q$) of a rational Padé approximation of the exponential are all simple (of multiplicity one) (see Theorem 4.12 of [16]). Furthermore, a rational Padé approximation of the exponential has all of its poles in an open right half plane if and only if $q - 4 \leq p \leq q$ (see [3]). Furthermore, Padé approximations are $\mathcal{A}$-stable if and only if $q - 2 \leq p \leq q$ (this result can be found in pages 475-489 of [15] or see [12]). A rational Padé approximation $r = \frac{P}{Q}$ is said to be subdiagonal if $p = q - 1$. In this case, a subdiagonal Padé approximation is always $\mathcal{A}$-stable and of odd approximation order $m = 2q - 1 = 2p + 1$. The first part of the following theorem is once again a result of Padé and the proof of this theorem follows [35] and [32]. The proof of the Perron-representation (2.5) follows Simone Flory’s dissertation [14].

**Theorem 2.1.3.** (Padé Approximation II) Let $r = \frac{P}{Q}$ with $Q(0) = 1$ be a rational approximation of the exponential of order $m = p + q$, then

$$P(z) = \sum_{j=0}^{p} \frac{(m-j)!p!}{m!j!(p-j)!} z^{j} = \sum_{j=0}^{p} M^{(m-j)}(1) z^{j}$$

where $M(z) = \frac{(-1)^{q}}{m!} z^{p}(1 - z)^{q}$. Furthermore,

$$Q(z) = \sum_{j=0}^{q} \frac{(m-j)!q!}{m!j!(q-j)!} (-z)^{j} = \sum_{j=0}^{q} M^{(m-j)}(0) z^{j} = c(\lambda_{1} - z) \cdot \ldots \cdot (\lambda_{q} - z) = c\lambda_{1}\lambda_{2} \cdot \ldots \cdot \lambda_{q} + \ldots + (-1)^{q-1} c(\lambda_{1} + \lambda_{2} + \ldots + \lambda_{q}) z^{q-1} + (-1)^{q} cz^{q}$$
where \( c = \frac{(m-q)!}{m!} = \frac{p!}{m!} \) and \( \lambda_i \) (\( 1 \leq i \leq q \)) are the distinct roots of \( Q \). Moreover,
\[
  r(z) - e^z = \frac{(-1)^q+1}{Q(z)} \frac{1}{m!} z^{m+1} e^z \int_0^1 s^p(1-s)^q e^{-sz} ds
\]
(2.5)
for \( z \in \mathbb{C} \) with \( z \neq \lambda_1, \ldots, \lambda_q \).

Proof. By Lemma 2.1.1, there exists a unique polynomial \( M \) with \( \deg(M) \leq m \) such that \( M^{(m-i)}(1) = a_i \) and \( M^{(m-i)}(0) = b_i \) for all \( 0 \leq i \leq m \). Furthermore, in the proof of Theorem 2.1.2, it was shown that \( z^{m-q}(z-1)^{m-p} \) must divide \( M \).

Now, with \( m = p + q \), it follows that
\[
  M(z) = cz^{m-q}(z-1)^{m-p} = cz^p(z-1)^q = cz^{p+q} + \ldots
\]
for some \( c \in \mathbb{R} \). One can calculate \( c \) directly by observing that \( 1 = Q(0) = b_0 = M^{(m)}(0) = c(p+q)! \) and thus \( c = \frac{1}{(p+q)!} = \frac{1}{m!} \). Using the binomial formula, one has that
\[
  M(z) = \frac{1}{m!} z^p(z-1)^q = \frac{1}{m!} z^p \sum_{i=0}^{q} \binom{q}{i} z^{q-i}(-1)^i = \sum_{i=0}^{q} \frac{q!}{m!i!(q-i)!} z^{m-i}(-1)^i.
\]
By taking derivatives,
\[
  M^{(i)}(z) = \sum_{j=i}^{q} \frac{q!(m-j)!}{m!j!(q-j)!(m-j-i)!} z^{m-j-i}(-1)^j
\]
and thus
\[
  M^{(m-i)}(z) = \sum_{j=m-i}^{q} \frac{q!(m-j)!}{m!j!(q-j)!(i-j)!} z^{i-j}(-1)^j.
\]
It follows that \( b_i = M^{(m-i)}(0) = \frac{q!(m-i)!}{m!i!(q-i)!}(-1)^i \) for all \( 0 \leq i \leq m \). To calculate the coefficients \( a_i = M^{(m-i)}(1) \) for \( 1 \leq i \leq m \), it is helpful to consider the shifted polynomial
\[
  \tilde{M}(z) := M(z+1) = \frac{1}{m!} (z+1)^p z^q.
\]
The chain rule gives that \( a_i = M^{(m-i)}(1) = \tilde{M}^{(m-i)}(0) \) for \( 0 \leq i \leq m \). Thus the derivative \( \tilde{M}^{(m-i)} \) for \( 1 \leq i \leq m \) can be determined using similar methods as was
used for calculating derivative of $M(z)$ above. The difference in the methods will be that the $p$ and $q$ are interchanged in the result and, since the binomial formula will now be applied to $(z+1)^q$, the $(-1)^i$ term will no longer be present. Therefore,

$$a_i = \tilde{M}^{(m-i)}(0) = \frac{p!(m-i)!}{i!(p-i)!}$$

for $0 \leq 1 \leq m$. Thus, it has been shown that

$$P(z) = \sum_{j=0}^{p} \frac{(m-j)!p!}{m!j!(p-j)!}z^j = \sum_{j=0}^{p} M^{(m-j)}(1)z^j$$

where $M(z) = \frac{(-1)^q}{m!}z^p(1-z)^q$, and

$$Q(z) = \sum_{j=0}^{q} \frac{(m-j)!q!}{m!j!(q-j)!}(-z)^j = \sum_{j=0}^{q} M^{(m-j)}(0)z^j.$$  

Focusing the investigation on $Q(z)$, one sees that by applying the Fundamental Theorem of Algebra,

$$Q(z) = c(\lambda_1 - z) \ldots (\lambda_q - z)$$

$$= c\lambda_1\lambda_2 \ldots \lambda_q + \ldots + (-1)^{q-1}c(\lambda_1 + \lambda_2 + \ldots + \lambda_q)z^{q-1} + (-1)^qc z^q$$

where $\lambda_i$ ($1 \leq i \leq q$) are the distinct roots of $Q$ and $c \in \mathbb{R}$. Notice that $c$ must be the coefficient of $(-z)^q$ in the expression $Q(z) = \sum_{j=0}^{q} \frac{(m-j)!q!}{m!j!(q-j)!}(-z)^j$. Thus $c = \frac{(m-q)!}{m!} = \frac{p!}{m!}$.

Finally, to see that (2.5) holds, let $F(\cdot)$ be a polynomial of degree $m$. Then integration by parts allows one to write

$$\tilde{Q}(z)e^z - \tilde{P}(z) = z^{m+1} \int_0^1 F(s)e^{z(1-s)}ds$$

where

$$\tilde{P}(z) = F(1)z^m + F'(1)z^{m-1} + \ldots + F^{(m)}(1) = \sum_{j=0}^{m} F^{(j)}(1)z^{m-j}$$

$$\tilde{Q}(z) = F(0)z^m + F'(0)z^{m-1} + \ldots + F^{(m)}(0) = \sum_{j=0}^{m} F^{(j)}(0)z^{m-j}.$$
Then consider

$$F(s) = cs^p(1-s)^q = c(-1)^qs^{p+q} + \sum_{i=0}^{p+q-1} a_is^{p+q-i}$$

for some constant $c \neq 0$. Since $m - p - 1 = q - 1$ and $m - q - 1 = p - 1$, it follows that

$$F(1) = F'(1) = \ldots = F^{(m-p-1)}(1) = 0, \quad F^{m-p}(1) \neq 0, \quad \text{and}$$

$$F(0) = F'(0) = \ldots = F^{(m-q-1)}(0) = 0, \quad F^{m-q}(0) \neq 0.$$  

Thus $\tilde{P}(z) = \sum_{j=m-p}^{m} F(j)(1)z^{m-j}$ and $\tilde{Q}(z) = \sum_{j=m-p}^{m} F(j)(0)z^{m-j}$ and this implies that $\tilde{P}$ has degree $p$ and $\tilde{Q}$ has degree $q$. Now to ensure that $\tilde{Q}(0) = 1 = F^{(m)}(0) = (-1)^q(p + q)!c$, choose $c = (-1)^q(1)\frac{1}{(p+q)!}$. Hence,

$$\tilde{Q}(z)e^z - \tilde{P}(z) = \frac{(-1)^q}{(p + q)!} z^{p+q+1} \int_0^1 s^{p}(1-s)^q e^{z(1-s)}ds,$$

and therefore, $r = \frac{\tilde{P}}{\tilde{Q}}$ is a rational approximation of the exponential of order $p + q$. Thus, by the first part of the theorem, $\tilde{P} = P$ and $\tilde{Q} = Q$ where $P$ and $Q$ are as previously defined.

Let $r = \frac{P}{Q}$ be an $A$-stable, subdiagonal Padé approximation. In this case, if $q$ is even then all of the roots of $Q(z)$ are pairwise complex conjugates, and if $q$ is odd then the largest of roots is real the other roots will all be pairwise complex conjugates.

**Proposition 2.1.4.** Let $r = \frac{P}{Q}$ be a subdiagonal Padé approximation with $Q(0) = 1$ and let $\lambda_i \ (1 \leq i \leq q)$ be the zeros of $Q(z)$. Then, for $1 \leq q \leq 50$,

(a) $Re(\lambda_i) > 0$,

(b) $\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_q = |\lambda_1| \cdot |\lambda_2| \cdot \ldots \cdot |\lambda_q| = \frac{m!}{p!}$,

(c) $(Re(\lambda_1) \cdot \ldots \cdot Re(\lambda_q))^\frac{1}{q} \leq \frac{Re(\lambda_1) + \ldots + Re(\lambda_q)}{q} = p + 1$, and

72
(d) \(1 \leq \Re(\lambda_1) \cdot \Re(\lambda_2) \cdot \ldots \cdot \Re(\lambda_q).\)

(e) \(\left| \frac{1}{Q(is)} \right| \leq 1 \text{ for all } s \in \mathbb{R}.

(f) \(\left| \frac{Q'(is)}{Q(is)} \right| \leq 1 \text{ for all } s \in \mathbb{R}.

(g) \(\frac{p!}{m!} \left| \frac{s^q}{Q(is)} \right| \leq 1 \text{ for all } s \in \mathbb{R}.

(h) |r'(is)| \leq 1 \text{ for all } s \in \mathbb{R}.

Furthermore, if \(q \leq 15\) and \(q - 2 \leq p \leq q\), then \(p \leq (\Re(\lambda_1) \cdot \ldots \cdot \Re(\lambda_q))^\frac{1}{q}\), and if \(16 \leq q \leq 28\) and \(q - 2 \leq p \leq q\), then \(p - 1 \leq (\Re(\lambda_1) \cdot \ldots \cdot \Re(\lambda_q))^\frac{1}{q}\).

Proof. To see (a), note that because \(r\) is \(A\)-stable, then \(|r(z)| \leq 1\) for \(\Re(z) \leq 0\). This means that if \(\lambda_i\) is a zero of \(Q(z)\) and thus a pole of \(r\), then \(\Re(\lambda_i) > 0\). For (b), recall from Theorem 2.1.3 that

\[ Q(z) = c\lambda_1\lambda_2 \cdot \ldots \cdot \lambda_q + \ldots + (-1)^{q-1}c(\lambda_1 + \lambda_2 + \ldots + \lambda_q)z^{q-1} + (-1)^q cz^q \quad (2.6) \]

where \(c = \frac{p!}{m!}\). Because \(Q(0) = 1\), then \(\frac{m!}{p!} = \lambda_1\lambda_2 \cdot \ldots \cdot \lambda_q\). Furthermore, since \(\lambda_i\) exist in complex pairs and \(\lambda_i \cdot \overline{\lambda}_i = |\lambda_i|^2\), it follows that

\[ \frac{m!}{p!} = \lambda_1\lambda_2 \cdot \ldots \cdot \lambda_q = |\lambda_1| \cdot |\lambda_2| \cdot \ldots \cdot |\lambda_q|. \]

To verify (c), recall again (2.6) from Theorem 2.1.3. Since the \(\lambda_i\) exist in complex pairs and \(Q(z) = \sum_{j=0}^{q} \frac{(m-j)!q!}{m!(q-j)!} (-z)^j\), then

\[ \lambda_1 + \lambda_2 + \ldots + \lambda_q = \Re(\lambda_1) + \Re(\lambda_2) + \ldots + \Re(\lambda_q) = \frac{(m-q+1)!q!}{(m!)(q-1)!} \cdot \frac{m!}{p!} = (p+1)q. \]

Because the geometric mean \(\sqrt[n]{a_1 \cdot \ldots \cdot a_n}\) with \(a_i \geq 0\) is always less than or equal to the arithmetic means \(\frac{a_1 + \ldots + a_n}{n}\), one has that

\[ (\Re(\lambda_1) \cdot \ldots \cdot \Re(\lambda_q))^\frac{1}{q} \leq \frac{\Re(\lambda_1) + \ldots + \Re(\lambda_q)}{q} = p + 1. \]
The proof for statement (d) follows from Theorem 2.4 of [34]. Statements (e), (f), (g), and (h) for \(1 \leq q \leq 50\) can be shown with Mathematica.

Figures 2.1 and 2.2 help to verify the claim that \(\left| \frac{1}{Q(is)} \right| \leq 1\) for all \(s \in \mathbb{R}\).

\[
\begin{align*}
&\text{FIGURE 2.1.} \quad \left| \frac{1}{Q(is)} \right| \text{ for } -50 < s < 50.
\
&\text{(a) } q = 10, \ m = 19 \\
&\text{(b) } q = 20, \ m = 39
\end{align*}
\]

\[
\begin{align*}
&\text{FIGURE 2.2.} \quad \left| \frac{1}{Q(is)} \right| \text{ for } -50 < s < 50.
\
&\text{(a) } q = 35, \ m = 69 \\
&\text{(b) } q = 50, \ m = 99
\end{align*}
\]

To add validation to the claim that \(\left| \frac{Q'(is)}{Q(is)} \right| \leq 1\) for all \(s \in \mathbb{R}\), see Figures 2.3 and 2.4 below.
For evidence that \( \frac{p^q}{m^q} \left| \frac{s^p}{Q_q(s)} \right| \leq 1 \) for all \( s \in \mathbb{R} \), see Figures 2.5 and 2.6 below.

\[ q = 10, m = 19 \]  
\[ q = 20, m = 39 \]

FIGURE 2.5. \( \frac{p^q}{m^q} \left| \frac{s^p}{Q_q(s)} \right| \) for \(-100 < s < 100\).
For evidence that $|r'(is)| \leq 1$ for all $s \in \mathbb{R}$, see Figures 2.7 and 2.8 below.

Finally, the statements that for $q \leq 15$ and $q - 2 \leq p \leq q$, $p \leq (Re(\lambda_1) \ldots Re(\lambda_q))^{\frac{1}{q}}$
and for $16 \leq q \leq 28$ and $q - 2 \leq p \leq q$, $p - 1 \leq (Re(\lambda_1) \ldots \cdot Re(\lambda_q))^{\frac{1}{q}}$ are found
in Theorem 2.5 of [35] and can also be checked with Mathematica.
2.2 Hille-Phillips Functional Calculus

Recall that $r$ is an $\mathcal{A}$-stable approximation of the exponential of order $m$ and $\text{Re}(a) \leq 0$, then from (2.4)

$$\left| r\left(\frac{tn}{n}\right)^n - e^{ta} \right| \leq C \frac{1}{n^m} t^{m+1} |a^{m+1}|$$

for all $t \geq 0$ and $n \in \mathbb{N}$ sufficiently large. The following representation of $\mathcal{A}$-stable, rational approximations of the exponential is needed to be able to replace the number $a$ in the above inequality by an operator $A$ which generates a bounded, strongly continuous semigroup.

If $r = \frac{P}{Q}$ is a subdiagonal Padé approximation of the exponential, then all the distinct poles $\lambda_i$ of $r$ lie in the right half-plane. This means that by using partial fractions

$$r(z) = \frac{P(z)}{Q(z)} = \frac{b_1}{(\lambda_1 - z)} + \frac{b_2}{(\lambda_2 - z)} + \ldots + \frac{b_q}{(\lambda_q - z)} \quad (2.7)$$

for $\text{Re}(z) \leq 0$ where the $\lambda_i$ are the roots of $Q(z)$ and

$$b_i := \frac{m!P(\lambda_i)}{p! \prod_{j=1}^{q}_{j \neq i} (\lambda_j - \lambda_i)}.$$ 

For $t > 0$, let $H_t$ be the normalized Heaviside function defined on $[0, \infty)$ with

$$H_t(s) = \begin{cases} 
1 & : s > t \\
\frac{1}{2} & : s = t \\
0 & : 0 \leq s \leq t
\end{cases}$$

and let $H_0(s) = 0$ for $s = 0$ and $H_0(s) = 1$ for $s > 0$. The Laplace Stieltjes transform of the normalized Heaviside functions are

$$1 = \int_0^\infty e^{zs} dH_0(s)$$

$$e^{zt} = \int_0^\infty e^{zs} dH_t(s)$$
for all $z$ with $\text{Re}(z) \leq 0$ and all $t \geq 0$. Then, if $\text{Re}(z) \leq 0$ and $\text{Re}(\lambda) > 0$, it follows that

$$
\frac{1}{\lambda - z} = \int_0^\infty e^{zs} e^{-\lambda s} \, ds = \int_0^\infty e^{zs} \, d\alpha_\lambda(s)
$$

where $\alpha_\lambda(s) := \frac{1}{\lambda} (1 - e^{-\lambda s}) = \int_0^s e^{-\lambda r} \, dr$ is a normalized function of bounded variation in $NBV^0[0, \infty)$. (See Chapter I of [38] or Section 1.9–1.10 of [2] for details on the Laplace-Stieltjes integral.) The space $NBV^0[0, \infty)$ of functions of bounded variation normalized at zero is a Banach algebra with multiplication defined by the Stieltjes convolution

$$(\alpha * \beta)(t) := \int_0^t \alpha(t - s) \, d\beta(s)$$

(see [21] page 8). The Laplace-Stieltjes transformation maps convolution onto multiplication which implies that

$$
\frac{1}{(\lambda - z)^j} = \int_0^\infty e^{zs} \, d\alpha_\lambda^j(s)
$$

where $\alpha_\lambda^j(s) := (\alpha_\lambda * \cdots * \alpha_\lambda)(s)$ is again a function in $NBV^0[0, \infty)$.

If $r$ is a subdiagonal Padé approximation of the exponential, then there exists an $\alpha \in NBV^0[0, \infty)$ such that $r(z) = \int_0^\infty e^{zs} \, d\alpha(s)$, where $\alpha(s) = \sum_{j=1}^q \alpha_{\lambda_j}(s)$. Then for $\text{Re}(z) \leq 0$, $r$ has the following representations:

$$
r(z) = \int_0^\infty e^{zs} \, d\alpha(s) \text{ and } r^n(z) = \int_0^\infty e^{zs} \, d\alpha_n^s(s).
$$

With substitution, it now follows that $r\left(\frac{t_z}{n}\right)^n = \int_0^\infty e^{zs} \, d\alpha_n(s)$, where $\alpha_n(s) := \alpha_{n}(\frac{n s}{t})$ is in $NBV^0[0, \infty$). This means that $\alpha_n - H_t$ is also in $NBV^0[0, \infty)$ and

$$
r\left(\frac{t_z}{n}\right)^n - e^{t_z} = \int_0^\infty e^{zs} \, d[\alpha_n - H_t](s). \tag{2.8}
$$

With the definition

$$
\mathcal{G}_0 := \{ f | f(z) = \int_0^\infty e^{zs} \, d\alpha(s) \text{ if } \text{Re}(z) \leq 0 \text{ for some } \alpha \in NBV^0[0, \infty) \},
$$

78
then one can verify that $\mathcal{G}_0$ is a Banach algebra with norm $\|f\|_0 := \|\alpha\|_{var} = V_\alpha(\infty)$. Furthermore, the following result of Einar Hille and Ralph Phillips (see [21]) provides the framework and justification to replace the number $z$ with $\text{Re}(z) \leq 0$ by a suitable operator $A$ in (2.8).

**Theorem 2.2.1. (Hille-Phillips Functional Calculus)** Let $(A, D(A))$ generate a strongly continuous semigroup $T$ of type $(M, 0)$, i.e. there exists a constant $M \geq 1$ such that $\|T(t)\| \leq M$. Consider $\Psi : \mathcal{G}_0 \rightarrow \mathcal{L}(X)$ defined by $\Psi(f) := f(A)$, where

$$f(A)x := \int_0^\infty T(s)x\,d\alpha(s)$$

for $x \in X$ and where $f(z) = \int_0^\infty e^{zs}d\alpha(s)$ for $\text{Re}(z) \leq 0$ and $\alpha \in NBV^0[0, \infty)$. Then $\Psi$ is an algebra homomorphism and $\|\Psi(f)\| = \|f(A)\| \leq M\|\alpha\|_{var}$. Moreover the statement of the theorem remains valid if $(A, D(A))$ generates a bi-continuous semigroup of type $(M, 0)$ (see [18]).

If $A$ is the generator of a bounded, strongly continuous semigroup and $r$ is an $\mathcal{A}$-stable rational approximation of the exponential, then the Hille-Phillips functional calculus along with (2.8) enables one to assert that

$$r\left(\frac{t}{n}A\right)^nx - T(t)x = \int_0^\infty T(s)x\,d[\alpha_n - H_t](s).$$

This representation is crucial to the proof of the Brenner-Thomée Theorem, which, through careful analysis of the functions $\alpha_n - H_t$, allow for estimates on

$$\left\|r\left(\frac{t}{n}A\right)^nx - T(t)x\right\|$$

and thus the effectiveness of the rational inversion procedures.
2.3 Approximations of Semigroups without Scaling and Squaring, Part I

Let \( r = \frac{P}{Q} \) be a subdiagonal Padé approximation of the exponential of order \( m = p + q \). In this case, through partial fraction decomposition, \( r \) must have the representation

\[
r(z) = \frac{P(z)}{Q(z)} = \frac{b_1}{(\lambda_1 - z)} + \frac{b_2}{(\lambda_2 - z)} + \ldots + \frac{b_q}{(\lambda_q - z)}
\]

where the \( \lambda_i \) are the roots of \( Q(z) \) and

\[
b_i := \frac{m!P(\lambda_i)}{p! \prod_{j=1}^{q} (\lambda_j - \lambda_i)}.
\]

Now, let \( A \) is the generator of a strongly continuous semigroup \( T(t) \) of of type \((M,0)\), i.e. there exists a constant \( M \geq 1 \) such that \( \|T(t)\| \leq Me^{\omega t} \). Using this representation for \( r \) and the Hille-Phillips functional calculus, then

\[
r(tA) = \frac{b_1}{t} R\left(\frac{\lambda_1}{t}, A\right) + \frac{b_2}{t} R\left(\frac{\lambda_2}{t}, A\right) + \ldots + \frac{b_q}{t} R\left(\frac{\lambda_q}{t}, A\right). \tag{2.9}
\]

Recall from Theorem 2.1.3 that \( Q(z) = c(\lambda_1 - z) \cdots (\lambda_q - z) \) where \( c = \frac{p!}{m!} \). Then, by using the Hille-Phillips functional calculus again and the fact that \( (\lambda_i - tA)^{-1} = \frac{1}{t} R(\frac{\lambda_i}{t}, A) \), one obtains

\[
\frac{1}{Q(tA)} = \frac{m!}{p! \cdot t^q} R\left(\frac{\lambda_1}{t}, A\right) \cdots R\left(\frac{\lambda_q}{t}, A\right).
\]

Therefore, by (2.5), \( r(tA)x - T(t)x = \)

\[
A^{m+1} R\left(\frac{\lambda_1}{t}, A\right) \cdots R\left(\frac{\lambda_q}{t}, A\right) \frac{(-1)^{q+1}}{p! t^{p+1}} \int_0^1 s^p (1-s)^q T((1-s)t)x \, ds \tag{2.10}
\]

for all \( x \in X \). The following lemma is needed.

Lemma 2.3.1. Let \( A \) be the generator of a strongly continuous (or bi-continuous) semigroup \( T(t) \) with \( \|T(t)\| \leq M \) \((t \geq 0)\). Then

\[
\|R(\lambda_1, A) \cdots R(\lambda_q, A)T(t)\| \leq \frac{M}{Re(\lambda_1) \cdots Re(\lambda_q)}
\]
for all \( t \geq 0 \) and all \( \lambda_i \in \mathbb{C} \) with \( \text{Re}(\lambda_i) > 0 \) (1 \( \leq i \leq q \)).

**Proof.** The statement follows from

\[
\|R(\lambda_1, A)R(\lambda_2, A) \cdots R(\lambda_q, A)T(t)x\|
= \|\int_0^\infty e^{-\lambda_1 s_1}T(s_1) \int_0^\infty e^{-\lambda_2 s_2}T(s_2) \int_0^\infty \cdots \int_0^\infty e^{-\lambda_q s_q}T(s_q)T(t)x ds_q \cdots ds_1\|
= \|\int_0^\infty e^{-\lambda_1 s_1} \int_0^\infty e^{-\lambda_2 s_2} \int_0^\infty \cdots \int_0^\infty e^{-\lambda_q s_q}T(s_1 + s_2 + \cdots + s_q + t)x ds_q \cdots ds_1\|
\leq \int_0^\infty e^{-\text{Re}(\lambda_1)} ds_1 \cdot \int_0^\infty e^{-\text{Re}(\lambda_2)} ds_2 \cdots \int_0^\infty e^{-\text{Re}(\lambda_q)} ds_q M \|x\|
= \frac{M}{\text{Re}(\lambda_1) \cdot \text{Re}(\lambda_2) \cdots \text{Re}(\lambda_q)} \|x\|.
\]

\( \square \)

Lemma 2.3.1 and (2.10) imply that

\[
\|r(tA)x - T(t)x\| \leq \frac{M}{\text{Re}(\lambda_1) \cdots \text{Re}(\lambda_q) p!} \frac{1}{t^{m+1}} \int_0^1 s^p (1 - s)^q ds \|A^{m+1}x\| \tag{2.11}
\]

for all \( x \in D(A^{m+1}) \) and \( t > 0 \). For \( p, q > -1 \), the Beta function has the following representation (see [1] Section 4.21):

\[
B(p + 1, q + 1) = \int_0^1 s^p (1 - s)^q ds = \frac{\Gamma(p + 1)\Gamma(q + 1)}{\Gamma(p + q + 2)} = \frac{p! \cdot q!}{(p + q + 1)!}.
\]

Recall from Section 2.1 that if \( r \) is a subdiagonal Padé approximation of the exponential of order \( m = 2q - 1 \) and with singularities \( \lambda_1, \ldots, \lambda_q \), then \( 1 \leq \text{Re}(\lambda_1) \cdot \text{Re}(\lambda_2) \cdots \text{Re}(\lambda_q) \). In addition, for subdiagonal Padé approximation of the exponential one also has that \( p^q \leq \text{Re}(\lambda_1) \cdots \text{Re}(\lambda_q) \) for \( 1 \leq p \leq 15 \) and \( (p - 1)^q \leq \text{Re}(\lambda_1) \cdots \text{Re}(\lambda_q) \) for \( 16 \leq p \leq 28 \). These observations and facts can be summerized as follows.

Let \( r \) be a subdiagonal rational Padé approximation of the exponential of order \( m = p + q \) and with singularities \( \lambda_1, \ldots, \lambda_q \). If \( A \) is the generator of a strongly continuous (or bi-continuous) semigroup \( T(t) \) of type \((M, 0)\) (i.e. \( \|T(t)\| \leq M \),
then
\[
\left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) x + \cdots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) x - T(t)x \right\| \leq \frac{M}{\text{Re}\lambda_1 \cdots \text{Re}\lambda_q (m+1)!} t^{m+1} \| A^{m+1}x \| \tag{2.12}
\]
for all \( x \in D(A^{m+1}) = D(A^{2q}) \) and \( t > 0 \). In particular, since \( \text{Re}(\lambda_1) \cdots \text{Re}(\lambda_q) \geq 1 \),
\[
\left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) x + \cdots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) x - T(t)x \right\| \leq M q! t^{2q} \| A^{2q}x \|. \tag{2.13}
\]
By Proposition 2.1.4, one has that
\[
\left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) x + \cdots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) x - T(t)x \right\| \leq M \frac{q!}{p^q(2q)!} t^{2q} \| A^{2q}x \| \tag{2.14}
\]
for all \( x \in D(A^{2q}) \) and \( t > 0 \) whenever \( 1 \leq p \leq 15 \) and
\[
\left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) x + \cdots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) x - T(t)x \right\| \leq M \frac{q!}{(p-1)^q(2q)!} t^{2q} \| A^{2q}x \| \tag{2.15}
\]
for all \( x \in D(A^{2q}) \) and \( t > 0 \) whenever \( 16 \leq p \leq 28 \).

**Theorem 2.3.2.** If \( A \in \mathcal{L}(X) \), and if
\[
r(z) = \frac{P(z)}{Q(z)} = \frac{b_1}{\lambda_1 - z} + \cdots + \frac{b_q}{\lambda_q - z}
\]
is a rational subdiagonal \( \text{Padé} \) approximation of the exponential where the \( \lambda_i \) are the roots of \( Q(z) \) and
\[
b_i := \frac{m! P(\lambda_i)}{p! \prod_{j \neq i} (\lambda_j - \lambda_i)},
\]
then
\[
\frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) + \cdots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) \to e^{tA}
\]
uniformly on compacts as \( q \to \infty \).
Proof. From 2.13, since $A \in \mathcal{L}(X)$, it follows that
\[
\left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) x + \cdots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) x - T(t)x \right\| \leq M \frac{q!}{(2q)!} t^{2q} \|A\|^{2q} \|x\|.
\]
Now the statement follows from the fact that
\[
M \frac{(q+1)!}{(2(q+1))!} t^{2(q+1)} \|A\|^{2(q+1)} \|x\| = \frac{1}{2(2q+1)} t^2 \|A\|^2 \to 0
\]
as $q \to \infty$. 

The following investigation will try to address some open problems concerning these “approximation without scaling and squaring” that are crucial for their applicability to the asymptotic setting. A major limitation of (2.13) is the fact that the convergence estimate holds only on $D(A^{2q})$. It will become evident in Section 1.4 that for applications to generalized functions and asymptotic Laplace transforms, it is crucial to obtain an estimate that holds for $x \in D(A)$ (and not on $D(A^{2q})$) for all $q \geq 1$. To obtain such estimates, one must look more closely at M. Kovács’ proof of the following key result of semigroup theory based of work of Hersh and Kato [17] and, in final form, of Brenner and Thomée [9].

### 2.4 The Constant in the Brenner and Thomée Theorem
The following fundamental result concerns the approximation of semigroups in terms of their resolvent.

**Theorem 2.4.1.** (Hersh-Kato, Brenner-Thomée) Let $r$ be an $A$-stable rational approximation scheme of the exponential of order $m$ and $(A, D(A))$ be the generator of a strongly continuous semigroup $T$ of type $(M, 0)$. Then
\[
\left\| r \left( \frac{t}{n} A \right)^n x - T(t)x \right\| \leq M C t^{m+1} \frac{1}{n^m} \|A^{m+1}x\|
\]
for all \( n \in \mathbb{N}, \ t \geq 0, \) and \( x \in D(A^{m+1}). \)

Originally, a slightly less sharp version of the result (assuming \( x \in D(A^{m+2}) \)) was shown in 1979 by Reuben Hersh and Tosio Kato (see [17]). The above version was shown in the same year by Philip Brenner and Vidar Thomée (see [9]). The result was extend to fractional powers of \( A \) by Mihály Kovács in his LSU dissertation [21]. One of the main contributions of this dissertation is to specify the size of the constant \( C. \)

**Theorem 2.4.2.** Let \( r = \frac{P}{Q}, \) where \( P \) and \( Q \) are polynomials of orders \( p \) and \( q \) respectively, be a subdiagonal Padé approximation scheme of the exponential of order \( m = 2q - 1 \) and \( (A, D(A)) \) be the generator of a strongly continuous semigroup \( T \) of type \( (M,0) \). Then

\[
\left\| r \left( \frac{t}{n} A \right)^n x - T(t)x \right\| \leq \frac{M\sqrt{2\pi}}{m!} \frac{1}{n^{m-\frac{1}{2}}} \left( \frac{(m-1)!(m+1)!}{(2m+1)!} \right)^{\frac{1}{2}} t^{m+1} \| A^{m+1}x \|
\]

for all \( n \in \mathbb{N}, \ t \geq 0, \) and \( x \in D(A^{m+1}) \) with \( 1 \leq q \leq 50. \)

**Proof.** The first part of the proof follows exactly the proof given by Kovács [21] (for the bi-continuous case see Jara [18]) up to shortly after (2.22). The new part of the proof starts with the Perron representation of rational Padé approximation in (2.23). Recall from Section 2.2 that with the Hille-Phillips functional calculus, it follows that

\[
r \left( \frac{t}{n} A \right)^n x - T(t)x = \int_0^\infty T(s)x \, d[\alpha_n - H_t](s)
\]

where \( r(z) = \int_0^\infty e^{zs} \, d\alpha(s) \) and \( r^n(z) = \int_0^\infty e^{zs} \, d\alpha^n(s) \) \((Re(z) \leq 0), \alpha_\lambda(s) := \frac{1}{\lambda}(1 - e^{-\lambda s}) = \int_0^s e^{-\lambda r} \, dr, \alpha_n(s) := \alpha^n(\frac{ns}{1}), \) and \( H_t \) is the normalized Heaviside function defined on \([0, \infty)\). Since \( \alpha_n(0) - H_t(0) = \alpha_n(\infty) - H_t(\infty) = 0, \) integration by parts yields that

\[
r \left( \frac{t}{n} A \right)^n x - T(t)x = -\int_0^\infty [\alpha_n - H_t](s) \, dT(s)x
\]
for all $x \in X$. If $x \in D(A)$ then $s \to T(s)x$ is continuously differentiable with $\frac{d}{ds}T(s)x = T(s)Ax$. Therefore, for $x \in D(A)$

$$r \left( \frac{t}{n}A \right)^n x - T(t)x = - \int_0^\infty [\alpha_n - H_t](s) T(s)Ax \, ds.$$ 

For $1 \leq k \leq m$ let $I_k[\alpha_n - H_t](s)$ denote the $k$-th antiderivative of $\alpha_n - H_t$: i.e.

$$I_k[\alpha_n - H_t](s) := \int_0^s \int_0^{s_1} \cdots \int_0^{s_k} [\alpha_n - H_t](s_1) \, ds_1 \, ds_2 \cdots ds_k$$

$$= \int_0^s \frac{(s-r)^{k-1}}{(k-1)!} [\alpha_n - H_t](r) \, dr.$$ 

It can be shown that

$$I_k[\alpha_n - H_t](0) = I_k[\alpha_n - H_t](\infty) = 0$$

for $1 \leq k \leq m$ (see [21] or Lemma III.6, [32]). Thus for $1 \leq k \leq m$ and $x \in D(A^{k+1})$, $k$ consecutive integrations by parts yields

$$r \left( \frac{t}{n}A \right)^n x - T(t)x = (-1)^{k+1} \int_0^\infty I_k[\alpha_n - H_t](s) T(s)A^{k+1}x \, ds. \quad (2.16)$$

As a consequence of (2.16), one obtains the following estimate:

$$\left\| r \left( \frac{t}{n}A \right)^n x - T(t)x \right\| = M \| A^{k+1}x \| \| I_k[\alpha_n - H_t] \|_{L^1(\mathbb{R}^+)} \quad (2.17)$$

for $t \geq 0$, $n \in \mathbb{N}$, and $x \in D(A^{k+1})$.

In order to estimate $\| I_k[\alpha_n - H_t] \|_{L^1(\mathbb{R}^+)}$, it is necessary to use the following Fourier representation of $I_k[\alpha_n - H_t]$ (for details see [21], [22], or Lemma III.6, [27]):

$$I_k[\alpha_n - H_t](s) = \left( \frac{-1}{i} \right)^{k+1} \frac{1}{\sqrt{2\pi}} \mathcal{F} \left[ r^n \left( \frac{l_n(\cdot)}{\cdot^{k+1}} \right) - e^{it\cdot} \right](s) \quad (2.18)$$

for all $n \in \mathbb{N}$ and $s \in (0, \infty)$ where

$$\mathcal{F}[f](s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isv} f(v) \, dv.$$
Following from (2.8), the representation

\[ r\left(-\frac{t}{n}z\right)^n e^{-tz} = \int_0^\infty e^{-zs} d[\alpha_n - H_t](s) \]  

exists for \( \Re(z) \geq 0 \) where \( \alpha_n - H_t \in NBV^0[0, \infty) \). Recall the complex inversion formula (see [38], Chapter II, Theorem 7.6a) states that if \( \alpha \in NBV^0[0, \infty) \) and \( f(z) = \int_0^\infty e^{-zs} d\alpha(s) \) converges for all \( \Re(z) > \sigma_c \), then for \( c > \max[\sigma_c, 0] \),

\[
\frac{1}{2\pi i} \lim_{R \to \infty} \int_{c-iR}^{c+iR} \frac{f(z)}{z} e^{zs} dz = \begin{cases} 
\alpha(s) & : n > 0 \\
\frac{\alpha(0^+)}{2} & : s = 0 \\
0 & : s < 0
\end{cases}.
\]

Then from (2.19) the above complex inversion formula yields

\[
\alpha_n(s) - H_t(s) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{c-iR}^{c+iR} \frac{r\left(-\frac{t}{n}z\right)^n e^{-tz}}{z} e^{zs} dz.
\]

Recall from Section 2.1 that a subdiagonal Padé approximation is always \( \mathcal{A} \)-stable. Using the \( \mathcal{A} \)-stability of \( r \) as well as Cauchy’s Theorem, it follows that

\[
\alpha_n(s) - H_t(s) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{c-iR}^{c+iR} \frac{r\left(-\frac{t}{n}z\right)^n e^{-tz}}{z} e^{zs} dz
= i \frac{1}{\sqrt{2\pi}} \mathcal{F} \left[ \frac{r(i\frac{t}{n}z)^n e^{it}}{z} \right](s),
\]

(see [21] or [27] Lemma III.6 for details). With an induction argument, one can show that (2.20) implies (2.18). The below estimate now follows

\[
\| I_m[\alpha_n - H_t] \|_{L^1(\mathbb{R}^+)} = \frac{1}{\sqrt{2\pi}} \| \mathcal{F} \left[ \frac{r(i\frac{t}{n}z)^n e^{it}}{n+1} \right] \|_{L^1(\mathbb{R}^+)} \leq \frac{1}{\sqrt{2\pi}} \| \mathcal{F} \left[ \frac{e^{-in\pi t}}{n+1} r(i\frac{t}{n}z)^n e^{it}} \right] \|_{L^1(\mathbb{R})},
\]

and through two simple substitutions (see [21], [22], or [27] for details), one obtains

\[
\| I_m[\alpha_n - H_t] \|_{L^1(\mathbb{R}^+)} \leq \frac{1}{\sqrt{2\pi}} \frac{n}{n+1} \| \mathcal{F} \left[ \frac{e^{-in\pi t}}{n+1} r(i\frac{t}{n}z)^n e^{it}} \right] \|_{L^1(\mathbb{R})}.
\]
Now define
\[
h_n(s) := \left( e^{-in\frac{1}{m+1}} - 1 \right),
\]
for \( s \in \mathbb{R} \) which means that
\[
\| I_m[\alpha_n - H_t] \|_{L^1(\mathbb{R}^+)} \leq \frac{1}{\sqrt{2\pi}} \frac{1}{n^m} \frac{t^{m+1}}{m+1} \| \mathcal{F}[h_n] \|_{L^1(\mathbb{R})}.
\] (2.21)

For all functions \( f \in L^2(\mathbb{R}) \) and \( g : s \to sf(s) \in L^2(\mathbb{R}) \), Carlson’s inequality states that \( f \in L^1(\mathbb{R}) \) and
\[
\| f \|_{L^1(\mathbb{R})} \leq \sqrt{\pi} \| f \|_{L^2(\mathbb{R})} \| g \|_{L^2(\mathbb{R})}^{\frac{1}{2}}.
\]

Now recall the following well-known property of the Fourier transform. If \( g \in L^2(\mathbb{R}) \) is absolutely continuous with \( g' \in L^2 \), then
\[
is \mathcal{F}[g](s) = \mathcal{F}[g'](s)
\]
almost everywhere. If \( f = \mathcal{F}[g] \in L^2(\mathbb{R}) \) for some absolutely continuous \( g \in L^2(\mathbb{R}) \) with \( g' \in L^2 \), Carlson’s inequality implies that
\[
\| F(g) \|_1 \leq \sqrt{\pi} \| \mathcal{F}[g] \|_2 \| \mathcal{F}[g'] \|_2^{\frac{1}{2}}.
\]

Furthermore, Parseval’s identity, \( \| \mathcal{F}[g] \|_2 = \| g \|_2 \) for \( g \in L^2(\mathbb{R}) \), shows that
\[
\| \mathcal{F}[g] \|_1 \leq \sqrt{\pi} \| g \|_2 \| g' \|_2^{\frac{1}{2}}.
\]

Using these properties and inequalities, it follows from (2.21) that
\[
\| I_m[\alpha_n - H_t] \|_{L^1(\mathbb{R}^+)} \leq \frac{1}{\sqrt{2}} \frac{1}{n^m} \frac{t^{m+1}}{m+1} \| h_n \|_{L^2(\mathbb{R})} \| h'_n \|_{L^2(\mathbb{R})}.
\]

Now, returning to (2.17) it now follows that
\[
\| r \left( \frac{t}{n} A \right)^n x - T(t)x \| \leq M \| A^{n+1}x \| \frac{1}{\sqrt{2}} \frac{1}{n^m} \frac{t^{m+1}}{m+1} \| h_n \|_{L^2(\mathbb{R})} \| h'_n \|_{L^2(\mathbb{R})} \] (2.22)
where
\[
h_n(s) = \left( e^{-in\frac{1}{m+1}} - 1 \right) = \frac{[e^{-in\frac{1}{m+1}} r(in\frac{1}{m+1})]^n - 1}{s^{m+1}} = \frac{[e^{-in\frac{1}{m+1}} r(in\frac{1}{m+1})]^n - 1}{(in\frac{1}{m+1})^{m+1}} \frac{1}{n}.
\]

87
By letting \( u(s) = n^{-\frac{1}{m+1}} s \), one has that
\[
h_n(s) = \left[ \frac{e^{-iu(s)} r(iu(s))^{n}}{[iu(s)]^{m+1}} \right]^{n-1} \frac{1}{m+1} \frac{1}{n} = \frac{e^{iu(s)} r(iu(s))}{[iu(s)]^{m+1}} \frac{1}{n} \sum_{j=0}^{n-1} e^{-iju(s)} r(iu(s))^j.
\]

Recall the following representation of \( r \) from Theorem 2.1.3:
\[
r(z) - e^{z} = \frac{(-1)^{q+1}}{Q(z)} \frac{1}{m!} z^{m+1} e^{z} \int_{0}^{1} sp(1-s)^{q} e^{-sz} ds
\]
where \( z \in \mathbb{C} \) and \( z \neq \lambda_1, \ldots, \lambda_q \). Thus
\[
\frac{e^{-z} r(z) - 1}{z^{m+1}} = \frac{(-1)^{q+1}}{Q(z)} \frac{1}{m!} \int_{0}^{1} t^{p}(1-t)^{q} e^{-tz} dt,
\]
and therefore
\[
h_n(s) = (-1)^{q+1} i^{m+1} \frac{1}{m!} \frac{1}{Q(iu(s))} \int_{0}^{1} t^{p}(1-t)^{q} e^{-tiu(s)} dt \left[ \frac{1}{n} \sum_{j=0}^{n-1} e^{-iju(s)} r(iu(s))^j \right]
= g_n(s) \cdot f_n(s)
\]
(2.24)
with
\[
g_n(s) = (-1)^{q+1} i^{m+1} \frac{1}{m!} \frac{1}{Q(iu(s))} \int_{0}^{1} t^{p}(1-t)^{q} e^{-tiu(s)} dt
\]
\[
f_n(s) = \frac{1}{n} \sum_{j=0}^{n-1} e^{-iju(s)} r(iu(s))^j
\]
and \( u(s) = n^{-\frac{1}{m+1}} s \).

Since \( r \) is \( \mathcal{A} \)-stable, \( \sum_{j=0}^{n-1} e^{-iju(s)} r(iu(s))^j \) is bounded by \( n \) which implies that \( \|f_n(s)\| \leq 1 \) for all \( s \in \mathbb{R} \). In order to estimate \( g_n(s) \), recall from Proposition 2.1.4, that \( \left| \frac{1}{Q(is)} \right| \leq 1 \) for all \( s \in \mathbb{R} \) and \( 1 \leq q \leq 50 \). Thus
\[
|g_n(s)| \leq \frac{1}{m!} \left| \int_{0}^{1} t^{p}(1-t)^{q} e^{-tiu(s)} dt \right|.
\]
Now
\[
\int_{0}^{1} t^{p}(1-t)^{q} e^{-tiu(s)} dt = \int_{0}^{1} t^{p}(1-t)^{q} e^{-t \frac{1}{m+1} s} dt,
\]
(88)
and by the substitution $\tau = n^{\frac{1}{m+1}} t$, then

$$
\begin{align*}
\int_0^1 t^p(1-t)^q e^{-t i n^{\frac{1}{m+1}} s} \, dt &= n^{\frac{p+1}{m+1}} \int_0^{n^{\frac{-1}{m+1}}} \tau^p (1 - n^{\frac{1}{m+1}} \tau)^q e^{-i \tau s} \, d\tau \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi_{n,m}(\tau) e^{i \tau s} \, d\tau
\end{align*}
$$

with

$$
\Psi_{n,m}(\tau) := \sqrt{2\pi n^{\frac{p+1}{m+1}}} \left\{ \begin{array}{ll}
\tau^p (1 - n^{\frac{1}{m+1}} \tau)^q & : 0 \leq \tau < n^{\frac{-1}{m+1}} \\
0 & : \text{else}
\end{array} \right.
$$

Then

$$
\left\| \int_0^1 t^p(1-t)^q e^{-t i u(s)} \, dt \right\|_2 = \| \mathcal{F}(\Psi_{n,m}(t)) \|_2 = \| \Psi_{n,m} \|_2
$$

which means

$$
\left\| \int_0^1 t^p(1-t)^q e^{-t i u(s)} \, dt \right\|_2^2 = \| \mathcal{F}(\Psi_{n,m}(t)) \|_2^2 = \| \Psi_{n,m} \|_2^2.
$$

In order to estimate $\| \Psi_{n,m} \|_2^2$, the following representation of the Beta function from [1] Section 4.21 is needed. For $(p, q > -1)$ and $\Gamma(n+1) = n!$,

$$
B(p+1, q+1) = \int_0^1 x^p (1-x)^q \, dx = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}.
$$

Then

$$
\| \Psi_{n,m} \|_2^2 = \int_{-\infty}^{\infty} |\Psi_{n,m}(\tau)|^2 \, d\tau = 2\pi n^{\frac{2(p+1)}{m+1}} \int_0^{n^{\frac{-1}{m+1}}} \tau^{2p} (1 - n^{\frac{1}{m+1}} \tau)^{2q} \, d\tau.
$$

Then, with the substitution $t = n^{\frac{1}{m+1}} \tau$,

$$
2\pi n^{\frac{2(p+1)}{m+1}} \int_0^{n^{\frac{-1}{m+1}}} \tau^{2p} (1 - n^{\frac{1}{m+1}} \tau)^{2q} \, d\tau = 2\pi n^{\frac{-1}{m+1}} \int_0^1 \tau^{2p} (1-t)^{2q} \, dt = 2\pi n^{\frac{-1}{m+1}} \frac{(2p)! (2q)!}{(2p + 2q + 1)!}
$$

which implies

$$
\left\| \int_0^1 t^p(1-t)^q e^{-t i u(s)} \, dt \right\|_2^2 = 2\pi n^{\frac{1}{m+1}} \frac{(2p)! (2q)!}{(2p + 2q + 1)!}.
$$

(2.25)
Hence, \( \|h_n\|_{L^2(\mathbb{R})} \leq \int_{-\infty}^{\infty} |g_n(s)|^2 \, ds \)

\begin{align*}
\leq \frac{1}{m^2} \|f_0^1 t^p(1 - t)^q e^{-t \text{i}u(s)} \, dt\|^2 \\
= (\frac{1}{m^2})^2 \|f_0^1 t^p(1 - t)^q e^{-t \text{i}u(s)} \, dt\|^2 \\
= (\frac{1}{m^2})^2 [2\pi n \frac{1}{m+1} \frac{(2p)! (2q)!}{(2p + 2q + 1)!}].
\end{align*}

Hence,

\( \|h_n\|_{L^2(\mathbb{R})}^2 \leq \|g_n\|_{L^2(\mathbb{R})}^2 \leq \frac{1}{m^2} \left[ 2\pi n \frac{1}{m+1} \frac{(2p)! (2q)!}{(2p + 2q + 1)!} \right]^2 \) (2.26)

Working towards an estimate for \( \|h'_n\|_{L^2(\mathbb{R})} \), first recall that \( h_n(s) = g_n(s) \cdot f_n(s) \)
which implies \( h'_n(s) = g'_n(s) f_n(s) + f'_n(s) g_n(s) \). Since \( |a + b|^2 \leq 2|a|^2 + 2|b|^2 \), it follows that

\begin{align*}
\|h'_n\|_2^2 &= \int_{-\infty}^{\infty} |h'_n(s)|^2 \, ds \\
&\leq \int_{-\infty}^{\infty} |g'_n(s)|^2 \, ds + \int_{-\infty}^{\infty} |f'_n(s) g_n(s)|^2 \, ds \\
&= 2\|g'_n\|_2^2 + 2\|f'_n g_n\|_2^2.
\end{align*}

To estimate \( \|g'_n\|_2 \), notice that \( g'_n(s) = \frac{(-1)^{q+2} - 1}{m!} \frac{1}{Q'(i\text{u}(s))} \left[ Q'(i\text{u}(s)) \int_{0}^{1} t^p(1 - t)^q e^{-t \text{i}u(s)} \, dt + \int_{0}^{1} t^{p+1}(1 - t)^q e^{-t \text{i}u(s)} \, dt \right] \).

Recall from Proposition 2.1.4 that \( \left| \frac{Q'(i\text{u}(s))}{Q(i\text{u}(s))} \right| \leq 1 \) and \( \left| \frac{1}{Q(i\text{u}(s))} \right| \leq 1 \) for all \( s \in \mathbb{R} \) and \( 1 \leq q \leq 50 \). Using these facts, with \( c_1 := \left( \frac{n-1}{n+1} \right)^2 \), it follows from (2.25) that

\begin{align*}
\|g'_n\|_2^2 &\leq c_1 \int_{-\infty}^{\infty} \left| \int_{0}^{1} t^p(1 - t)^q e^{-t \text{i}u(s)} \, dt + \int_{0}^{1} t^{p+1}(1 - t)^q e^{-t \text{i}u(s)} \, dt \right|^2 \, ds \\
&\leq c_1 \int_{-\infty}^{\infty} \left[ \int_{0}^{1} t^p(1 - t)^q e^{-t \text{i}u(s)} \, dt \right]^2 + 2 \left| \int_{0}^{1} t^{p+1}(1 - t)^q e^{-t \text{i}u(s)} \, dt \right|^2 \, ds \\
&\leq c_1 \left[ 2(2\pi n) \frac{1}{m+1} \frac{(2p)! (2q)!}{(2p + 2q + 1)!} + 2(2\pi n) \frac{1}{m+1} \frac{(2p + 2)! (2q)!}{(2p + 2q + 3)!} \right].
\end{align*}

Then

\( \|g'_n\|_2^2 \leq \frac{4\pi}{(m!)^2} \frac{1}{n} \frac{(2p)! (2q)!}{(2p + 2q + 1)!} \left[ 1 + \frac{(2p + 2)(2p + 1)}{(2p + 2q + 3)(2p + 2q + 2)} \right] \).

To estimate \( \|f'_n g_n\|_2^2 \), recall that \( f_n(s) = \frac{1}{n} \sum_{j=0}^{n-1} e^{-j \text{i}u(s)} r(i\text{u}(s))^j \) which implies that

\( f'_n(s) = \frac{1}{n} \sum_{j=0}^{n-1} e^{-j \text{i}u(s)} (-j \text{u'}(s)) r(i\text{u}(s))^{j+1} + \frac{1}{n} \sum_{j=0}^{n-1} e^{-j \text{i}u(s)} j r(i\text{u}(s))^{j-1} \text{u'}(i\text{u}(s)) \text{u}(s) \).

90
By Proposition 2.1.4, \(|r'(iu(s))| \leq 1\) for all \(s \in \mathbb{R}\) and \(1 \leq q \leq 50\). Thus
\[
|f_n'(s)| \leq \frac{2}{n} |u'(s)| \sum_{j=0}^{n-1} \frac{j}{n^{\frac{1}{m+1}}} \left( n - 1 \right) n^{\frac{1}{2}} = (n - 1) n^{\frac{1}{m+1}}.
\]

It follows from (2.25) that
\[
\|f_n'g_n\|^2 \leq \left( (n - 1) n^{\frac{1}{m+1}} \right)^2 \|g_n\|^2 \leq \left( (n - 1) n^{\frac{1}{m+1}} \right)^2 \cdot \frac{1}{(m!)^2} \frac{1}{2\pi n^{\frac{1}{m+1}}} \frac{(2p)!(2q)!}{(2p + 2q + 1)!}.
\]

This means that
\[
\|h_n\|^2 \leq 2\|g_n\|^2 + 2\|f_n'g_n\|^2
\]
\[
\leq \frac{8\pi}{(m!)^2} \frac{1}{n^{\frac{1}{m+1}}} \frac{(2p)!(2q)!}{(2p + 2q + 1)!} \left[ 1 + \frac{(2p + 2)(2p + 1)}{(2p + 2q + 3)(2p + 2q + 2)} \right]
\]
\[
+ \frac{4\pi}{(m!)^2} \left[ (n - 1) n^{\frac{1}{m+1}} \right]^2 n^{\frac{1}{m+1}} \frac{(2p)!(2q)!}{(2p + 2q + 1)!}
\]
\[
\leq \frac{8\pi}{(m!)^2} \frac{(2p)!(2q)!}{(2p + 2q + 1)!} \frac{1}{n^{\frac{1}{m+1}}} \left[ \left( 1 + \frac{(2p + 2)(2p + 1)}{(2p + 2q + 3)(2p + 2q + 2)} + \frac{1}{2} (n - 1)^2 \right) \right]
\]
\[
\leq \frac{8\pi}{(m!)^2} \frac{(2p)!(2q)!}{(2p + 2q + 1)!} \frac{1}{n^{\frac{1}{m+1}}} \left[ n^2 \right].
\]

Recalling the estimate for \(\|h_n\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\) from (2.26), one now has that
\[
\|h_n\|^2 \|h_n'\|^2 \leq \frac{1}{\sqrt{m!}} \left[ \frac{2\pi n^{\frac{1}{m+1}} (2p)!(2q)!}{(2p + 2q + 1)!} \right] \frac{1}{\sqrt{m!}} \left[ \frac{8\pi n^2 (2p)!(2q)!}{n^{\frac{1}{m+1}} (2p + 2q + 1)!} \right]^{\frac{1}{2}}
\]
\[
\leq \frac{2\sqrt{\pi}}{m!} n^{\frac{1}{2}} \left( \frac{(2p)!(2q)!}{(2p + 2q + 1)!} \right)^{\frac{1}{2}}
\]
\[
\leq \frac{2\sqrt{\pi}}{m!} n^{\frac{1}{2}} \left( \frac{(m - 1)!(m + 1)!}{(m + 1)!} \right)^{\frac{1}{2}}
\]
since \(m = p + q = 2q - 1 = 2p + 1\). Thus, by (2.22),
\[
\left\| r \left( \frac{t}{n} \right)^n A x - T(t) x \right\| \leq M \sqrt{2\pi} \frac{1}{m!} \frac{1}{n^{m - \frac{1}{2}}} \left( \frac{(m - 1)!(m + 1)!}{(2m + 1)!} \right)^{\frac{1}{2}} t^{m + 1} \|A^{m+1}x\|.
\]

$$\square$$

## 2.5 Approximations of Semigroups without Scaling and Squaring, Part II

Using the methods of the previous section, it will be shown in this section how the estimate (2.13) could possibly be adjusted in order to show that
\[
\frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) x + \ldots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) x \to T(t)x
\]
for all \( x \in D(A) \). First, define \( f_{p,q}^{(k)}(t) := t^p(1 - t)^q \). Notice that

\[
\begin{align*}
    f_{p,q}^{(k)}(t) &= \sum_{j=0}^{k} \binom{k}{j} \frac{p!}{(p-j)!} t^{p-j} (-1)^{k-j} \frac{q!}{(q-k+j)!} (1 - t)^{q-k+j} \\
    &= (-1)^k k! \sum_{j=0}^{k} (-1)^j \binom{p}{j} \binom{q}{k-j} f_{p-j,q-k+j}(t).
\end{align*}
\]

This means that \( f_{p,q}^{(k)}(0) = f_{p,q}^{(k)}(1) = 0 \) for all \( 0 \leq k < p \), and

\[
\int_0^1 f_{p,q}(t) z^p e^{-tz} dt = \int_0^1 f_{p,q}^{(p)}(t) e^{-tz} dt
\]

by using integration by parts \( p \)-times.

**Theorem 2.5.1.** Let \( r(z) = \frac{b_1}{\lambda_1 - z} + \ldots + \frac{b_q}{\lambda_q - z} \) be a subdiagonal Padé approximation of the exponential of order \( m = p + q = 2q - 1 \). If \( A \) is the generator of a strongly continuous \( T(t) \) of type \((M, 0)\), then

\[
\left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) x + \ldots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) x - T(t)x \right\| \leq M t \|Ax\| \sqrt{5} \cdot c(r, m)
\]

for all \( x \in D(A) \) and \( t > 0 \) where

\[
c(r, m) := \frac{1}{p!} (M_{p,q})^{\frac{1}{4}} \left( \tilde{M}_{p,q} + M_{p,q} \right)^{\frac{1}{4}}
\]

with \( M_{p,q} := \int_0^1 \left| f_{p,q}^{(p)}(t) \right|^2 dt \) and \( \tilde{M}_{p,q} := \int_0^1 \left| (tf_{p,q}^{(p)}(t) - qf_{p,q}^{(p-1)}(t)) \right|^2 dt \).

Moreover, for \( 2 \leq q \leq 50 \) and \( 3 \leq m \leq 99 \),

\[
c(r, m) \leq 0.7476 \left( \frac{1}{\sqrt{m}} \right).
\]

**Proof.** For the start of this proof, the argument follows exactly the proof of Theorem 2.4.2 with \( n = 1 \). In particular, it was shown in the beginning of the proof of Theorem 2.4.2 that

\[
\|r(tA)x - T(t)x\| \leq M_A \|Ax\| \frac{1}{\sqrt{2}} t \|h\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|h'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}
\]

where

\[
h(s) = \frac{e^{-isr(is)}}{s} - 1 = \frac{e^{-isr(is)}}{is} - 1i.
\]
Recall the following representation of \( r \) from Theorem 2.1.3:

\[
    r(z) = e^z \frac{(-1)^q + 1}{Q(z)} \frac{1}{m!} z^{m+1} e^z \int_0^1 t^p (1 - t)^q e^{-tz} dt
\]

where \( z \in \mathbb{C} \) and \( z \neq \lambda_1, \ldots, \lambda_q \). (This is the point in the proof where the argument significantly departs from the proof of Theorem 2.4.2.) Using the representation from Theorem 2.1.3, it follows that

\[
    \frac{e^{-z}r(z) - 1}{z} = (-1)^q + 1 \frac{z^q}{Q(z)} \frac{1}{m!} \int_0^1 t^p (1 - t)^q z^p e^{-tz} dt.
\]  \hfill (2.29)

Using (2.27), one has that (2.29) now becomes

\[
    \frac{e^{-z}r(z) - 1}{z} = (-1)^q + 1 \frac{z^q}{Q(z)} \frac{1}{m!} \int_0^1 f_{p,q}(t) e^{-tz} dt.
\]

Therefore,

\[
    h(s) = \frac{\left[ e^{-is}r(is) \right] - 1}{is} = (-i)^q + 1 \frac{s^q}{Q(is)} \frac{1}{m!} \int_0^1 f_{p,q}(t) e^{-tis} dt.
\]

Recall from Proposition 2.1.4, that \( \frac{p!}{m!} \left| \frac{s^q}{Q(is)} \right| \leq 1 \) for all \( s \in \mathbb{R} \) and \( 1 \leq q \leq 50 \). This means that \( \left| \frac{s^q}{Q(is)} \right| \leq \frac{m!}{p!} \), and it follows that

\[
    |h(s)| \leq \frac{1}{p!} \left| \int_0^1 f_{p,q}(t) e^{-tis} dt \right|.
\]  \hfill (2.30)

Now

\[
    \int_0^1 f_{p,q}(t) e^{-tis} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi_m(t) e^{-its} dt = \mathcal{F}(\Psi_m)(s)
\]

with

\[
    \Psi_m(t) := \sqrt{2\pi} \begin{cases} 
        f_{p,q}(t) & 0 \leq t < 1 \\
        0 & \text{else}
    \end{cases}
\]

Then

\[
    \left\| \int_0^1 f_{p,q}(t) e^{-tis} dt \right\|_2 = \|\mathcal{F}(\Psi_m(t))\|_2 = \|\Psi_m\|_2
\]
techniques similar to those used to estimate \( \| \mathcal{F}(\Psi_m(t)) \|_2 = \| \Psi_m \|_2 = 2\pi \int_0^1 |f_{p,q}^{(p)}(t)|^2 \, dt := 2\pi M_{p,q} \).

In order to calculate \( M_{p,q} \), the following representation of the Beta function from [1] Section 4.21 is needed. For \( (p, q > -1) \) and \( \Gamma(n + 1) = n! \),

\[
B(p + 1, q + 1) = \int_0^1 x^p(1 - x)^q \, dx = \frac{\Gamma(p + 1)\Gamma(q + 1)}{\Gamma(p + q + 2)}.
\]

Now, returning to (2.30), one has that \( \|h\|_{L^2(\mathbb{R})}^2 \leq \frac{2\pi}{(p+q)!} M_{p,q} \).

Working towards an estimate for \( \|h'\|_2 \), notice that

\[
h'(s) = \frac{(-i)^{q+1}}{m!} \left[ \frac{s q^q - 1}{Q(is)^2} \int_0^1 f_{p,q}^{(p)}(t)e^{-tis} \, dt + \frac{(-i)^q}{m!} \int_0^1 tf_{p,q}^{(p)}(t)e^{-tis} \, dt \right] = \frac{(-i)^{q+1}}{m!} \left[ \frac{s q^q - 1}{Q(is)^2} \int_0^1 f_{p,q}^{(p)}(t)e^{-tis} \, dt + \frac{(-i)^q}{m!} \int_0^1 tf_{p,q}^{(p)}(t)e^{-tis} \, dt \right] = \frac{(-i)^{q+1}}{m!} \left[ \frac{s q^q - 1}{Q(is)^2} \int_0^1 f_{p,q}^{(p-1)}(t)e^{-tis} \, dt + \frac{(-i)^q}{m!} \int_0^1 tf_{p,q}^{(p)}(t)e^{-tis} \, dt \right] = \frac{(-i)^{q+1}}{m!} \left[ \frac{s q^q - 1}{Q(is)^2} \int_0^1 f_{p,q}^{(p)}(t) - q f_{p,q}^{(p-1)}(t) e^{-tis} \, dt + \frac{(-i)^q}{m!} \int_0^1 tf_{p,q}^{(p)}(t)e^{-tis} \, dt \right].
\]

Again recall from Proposition 2.1.4 that \( \left| \frac{Q'(is)}{Q(is)} \right| \leq 1 \) and \( \frac{p^l}{m!} \left| \frac{s^q}{Q(is)} \right| \leq 1 \) for all \( s \in \mathbb{R} \)
and \( 1 \leq q \leq 50 \). Using these statements and the fact that \( |a + b|^2 \leq 2|a|^2 + 2|b|^2 \), it now follows that

\[
\|h'\|_2^2 \leq \frac{1}{(p+q)^2} \int_{-\infty}^{\infty} \left[ \int_0^1 (tf_{p,q}^{(p)}(t) - q f_{p,q}^{(p-1)}(t)) e^{-tis} \, dt + \frac{Q'(is)}{Q(is)} \int_0^1 f_{p,q}^{(p)}(t)e^{-tis} \, dt \right]^2 \, ds
\]

\[
\leq \frac{1}{(p+q)^2} \int_{-\infty}^{\infty} 2 \left[ \int_0^1 (tf_{p,q}^{(p)}(t) - q f_{p,q}^{(p-1)}(t)) e^{-tis} \, dt \right]^2 + 2 \left[ \int_0^1 f_{p,q}^{(p)}(t)e^{-tis} \, dt \right]^2 \, ds
\]

\[
\leq 2 \frac{1}{(p+q)^2} \int_0^1 (t f_{p,q}^{(p)}(t) - q f_{p,q}^{(p-1)}(t)) e^{-tis} \, dt \| h' \|_2^2 + 2 \frac{1}{(p+q)^2} \int_0^1 f_{p,q}^{(p)}(t)e^{-tis} \, dt \|_2^2.
\]

Taking advantage again of the properties of the Fourier transform, proceed with techniques similar to those used to estimate \( \|h\|_2^2 \). Notice that

\[
\left\| \int_0^1 (tf_{p,q}^{(p)}(t) - q f_{p,q}^{(p-1)}(t)) e^{-tis} \, dt \right\|_2^2 = 2\pi \int_0^1 \left| (tf_{p,q}^{(p)}(t) - q f_{p,q}^{(p-1)}(t)) \right|^2 \, dt
\]

\[
:= 2\pi \tilde{M}_{p,q}.
\]

94
and
\[ \left\| \int_0^1 f_{p,q}^{(p)}(t) e^{-it}\,dt \right\|_2^2 = 2\pi \int_0^1 |f_{p,q}^{(p)}(t)|^2 \,dt = 2\pi M_{p,q}. \]

Therefore,
\[ \|h'\|_2^2 \leq \frac{4\pi}{(p!)^2} \tilde{M}_{p,q} + \frac{4\pi}{(p!)^2} M_{p,q}, \]

and
\[ \|h\|_{L^2(\mathbb{R})}^2 \|h'\|_{L^2(\mathbb{R})}^2 \leq \frac{8\pi^2}{(p!)^4} M_{p,q} \left( \tilde{M}_{p,q} + M_{p,q} \right). \]

Finally, it follows from (2.28) and the fact that
\[ \|r(tA)x - T(t)x\| \leq M t \|Ax\| \left( \frac{1}{\sqrt{2}} \left( \frac{8\pi^2}{p!} \right)^{\frac{1}{4}} (M_{p,q} + \tilde{M}_{p,q}) \right)^{\frac{1}{4}}. \]

Now, define
\[ c(r, m) := \frac{1}{p!} (M_{p,q})^{\frac{1}{4}} \left( \tilde{M}_{p,q} + M_{p,q} \right)^{\frac{1}{4}}. \]

Then using the representation for \( r(tA) \) from (2.9), since \( \frac{1}{\sqrt{2}} \left( \frac{8\pi^2}{p!} \right)^{\frac{1}{4}} < \sqrt{5} \), it follows that
\[ \left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) x + \ldots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) x - T(t)x \right\| \leq M t \|Ax\| \sqrt{5} \cdot c(r, m) \]

for all \( x \in D(A) \) and \( t > 0 \). Mathematica is very helpful for computing \( c(r, m) \) directly. Since \( M_{p,q} := \int_0^1 |f_{p,q}^{(p)}(t)|^2 \,dt \) and \( \tilde{M}_{p,q} := \int_0^1 \left| (t f_{p,q}^{(p)}(t) - q f_{p,q}^{(p-1)}(t)) \right|^2 \,dt \), Mathematica is able to compute \( c(r, m) \) symbolically and give exact results for \( c(r, m) \) given a specific value of \( m \). We used the following Mathematica code to compute \( c(r, m) \). In the following code, \( q := \text{Deg} q \) (the degree of \( Q(z) \)), \( f_{p,q}^{(k)}(t) := f[p, q, k, t] \), \( M_{p,q} := \text{Mpq} \), \( \tilde{M}_{p,q} := \text{TMPQ} \), and \( c(r, m) := \text{Cr} \) for the defined \( q \).
(q = 56 in this case):

\[
\text{Degq} = 56
\]

\[
f[p, q, k, t] := (-1)^k * k! * \text{Sum}[(-1)^j * \text{Binomial}[p, j] * \text{Binomial}[q, k - j] * t^{(p-j)}(1-t)^{(q-k+j)}, \{j, 0, k\}];
\]

\[
\text{Mpq} = \text{Integrate}[(f[\text{Degq} - 1, f[\text{Degq} - 1, t])^2, \{t, 0, 1\}];
\]

\[
\text{TMPQ} = \text{Integrate}[(t * f[\text{Degq} - 1, f[\text{Degq} - 1, t] - \text{Degq} * f[\text{Degq} - 1, f[\text{Degq} - 2, t)]^2, \{t, 0, 1\}];
\]

\[
\text{Cr} = N[(1/(\text{Degq} - 1)! \cdot (\text{Mpq})^{(1/4)} \cdot (\text{TMPQ} + \text{Mpq})^{(1/4)}].
\]

The results indicate that \( c(r, m) \rightarrow 0 \) like \( \left( \frac{1}{\sqrt{m}} \right) \) for \( 3 \leq m \leq 1999 \). Some values of \( c(r, m) \) and \( 0.7476 \left( \frac{1}{\sqrt{m}} \right) \) as computed by Mathematica for \( 3 \leq m \leq 1999 \) are shown in the following table for certain values of \( m \) to support this claim.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( c(r, m) )</th>
<th>( 0.7476 \cdot \frac{1}{\sqrt{m}} )</th>
<th>( m )</th>
<th>( c(r, m) )</th>
<th>( 0.7476 \cdot \frac{1}{\sqrt{m}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.40883</td>
<td>0.431627</td>
<td>199</td>
<td>0.0528828</td>
<td>0.052996</td>
</tr>
<tr>
<td>13</td>
<td>0.201346</td>
<td>0.207347</td>
<td>399</td>
<td>0.0373886</td>
<td>0.0374268</td>
</tr>
<tr>
<td>27</td>
<td>0.141687</td>
<td>0.143876</td>
<td>599</td>
<td>0.0305263</td>
<td>0.0305461</td>
</tr>
<tr>
<td>41</td>
<td>0.115556</td>
<td>0.116756</td>
<td>799</td>
<td>0.026436</td>
<td>0.0264482</td>
</tr>
<tr>
<td>55</td>
<td>0.100025</td>
<td>0.100806</td>
<td>999</td>
<td>0.0236447</td>
<td>0.023653</td>
</tr>
<tr>
<td>69</td>
<td>0.0894417</td>
<td>0.0900005</td>
<td>1199</td>
<td>0.0215844</td>
<td>0.0215904</td>
</tr>
<tr>
<td>83</td>
<td>0.0816352</td>
<td>0.0820598</td>
<td>1399</td>
<td>0.0199832</td>
<td>0.0199876</td>
</tr>
<tr>
<td>97</td>
<td>0.0755571</td>
<td>0.0759073</td>
<td>1599</td>
<td>0.0186924</td>
<td>0.0186958</td>
</tr>
<tr>
<td>111</td>
<td>0.0706844</td>
<td>0.0709591</td>
<td>1999</td>
<td>0.0167189</td>
<td>0.016721</td>
</tr>
</tbody>
</table>

\( \square \)
2.6 Inversion of Asymptotic Laplace Transforms of Generalized Functions

Theorem 2.5.1 has the following consequence. Assuming for the moment that $A^{-1}$ exists, it follows from

$$\left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) x + \ldots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) x - T(t)x \right\| \leq C \frac{t}{\sqrt{m}} \|x\|$$

for all $x \in D(A)$ that

$$\left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) A^{-1}x + \ldots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) A^{-1}x - T(t)A^{-1}x \right\| \leq C \frac{t}{\sqrt{m}} \|x\|.$$ 

Since

$$T(t)x - x = A \int_0^t T(s)x \, ds$$

$$T(t)A^{-1}x - A^{-1}x = \int_0^t T(s)x \, ds,$$

then it follows from above that

$$\left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) A^{-1}x + \ldots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) A^{-1}x - f_0^t T(s)x \, ds \right\| \leq \frac{Ct}{\sqrt{m}} \|x\|.$$ 

Furthermore, because $(\mu I - A)R(\mu, A) = I$ implies that $\mu R(\mu, A) = AR(\mu, A) + I$ or $\mu R(\mu, A)A^{-1} = R(\mu, A) + A^{-1}$, it follows that

$$\left\| \frac{b_1}{t} R \left( \frac{\lambda_1}{t}, A \right) A^{-1}x + \ldots + \frac{b_q}{t} R \left( \frac{\lambda_q}{t}, A \right) A^{-1}x - \int_0^t T(s)x \, ds \right\| =$$

$$\left\| \frac{b_1}{\lambda_1} \cdot R \left( \frac{\lambda_1}{t}, A \right) A^{-1}x + \ldots + \frac{b_q}{\lambda_q} \cdot R \left( \frac{\lambda_q}{t}, A \right) A^{-1}x - A^{-1}x - \int_0^t T(s)x \, ds \right\| =$$

$$\left\| \frac{b_1}{\lambda_1} R \left( \frac{\lambda_1}{t}, A \right) x + \ldots + \frac{b_q}{\lambda_q} R \left( \frac{\lambda_q}{t}, A \right) x + \left( \frac{b_1}{\lambda_1} + \ldots + \frac{b_q}{\lambda_q} \right) A^{-1}x - A^{-1}x$$

$$- \int_0^t T(s)x \, ds \right\| =$$

$$\left\| \frac{b_1}{\lambda_1} R \left( \frac{\lambda_1}{t}, A \right) x + \ldots + \frac{b_q}{\lambda_q} R \left( \frac{\lambda_q}{t}, A \right) x - f_0^t T(s)x \, ds \right\| \leq \frac{Ct}{\sqrt{m}} \|x\|$$

for all $x \in X$ since $Q(0) = \frac{b_1}{\lambda_1} + \ldots + \frac{b_q}{\lambda_q} = 1$.

In order to remove the assumption that $A^{-1}$ exists, let $A$ be the generator of a strongly continuous or bi-continuous semigroup $T(t)$ with $\|T(t)\| \leq M$. Let $\varepsilon > 0$ and define $A_\varepsilon := A - \varepsilon I$. Then $A_\varepsilon$ generates the semigroup $T_\varepsilon(t) := e^{-\varepsilon t}T(t)$ with
\[ \|T_\varepsilon(t)\| \leq M e^{-\varepsilon t}. \]

It follows that \( R(\lambda + \varepsilon, A)x = R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T_\varepsilon(t)x \, dt \)
exists for all \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) > -\varepsilon \). In particular, \( A_\varepsilon^{-1} \) exists. By letting \( \varepsilon \to 0 \),
it follows that the inequality
\[
\left\| \frac{b_1}{\lambda_1} R \left( \frac{\lambda_1}{t}, A_\varepsilon \right) x + \ldots + \frac{b_q}{\lambda_q} R \left( \frac{\lambda_q}{t}, A_\varepsilon \right) x - \int_0^t T_\varepsilon(s)x \, ds \right\| \leq \frac{C t}{\sqrt{m}} \|x\|
\]
implies that
\[
\left\| \frac{b_1}{\lambda_1} R \left( \frac{\lambda_1}{t}, A \right) x + \ldots + \frac{b_q}{\lambda_q} R \left( \frac{\lambda_q}{t}, A \right) x - \int_0^t T(s)x \, ds \right\| \leq \frac{C t}{\sqrt{m}} \|x\|
\]
is valid for all generators of semigroups with \( \|T(t)\| \leq M \) (without the assumption
that \( A^{-1} \) exists). Moreover, by the Transference Principle (Proposition 2.0.7), it
follows that for all \( f \in C_b([0, \infty), X) \):
\[
\left\| \frac{b_1}{\lambda_1} \hat{f} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{\lambda_q} \hat{f} \left( \frac{\lambda_q}{t} \right) - \int_0^t f(s) \, ds \right\| \leq \frac{C t}{\sqrt{m}} \|f\|.
\]

Let \( u \in C^{-k}([0, \infty], X) \) be a \( k \)-generalized function for some \( k \in L^1[0, \infty) \) such
that its continuous representative \( f := k \ast u \) is in \( C_b([0, \infty], X) \). Then \( u \) is Laplace transformable (in the classical sense), \( \hat{u}(\lambda) := \frac{\hat{f}(\lambda)}{k(\lambda)} \) for \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) > 0 \), and
\[
\left\| \frac{b_1}{\lambda_1} \hat{u} \left( \frac{\lambda_1}{t} \right) \hat{k} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{\lambda_q} \hat{u} \left( \frac{\lambda_q}{t} \right) \hat{k} \left( \frac{\lambda_q}{t} \right) - (1 \ast f)(t) \right\| \leq \frac{C t}{\sqrt{2q - 1}} \|f\|
\]
for all \( 2 \leq q \leq 1000 \). Statement (2.31) could be extended to all generalized func-
tions \( u \) (and their continuous representatives \( f \)) if one could show that
\[
\frac{b_1}{\lambda_1} a \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{\lambda_q} a \left( \frac{\lambda_q}{t} \right) \to 0 \quad (2.32)
\]
on \( [0, T] \) as \( q \to \infty \) for all \( a \in O(\Sigma, X) \) with \( a \approx_T 0 \).

The following numerical explorations concern the approximation of \( 1 \ast f = 1 \ast k \ast u \)
in terms of
\[
\frac{b_1}{\lambda_1} \hat{u} \left( \frac{\lambda_1}{t} \right) \hat{k} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{\lambda_q} \hat{u} \left( \frac{\lambda_q}{t} \right) \hat{k} \left( \frac{\lambda_q}{t} \right)
\]
(2.33)
Example 2.6.1. Consider (2.33) for the delta function $u = \delta_1$ at $x_0 = 1$. To begin, let $k(t) = \delta'(t)$. Then $1 * f = 1 * k * u = u$ and since $\hat{k}(\lambda) = \lambda$, it follows that

$$f_a(t) := \frac{b_1}{t} \hat{\lambda_1} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{t} \hat{\lambda_q} \left( \frac{\lambda_q}{t} \right) = \frac{b_1}{t} e^{-\frac{\lambda_1}{t}} + \ldots + \frac{b_q}{t} e^{-\frac{\lambda_q}{t}}$$

should be an approximation of $u = \delta_1$. Below are the numerical results for $q = 19$.

![Graph showing the approximation $f_a(t)$ with $q = 19$.]

This approximation is an “approximate identity” in the sense that

$$0 \leq \int_0^{0.9456387} f_a(t) \, dt \leq 10^{-5},$$
$$\int_0^{1.11531} f_a(t) \, dt \approx 1,$$ and
$$0 \leq \int_{1.054613}^{2.11531} f_a(t) \, dt \leq 10^{-5}.$$ 

Next take $k(t) = \delta(t)$. Then $1 * f = 1 * k * u = 1 * u = H_1$ where $H_1$ is the Heaviside function with the jump at 1. Since $\hat{k} = 1$, it follows that

$$F_a(t) := \frac{b_1}{\lambda_1} \hat{\lambda_1} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{\lambda_q} \hat{\lambda_q} \left( \frac{\lambda_q}{t} \right) = \frac{b_1}{\lambda_1} e^{-\frac{\lambda_1}{t}} + \ldots + \frac{b_q}{\lambda_q} e^{-\frac{\lambda_q}{t}}$$

should be an approximation of $H_1$. Notice that the numerical result below supports (2.32) for $a(\lambda) = e^{-\lambda} = \hat{\lambda}(\lambda)$. Below are the results for $q = 19$. 

99
FIGURE 2.10. $F_a(t) = \frac{b_1}{\lambda_1} e^{-\frac{\lambda_1 t}{2}} + \ldots + \frac{b_q}{\lambda_q} e^{-\frac{\lambda_q t}{2}}$ with $q = 19$.

Now suppose that $k(t) = 1$. Then $1* f = 1*k*u = 1*1*u = 1*H_1 = (\cdot-1)\chi_{[1, \infty)}(\cdot)$. Since $\hat{k}(\lambda) = \frac{1}{\lambda}$ it follows that

$$F_a^{[1]}(t) := \frac{b_1 t}{\lambda_1^2} \hat{u} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q t}{\lambda_q^2} \hat{u} \left( \frac{\lambda_q}{t} \right) = \frac{b_1 t}{\lambda_1^2} e^{-\frac{\lambda_1 t}{2}} + \ldots + \frac{b_q t}{\lambda_q^2} e^{-\frac{\lambda_q t}{2}}$$

should be an approximation of

$$F^{[1]}(t) \begin{cases} 
0 & 0 \leq t \leq 1 \\
 t - 1 & t > 1
\end{cases}$$

Below are the results for $q = 19$.

FIGURE 2.11. $F_a^{[1]}(t) = \frac{b_1 t}{\lambda_1^2} e^{-\frac{\lambda_1 t}{2}} + \ldots + \frac{b_q t}{\lambda_q^2} e^{-\frac{\lambda_q t}{2}}$ with $q = 19$. 

100
Here is the graph of the error obtained when approximating $1 \ast 1 \ast u = F^{[1]}(t)$ by $F^{[1]}_a(t)$.

![Graph of error]

FIGURE 2.12. Error when approximating $1 \ast 1 \ast u = F^{[1]}(t)$ by $F^{[1]}_a(t)$ with $q = 19$.

Finally, to drive home the point that the integration dramatically improves the quality of the approximation, now consider $k(t) = t$. Then

$$1 \ast f = 1 \ast k \ast u = 1 \ast 1 \ast 1 \ast u = 1 \ast 1 \ast H_1 = 1 \ast F^{[1]} = F^{[2]},$$

where

$$F^{[2]}(t) \begin{cases} 
0 & 0 \leq t \leq 1 \\
\frac{1}{2} (t - 1)^2 & t > 1
\end{cases}.$$

Since $\hat{k}(\lambda) = \frac{1}{\lambda^2}$, it follows that

$$F^{[2]}_a(t) := \frac{b_1 t^2}{\lambda_1^2} u \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q t^2}{\lambda_q^3} \hat{u} \left( \frac{\lambda_q}{t} \right) = \frac{b_1 t^2}{\lambda_1^3} e^{-\frac{\lambda_1}{t}} + \ldots + \frac{b_q t^2}{\lambda_q^3} e^{-\frac{\lambda_q}{t}}$$

should be an approximation of $F^{[2]}(t)$. Below are the results for $q = 19$. 

101
Here is the graph of the error obtained when approximating $1 \ast 1 \ast 1 \ast u = F^{[2]}(t)$ by $F^{[2]}_{a}(t)$.

**Example 2.6.2.** Consider the inverse Laplace transform $\hat{u}(\lambda) = \frac{1}{1 + \lambda e^{-\lambda}}$ (see Theorem 1.4.6). Recall that because Proposition 1.4.2 shows that since $\hat{u}$ has infinitely many singularities which grow exponentially in the right half-plane, then $\hat{u}(\lambda)$ cannot be the Laplace transform of a function in the classical sense. The following numerical explorations will attempt to indicate whether the methods of (2.31) can be extended to all generalized functions $u$ (and their continuous representatives.
\( \hat{u}(\lambda) = \frac{1}{1 + \lambda e^{-\lambda}} = 1 - \lambda e^{-\lambda} + \lambda^2 e^{-2\lambda} - \lambda^3 e^{-3\lambda} + \ldots \)

is formally the Laplace transform of

\[ u(t) = \delta(t) - \delta'_1(t) + \delta''_2(t) - \delta''''_3(t) + \ldots. \]

Again consider the approximation of \( 1 \ast f = 1 \ast k \ast u \) in terms of

\[ \frac{b_1}{\lambda_1} \hat{u} \left( \frac{\lambda_1}{t} \right) \hat{k} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{\lambda_q} \hat{u} \left( \frac{\lambda_q}{t} \right) \hat{k} \left( \frac{\lambda_q}{t} \right). \]  

(2.34)

To begin, start with \( k(t) = \delta'(t) \). Then \( 1 \ast f = u \) and (2.34) become

\[ u_a(t) = \frac{b_1}{t} \hat{u} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{t} \hat{u} \left( \frac{\lambda_q}{t} \right) \]

since \( \hat{k}(\lambda) = \lambda \). One hopes that \( u_a(t) \) approximates in a certain sense \( u(t) = \delta(t) - \delta'_1(t) + \delta''_2(t) - \delta''''_3(t) + \ldots \). Below are the results for \( q = 10, 13, 16, \) and 19:

**FIGURE 2.15.** \( u_a(t) = \frac{b_1}{t} \hat{u} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{t} \hat{u} \left( \frac{\lambda_q}{t} \right). \)

**FIGURE 2.16.** \( u_a(t) = \frac{b_1}{t} \hat{u} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{t} \hat{u} \left( \frac{\lambda_q}{t} \right). \)
The above approximations in Figures 2.15 and 2.16 are quite poor and unrealible. Now take \( k(t) = \delta(t) \) with \( \hat{\lambda}(\lambda) = 1 \). Then \( 1 \ast f = 1 \ast u \) and (2.34) become

\[
(1 \ast u)_a(t) = \frac{b_1}{\lambda_1} \hat{u} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{\lambda_q} \hat{u} \left( \frac{\lambda_q}{t} \right)
\]

and one again hopes that \( (1 \ast u)_a(t) \) approximates in a certain sense \( (1 \ast u)(t) = 1 - \delta_1(t) + \delta'_2(t) - \delta''_3(t) + \ldots \). Below are the results for \( q = 10 \) and 19 which show a dramatic improvement from the “approximations” shown in Figures 2.15 and 2.16.

**FIGURE 2.17.** \( (1 \ast u)_a(t) = \frac{b_1}{\lambda_1} \hat{u} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q}{\lambda_q} \hat{u} \left( \frac{\lambda_q}{t} \right) \) with \( q = 10 \).

**FIGURE 2.18.** \( (1 \ast u)_a \) and \( 1 \)-function for \([0, 1]\) with \( q = 19 \).
Now suppose that \( k(t) = 1 \) with \( \hat{k}(\lambda) = \frac{1}{\lambda} \). It follows that

\[
1 \ast f = 1 \ast 1 \ast u : t \to t - H_1(t) + \delta_2(t) - \delta_3'(t) + \ldots
\]

should be approximated by

\[
(1 \ast 1 \ast u)_a(t) = \frac{b_1 t}{\lambda_1^2} \hat{u} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q t^2}{\lambda_q^3} \hat{u} \left( \frac{\lambda_q}{t} \right).
\]

The results below for \( q = 19 \) show further improvement from the previous approximation. Specifically, approximation appears to be remain accurate for larger values of \( t \).

![Figure 2.19. (1 \ast 1 \ast u)_a and 1 \ast 1 \ast u with q = 19.](image)

Now, with \( k(t) = t = (1 \ast 1)(t) \) with \( \hat{k}(\lambda) = \frac{1}{\lambda^2} \), one should be able to approximate

\[
1 \ast f = (1 \ast 1 \ast 1 \ast u) : t \to \frac{t^2}{2} - (t - 1)H_1(t) + H_2(t) - \delta_3(t) + \ldots
\]

with

\[
(1 \ast 1 \ast 1 \ast u)_a(t) = \frac{b_1 t^2}{\lambda_1^3} \hat{u} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q t^2}{\lambda_q^3} \hat{u} \left( \frac{\lambda_q}{t} \right).
\]

Here is the result for \( q = 19 \). Again, notice the the approximation is remaining accurate for increasing intervals \([0, t]\) as compared to the accuracy of previous approximation.
Continuing, with $k(t) = \frac{t^2}{2} = (1 \ast 1 \ast 1)(t)$ with $\hat{k}(\lambda) = \frac{1}{\lambda^3}$, one should be able to approximate

$$1 \ast f = (1 \ast 1 \ast 1 \ast u) : t \rightarrow \frac{t^3}{6} - \frac{1}{2}(t-1)^2H_1(t) + (t-2)H_2(t) - H_3(t) + \delta_4(t) - \ldots$$

with

$$(1 \ast 1 \ast 1 \ast u)_a(t) = \frac{b_1}{\lambda_1^4} \hat{u}\left(\frac{\lambda_1}{t}\right) + \ldots + \frac{b_q}{\lambda_q^4} \hat{u}\left(\frac{\lambda_q}{t}\right).$$

Here is the result for $q = 19$, which seems to be a fairly nice approximation of $1 \ast 1 \ast 1 \ast u$ for $t \in [0, 3)$. 

FIGURE 2.21. $(1 \ast 1 \ast 1 \ast u)_a$ and $1 \ast 1 \ast 1 \ast u$ with $q = 19.$
Finally, with \( k(t) = \frac{t^3}{6} = (1 * 1 * 1 * 1)(t) \) with \( \hat{k}(\lambda) = \frac{1}{\lambda^4} \), one should be able to approximate \( 1 * f = \)

\[
(1*1*1*1*1*u) : t \rightarrow \frac{t^4}{4!} - \frac{1}{3!} (t-1)^3 H_1(t) + \frac{1}{2!} (t-2)^2 H_2(t) - (t-3) H_3(t) + H_4(t) - \ldots
\]

with

\[
(1 * 1 * 1 * 1 * 1 * u)_a(t) = \frac{b_1 t^4}{\lambda_1^5} \hat{u} \left( \frac{\lambda_1}{t} \right) + \ldots + \frac{b_q t^4}{\lambda_q^5} \hat{u} \left( \frac{\lambda_q}{t} \right).
\]

Below are the result and the corresponding error for \( q = 19 \).

**FIGURE 2.22.** \((1 * 1 * 1 * 1 * 1 * u)_a \) and \((1 * 1 * 1 * 1 * 1 * u)\) with \( q = 19 \).

**FIGURE 2.23.** Error when approximating \( 1 * 1 * 1 * 1 * 1 * u \) by \((1 * 1 * 1 * 1 * 1 * u)_a(t)\) with \( q = 19 \).
References


Vita

Lee Gregory Windsperger was born in Saint Paul, Minnesota, and grew up in the Twin Cities area before beginning his college career in Saint Louis, Missouri. He finished his undergraduate studies at Saint Louis University in May of 2005. In August of 2005, he came to Louisiana State University and earned a master of science in mathematics in 2007. During his time in Baton Rouge, he met the love of his life, Jayci Beckett, and they were married in the summer of 2011. Currently, he is a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2012. He and his new family plan to move back up the Mississippi to his home state for a position in Winona State University’s Department of Mathematics and Statistics.