

STABILITY IN DYNAMICAL POLYSYSTEMS

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Abstract

A dynamical polysystem consists of a family of continuous dynamical systems, all acting on a given metric space. The first chapter of the present thesis shows a generalization of control systems via dynamical polysystems and establishes the equivalence of the two notions under certain lipschitz condition on the function defining the dynamics. The remaining chapters are focused on a basic theory of dynamical polysystems. Some topological properties of limit sets are described in Chapter 2. Chapters 3 and 4 provide characterizations for various notions of strong stability. Chapter 5 makes use of the theory of closed relations to study Lyapunov functions. Prolongations and absolute stability make the object of the last chapter.

Introduction

The notion of *dynamical polysystem* appeared in the 1970's, being introduced by C. Lobry, [Lo]. It had the following meaning: A dynamical polysystem on a manifold M is a family

$$\mathcal{F}_{pc} = \{\mathcal{F}(\cdot, u) : u \in \mathcal{U}_{pc}\}$$

of smooth vector fields depending on a piecewise constant parameter u , called input. A similar meaning was given to dynamical polysystems in the work of J. Tsiniias and N. Kalouptsidis, [TK].

In this paper, a dynamical polysystem is regarded in a slightly more general way, as a family of continuous dynamical systems, all defined on the same metric space X , not necessarily by means of differential equations. The analogy between dynamical polysystems and control systems with piecewise constant inputs is quite natural. Intuitively, a motion in a dynamical polysystem means starting at a point $x \in X$, traveling for a time t_1 according to a dynamical system Φ_1 , then switching to another dynamical system Φ_2 and traveling for a time t_2 , and so forth.

The concept of stability in dynamical systems has a rich variety of forms as well as a vast history. Ranging from weak stability to asymptotic stability and from Poisson stability to Lyapunov stability, broad developments on the idea of stability can be found in [BS] and [AH].

The present thesis has two main objectives, both touching historical as well as modern concerns in the theory of dynamical systems.

The first goal is to create a bridge between continuous dynamical systems and control systems, by means of dynamical polysystems. In this respect, Chapter 1 shows, in its first part, how a generalized control system can arise from a family

of continuous dynamical systems on a metric space X , provided the family of dynamical systems satisfies a condition of smooth dependence on indices. Definition 1.12 shows how this generalized control system occurs and Proposition 1.14 proves its consistency. The second part of Chapter 1 is dedicated to proving that, indeed, such method allows a consistent generalization of control systems that are usually defined by differential equations with parameters (Theorem 1.20), under the Lipschitz condition

$$|f(x, m) - f(y, m')| \leq \alpha|x - y| + \beta d(m, m').$$

on the function f defining the dynamics (Proposition 1.19).

The second objective, inspired by the work of Bhatia and Szegö, [BS], is to develop a basic theory of stability and attraction in dynamical polysystems, with the future perspective of translating some stability results from dynamical polysystems to control systems. We focus on strong notions of stability and attraction, properties that hold for all possible trajectories, not just some. Chapter 2 presents topological properties of somewhat relaxed limit sets in the context of general dynamical polysystems. Chapters 3 and 4 deal with more refined versions of notions like limit set and stability, the main goal being to characterize stability by means of Lyapunov-like functions. In trying to make Lyapunov functions a reliable tool for handling stability, Chapter 5 takes a different approach and studies dynamical polysystems in the light of the theory of closed relations developed by E. Akin, [Ak]. Prolongations and prolongational limit sets are then studied in Chapter 6 and a characterization of absolute stability is given in Theorem 6.17.

Chapter 1

Dynamical Polysystems and Control Systems

1.1 Definitions

Consider a family \mathcal{F} of continuous dynamical systems, all defined on a metric space X . For any $\Phi \in \mathcal{F}$ and $t \in \mathbb{R}$, $\Phi_t(x) = \Phi(t, x)$ defines a homeomorphism Φ_t on X , having inverse Φ_{-t} .

Definition 1.1. Let \mathcal{G} be the subgroup of $(\mathbb{R} \times \text{Homeo}(X), (+, \circ))$ generated by

$$\{(t, \Phi_t) : \Phi \in \mathcal{F}, t \in \mathbb{R}\}$$

The pair (\mathcal{G}, X) is called a **dynamical polysystem** on X .

The **accessibility semigroup** of \mathcal{G} , denoted by \mathcal{S} , is the subsemigroup of \mathcal{G} generated by

$$\{(t, \Phi_t) : \Phi \in \mathcal{F}, t \geq 0\}$$

The pair (\mathcal{S}, X) is called the **accessibility polysystem** on X generated by \mathcal{F} .

We note a similarity between dynamical polysystems and control systems. It is natural then to have a closer look at this similarity, before continuing with a theory of dynamical polysystems.

For the rest of this chapter, let X be a complete metric space, M another metric space, and $\Phi : \mathbb{R} \times X \times M \rightarrow X$ a continuous function with the property that every $\Phi_m : \mathbb{R} \times X \rightarrow X$ defined by $\Phi_m(t, x) := \Phi(t, x, m)$ is a continuous dynamical system on X .

The main objective of this chapter is to construct a control system, in the sense of Sontag (see [So]), with a transition function Ψ naturally arising from Φ , under certain hypotheses on Φ .

1.2 Regulated functions

Definition 1.2. Let $u : [a, b] \rightarrow M$. We say that u is a **regulated function** if it is the uniform limit of a sequence of piecewise constant functions.

It is shown (see [Bo], p. 60) that a necessary and sufficient condition for a function to be regulated is that it has a limit from the left and a limit from the right at every point in its domain (at end points, just the applicable one-sided limit).

Definition 1.3. Let $u : [0, T] \rightarrow M$ and $v : [0, S] \rightarrow M$. The **concatenation** $u * v : [0, T + S] \rightarrow M$ (also called the **Myhill product**) of u and v is defined by

$$u * v(t) := \begin{cases} u(t) & , \text{ if } 0 \leq t < T ; \\ v(t - T) & , \text{ if } T \leq t \leq T + S. \end{cases}$$

Remark 1.4. The operation of concatenation preserves the uniform limit. That is, if $u_n \xrightarrow{u} u$ and $v_n \xrightarrow{u} v$ then $u_n * v_n \xrightarrow{u} u * v$.

Proof. Assume that u_n, u are defined on $[0, T]$ and v_n, v are defined on $[0, S]$. Let $\epsilon > 0$ and $N_1, N_2 \in \mathbb{N}$ be such that

$$n \geq N_1 \Rightarrow d(u_n(t), u(t)) < \epsilon, \text{ for all } t$$

and

$$n \geq N_2 \Rightarrow d(v_n(t), v(t)) < \epsilon, \text{ for all } t.$$

Set $N := \max(N_1, N_2)$ and let $t \in [0, T + S]$. If $n \geq N$ then $d(u_n * v_n(t), u * v(t)) = d(u_n(t), u(t))$ if $0 \leq t < T$ and $d(u_n * v_n(t), u * v(t)) = d(v_n(t - T), v(t - T))$ if $T \leq t \leq T + S$. In either case, $d(u_n * v_n(t), u * v(t)) < \epsilon$.

□

1.3 Piecewise constant controls

Definition 1.5. Let $T > 0$ and consider $u : [0, T] \rightarrow M$, a piecewise constant function defined by a finite partition $0 = t_0 < t_1 < t_2 < \dots < t_k = T$ of the interval $[0, T]$, and elements m_1, m_2, \dots, m_k of M with $u(t) = m_i$ whenever $t \in (t_{i-1}, t_i)$, for $i \in \{1, 2, \dots, k\}$. Note that the values of u at the points t_0, t_1, \dots, t_k are ignored, for they are unimportant when defining the transition function.

For $x \in X$ and u as above, define the sequence:

$$\begin{aligned} x_1 &:= \Phi(t_1, x, m_1), \\ x_2 &:= \Phi(t_2 - t_1, x_1, m_2), \\ &\dots \dots \dots \\ x_k &:= \Phi(t_k - t_{k-1}, x_{k-1}, m_k), \end{aligned}$$

and set

$$\Psi(T, x, u) := x_k.$$

Remark 1.6. The function Ψ is well defined, that is $\Psi(T, x, u)$ is independent of the representation of the piecewise constant function u .

Proof. Let u have an arbitrary representation as above. We consider also the unique representation u^* given by maximal intervals of constancy. Express the later using $0 = s_0 < s_1 < s_2 < \dots < s_l = T$ so that $u^*(t) = m'_i$, when $t \in (s_{i-1}, s_i)$ and $m'_i \neq m'_{i+1}$ for all i . Now, construct the sequence:

$$\begin{aligned} x'_1 &:= \Phi(s_1, x, m'_1), \\ x'_2 &:= \Phi(s_2 - s_1, x'_1, m'_2), \\ &\dots \dots \dots \\ x'_l &:= \Phi(s_l - s_{l-1}, x'_{l-1}, m'_l). \end{aligned}$$

Then there exist indices $k_1 < k_2 < \dots < k_l$ for which $t_{k_1} = s_1, t_{k_2} = s_2, \dots, t_{k_l} = s_l$.

Using induction, we can easily prove that $x_{k_i} = x'_i$ for all $i \in \{1, 2, \dots, l\}$. It suffices to show it for $i = 1$, since the same verification is used for the inductive step. Noting that $m_1 = m_2 = \dots = m_{k_1} = m'_1$, we have:

$$\begin{aligned} x'_1 &= \Phi(s_1, x, m'_1) = \Phi(s_1 - t_1, \Phi(t_1, x, m'_1), m'_1) = \Phi(s_1 - t_1, \Phi(t_1, x, m_1), m'_1) = \\ &\Phi(s_1 - t_1, x_1, m'_1) = \Phi(s_1 - t_2, \Phi(t_2 - t_1, x_1, m'_1), m'_1) = \\ &\Phi(s_1 - t_2, \Phi(t_2 - t_1, x_1, m_2), m'_1) = \Phi(s_1 - t_2, x_2, m'_1) = \dots = x_{k_1}. \end{aligned}$$

The only fact used in this sequence of equalities is the semigroup property of the dynamical system $\Phi_{m'_1}$. □

Proposition 1.7. *The function Ψ defined above satisfies the semigroup property for piecewise constant controls:*

$$\Psi(T + S, x, u * v) = \Psi(S, \Psi(T, x, u), v), \quad (1.1)$$

where $u : [0, T] \rightarrow M$ and $v : [0, S] \rightarrow M$ are piecewise constant functions.

Proof. Note that if $u(T) \neq v(0)$ then the property holds by the mere construction of $\Psi(T + S, x, u * v)$.

Without loss of generality, we can then reduce the problem to the case when u and v are both constant functions and, moreover, they are defined by the same constant, say m . The property becomes, in this case,

$$\Phi(T + S, x, m) = \Phi(S, \Phi(T, x, m), m).$$

This is true indeed, by the semigroup property of the dynamical system Φ_m . □

1.4 Regulated controls

In order to extend the definition of the function Ψ , we will assume that the given polysystem satisfies the following hypothesis:

H 1.8. For every $T > 0$ there exists a continuous function $K : [0, T] \rightarrow [1, \infty)$ such that

$$K(t_1)K(t_2) \leq K(t_1 + t_2) \text{ for every } t_1 \text{ and } t_2 \quad (1.2)$$

and

for every $\epsilon > 0$ there exists a $\delta > 0$ such that $x_0, y_0 \in X$, $t \in [0, T]$, and $d(m_0, n_0) < \delta$ imply

$$d(\Phi(t, x_0, m_0), \Phi(t, y_0, n_0)) \leq K(t)d(x_0, y_0) + \epsilon \int_0^t K(s)ds. \quad (1.3)$$

Consider $x \in X, T > 0, u : [0, T] \rightarrow M$, a regulated function, and $\{u_n\}_n$ a sequence of piecewise constant functions on $[0, T]$, converging uniformly to u .

Fix a positive integer n and assume that u_n is defined by a partition $0 = t_0^n < t_1^n < t_2^n < \dots < t_{k_n}^n = T$ of the interval $[0, T]$ and elements $m_1^n, m_2^n, \dots, m_{k_n}^n$ of M with $u_n(t) = m_i^n$ whenever $t \in (t_{i-1}^n, t_i^n)$, for $i \in \{1, 2, \dots, k_n\}$.

Following the procedure described in Definition 1.5, construct the sequence:

$$\begin{aligned} x_1^n &:= \Phi(t_1^n, x, m_1^n), \\ x_2^n &:= \Phi(t_2^n - t_1^n, x_1^n, m_2^n), \\ &\dots \dots \dots \\ x_{k_n}^n &:= \Phi(t_{k_n}^n - t_{k_n-1}^n, x_{k_n-1}^n, m_{k_n}^n). \end{aligned}$$

Using Definition 1.5, we can rename the last element of this sequence:

$$x_n := x_{k_n}^n = \Psi(T, x, u_n). \quad (1.4)$$

Lemma 1.9. Assume that Φ satisfies hypothesis 1.8. Given $T > 0, x \in X, u$, and $\{u_n\}_n$ as above, the sequence $\{x_n\}_n$, defined by equation 1.4, is Cauchy (hence convergent, by the completeness of X).

Proof. Let K be a function as in hypothesis 1.8 and set $A := \int_0^T K(s)ds$.

Let $\epsilon > 0$. Find $\delta > 0$ such that equation 1.3 is satisfied for $\frac{\epsilon}{A}$, that is:

$$d(\Phi(t, x_0, m_0), \Phi(t, y_0, n_0)) \leq K(t)d(x_0, y_0) + \frac{\epsilon}{A} \int_0^t K(s)ds,$$

whenever $d(m_0, n_0) < \delta, x_0, y_0 \in X, t \in [0, T]$.

Since $u_n \xrightarrow{n} u$ uniformly, there exists $N \in \mathbb{N}$ such that $p, q \geq N$ implies $d(u_p(t), u_q(t)) < \delta$, for all $t \in [0, T]$.

Fix $p, q \geq N$.

As in the discussion preceding equation 1.4, u_p and u_q are defined using partitions of the interval $[0, T]$: $0 = t_0^p < t_1^p < t_2^p < \dots < t_{k_p}^p = T$ and $0 = t_0^q < t_1^q < t_2^q < \dots < t_{k_q}^q = T$, respectively. Letting $0 = s_1 < s_2 < \dots < s_k = T$ be the common refinement of the two partitions, we see that $u_p(t) = m_i^p$ and $u_q(t) = m_i^q$, whenever $t \in (s_{i-1}, s_i)$, for all $i \in \{1, 2, \dots, k\}$. Also, $d(m_i^p, m_i^q) < \delta$, for all $i \in \{1, 2, \dots, k\}$.

Now, we prove by induction on j that

$$d(x_j^p, x_j^q) \leq \frac{\epsilon}{A} \int_0^{s_j} K(s)ds, \text{ for all } j \in \{1, 2, \dots, k\}. \quad (1.5)$$

For $j = 1$ we have:

$$d(x_1^p, x_1^q) = d(\Phi(s_1, x, m_1^p), \Phi(s_1, x, m_1^q)) \leq \frac{\epsilon}{A} \int_0^{s_1} K(s)ds,$$

by property 1.3.

Now we prove that if inequality 1.5 holds for j , then it holds for $j + 1$.

$$\begin{aligned} d(x_{j+1}^p, x_{j+1}^q) &= d(\Phi(s_{j+1} - s_j, x_j^p, m_{j+1}^p), \Phi(s_{j+1} - s_j, x_j^q, m_{j+1}^q)) \leq \\ &K(s_{j+1} - s_j)d(x_j^p, x_j^q) + \frac{\epsilon}{A} \int_0^{s_{j+1} - s_j} K(s)ds, \end{aligned}$$

by property 1.3. Using the inductive hypothesis, we then have:

$$d(x_{j+1}^p, x_{j+1}^q) \leq K(s_{j+1} - s_j) \frac{\epsilon}{A} \int_0^{s_j} K(s) ds + \frac{\epsilon}{A} \int_0^{s_{j+1} - s_j} K(s) ds =$$

$$\frac{\epsilon}{A} \left[\int_0^{s_j} K(s_{j+1} - s_j) K(s) ds + \int_0^{s_{j+1} - s_j} K(s) ds \right].$$

Now, using property 1.2 of $K(\cdot)$, we may write:

$$d(x_{j+1}^p, x_{j+1}^q) \leq \frac{\epsilon}{A} \left[\int_0^{s_j} K(s_{j+1} - s_j + s) ds + \int_0^{s_{j+1} - s_j} K(s) ds \right] =$$

$$\frac{\epsilon}{A} \left[\int_{s_{j+1} - s_j}^{s_{j+1}} K(t) dt + \int_0^{s_{j+1} - s_j} K(s) ds \right] = \frac{\epsilon}{A} \int_0^{s_{j+1}} K(s) ds.$$

It follows that $d(x_p, x_q) \leq \frac{\epsilon}{A} \int_0^T K(s) ds = \epsilon$, proving that $\{x_n\}_n$ is a Cauchy sequence. \square

We summarize the results in Lemma 1.9 by the following

Corollary 1.10. *Under hypothesis 1.8, if $T > 0, x \in X, u : [0, T] \rightarrow M$ is a regulated function, and $\{u_n\}_n$ is a sequence of piecewise constant functions uniformly converging to u , then*

$$\lim_{n \rightarrow \infty} \Psi(T, x, u_n) \text{ exists in } X.$$

Lemma 1.11. *If Φ satisfies hypothesis 1.8, $T > 0, x \in X, u : [0, T] \rightarrow M$ is a regulated function, and $\{u_n\}_n, \{v_n\}_n$ are two sequences of piecewise constant functions, both uniformly converging to u , then*

$$\lim_{n \rightarrow \infty} \Psi(T, x, u_n) = \lim_{n \rightarrow \infty} \Psi(T, x, v_n).$$

Proof. Let $\{w_n\}_n$ be the sequence constructed by:

$$w_n := \begin{cases} u_k & , \text{ if } n = 2k + 1 ; \\ v_k & , \text{ if } n = 2k. \end{cases} \quad (1.6)$$

It is not difficult to show that $w_n \xrightarrow{n} u$ uniformly. To see this, let $\epsilon > 0$ and let $N_1, N_2 \in \mathbb{N}$ be such that

$$n \geq N_1 \text{ implies } d(u_n(t), u(t)) < \epsilon, \text{ for all } t \in [0, T]$$

and

$$n \geq N_2 \text{ implies } d(v_n(t), u(t)) < \epsilon, \text{ for all } t \in [0, T].$$

Set $N := 2 \cdot \max(N_1, N_2)$.

If $n \geq N$ then $d(w_n(t), u(t)) < \epsilon$, for all $t \in [0, T]$.

Then,

$$\lim_{k \rightarrow \infty} \Psi(T, x, u_k) = \lim_{k \rightarrow \infty} \Psi(T, x, w_{2k+1}) = \lim_{k \rightarrow \infty} \Psi(T, x, w_{2k}) = \lim_{k \rightarrow \infty} \Psi(T, x, v_k).$$

□

Definition 1.12. *If Φ satisfies hypothesis 1.8, $T > 0, x \in X$, and $u : [0, T] \rightarrow M$ is a regulated function, define*

$$\Psi(T, x, u) := \lim_{n \rightarrow \infty} \Psi(T, x, u_n),$$

where $\{u_n\}_n$ is any sequence of piecewise constant functions uniformly converging to u .

Note that the consistency of this definition is assured by Lemma 1.11.

Lemma 1.13. *If $u : [0, T] \rightarrow M$ is a piecewise constant function then the map*

$$x \mapsto \Psi(T, x, u)$$

is continuous on X .

Proof. Let u have the unique representation given by maximal intervals of constancy and let k be the number of these intervals. We will prove the result inductively by k .

If $k = 1$ then u is constant on $[0, T]$, say $u \equiv m \in M$. Then $\Psi(T, x, u) = \Phi(T, x, m)$ and the result follows from the continuity of Φ in the second argument.

Now suppose that the result is true for any piecewise constant function with k maximal intervals of constancy and let u have $k+1$ maximal intervals of constancy. Specifically, assume that $0 = t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} = T$ are such that $u|_{(t_{i-1}, t_i)} \equiv m_i$ for $i \in \{1, 2, \dots, k+1\}$. Then

$$\Psi(T, x, u) = \Phi(t_{k+1} - t_k, \Psi(t_k, x, u|_{[0, t_k]}), m_{k+1})$$

and the result follows from the inductive hypothesis combined with the continuity of Φ in the second variable. □

Proposition 1.14. *The function Ψ satisfies the semigroup property for regulated controls:*

$$\Psi(T + S, x, u * v) = \Psi(S, \Psi(T, x, u), v), \quad (1.7)$$

where $u : [0, T] \rightarrow M$ and $v : [0, S] \rightarrow M$ are regulated functions.

Proof. Let $u : [0, T] \rightarrow M$ and $v : [0, S] \rightarrow M$ be regulated functions. Consider two sequences of piecewise constant functions, $\{u_n\}_n$ and $\{v_k\}_k$, with $u_n \xrightarrow{u} u$ and $v_k \xrightarrow{v} v$. Then, fixing a $k \in \mathbb{N}$, note that $u_n * v_k \xrightarrow{n} u * v_k$ uniformly, by Remark 1.4. We have:

$$\Psi(T + S, x, u * v_k) = \lim_{n \rightarrow \infty} \Psi(T + S, x, u_n * v_k),$$

by the definition of $\Psi(T + S, x, u * v_k)$. Then, using Proposition 1.7, we may continue and write

$$\lim_{n \rightarrow \infty} \Psi(T + S, x, u_n * v_k) = \lim_{n \rightarrow \infty} \Psi(S, \Psi(T, x, u_n), v_k).$$

Using Lemma 1.13, we see that

$$\lim_{n \rightarrow \infty} \Psi(S, \Psi(T, x, u_n), v_k) = \Psi(S, \lim_{n \rightarrow \infty} \Psi(T, x, u_n), v_k) = \Psi(S, \Psi(T, x, u), v_k).$$

Thus,

$$\Psi(T + S, x, u * v_k) = \Psi(S, \Psi(T, x, u), v_k), \text{ for every } k.$$

Now, taking the limit as k tends to infinity of both sides, we conclude that

$$\Psi(T + S, x, u * v) = \Psi(S, \Psi(T, x, u), v).$$

□

1.5 Control systems via dynamical polysystems

Let $X = \mathbb{R}^n$, let M be a metric space, and let $f : X \times M \rightarrow \mathbb{R}^n$ be a function satisfying the conditions of being a *right hand side*, (*rhs*, see [So]), that is:

$$f(\cdot, m) \text{ is of class } C^1 \text{ for each fixed } m$$

and

$$f \text{ and } f_x \text{ are continuous on } X \times M.$$

This *rhs* gives rise to a time-invariant continuous-time control system, whose transition function $\phi(t, x_0, u)$ is defined to be the unique maximal solution of the initial value problem

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(0) = x_0. \end{cases} \quad (1.8)$$

For details on this control system one may consult [So], pp. 44.

We would like to define a family of dynamical systems on X , using the above system and constant controls. The control system itself can be defined on arbitrary open subsets X of \mathbb{R}^n ; but in order to avoid additional constraints, for the family of

dynamical systems constructed below, we want to assume that X is the whole of \mathbb{R}^n . Then the existence of global solutions for the initial value problem 1.8 is required. And this existence is guaranteed by the following result, a direct consequence of Proposition C.3.8 in [So].

Proposition 1.15. *Assume that the rhs f satisfies the property that for every $m \in M$ there exists a constant $\alpha > 0$ so that*

$$|f(x, m) - f(y, m)| \leq \alpha|x - y|, \quad (1.9)$$

for all x and y in $X = \mathbb{R}^n$. Then the initial value problem 1.8 admits global solutions, for every constant control $u \equiv m$.

Now, Proposition 1.15 allows us to construct a family of dynamical systems on $X = \mathbb{R}^n$. For $t > 0, x \in X$, and $m \in M$, set

$$\Phi(t, x, m) := \phi(t, x, u), \text{ where } u : [0, t] \rightarrow M, u \equiv m. \quad (1.10)$$

In other words, $\Phi(t, x, m)$ is defined to be the unique solution of the initial value problem

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}, m) \\ \mathbf{x}(0) = x. \end{cases} \quad (1.11)$$

The continuity of Φ involves smooth dependence on initial conditions and parameters. The following result is needed.

Theorem 1.16. *(Brauer-Nohel, see [BN], pp. 331) Let $g, h : D \rightarrow \mathbb{R}^n$ be bounded functions of class \mathbf{C}^1 and let $K > 0$ be an upper bound for $|g'(x)|$ as well as for $|h'(x)|$ on D . Let ϕ and ψ be solutions of $\dot{x} = g(x)$ and $\dot{x} = h(x)$, respectively, with $\phi(0) = x_0$ and $\psi(0) = y_0$, existing on a common interval $[0, T]$. Suppose*

$|g(x) - h(x)| \leq \eta$, for all $x \in D$. Then ϕ and ψ satisfy the estimate

$$|\phi(t) - \psi(t)| \leq |x_0 - y_0|e^{Kt} + \eta T e^{Kt},$$

for all $t \in [0, T]$.

Proposition 1.17. *The map Φ is continuous on $[0, \infty) \times X \times M$.*

Proof. Fix $t_0 > 0, x_0 \in X$ and $m_0 \in M$. We will show that Φ is continuous at (t_0, x_0, m_0) .

Consider $\epsilon > 0$.

If ξ is the (global) solution of the initial value problem 1.11, with $x = x_0$ and $m = m_0$, by the continuity of ξ at t_0 it follows that there exists a $\nu > 0$ such that, for every $t \in (t_0 - \nu, t_0 + \nu)$,

$$|\xi(t) - \xi(t_0)| \leq \frac{\epsilon}{2},$$

or, equivalently,

$$|\Phi(t, x_0, m_0) - \Phi(t_0, x_0, m_0)| \leq \frac{\epsilon}{2}.$$

By the continuity of $f_x(\cdot, \cdot)$, there exist neighborhoods A of x_0 and B of m_0 and a positive number K such that $|f_x(x, m)| \leq K$ for all $x \in A$ and $m \in B$.

Let $\eta := \epsilon \frac{e^{-K(t_0+\nu)}}{4(t_0+\nu)}$.

Let $\delta > 0$ be such that $\delta < \eta(t_0 + \nu)$, $B(x_0, \delta) \subset A$, $B(m_0, \delta) \subset B$, and $|f(x, m) - f(x, m_0)| \leq \eta$, whenever $x \in B(x_0, \delta)$ and $m \in B(m_0, \delta)$. We may also assume that $\delta < \nu$.

Let $t \in B(t_0, \delta)$, $x \in B(x_0, \delta)$, and $m \in B(m_0, \delta)$.

Now, set $g(\cdot) := f(\cdot, m)$, $h(\cdot) := f(\cdot, m_0)$, $D := B(x_0, \delta)$, and $T := t$, and apply Theorem 1.16 to obtain

$$|\Phi(t, x, m) - \Phi(t, x_0, m_0)| \leq |x - x_0|e^{Kt} + \eta t e^{Kt}.$$

Since $t < t_0 + \nu$ and $|x - x_0| \leq \delta < \eta(t_0 + \nu)$, we have

$$|\Phi(t, x, m) - \Phi(t, x_0, m_0)| \leq e^{K(t_0 + \nu)}[\eta(t_0 + \nu) + \eta(t_0 + \nu)] \leq 2e^{K(t_0 + \nu)}\eta(t_0 + \nu) = \frac{\epsilon}{2}.$$

Using the triangle inequality, we conclude that

$$\begin{aligned} |\Phi(t, x, m) - \Phi(t_0, x_0, m_0)| &\leq |\Phi(t, x, m) - \Phi(t, x_0, m_0)| + |\Phi(t, x_0, m_0) - \Phi(t_0, x_0, m_0)| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

The following result, known as Gronwall's Lemma, is needed for the main result in this section. For a detailed proof, see [CLSW], pp. 179.

Lemma 1.18. *Let $z : [0, T] \rightarrow \mathbb{R}^n$ satisfy*

$$|\dot{z}(t)| \leq \gamma|z(t)| + c, \text{ for all } t \in [0, T],$$

where γ and c are non-negative constants.

Then, for every $t \in [0, T]$, we have

$$|z(t) - z(0)| \leq (e^{\gamma t} - 1)|z(0)| + c \int_0^t e^{\gamma(t-s)} ds.$$

Proposition 1.19. *Let the rhs f satisfy the Lipschitz condition that there exist positive constants α and β such that for every $x, y \in X, m, m' \in M$,*

$$|f(x, m) - f(y, m')| \leq \alpha|x - y| + \beta d(m, m'). \quad (1.12)$$

Then Φ satisfies Hypothesis 1.8.

Proof. Note, first, that condition 1.9 is automatically satisfied, so the definition of Φ makes sense.

Let $T > 0$ and define $K : [0, T] \rightarrow [1, \infty)$ by $K(t) := e^{\alpha t}$. Clearly,

$$K(t_1)K(t_2) = K(t_1 + t_2), \text{ for all } t_1, t_2.$$

Now, let $\epsilon > 0$ and set $\delta := \epsilon/\beta$.

Fix $x_0, y_0 \in X$, $t \in [0, T]$, and $m, m' \in M$ with $d(m, m') \leq \delta$. For simplicity, denote $x(t) := \Phi(t, x_0, m)$ and $y(t) := \Phi(t, y_0, m')$ and set $z(t) := x(t) - y(t)$. Then $\dot{x}(t) = f(x(t), m)$, $\dot{y}(t) = f(y(t), m)$, $x(0) = x_0$, and $y(0) = y_0$. So we may write

$$|\dot{z}(t)| = |\dot{x}(t) - \dot{y}(t)| = |f(x(t), m) - f(y(t), m')|,$$

and then, using 1.12,

$$|\dot{z}(t)| \leq \alpha|x(t) - y(t)| + \beta d(m, m') \leq \alpha|z(t)| + \epsilon.$$

Now using Lemma 1.18, with α and ϵ as γ and c , respectively, it follows that

$$|z(t) - z(0)| \leq (e^{\alpha t} - 1)|z(0)| + \epsilon \int_0^t e^{\alpha(t-s)} ds.$$

Making the change of variable $u := t - s$, we observe that

$$\int_0^t e^{\alpha(t-s)} ds = - \int_t^0 e^{\alpha u} du = \int_0^t e^{\alpha s} ds,$$

and therefore

$$|z(t)| \leq |z(0)| + |z(t) - z(0)| \leq K(t)|z(0)| + \epsilon \int_0^t K(s) ds,$$

or, equivalently,

$$|\Phi(t, x_0, m) - \Phi(t, y_0, m')| \leq K(t)|x_0 - y_0| + \epsilon \int_0^t K(s) ds.$$

□

Theorem 1.20. *Assume that the metric space M is separable and consider the control system with regulated controls given by a rhs f that satisfies the Lipschitz condition 1.12. Then its transition function ϕ can be constructed from the polysystem Φ defined by Equation 1.10, according to the procedure described in the previous section. In other words, ϕ and the function Ψ from Definition 1.12 are one and the same.*

Proof. We first prove that ϕ and Ψ agree on piecewise constant controls.

Let $T > 0$, $x \in X$, and $u : [0, T] \rightarrow M$, a piecewise constant function defined by a finite partition $0 = t_0 < t_1 < t_2 < \dots < t_k = T$ of the interval $[0, T]$, and elements m_1, m_2, \dots, m_k of M with $u(t) = m_i$ whenever $t \in (t_{i-1}, t_i)$, for $i \in \{1, 2, \dots, k\}$.

By Proposition 1.7, we may write:

$$\begin{aligned} \Psi(T, x, u) &= \Psi(t_k - t_{k-1}, \Psi(t_{k-1} - t_{k-2}, \dots, \Psi(t_1, x, m_1), m_2), \dots, m_k) = \\ &= \Psi(t_k - t_{k-1}, \Psi(t_{k-1} - t_{k-2}, \dots, \Phi(t_1, x, m_1), m_2), \dots, m_k). \end{aligned}$$

Using equation 1.10 repeatedly and the semigroup property of ϕ , we obtain:

$$\begin{aligned} \Psi(T, x, u) &= \Psi(t_k - t_{k-1}, \Psi(t_{k-1} - t_{k-2}, \dots, \phi(t_1, x, m_1), m_2), \dots, m_k) = \dots \\ &= \phi(t_k - t_{k-1}, \phi(t_{k-1} - t_{k-2}, \dots, \phi(t_1, x, m_1), m_2), \dots, m_k) = \phi(T, x, u). \end{aligned}$$

Now let t and x be as before and let u be a regulated control. Lema 2.8.2 in [So] implies that if M is separable and $\phi(t, x, u) = z$, then there exists a sequence of piecewise constant controls $\{u_n\}_n$ converging uniformly to u so that, if $z_n := \phi(t, x, u_n)$, then z_n converges to z . This means that

$$\phi(t, x, u) = \lim_n \phi(t, x, u_n).$$

It follows by the first part of the proof that

$$\phi(t, x, u_n) = \Psi(t, x, u_n).$$

So

$$\phi(t, x, u) = \lim_n \Psi(t, x, u_n) = \Psi(t, x, u),$$

by Definition 1.12. □

Chapter 2

Limit Sets

2.1 Definition and related notations

Let \mathcal{F} be a family of continuous dynamical systems, all defined on a metric space X and define a dynamical polysystem (\mathcal{G}, X) as in Definition 1.1. Recall that the pair (\mathcal{S}, X) is called the **accessibility polysystem** on X generated by \mathcal{F} .

Remark 2.1. *An element of \mathcal{G} has form*

$$g = (t, h) = (t_1 + t_2 + \dots + t_k, \Phi_{t_1}^1 \circ \Phi_{t_2}^2 \circ \dots \circ \Phi_{t_k}^k)$$

with $t_i \in \mathbb{R}$ and $\Phi^i \in \mathcal{F}$, for $0 \leq i \leq k$.

The group \mathcal{G} acts on X by

$$(t, h).x = h(x)$$

This is indeed an action, since $(t_1, h_1).((t_2, h_2).x) = (t_1, h_1).h_2(x) = h_1(h_2(x)) = (t_1 + t_2, h_1 \circ h_2).x = ((t_1, h_1).(t_2, h_2)).x$.

Moreover, any subsemigroup of \mathcal{G} acts on X by restricting this action.

A **subsystem** of (\mathcal{G}, X) can be defined in a natural way, by restricting \mathcal{F} to a subset.

The polysystem (\mathcal{G}, X) can be considered (and, in fact, is) a \mathcal{G} -dynamical system. In what follows, though, notions related to dynamical systems in general may be defined or approached differently, given the concern for regarding polysystems in close connection with continuous-time dynamical systems.

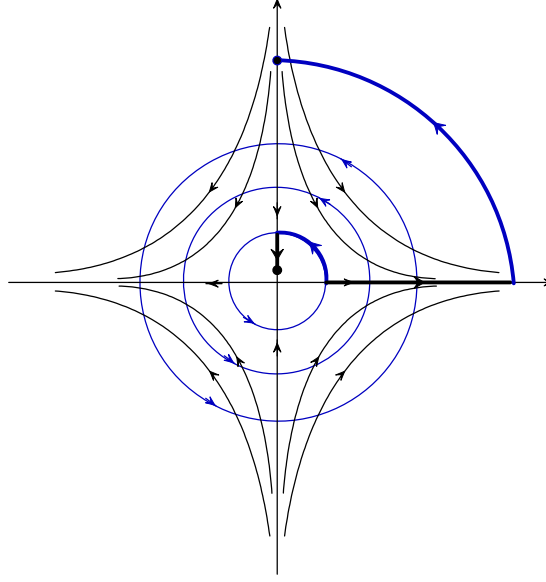
Remark 2.2. *The group \mathcal{G} is generated by the accessibility semigroup \mathcal{S} , in the sense that \mathcal{G} is the smallest subgroup containing \mathcal{S} .*

In general, \mathcal{G} and \mathcal{S} are not commutative, as shown in the following

Example 2.3. Let \mathcal{F} contain a rotation on the plane, Φ , and a hyperbolic flow on the plane, Ψ , defined by

$$\Phi(t, z) := re^{i(t+\theta)} \quad \text{and} \quad \Psi(t, z) := (e^t x, e^{-t} y)$$

where $z = (x, y) = re^{i\theta}$ is given in cartesian and polar coordinates.



To show that $\Phi_s \circ \Psi_t \neq \Psi_t \circ \Phi_s$, let $s = t = \frac{\pi}{2}$ and $z = (1, 0) = 1e^{i(0)}$. Then

$$(\Psi_{\frac{\pi}{2}} \circ \Phi_{\frac{\pi}{2}})(z) = \Phi_{\frac{\pi}{2}}(e^{i\frac{\pi}{2}}) = \Phi_{\frac{\pi}{2}}(0, 1) = (0, e^{-\frac{\pi}{2}})$$

and

$$(\Phi_{\frac{\pi}{2}} \circ \Psi_{\frac{\pi}{2}})(z) = \Psi_{\frac{\pi}{2}}(e^{\frac{\pi}{2}}, 0) = (0, e^{\frac{\pi}{2}}).$$

Definition 2.4. For $x \in X$, define the \mathcal{G} – **orbit** of x to be $\mathcal{G}.x := \{g.x : g \in \mathcal{G}\}$. A point $x \in X$ is called a **rest point** if $\mathcal{G}.x = \{x\}$. Similarly, define the \mathcal{S} – **orbit** (or the **accessible orbit**) of x to be $\mathcal{S}.x := \{g.x : g \in \mathcal{S}\}$.

Remark 2.5. The point x is a rest point if and only if x is a rest point for each dynamical system in \mathcal{F} .

Proof. If x is a rest point for (\mathcal{G}, X) and $\Phi \in \mathcal{F}$, then $\Phi_t(x) = \Phi(t, x) = x$ for any $t \in \mathbb{R}$, so x is a rest point for Φ .

Conversely, if x is a rest point for all dynamical systems in \mathcal{F} and $g = (t_1 + t_2 + \dots + t_k, \Phi_{t_1}^1 \circ \Phi_{t_2}^2 \circ \dots \circ \Phi_{t_k}^k) \in \mathcal{G}$, then $g.x = (\Phi_{t_1}^1 \circ \Phi_{t_2}^2 \circ \dots \circ \Phi_{t_k}^k)(x) = (\Phi_{t_1}^1 \circ \Phi_{t_2}^2 \circ \dots \circ \Phi_{t_{k-1}}^{k-1})(x) = \dots = \Phi_{t_1}^1(x) = x$.

□

Definition 2.6. Let \mathcal{T} be a subsemigroup of \mathcal{G} . A subset M of X is said to be \mathcal{T} -invariant if $\mathcal{T}.M \subseteq M$.

Proposition 2.7. The closure of a \mathcal{T} -invariant set is \mathcal{T} -invariant.

Proof. If M is invariant and x is in \overline{M} , then $x_n \xrightarrow{n} x$ for some sequence $\{x_n\}_n \subseteq M$. If $(t, h) \in \mathcal{T}$ then h is continuous, so $h(x_n) \xrightarrow{n} h(x)$. By the invariance of M , $h(x_n) \in M$ for all n , therefore $h(x) \in \overline{M}$.

□

Proposition 2.8. If $\{M_i\}_i$ is a collection of \mathcal{T} -invariant subsets of X , then $\bigcup_i M_i$ and $\bigcap_i M_i$ are \mathcal{T} -invariant sets.

Proof. Using the fact that function evaluation commutes with unions and intersections, we see that

$$\mathcal{T}.\left(\bigcup_i M_i\right) = \bigcup_i (\mathcal{T}.M_i) \subseteq \bigcup_i M_i$$

and

$$\mathcal{T}.\left(\bigcap_i M_i\right) = \bigcap_i (\mathcal{T}.M_i) \subseteq \bigcap_i M_i.$$

□

2.2 Positive limit sets

In what follows, \mathcal{G} and \mathcal{S} are given the relative product topology from the usual topology on \mathbb{R} and the relative topology of pointwise convergence from $\text{Homeo}(X)$.

So $(t_n, h_n) = g_n \xrightarrow{n} g = (t, h)$ means $t_n \xrightarrow{n} t$ in \mathbb{R} and $h_n \xrightarrow{n} h$ pointwise in $\text{Homeo}(X)$.

Proposition 2.9. *\mathcal{G} and \mathcal{S} are separately continuous topological semigroups (i.e. left and right translations in \mathcal{G} and \mathcal{S} are continuous).*

Proof. For $a \in \mathcal{G}$, define:

$$L_a : \mathcal{G} \rightarrow \mathcal{G}, L_a(g) := ag$$

and

$$R_a : \mathcal{G} \rightarrow \mathcal{G}, R_a(g) := ga.$$

Suppose $a = (t, h) \in \mathcal{G}$ and $g_n \xrightarrow{n} g$. Then $g_n \cdot x \xrightarrow{n} g \cdot x$ for any $x \in X$, and by the continuity of h at $g \cdot x$, $a \cdot (g_n \cdot x) \xrightarrow{n} a \cdot (g \cdot x)$. Thus, $ag_n \xrightarrow{n} ag$, so L_a is continuous at g . Also, $g_n \cdot (a \cdot x) \xrightarrow{n} g \cdot (a \cdot x)$ for any $x \in X$, and so $g_n a \xrightarrow{n} ga$, so R_a is continuous at g .

The same argument functions for \mathcal{S} . □

For $T \geq 0$, consider

$$\mathcal{S}_{\geq T} := \{(t_1 + \dots + t_k, \Phi_{t_k}^k \circ \dots \circ \Phi_{t_1}^1) : \Phi^1, \dots, \Phi^k \in \mathcal{F}, t_1 + \dots + t_k \geq T\},$$

the elements in \mathcal{S} having total time at least T .

Note that $\mathcal{S}_{\geq T}$ is an ideal of \mathcal{S} , since $\mathcal{S}\mathcal{S}_{\geq T} \cup \mathcal{S}_{\geq T}\mathcal{S} \subseteq \mathcal{S}_{\geq T}$.

Definition 2.10. *If $x \in X$, the **positive limit set** of x is defined by*

$$\lambda^+(x) := \bigcap_{T \geq 0} (\overline{\mathcal{S}_{\geq T} \cdot x}).$$

Lemma 2.11. *If $\mathcal{A} \subseteq \mathcal{G}$ and $B \subseteq X$ then $\mathcal{A}.\overline{B} \subseteq \overline{\mathcal{A}.B}$.*

Proof. Knowing that a homeomorphism carries closed sets into closed sets, we may write

$$\mathcal{A}.\overline{B} = \bigcup_{g \in \mathcal{A}} g.\overline{B} = \bigcup_{g \in \mathcal{A}} \overline{g.B} \subseteq \overline{\mathcal{A}.B}$$

□

Proposition 2.12. *For $x \in X$, $\lambda^+(x)$ is closed and invariant.*

Proof. We see that $\lambda^+(x)$ is automatically closed, as an intersection of closed sets.

For the \mathcal{S} -invariance of $\lambda^+(x)$, apply Lemma 2.11 with $\mathcal{A} := \mathcal{S}$ and $B := \mathcal{S}_{\geq T}.x$:

$$\mathcal{S}.\overline{(\mathcal{S}_{\geq T}.x)} \subseteq \overline{\mathcal{S}.\mathcal{S}_{\geq T}.x}.$$

Then

$$\mathcal{S}.\lambda^+(x) = \mathcal{S}.\bigcap_{T \geq 0} \overline{\mathcal{S}_{\geq T}.x} \subseteq \bigcap_{T \geq 0} \mathcal{S}.\overline{(\mathcal{S}_{\geq T}.x)} \subseteq \bigcap_{T \geq 0} \overline{\mathcal{S}.\mathcal{S}_{\geq T}.x} = \bigcap_{T \geq 0} \overline{\mathcal{S}_{\geq T}.x} = \lambda^+(x).$$

□

2.3 Connectedness of limit sets

Lemma 2.13. *If T is a left topological semigroup with 1 and A is a connected subset of T containing 1, then A generates a connected subsemigroup.*

Proof. Let $\langle A \rangle$ be the subsemigroup generated by A . Then $\langle A \rangle = \bigcup_{n \geq 1} A^n$. It is enough to show (by induction) that A^n is connected for each n , for then $\langle A \rangle$ is a union of connected sets, all containing 1, and the result follows from the Clover Leaf Theorem.

Assume that A^n is connected. Then

$$A^{n+1} = AA^n = \bigcup_{a \in A} aA^n$$

Since left translations in T are continuous, each aA^n is the continuous image of a connected set, so it is connected. Also, each aA^n intersects the connected set

A , because $1 \in A^n$. Again from the Clover Leaf Theorem, it follows that A^{n+1} is connected. \square

Proposition 2.14. *\mathcal{G} and \mathcal{S} are connected.*

Proof. Observe that $\mathcal{G} = \langle A \rangle$, where

$$A = \{(t, \Phi_t) : t \in \mathbb{R}, \Phi \in \mathcal{F}\}$$

So, by Lemma 2.13 and Proposition 2.9, it suffices to show that A is connected.

We may write

$$A = \bigcup_{\Phi \in \mathcal{F}} \{(t, \Phi_t) : t \in \mathbb{R}\} = \bigcup_{\Phi \in \mathcal{F}} \text{Im}(\omega_\Phi),$$

where $\omega_\Phi : \mathbb{R} \rightarrow \mathbb{R} \times \text{Homeo}(X)$ is the map defined by $\omega_\Phi(t) := (t, \Phi_t)$.

To show that $\text{Im}(\omega_\Phi)$ is connected for every $\Phi \in \mathcal{F}$, it suffices to show that ω_Φ is continuous.

Let $t_n \xrightarrow{n} t$ in \mathbb{R} . For any $x \in X$, $\Phi_{t_n}(x) = \Phi(t_n, x) \xrightarrow{n} \Phi(t, x)$, by the continuity (in the first argument) of the dynamical system Φ . So $\Phi_{t_n} \xrightarrow{n} \Phi_t$ pointwise, that is $\omega_\Phi(t_n) \xrightarrow{n} \omega_\Phi(t)$.

Thus A is a union of connected sets, all containing 1_X , so it is connected.

Replacing ω_Φ by its restriction to the connected interval $[0, \infty)$, the same proof works for \mathcal{S} . \square

Lemma 2.15. *An ideal of a connected, separately continuous topological semigroup with identity is connected.*

Proof. If I is an ideal of a connected, separately continuous topological semigroup with identity T , we may write:

$$I = IT = TI = \bigcup_{a \in I} aT = \bigcup_{a \in I} Ta.$$

If we fix $a_0 \in I$ then, for any $a \in I$ we have $a_0T \cap Ta \neq \emptyset$, since $a_0a \in a_0T \cap Ta$.

Since left and right translations in T are continuous and T is connected, a_0T is connected and Ta is connected, for all $a \in I$. So I is a union of connected sets, each intersecting the connected set a_0T , and thus I is connected. \square

Corollary 2.16. $\mathcal{S}_{\geq T}$ is a connected ideal of \mathcal{S} .

Lemma 2.17. If $\{C_i\}_i$ is a descending family of non-empty closed connected sets in a locally compact space and $\bigcap_i C_i$ is non-empty and compact, then $\bigcap_i C_i$ is connected.

Proof. By local compactness, there exists a compact set K and an open set V such that $\bigcap_i C_i \subseteq V \subseteq K$.

For some j we have $C_j \cap K \subseteq V$, otherwise $(C_i \cap K) \setminus V$ is a closed non-empty set for all i , so $\bigcap_i ((C_i \cap K) \setminus V) = (\bigcap_i (C_i \cap K)) \setminus V \neq \emptyset$, by the finite intersection property of the compact subspace K , contradiction. Further, C_j being connected, we must have $C_j \setminus K = \emptyset$, or else $C_j = (C_j \cap K) \cup (C_j \setminus K)$ is a separation. So $C_j \subseteq V$. Then, without loss of generality, we may assume that $C_i \subseteq V \subseteq K$ for all i .

Then $\bigcap_i C_i$ is connected, as the intersection of a descending family of non-empty connected closed sets in the compact space K (see [Ku], pp. 170).

\square

To prove the main result of this section, we use the following theorem.

Theorem 2.18. ([Ku], pp. 172) Let Y be a Hausdorff continuum (a compact connected Hausdorff space) and let U be open in Y . Then every component of U has a limit point in $\bar{U} \setminus U$.

Theorem 2.19. *If X is locally compact and $x \in X$ then $\lambda^+(x)$ is connected whenever it is compact. If $\lambda^+(x)$ is not compact then none of its components is compact.*

Proof. From the Corollary 2.16 it follows that $\mathcal{S}_{\geq T}.x$ is connected, as the continuous image of $\mathcal{S}_{\geq T}$ under the evaluation map at x . Then $\overline{\mathcal{S}_{\geq T}.x}$ is connected, as the closure of a connected set. Now $\lambda^+(x) = \bigcap_{T \geq 0} \overline{(\mathcal{S}_{\geq T}.x)}$, the intersection of a descending family of non-empty closed connected sets.

If $\lambda^+(x)$ is compact, then it is connected by Lemma 2.17.

For the second part of the theorem, consider $\tilde{X} = X \cup \{\infty\}$, the one-point compactification of X . Extend each dynamical system Φ in \mathcal{F} to a dynamical system $\tilde{\Phi}$ on \tilde{X} , where $\tilde{\Phi}(t, x) = \Phi(t, x)$ for $x \in X$ and $\tilde{\Phi}(t, \infty) = \infty$. Now call $\tilde{\lambda}^+(x)$ the positive limit set of x in the polysystem defined by $\tilde{\mathcal{F}} = \{\tilde{\Phi} : \Phi \in \mathcal{F}\}$.

Then

$$\tilde{\lambda}^+(x) = \bigcap_{T \geq 0} \overline{\mathcal{S}_{\geq T}.x}^{\tilde{X}}$$

We see that $\lambda^+(x) \subseteq \tilde{\lambda}^+(x) \subseteq \lambda^+(x) \cup \{\infty\}$, since the only possible addition to the closure in X to get the closure in \tilde{X} is ∞ .

If $\lambda^+(x)$ is not compact then $\tilde{\lambda}^+(x) \neq \lambda^+(x)$, since $\tilde{\lambda}^+(x)$ is compact, as closed set in a compact space. So we must have

$$\tilde{\lambda}^+(x) = \lambda^+(x) \cup \{\infty\}$$

Now, $\tilde{\lambda}^+(x)$ being compact in \tilde{X} , by the first part of the theorem, it is connected. Therefore we may apply Theorem 2.18, observing that $\lambda^+(x)$ is open in $\tilde{\lambda}^+(x)$. It follows that every component of $\lambda^+(x)$ has ∞ as a limit point, and thus is not compact.

□

Chapter 3

Stability of Compact Sets

3.1 More limit sets

Throughout this chapter, we consider a dynamical polysystem on X , generated by a family of continuous dynamical systems \mathcal{F} .

To study various notions related to the stability of dynamical polysystems, one may consider a stronger version of the positive limit set. For the purpose of simplifying formulas, let $\mathcal{N}(x)$ denote the set of all open neighborhoods of x in X .

Definition 3.1. *Let \mathcal{T} be a subsemigroup of \mathcal{G} .*

*The **positive \mathcal{T} – limit set** of a point $x \in X$ is defined by*

$$\Lambda_{\mathcal{T}}^+(x) = \bigcap_{g \in \mathcal{T}} \overline{\mathcal{S}g.x}$$

*The **first \mathcal{T} – prolongational limit set** of a point $x \in X$ is defined by*

$$J_{\mathcal{T}}^+(x) = \bigcap_{\substack{g \in \mathcal{T} \\ U \in \mathcal{N}(x)}} \overline{\mathcal{S}g.U}$$

*The **first \mathcal{T} – prolongation** of a point $x \in X$ is defined by*

$$D_{\mathcal{T}}^+(x) = \bigcap_{U \in \mathcal{N}(x)} \overline{\mathcal{S}\mathcal{T}.U}$$

If M is a subset of X , we can define

$$\Lambda_{\mathcal{T}}^+(M) = \bigcup_{x \in M} \Lambda_{\mathcal{T}}^+(x), \quad J_{\mathcal{T}}^+(M) = \bigcup_{x \in M} J_{\mathcal{T}}^+(x) \quad \text{and} \quad D_{\mathcal{T}}^+(M) = \bigcup_{x \in M} D_{\mathcal{T}}^+(x).$$

For simplicity, if $\mathcal{T} = \mathcal{S}$, the sets defined above will be denoted by $\Lambda^+(x)$, $J^+(x)$, and $D^+(x)$, respectively.

Remark 3.2. $\Lambda^+(x) \subset \lambda^+(x)$.

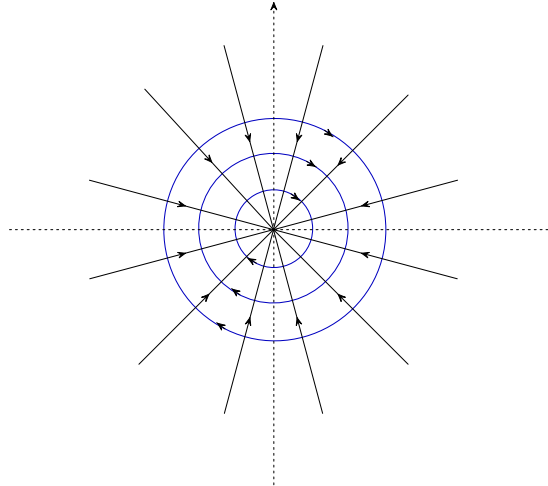
Proof. Assume that $y \in \Lambda^+(x)$ and let $T \geq 0$. Let $\Phi \in \mathcal{F}$ and denote

$$g := (T, \Phi_T) \in \mathcal{S}_{\geq T}.$$

Then $y \in \overline{\mathcal{S}g.x} \subset \overline{\mathcal{S}_{\geq T}.x}$. Thus, $y \in \bigcap_{T \geq 0} (\overline{\mathcal{S}_{\geq T}.x}) = \lambda^+(x)$. \square

This containment may be strict, as shown in the following example.

Example 3.3. Let \mathcal{F} contain two dynamical systems on the plane: a rotation and a radial collapse to the origin.



If x is any point in the plane, different from the origin, then $\Lambda^+(x)$ is the origin and $\lambda^+(x)$ is the closed disc centered at the origin, with radius $|x|$.

Proposition 3.4. The sets in Definition 3.1 satisfy:

1. $\Lambda_T^+(x) \subseteq J_T^+(x) \subseteq D_T^+(x)$
2. $\Lambda_T^+(x)$, $J_T^+(x)$ and $D_T^+(x)$ are closed \mathcal{S} -invariant sets.

Proof. 1. If $y \in \Lambda_{\mathcal{T}}^+(x)$ then, for any $g \in \mathcal{T}$ and $U \in \mathcal{N}(x)$, $y \in \overline{\mathcal{S}g.x} \subseteq \overline{\mathcal{S}g.U}$.

So $\Lambda_{\mathcal{T}}^+(x) \subseteq J_{\mathcal{T}}^+(x)$. Further,

$$J_{\mathcal{T}}^+(x) = \bigcap_{\substack{g \in \mathcal{T} \\ U \in \mathcal{N}(x)}} \overline{\mathcal{S}g.U} \subseteq \bigcap_{\substack{g \in \mathcal{T} \\ U \in \mathcal{N}(x)}} \overline{\mathcal{S}\mathcal{T}.U} = \bigcap_{U \in \mathcal{N}(x)} \overline{\mathcal{S}\mathcal{T}.U} = D_{\mathcal{T}}^+(x)$$

2. For $x \in X$, $\Lambda_{\mathcal{T}}^+(x)$, $J_{\mathcal{T}}^+(x)$ and $D_{\mathcal{T}}^+(x)$ are closed, as intersections of closed sets.

To prove the invariance, suppose that $y \in \Lambda_{\mathcal{T}}^+(x)$ and let $(t, h) \in \mathcal{S}$. Since for any $g \in \mathcal{T}$, $y \in \overline{\mathcal{S}g.x}$, and h is a homeomorphism, it follows that

$$h(y) \in h(\overline{\mathcal{S}g.x}) = \overline{h(\mathcal{S}g.x)} \subseteq \overline{\mathcal{S}g.x}.$$

Then

$$h(y) \in \bigcap_{g \in \mathcal{T}} \overline{\mathcal{S}g.x} = \Lambda_{\mathcal{T}}^+(x).$$

The same argument holds if x is replaced by any $U \in \mathcal{N}(x)$, thus proving the \mathcal{S} -invariance of $J_{\mathcal{T}}^+(x)$.

For the \mathcal{S} -invariance of $D_{\mathcal{T}}^+(x)$, take $y \in D_{\mathcal{T}}^+(x)$ and $(t, h) \in \mathcal{S}$. Then

$$h(y) \in h(\overline{\mathcal{S}\mathcal{T}.U}) = \overline{h(\mathcal{S}\mathcal{T}.U)},$$

for any neighborhood U of x . Then

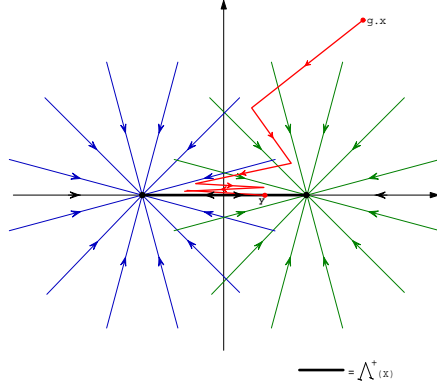
$$h(y) \in \bigcap_{U \in \mathcal{N}(x)} \overline{h(\mathcal{S}\mathcal{T}.U)} \subseteq \bigcap_{U \in \mathcal{N}(x)} \overline{\mathcal{S}\mathcal{T}.U} = D_{\mathcal{T}}^+(x).$$

□

The following example shows that $\Lambda^+(x)$, in general, is not \mathcal{G} -invariant.

Example 3.5. Let \mathcal{F} consist of two dynamical systems on the plane, defined by

$$\left\{ \begin{array}{l} \dot{x}_1 = -x_1 + 1 \\ \dot{x}_2 = -x_2 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \dot{x}_1 = -x_1 - 1 \\ \dot{x}_2 = -x_2 \end{array} \right.$$



If x is any point in the plane, then $\Lambda^+(x) = [-1, 1] \times \{0\}$. But any point y of this segment can be "pulled back" out of the segment, on the x -axis, under the action of \mathcal{G} .

Definition 3.6. A subset A of X is said to be **positively recursive** with respect to another subset B of X , if for each $s_1 \in \mathcal{S}$ there exists $s_2 \in \mathcal{S}$ and $x \in B$ such that $s_2 s_1 . x \in A$. A is said to be **self positively recursive** if it is positively recursive with respect to itself.

Definition 3.7. A point $x \in X$ is **positively Poisson stable** if every neighborhood of x is positively recursive with respect to $\{x\}$.

Proposition 3.8. A point x in X is positively Poisson stable if and only if $x \in \Lambda^+(x)$.

Proof. $x \in \Lambda^+(x) \Leftrightarrow$

$x \in \overline{\mathcal{S}g.x}$, for every $g \in \mathcal{S} \Leftrightarrow$

for every neighborhood U of x and every $g \in \mathcal{S}$, $U \cap (\mathcal{S}g.x) \neq \emptyset \Leftrightarrow$

for every neighborhood U of x and every $g \in \mathcal{S}$, there exists $g' \in \mathcal{S}$ such that

$g'g.x \in U \Leftrightarrow$

every neighborhood U of x is positively recursive with respect to $\{x\} \Leftrightarrow$

x is positively Poisson stable.

□

3.2 Attractors and stability

Definition 3.9. *If M is a non-empty compact subset of the metric space X , then the **region of weak attraction**, the **region of attraction** and the **region of uniform attraction** of M are defined, respectively, as follows:*

$$A_\omega(M) := \{x \in X : \Lambda^+(x) \cap M \neq \emptyset\}$$

$$A(M) := \{x \in X : \Lambda^+(x) \neq \emptyset \text{ and } \Lambda^+(x) \subseteq M\}$$

$$A_u(M) := \{x \in X : J^+(x) \neq \emptyset \text{ and } J^+(x) \subseteq M\}$$

Definition 3.10. *A non-empty compact set M in X is said to be:*

*a **weak attractor**, if $A_\omega(M)$ is a neighborhood of M ;*

*an **attractor**, if $A(M)$ is a neighborhood of M ;*

*a **uniform attractor**, if $A_u(M)$ is a neighborhood of M ;*

***stable**, if every neighborhood U of M contains an \mathcal{S} -invariant neighborhood V of M ;*

***asymptotically stable**, if it is stable and an attractor;*

***unstable**, if it is not stable.*

Theorem 3.11. *If X is locally compact then a non-empty compact subset M of X is stable if and only if $D^+(M) = M$.*

Proof. (\Rightarrow) Assume that M is stable. If $y \in D^+(M)$ then $y \in D^+(x)$ for some $x \in M$.

In particular, for any U open around M , $y \in \overline{\mathcal{S} \cdot U}$.

By the stability of M , for U open containing M , there exists V open such that $M \subseteq V \subseteq U$ and $\mathcal{S} \cdot V \subseteq V$. Then $y \in \overline{\mathcal{S} \cdot V} \subseteq \overline{V}$ for such V .

Therefore $y \in \overline{V}$ for any open V that contains M , that is

$$y \in \bigcap_{V \in \mathcal{N}(M)} \overline{V} = M.$$

(\Leftarrow) Suppose $D^+(M) = M$ and let V be any open neighborhood of M . By local compactness, there exists a compact set K such that $M \subseteq \text{Int}(K) \subseteq K \subseteq V$. Let x be any member of M . Then $D^+(x) \subseteq M$, and it follows that

$$\bigcap_{U \in \mathcal{B}} \overline{\mathcal{S}.U} \subseteq M,$$

where $\mathcal{B} = \{U \text{ open} : x \in U \subseteq \text{Int}(K)\}$.

Now, for the given x , there exists some $U_x \in \mathcal{B}$ such that $\overline{\mathcal{S}.U_x} \subseteq K$. This can be proven by contradiction. Suppose $\overline{\mathcal{S}.U} \not\subseteq K$ for all $U \in \mathcal{B}$. Then for every $U \in \mathcal{B}$, there exists $x_U \in U$ such that $\overline{\mathcal{S}.x_U} \not\subseteq K$. So $\mathcal{S}.x_U \not\subseteq K$, since K is closed. But $\mathcal{S}.x_U$ is a connected set satisfying:

$$(\mathcal{S}.x_U) \cap (\text{Int}(K)) \neq \emptyset \quad \text{and} \quad (\mathcal{S}.x_U) \cap (X \setminus K) \neq \emptyset.$$

Therefore $(\mathcal{S}.x_U) \cap (\partial K) \neq \emptyset$, and so $(\overline{\mathcal{S}.U}) \cap (\partial K) \neq \emptyset$. Take

$$\mathcal{A} := \{(\overline{\mathcal{S}.U}) \cap (\partial K) : U \in \mathcal{B}\}.$$

Then \mathcal{A} is a descending family of non-empty closed sets in the compact space ∂K , so $\bigcap \mathcal{A} \neq \emptyset$. This implies $M \cap (\partial K) \neq \emptyset$, contradicting the choice of K .

Take

$$W := \bigcup_{x \in M} \mathcal{S}.U_x.$$

Note that $M \subseteq W \subseteq K \subseteq V$. Also, W is \mathcal{S} -invariant, since

$$\mathcal{S}.W = \mathcal{S}. \bigcup_{x \in M} \mathcal{S}.U_x = \bigcup_{x \in M} \mathcal{S}.U_x = W.$$

For the given V , we found W open and invariant with $M \subseteq W \subseteq V$. This proves the stability of M . □

Lemma 3.12. *If $\omega \in \overline{\mathcal{S}.x}$ then $J^+(x) \subseteq J^+(\omega)$.*

Proof. Suppose $y \in J^+(x)$ and let V be open around ω and $g \in \mathcal{S}$. It suffices to show that $y \in \overline{\mathcal{S}g.V}$.

Since $\omega \in \overline{\mathcal{S}.x}$, there exists $g_1 \in \mathcal{S}$ such that $g_1.x \in V$, so $x \in g_1^{-1}.V$.

Now, $y \in J^+(x)$ implies $y \in \overline{\mathcal{S}s.U}$ for all $s \in \mathcal{S}$ and all neighborhoods U of x . In particular, for $s := gg_1$ and $U := g_1^{-1}.V$:

$$y \in \overline{\mathcal{S}gg_1.g_1^{-1}.V} = \overline{\mathcal{S}g.V}.$$

□

Corollary 3.13. *If M is a compact subset of X and $x \in A_\omega(M)$ then*

$$J^+(x) \subseteq J^+(M) \subseteq D^+(M).$$

Proof. From $x \in A_\omega(M)$, we have $\Lambda^+(x) \cap M \neq \emptyset$. So, if $\omega \in \Lambda^+(x) \cap M$, then $\omega \in \overline{\mathcal{S}.x}$, and applying the previous Lemma: $J^+(x) \subseteq J^+(\omega) \subseteq J^+(M) \subseteq D^+(M)$. □

Theorem 3.14. *Suppose X is locally compact. If a non-empty compact subset M of X is stable and a weak attractor then M is an attractor (so asymptotically stable).*

Proof. Since $A_\omega(M)$ is a neighborhood of M , it suffices to show that $A_\omega(M) \subseteq A(M)$. If $x \in A_\omega(M)$, then $\Lambda^+(x) \neq \emptyset$ and, from Corollary 3.13,

$$\Lambda^+(x) \subseteq J^+(x) \subseteq D^+(M).$$

M being stable, by theorem 3.11, $\Lambda^+(x) \subseteq M$. So M is an attractor.

□

Chapter 4

Stability of Closed Sets

4.1 Definitions for stability

We now generalize from compact sets to closed sets.

Definition 4.1. *A closed subset M of X is said to be*

stable, if for every $\epsilon > 0$ and $x \in M$, there exists $\delta > 0$ such that $\mathcal{S}.B(x, \delta) \subset B(M, \epsilon)$;

equi-stable, if for every $x \notin M$, there exists $\delta > 0$ such that $x \notin \overline{\mathcal{S}.B(M, \delta)}$;

uniformly stable, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\mathcal{S}.B(M, \delta) \subset B(M, \epsilon)$.

Lemma 4.2. (Lebesgue number lemma) *For every open cover \mathcal{U} of a compact subset M of a metric space X , there exists a real number $\delta > 0$ such that every open ball of radius δ , centered at a point in M is contained in some element of \mathcal{U} .*

Proof. By contradiction, suppose that no Lebesgue number existed. Then there exists an open cover \mathcal{U} of M such that for all $\delta > 0$ there exists an $x \in M$ such that no $U \in \mathcal{U}$ contains $B(x, \delta)$. Specifically, for each $n \in \mathbb{N}$, we can choose an $x_n \in M$ such that no $U \in \mathcal{U}$ contains $B(x_n, 1/n)$. Now, M is compact so there exists a subsequence (x_{n_k}) of the sequence of points (x_n) that converges to some $y \in M$. Also, \mathcal{U} being an open cover of M implies that there exists $\lambda > 0$ and $U \in \mathcal{U}$ such that $B(y, \lambda) \subseteq U$. Since the sequence (x_{n_k}) converges to y , for k large enough it is true that $d(x_{n_k}, y) < \lambda/2$ and $1/n_k < \lambda/2$. Thus after an application of the triangle inequality, it follows that

$$B(x_{n_k}, 1/n_k) \subseteq B(y, \lambda) \subseteq U,$$

contradicting the assumption that no $U \in \mathcal{U}$ contains $B(x_n, 1/n)$. Hence a Lebesgue number for \mathcal{U} does exist. \square

Remark 4.3. *Stability, as defined at the beginning of this section, is indeed a generalization of stability as in Definition 3.10.*

Proof. Let M be a compact subset of X such that for every $\epsilon > 0$ and $x \in M$, there exists $\delta > 0$ such that $\mathcal{S}.B(x, \delta) \subset B(M, \epsilon)$. We will prove that every neighborhood of M contains an \mathcal{S} -invariant neighborhood of M .

Let $\epsilon > 0$. By Lemma 4.2, there exists a $\delta > 0$ such that, for every $x \in M$, $B(x, \delta) \subseteq B(x, \epsilon)$. We may also assume that $\delta < \epsilon$. Now, for every $x \in M$, pick a $\delta_x \in (0, \delta)$ such that $\mathcal{S}.B(x, \delta_x) \subset B(M, \epsilon)$. Set $V := \cup\{B(x, \delta_x) : x \in M\}$. Then $M \subseteq V \subseteq B(M, \delta) \subseteq B(M, \epsilon)$ and $\mathcal{S}.V \subseteq V$. \square

Proposition 4.4. *If X is locally compact and M is compact, then M is uniformly stable whenever it is stable or equi-stable.*

Proof. 1. Suppose M is stable.

Let $\epsilon > 0$. For $x \in M$, pick $\delta_x > 0$ such that $\mathcal{S}.B(x, \delta_x) \subset B(M, \epsilon)$. Since $\cup\{B(x, \delta_x) : x \in M\}$ is an open cover of M , by Lemma 4.2, there exists a Lebesgue number $\delta > 0$ such that $B(z, \delta) \subset \cup\{B(x, \delta_x) : x \in M\}$, for every $z \in M$. Then

$$B(M, \delta) = \cup\{B(z, \delta) : z \in M\} \subset \cup\{B(x, \delta_x) : x \in M\}.$$

This implies that

$$\mathcal{S}.B(M, \delta) \subset \mathcal{S}.(\cup\{B(x, \delta_x) : x \in M\}) = \cup\{\mathcal{S}.B(x, \delta_x) : x \in M\} \subset B(M, \epsilon),$$

so M is uniformly stable.

2. Suppose M is equi-stable.

Let $\epsilon > 0$. Since M is compact in the locally compact space X , there exists $\eta > 0$ such that $B[M, \eta]$ is compact, hence $H(M, \eta) := \{x \in X : d(M, x) = \eta\}$ is compact. We may pick $0 < \eta < \epsilon$. By equi-stability, for each $x \in H(M, \eta)$, there exists a $\delta_x > 0$ such that $x \notin \overline{\mathcal{S}.B(M, \delta_x)}$. Set $S_x := \overline{\mathcal{S}.B(M, \delta_x)}$.

Then $H(M, \eta) \subset \bigcup\{X \setminus S_x : x \in H(M, \eta)\}$ is an open cover, so it admits a finite subcover

$$H(M, \eta) \subset \bigcup_{i=1}^n (X \setminus S_{x_i}) = X \setminus \left(\bigcap_{i=1}^n S_{x_i}\right).$$

Set $\delta := \min\{\delta_1, \dots, \delta_n, \eta\}$.

Then $H(M, \eta) \subset X \setminus (\mathcal{S}.B(M, \delta))$, so $\mathcal{S}.B(M, \delta) \subset X \setminus H(M, \eta)$.

Now, since $\mathcal{S}.B(M, \delta)$ consists of continuous paths originating in $B(M, \delta) \subset B(M, \eta)$, we must have $\mathcal{S}.B(M, \delta) \subset B(M, \eta) \subset B(M, \epsilon)$.

Thus, M is uniformly stable. □

Proposition 4.5. *If M is stable or equi-stable then it is positively invariant.*

Proof. 1. Suppose M is stable. Then for every $x \in M, g \in \mathcal{S}$ and $\epsilon > 0$, $gx \in B(M, \epsilon)$. So gx being in every ϵ -neighborhood of M , it must be in $\overline{M} = M$.

2. Suppose M is equi-stable and $gx \notin M$, for some $x \in M$ and $g \in \mathcal{S}$. Then there exists a $\delta > 0$ such that $gx \notin \overline{\mathcal{S}.B(M, \delta)}$. But $gx \in \mathcal{S}.M \subset \overline{\mathcal{S}.B(M, \delta)}$, a contradiction. □

4.2 Stability characterizations

Theorem 4.6. *A closed set M is stable if and only if there exists a non-negative function ϕ on X with the following properties:*

- (i) $\phi(x) = 0$ if and only if $x \in M$;
- (ii) For every $\epsilon > 0$ there exists a $\delta > 0$ such that $\phi(x) \geq \delta$ whenever $d(x, M) \geq \epsilon$;
- (iii) $\phi(x_n) \xrightarrow{n} 0$ whenever $\mathbf{x}_n \xrightarrow{n} x \in M$;
- (iv) $\phi(gx) \leq \phi(x)$, for all $x \in X$ and $g \in \mathcal{S}$.

Proof. (\Rightarrow) Suppose M is stable and define

$$\phi(x) := \sup_{g \in \mathcal{S}} \frac{d(gx, M)}{1 + d(gx, M)}.$$

Note that $\phi(x) \in [0, 1]$ for all x , so ϕ is well defined. This function has all properties listed above:

(i) If $\phi(x) = 0$ then $d(gx, M) = 0$ for all $g \in \mathcal{S}$. In particular, $d(x, M) = 0$, so $x \in \overline{M} = M$. If $x \in M$ then, for any $g \in \mathcal{S}$, $gx \in M$, by Proposition 4.5. This means $d(gx, M) = 0$ for all $g \in \mathcal{S}$, so $\phi(x) = 0$.

(ii) This part is proven by contradiction. Suppose that there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists an $x \in X$ with $d(x, M) \geq \epsilon$ and $\phi(x) < \delta$.

For this ϵ , set $\delta := \frac{\epsilon}{\epsilon+1}$.

There exists an $x \in X$ such that $d(x, M) \geq \epsilon$ and $\phi(x) < \delta$. Then

$$\phi(x) \geq \frac{d(x, M)}{1 + d(x, M)} = \frac{1}{\frac{1}{d(x, M)} + 1} \geq \frac{1}{\frac{1}{\epsilon} + 1} = \frac{\epsilon}{\epsilon + 1} = \delta,$$

a contradiction.

(iii) Let $x_n \xrightarrow{n} x \in M$. Note that, for every n ,

$$\phi(x_n) = \sup_{g \in \mathcal{S}} \frac{d(gx_n, M)}{1 + d(gx_n, M)} \leq \sup_{g \in \mathcal{S}} d(gx_n, M). \quad (4.1)$$

Consider $\epsilon > 0$. Since $x \in M$ stable, there exists a $\delta > 0$ such that $\mathcal{S}.B(M, \delta) \subset B(M, \epsilon)$. As $x_n \xrightarrow{n} x$, there exists N such that for all $n \geq N$, $x_n \in B(M, \delta)$.

So, for $n \geq N$ and $g \in \mathcal{S}$, $gx_n \in B(M, \epsilon)$, that is $d(gx_n, M) < \epsilon$.

Then

$$\sup_{g \in \mathcal{S}} d(gx_n, M) \leq \epsilon, \quad \text{for all } n \geq N. \quad (4.2)$$

Combining inequalities 4.1 and 4.2, we obtain $\phi(x_n) \leq \epsilon$, for all $n \geq N$.

Since ϵ was arbitrarily chosen, we conclude that $\phi(x_n) \xrightarrow{n} 0$.

(iv) Let $x \in X$ and $g \in \mathcal{S}$. We may write:

$$\phi(gx) = \sup_{g' \in \mathcal{S}} \frac{d(g'gx, M)}{1 + d(g'gx, M)} = \sup_{h \in \mathcal{S}g} \frac{d(hx, M)}{1 + d(hx, M)} \leq \sup_{h \in \mathcal{S}} \frac{d(hx, M)}{1 + d(hx, M)} = \phi(x).$$

(\Leftarrow) Assume that ϕ as in the text of the theorem exists. Let $\epsilon > 0$ and $x \in M$.

Set $m_0 := \inf\{\phi(z) : d(z, M) \geq \epsilon\}$.

By property (ii), $m_0 > 0$. By (iii), there exists a $\delta > 0$ such that for any $y \in B(x, \delta)$, $\phi(y) < m_0$.

It suffices now to show that $\mathcal{S}.B(x, \delta) \subset B(M, \epsilon)$.

Assume that the inclusion does not hold. Then there exist $y \in B(x, \delta)$ and $g \in \mathcal{S}$ with $d(gy, M) \geq \epsilon$. By property (iv), $\phi(gy) \leq \phi(y) < m_0$.

Also, $\phi(gy) \geq \inf\{\phi(z) : d(z, M) \geq \epsilon\}$, as gy is one of the z 's in the set. So $\phi(gy) \geq m_0$, and this is a contradiction that proves the stability of M .

□

Theorem 4.7. *A closed set M in X is equi-stable if and only if there exists a non-negative function ϕ such that:*

(i) $\phi(x) = 0$ if and only if $x \in M$;

(ii) for every $\epsilon > 0$ there exists a $\delta > 0$ with $\phi(x) \leq \epsilon$ whenever $d(x, M) \leq \delta$;

(iii) for every $x \in X$ and $\epsilon > 0$ there exists a $\delta > 0$ such that $\phi(y) > \phi(x) - \epsilon$ whenever $d(x, y) < \delta$ (ϕ is lower semi-continuous);

(iv) $\phi(gx) \leq \phi(x)$ for all $x \in X$ and $g \in \mathcal{S}$.

Proof. (\Rightarrow) Suppose M is equi-stable and define

$$\phi(x) = \begin{cases} 0, & \text{if } x \in M \\ \sup\{\delta > 0 : x \notin \overline{\mathcal{S}.B(M, \delta)}\}, & \text{if } x \notin M \end{cases}$$

Note that $\phi(x) \leq d(x, M)$, for every $x \in X$. So ϕ is well defined.

(i) Suppose $x \notin M$. By equi-stability, there exists a $\delta_0 > 0$ such that $x \notin \overline{\mathcal{S}.B(M, \delta_0)}$. Then $\phi(x) \geq \delta_0 > 0$.

Conversely, if $x \in M$, $\phi(x) = 0$.

(ii) For $\epsilon > 0$, let $\delta := \epsilon$. Now, if $d(x, M) \leq \delta$ then $\phi(x) \leq d(x, M) \leq \delta = \epsilon$.

(iii) Let $x \in X$ and $\epsilon > 0$. If $x \in M$ then $\phi(x) = 0$, so (iii) holds trivially.

If $x \notin M$, then $\phi(x) > 0$, so, by the definition of ϕ , there exists an $\eta > \phi(x) - \epsilon$ such that $x \notin \overline{\mathcal{S}.B(M, \eta)}$. The complement of $\overline{\mathcal{S}.B(M, \eta)}$ being an open set around x , there exists a $\delta > 0$ such that $B(x, \delta) \cap \overline{\mathcal{S}.B(M, \eta)} = \emptyset$. So, if $d(x, y) < \delta$ then $y \notin \overline{\mathcal{S}.B(M, \eta)}$, that is $\phi(y) \geq \eta > \phi(x) - \epsilon$.

(iv) If $gx \in M$ then $\phi(gx) = 0 \leq \phi(x)$.

If $gx \notin M$ then $x \notin M$, by the positive invariance of M . For the given g and any $\delta > 0$, we may write:

$$\overline{\mathcal{S}.B(M, \delta)} \subset \overline{g^{-1}\mathcal{S}.B(M, \delta)} = g^{-1}(\overline{\mathcal{S}.B(M, \delta)}),$$

the last equality being insured by the continuity of the action of g^{-1} on X .

We then have

$$\begin{aligned} \phi(gx) &= \sup\{\delta > 0 : gx \notin \overline{\mathcal{S}.B(M, \delta)}\} = \sup\{\delta > 0 : x \notin g^{-1}(\overline{\mathcal{S}.B(M, \delta)})\} \leq \\ &\sup\{\delta > 0 : x \notin \overline{\mathcal{S}.B(M, \delta)}\} = \phi(x). \end{aligned}$$

(\Leftarrow) Let ϕ be a non-negative function on X satisfying (i)-(iv) and $x \notin M$. By (ii), there exists a $\delta > 0$ such that $\phi(B(M, \delta)) \subset [0, \frac{\epsilon}{2}]$. Using (iv), we obtain $\phi(\mathcal{S}.B(M, \delta)) \subset [0, \frac{\epsilon}{2}]$ or, equivalently, $\mathcal{S}.B(M, \delta) \subset \phi^{-1}[0, \frac{\epsilon}{2}]$. Next, observe that $\phi^{-1}[0, \frac{\epsilon}{2}]$ is a closed set, since, by (iii), ϕ is lower-semicontinuous.

Now, $\overline{\mathcal{S}.B(M, \delta)} \subset \phi^{-1}[0, \frac{\epsilon}{2}]$, that is $\phi(\overline{\mathcal{S}.B(M, \delta)}) \subset [0, \frac{\epsilon}{2}]$. Since $\phi(x) = \epsilon$, we must have $x \notin \overline{\mathcal{S}.B(M, \delta)}$.

□

Theorem 4.8. *A closed set M in X is uniformly stable if and only if there exists a non-negative function ϕ such that:*

- (i) for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\phi(x) \geq \delta$ whenever $d(x, M) \geq \epsilon$;
- (ii) for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\phi(x) < \epsilon$ whenever $d(x, M) < \delta$;
- (iii) $\phi(gx) \leq \phi(x)$ for all $x \in X$ and $g \in \mathcal{S}$.

Proof. (\Rightarrow) If M is uniformly stable then it is stable. Construct ϕ by letting

$$\phi(x) := \sup_{g \in \mathcal{S}} \frac{d(gx, M)}{1 + d(gx, M)},$$

for every $x \in X$. Then (i) and (iii) are automatically satisfied, by Theorem 4.6. To prove (ii) we use the uniform stability of M : for $\epsilon > 0$ there exists $\delta > 0$ such that $\mathcal{S}.B(M, \delta) \subset B(M, \frac{\epsilon}{2-\epsilon})$. So if $d(x, M) < \delta$ and $g \in \mathcal{S}$ then $d(gx, M) < \frac{\epsilon}{2-\epsilon}$. Then

$$\phi(x) = \sup_{g \in \mathcal{S}} \frac{d(gx, M)}{1 + d(gx, M)} \leq \frac{1}{1 + \frac{2-\epsilon}{\epsilon}} = \frac{\epsilon}{2} < \epsilon.$$

(\Leftarrow) Let ϕ be a non-negative function on X satisfying (i), (ii) and (iii) and let $\epsilon > 0$.

By (i), there exists a $\delta_1 > 0$ such that $\phi(X \setminus B(M, \epsilon)) \subset [\delta_1, \infty)$. Then

$X \setminus B(M, \epsilon) \subset \phi^{-1}[\delta_1, \infty)$, or, equivalently, $\phi^{-1}[0, \delta_1) \subset B(M, \epsilon)$.

Now, using δ_1 instead of ϵ in (ii), there exists a δ such that $\phi(B(M, \delta)) \subset [0, \delta_1]$.
 By (iii) then, $\phi(\mathcal{S}.B(M, \delta)) \subset [0, \delta_1]$, or $\mathcal{S}.B(M, \delta) \subset \phi^{-1}[0, \delta_1]$.

We conclude that $\mathcal{S}.B(M, \delta) \subset B(M, \epsilon)$, so M is uniformly stable. □

4.3 Attractors and asymptotic stability

Definition 4.9. A *path* from x is any continuous function $\gamma : [0, \infty) \rightarrow \mathcal{S}.x$ with the properties that $\gamma(0) = x$ and whenever $t_1 < t_2$ there exists a $g \in \mathcal{S}$ with $\gamma(t_2) = g\gamma(t_1)$.

Note that the continuity of γ does not follow from the other two properties. We impose it in order to keep the notion of *path* close to what a *trajectory* means in the classical context of a single dynamical system and, moreover, in the one of control systems.

Definition 4.10. Given a closed set, M , and a path from x , γ , we say that:

1. γ **clusters** to M if

for every $\epsilon, T > 0$ there exists a $t \geq T$ with $\gamma(t) \in B(M, \epsilon)$;

2. γ **asymptotically approaches** M if

$$\lim_{t \rightarrow \infty} d(\gamma(t), M) = 0.$$

Definition 4.11. A closed set $M \subset X$ is said to be:

1. a **weak semi-attractor** if for every $x \in M$ there exists a $\delta_x > 0$ such that every path starting in $B(x, \delta_x)$ clusters to M ;

2. a **semi-attractor** if for every $x \in M$ there exists a $\delta_x > 0$ such that every path starting in $B(x, \delta_x)$ asymptotically approaches M ;

3. a **weak attractor** if there exists a $\delta > 0$ such that every path starting in $B(M, \delta)$ clusters to M ;
4. an **attractor** if there exists a $\delta > 0$ such that every path starting in $B(M, \delta)$ asymptotically approaches M ;
5. a **uniform attractor** if there exists a $\delta > 0$ such that for every $\epsilon > 0$ there exists a $T \geq 0$ such that for every path γ starting in $B(M, \delta)$ we have $\gamma([T, \infty)) \subset B(M, \epsilon)$;
6. an **equi-attractor** if it is an attractor and there exists a $\lambda > 0$ such that for every $\epsilon \in (0, \lambda)$ and for every $T > 0$ there exists a $\delta > 0$ such that for every path γ starting in $B[M, \lambda] \setminus B(M, \epsilon)$ we have $\gamma([0, T]) \cap B(M, \delta) \neq \emptyset$;
7. **semi-asymptotically stable** if it is stable and a semi-attractor;
8. **asymptotically stable** if it is uniformly stable and an attractor;
9. **uniformly asymptotically stable** if it is uniformly stable and a uniform attractor.

For a closed set M we define the **weak basin of attraction** by

$$A_w(M) := \{x \in X : \text{every path from } x \text{ clusters to } M\}$$

and the **basin of attraction** by

$$A(M) := \{x \in X : \text{every path from } x \text{ asymptotically approaches } M\}.$$

Theorem 4.12. *A closed set M is semi-asymptotically stable if and only if there exists a function $\phi : X \rightarrow [0, \infty)$ with the following properties:*

- (i) $\phi(x_n) \rightarrow 0$ whenever $x_n \rightarrow x \in M$;

(ii) $\phi(x) = 0$ if and only if $x \in M$;

(iii) there exists a strictly increasing function $\alpha : [0, \infty) \rightarrow [0, \infty)$ with $\alpha(0) = 0$ and $\phi(x) \geq \alpha(d(x, M))$ for all x ;

(iv) $\phi(gx) \leq \phi(x)$, for all $x \in X$ and $g \in \mathcal{S}$;

(v) for every $x \in M$ there exists a $\delta_x > 0$ such that for every path γ from $B(x, \delta_x) \setminus M$, ϕ decreases strictly along γ , to 0. [This means $\phi(\gamma(t_1)) > \phi(\gamma(t_2))$ whenever $t_1 < t_2$ and $\phi(\gamma(t)) \rightarrow 0$ as $t \rightarrow \infty$.]

Proof. (\Leftarrow) The stability of M follows from Theorem 4.6. The four conditions required for ϕ are satisfied by (ii), (iii), (i), (iv), in this order. That M is a semi-attractor follows from (iii) and (v), as shown below.

Fix $x \in M$ and let δ_x be given by condition (v). If γ is a path from $B(x, \delta_x)$, using (iii) we have

$$\phi(\gamma(t)) \geq \alpha(d(\gamma(t), M)).$$

From property (v),

$$\lim_{t \rightarrow \infty} \phi(\gamma(t)) = 0,$$

so

$$\lim_{t \rightarrow \infty} \alpha(d(\gamma(t), M)) = 0,$$

and thus

$$\lim_{t \rightarrow \infty} d(\gamma(t), M) = 0.$$

This proves that, given $x \in M$, there exists a $\delta_x > 0$ such that every path from from $B(x, \delta_x)$ asymptotically approaches M .

(\Rightarrow) For each $x \in X$, define

$$\psi(x) := \sup \left\{ \frac{d(\gamma(t), M)}{1 + d(\gamma(t), M)} : t \geq 0, \gamma \text{ path from } x \right\},$$

or, equivalently,

$$\psi(x) = \sup\{\psi_g(x) : g \in \mathcal{S}\}, \text{ where } \psi_g(x) := \frac{d(gx, M)}{1 + d(gx, M)}. \quad (4.3)$$

Next we verify that ψ satisfies conditions (i)-(iv).

(i) Let $x_n \rightarrow x \in M$ and $\epsilon > 0, \epsilon < 1$. We want to show that there exists some positive integer N such that $\psi(x_n) \leq \epsilon$ for all $n \geq N$.

M being stable, there exists some $\delta_x > 0$ for which $\mathcal{S}.B(x, \delta_x) \subset B(M, \frac{\epsilon}{1-\epsilon})$. Let N be such that for all $n \geq N$, $\mathcal{S}.x_n \subset B(M, \frac{\epsilon}{1-\epsilon})$. Then, for every $g \in \mathcal{S}$, $d(gx_n, M) < \frac{\epsilon}{1-\epsilon}$, implying

$$\frac{d(gx_n, M)}{1 + d(gx_n, M)} < \epsilon.$$

Taking the supremum over all $g \in \mathcal{S}$ and using (4.3), we obtain $\psi(x_n) \leq \epsilon$.

(ii) If $x \in M$ then $\psi(x) = 0$, by the positive invariance of M . If $x \notin M$, then $d(x, M) > 0$ and

$$\psi(x) \geq \frac{d(gx, M)}{1 + d(gx, M)} > 0.$$

(iii) Take $\alpha(\theta) := \frac{\theta}{1+\theta}$, for all $\theta \geq 0$. This function is strictly increasing and, for $x \in X$,

$$\psi(x) \geq \frac{d(x, M)}{1 + d(x, M)} = \alpha(d(x, M)).$$

(iv) If $g \in \mathcal{S}$ and $x \in X$ then

$$\begin{aligned} \psi(gx) &:= \sup\left\{\frac{d(g'gx, M)}{1 + d(g'gx, M)} : g' \in \mathcal{S}\right\} = \sup\left\{\frac{d(hx, M)}{1 + d(hx, M)} : h \in \mathcal{S}g\right\} \leq \\ &\leq \sup\left\{\frac{d(hx, M)}{1 + d(hx, M)} : h \in \mathcal{S}\right\} = \psi(x). \end{aligned}$$

To ensure property (v), we need to modify the function ψ by defining

$$\omega(x) := \sup\left\{\int_0^\infty \psi(\gamma(t))e^{-t} dt : \gamma \text{ path from } x\right\}.$$

Note that the integral in this definition is well-defined and bounded by 1, as ψ is bounded by 1 and lower semi-continuous.

Now take

$$\phi(x) := \psi(x) + \omega(x), \text{ for all } x \in X.$$

Then ϕ inherits properties (i)-(iv) from ψ .

To see that ϕ also satisfies (v), consider $x \in M$. Since M is a semi-attractor, there exists a $\delta_x > 0$ such that every path from $B(x, \delta_x)$ asymptotically approaches M . Let γ be a path starting at a point $y \in B(x, \delta_x) \setminus M$. Then

$$\lim_{t \rightarrow \infty} d(\gamma(t), M) = 0.$$

We now prove by contradiction that ϕ decreases strictly to 0 along γ .

Assume that $\phi(y) = \phi(\gamma(t))$, for some $t > 0$. This is possible only if $\psi(\gamma(t+s)) = \psi(\gamma(s))$, for all $s > 0$. Letting $s = t, 2t, 3t, \dots$, we conclude that $\psi(\gamma(nt)) = \psi(y)$, for all n . Then the stability of M implies that

$$\lim_{t \rightarrow \infty} \psi(\gamma(nt)) = 0,$$

so $\psi(y) = 0$. This contradicts property (ii), thus proving that ϕ satisfies (v). \square

Theorem 4.13. *A closed set M is asymptotically stable if and only if there exists a function $\phi : X \rightarrow [0, \infty)$ with the following properties:*

- (i) *for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\phi(x) < \epsilon$ whenever $d(x, M) < \delta$;*
- (ii) *there exists a strictly increasing function $\alpha : [0, \infty) \rightarrow [0, \infty)$ with $\alpha(0) = 0$ and $\phi(x) \geq \alpha(d(x, M))$ for all x ;*
- (iii) *$\phi(gx) \leq \phi(x)$, for all $x \in X$ and $g \in \mathcal{S}$;*
- (iv) *there exists an $\epsilon > 0$ such that for every path γ from $B(M, \epsilon) \setminus M$, ϕ decreases strictly along γ , to 0. [This means $\phi(\gamma(t_1)) > \phi(\gamma(t_2))$ whenever $t_1 < t_2$ and $\phi(\gamma(t)) \rightarrow 0$ as $t \rightarrow \infty$.]*

Proof. (\Rightarrow) This implication follows exactly as in the proof of Theorem 4.12.

(\Leftarrow) For the uniform stability of M , verify the three conditions in Theorem 4.8.

- (1) Let $\epsilon > 0$ and set $\delta := \alpha(\epsilon) > 0$. If $d(x, M) \geq \epsilon$ then $\alpha(d(x, M)) \geq \alpha(\epsilon) = \delta$, so, by (ii), $\phi(x) \geq \delta$.
- (2) is equivalent to (i).
- (3) is equivalent to (iii).

This shows that M is uniformly stable.

That M is an attractor follows from (ii) and (iv):

Let ϵ be given by condition (iv). If γ is a path from $B(M, \epsilon)$, using (iv) we have

$$\lim_{t \rightarrow \infty} \phi(\gamma(t)) = 0.$$

From property (ii), $\phi(\gamma(t)) \geq \alpha(d(\gamma(t), M))$, so

$$\lim_{t \rightarrow \infty} \alpha(d(\gamma(t), M)) = 0,$$

and thus

$$\lim_{t \rightarrow \infty} d(\gamma(t), M) = 0,$$

as α is increasing.

This proves that γ approaches M asymptotically. Therefore M is an attractor.

□

Chapter 5

Closed Relations and Polysystems

5.1 Preliminaries

The results in chapters 3 and 4 involve Lyapunov-like functions. This chapter follows the ideas of E. Akin in an attempt to ease the problem of finding strict Lyapunov functions for polysystems. In order to use these ideas, let us observe that a polysystem can be viewed as a closed relation, in the following sense. Define a closed relation on X by

$$f = \overline{\{(x, gx) \in X \times X : g \in \mathcal{S}_{[0,1]}\}}, \quad (5.1)$$

where $\mathcal{S}_{[0,1]}$ denotes all elements of \mathcal{S} with time component between 0 and 1. Note that if $y = gx$, with $g \in \mathcal{S}$, then $(x, y) \in f^k$, for some positive integer k .

The facts about closed relations listed below can be found in [Ak].

Definition 5.1. *Let X be a metric space and f a closed relation on X .*

*A **Lyapunov function** for f is a continuous real-valued function L on X with the property that $L(x) \leq L(y)$ whenever $(x, y) \in f$.*

*A point $x \in X$ is **regular** for L if*

$$L(y_1) < L(x) < L(y_2) \text{ whenever } (y_1, x) \in f \text{ and } (x, y_2) \in f$$

*and **critical** for L if it is not regular.*

Denote by $|L|$ the set of critical points for L .

*Also, $|f|$ denotes the **cyclic set** of f , that is*

$$|f| := \{x \in X : (x, x) \in f\}$$

Definition 5.2. Given a metric space X , a closed relation f on X , $x, y \in X$ and $\epsilon > 0$, an ϵ -**chain** from x to y is a sequence of points in X , $x = x_0, x_1, \dots, x_n = y$ with the property that

$$d(x_{i+1}, f(x_i)) < \epsilon, \text{ for all } i \in \{0, \dots, n-1\}.$$

Note that in the above definition $d(x_{i+1}, f(x_i))$ refers to the distance from a point to a set, which means, as usually, the infimum of distances from x_{i+1} to every point in $f(x_i)$.

Definition 5.3. Given a closed relation f on a metric space X , define the **chain relation** $\mathcal{C}f$ associated to f , by

$(x, y) \in \mathcal{C}f$ if for every $\epsilon > 0$, there exists an ϵ -chain from x to y .

Note that $\mathcal{C}f$ is a closed transitive relation containing f .

Theorem 5.4. (Akin, [Ak, pp. 33]) If F is a closed transitive relation on a compact metric space X then there exists a Lyapunov function L for F with $|L| = |F|$.

Corollary 5.5. (Akin, [Ak, pp. 34]) If f is a closed relation on a compact metric space X then there exists a Lyapunov function L for f with $|L| = |\mathcal{C}f|$.

5.2 Polysystems viewed as closed relations

Definition 5.6. Let X be a metric space and (\mathcal{S}, X) a polysystem, as defined in chapter 1. A **Lyapunov function** for the polysystem (\mathcal{S}, X) is a continuous real-valued function L on X with $L(x) \leq L(gx)$ for every $x \in X$ and $g \in \mathcal{S}$.

Remark 5.7. If f is defined by 5.1 and L is a Lyapunov function for f then L is a Lyapunov function for the polysystem (\mathcal{S}, X) .

Proof. Let L be a Lyapunov function for f , let $g \in \mathcal{S}$ and $x \in X$. Writing g as $g = g_1 g_2 \dots g_k$, with $g_i \in \mathcal{S}_{[0,1]}$ for all $i \in \{1, 2, \dots, k\}$, we have

$$L(gx) = L(g_1 g_2 \dots g_k \cdot x) \geq L(g_2 \dots g_k \cdot x) \geq \dots \geq L(g_k \cdot x) \geq L(x).$$

□

Definition 5.8. Given $\epsilon > 0$ and $x, y \in X$, an ϵ -**chain** from x to y in the polysystem (\mathcal{S}, X) is a sequence of pairs $(g_0, x_0), (g_1, x_1), \dots, (g_k, x_k)$ in (\mathcal{S}, X) with $x_0 = x, x_k = y, g_i \in \mathcal{S}_{[1,\infty)}$ for all i and $d(x_{i+1}, g_i \cdot x_i) < \epsilon$ for all $i \in \{0, 1, \dots, k\}$.

Note that the requirement $g_i \in \mathcal{S}_{[1,\infty)}$ is needed to avoid triviality in constructing ϵ -chains. Without it, any two points in X could be connected through an ϵ -chain, using the mere continuity of actions by elements in \mathcal{S} on X .

Finally, define a chain relation \mathcal{C} for the polysystem (\mathcal{S}, X) , by

$$(x, y) \in \mathcal{C} \text{ if for every } \epsilon > 0 \text{ there exists an } \epsilon\text{-chain from } x \text{ to } y, \quad (5.2)$$

(in the sense of polysystems).

Definition 5.9. A point x in X is said to be **chain-recurrent** (in the sense of polysystems) if $x \in |\mathcal{C}|$, (that is, for every $\epsilon > 0$ there exists an ϵ -chain from x to x).

Proposition 5.10. If f is defined by 5.1 and \mathcal{C} by 5.2 then $\mathcal{C} \subset \mathcal{C}f$.

Proof. Let $x, y \in \mathcal{C}$. For $\epsilon > 0$ there exists an ϵ -chain (in the sense of polysystems) from x to y , $(g_0, x_0), (g_1, x_1), \dots, (g_k, x_k)$. Every g_i in this chain can be written as

$$g_i = g_i^{j_1} g_i^{j_2} \dots g_i^{j_{k_i}}$$

with $g_i^{j_l} \in \mathcal{S}_{[0,1]}$, for all l . We can construct then an ϵ -chain from x to y (in the sense of relations), as follows:

$$x = x_0, \dots, g_{i-1} x_{i-1}, x_i, g_i^{j_{k_i}} x_i, g_i^{j_{k_i-1}} g_i^{j_{k_i}} x_i, \dots, g_i^{j_1} g_i^{j_2} \dots g_i^{j_{k_i}} x_i = g_i x_i, x_{i+1}, \dots, \dots, x_k.$$

It suffices to show now that $d(g_{i-1}x_{i-1}, f(x_i)) < \epsilon$ and $d(g_i^{j_{k_i}}x_i, f(x_i)) < \epsilon$. The first inequality is seen to be satisfied by noting that $d(g_{i-1}x_{i-1}, x_i) < \epsilon$ and $x_i \in f(x_i)$. The second one is true since $g_i^{j_{k_i}}x_i \in f(x_i)$ and so $d(g_i^{j_{k_i}}x_i, f(x_i)) = 0 < \epsilon$.

□

Theorem 5.11. *If (\mathcal{S}, X) is a polysystem defined on the compact metric space X then there exists a Lyapunov function L for the polysystem with $|L| = |\mathcal{C}f|$.*

Proof. The theorem follows from Corollary 5.5.

□

Corollary 5.12. *If (\mathcal{S}, X) is a polysystem defined on the compact metric space X then there exists a Lyapunov function L for the polysystem with $|\mathcal{C}| \subset |L|$.*

From this Corollary we draw the conclusion that, in trying to obtain a strict Lyapunov function L for the polysystem (\mathcal{S}, X) , the most one can hope is that the critical points for L are precisely the chain-recurrent points in the polysystem.

Chapter 6

Higher Prolongations

6.1 Two operators on multifunctions

Following the ideas of *Bathia* and *Szegö*, two operations on the class of functions from X into 2^X are needed in order to define higher prolongations. The results given without proof can be found in [BS].

Definition 6.1. If $F : X \rightarrow 2^X$, define $\mathcal{D}F$ by

$$\mathcal{D}F(x) := \cap \{ \overline{F(U)} : U \text{ neighborhood of } x \} \text{ for all } x \text{ in } X,$$

and $\mathcal{T}F$ by

$$\mathcal{T}F(x) := \cup \{ F^n(x) : n = 1, 2, \dots \} \text{ for all } x \text{ in } X,$$

where $F^1(x) = F(x)$ and $F^n(x) = F(F^{n-1}(x))$, for $n \geq 2$.

Lemma 6.2. For any $x \in X$ and $F : X \rightarrow 2^X$ we have:

(i) $y \in \mathcal{D}F(x)$ if and only if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n \in F(x_n)$ for every n , $x_n \rightarrow x$ and $y_n \rightarrow y$;

(ii) $y \in \mathcal{T}F(x)$ if and only if there exist points x_1, x_2, \dots, x_k such that $x_1 = x$, $x_k = y$ and $x_{i+1} \in F(x_i)$ for $i \in \{1, 2, \dots, k-1\}$.

Proof. (i) If $y \in \mathcal{D}F(x)$ then $y \in \overline{F(B(x, \frac{1}{n}))}$ for every n , implying that for every n there exists $x_n \in B(x, \frac{1}{n})$ and $y_n \in F(x_n)$ with $d(y_n, y) < \frac{1}{n}$.

Conversely, given sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n \in F(x_n)$, $x_n \rightarrow x$ and $y_n \rightarrow y$, given any neighborhood U of x , for n large enough, $x_n \in U$. Then $y_n \in F(x_n) \subset F(U)$ and, since $y_n \rightarrow y$, $y \in \overline{F(U)}$.

(ii) $y \in \mathcal{T}F(x)$ means $y \in F^k(x)$ for some k . Set $x_k = y$. There exists $x_{k-1} \in F^{k-1}(x)$ such that $x_k \in F(x_{k-1})$. Iterating this step, we find the desired sequence. \square

Lemma 6.3. (i) $\mathcal{D}^2 = \mathcal{D}$ and $\mathcal{T}^2 = \mathcal{T}$ (\mathcal{D} and \mathcal{T} are idempotent operators);

(ii) If M is a compact subset of X then $\mathcal{D}F(M) = \cup\{\mathcal{D}F(x) : x \in M\}$ is closed.

(iii) If ϕ is a continuous real-valued function on X with $\phi(y) \leq \phi(x)$ whenever $y \in F(x)$, then $\phi(y) \leq \phi(x)$ whenever $y \in \mathcal{D}F(x) \cup \mathcal{T}F(x)$.

Definition 6.4. A function $F : X \rightarrow 2^X$ is called **transitive** if $\mathcal{T}F = F$.

Proposition 6.5. (i) A function $F : X \rightarrow 2^X$ is transitive if and only if $F^2 = F$;

(ii) Given $F : X \rightarrow 2^X$, $\mathcal{T}F$ is transitive.

Proof. (i) (\Rightarrow) For $x \in X$ one has:

$$\begin{aligned} F^2(x) &= F(F(x)) = \mathcal{T}F(\mathcal{T}F(x)) = \mathcal{T}F\left(\bigcup_{n=1}^{\infty} F^n(x)\right) = \bigcup_{k=1}^{\infty} F^k\left(\bigcup_{n=1}^{\infty} F^n(x)\right) \\ &= \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} F^{k+n}(x) = \bigcup_{i=1}^{\infty} F^i(x) = \mathcal{T}F(x) = F(x). \end{aligned}$$

(\Leftarrow) If $F^2 = F$ then $F^n = F$ for any positive integer n . So, for $x \in X$ we have:

$$\mathcal{T}F(x) = \bigcup_{n=1}^{\infty} F^n(x) = \bigcup_{n=1}^{\infty} F(x) = F(x).$$

(ii) Since \mathcal{T} is an idempotent operator, $\mathcal{T}(\mathcal{T}F) = \mathcal{T}F$. Thus $\mathcal{T}F$ is transitive. \square

Definition 6.6. A map $F : X \rightarrow 2^X$ is called a **cluster map** if $\mathcal{D}F = F$.

Definition 6.7. A map $F : X \rightarrow 2^X$ is called a **c - c map** if it has the property that for any compact set $K \subset X$ and $x \in K$, either $F(x) \subset K$ or $F(x) \cup \partial K \neq \emptyset$.

Theorem 6.8. Assume X is locally compact and $F : X \rightarrow 2^X$ is a c - c map. If $F(x)$ is compact then it is connected.

Theorem 6.9. *Let X be locally compact, $F : X \rightarrow 2^X$ a $c - c$ map and M a compact subset of X . Then $\mathcal{D}F(M) = M$ if and only if for every neighborhood U of M there exists a neighborhood W of M such that $F(W) \subset U$.*

Lemma 6.10. (i) *If $\{F_\alpha\}$ is a family of $c - c$ maps then $F = \cup F_\alpha$ is a $c - c$ map;*
(ii) *If F_1 and F_2 are $c - c$ maps then so is the composition $F_1 \circ F_2$;*
(iii) *If F is a $c - c$ map then so are $\mathcal{D}F$ and $\mathcal{T}F$.*

Theorem 6.11. *Let X be locally compact and M a compact subset of X . Let $F : X \rightarrow 2^X$ be a $c - c$ map which is moreover a transitive map as well as a cluster map. Then $F(M) = M$ if and only if there exists a fundamental system of compact neighborhoods $\{U_n\}$ of M such that $F(U_n) = U_n$ for every n .*

6.2 Higher prolongations

Throughout this section (\mathcal{S}, X) is a dynamical polysystem. The symbol \mathcal{S} will have a double meaning: the admissible semigroup as defined in Chapter 1, and a multifunction naturally given by the action of this semigroup on the space X .

Let $\mathcal{S} : X \rightarrow 2^X$ be the map given by $\mathcal{S}(x) := \mathcal{S}.x$. It assigns to each point $x \in X$ the positive orbit of x , in the sense of dynamical polysystems. Since $\mathcal{S}(x)$ is connected, \mathcal{S} is a $c - c$ map. It is also transitive, since $\mathcal{S}(\mathcal{S}.x) = \mathcal{S}.x$ for every x . Then $\mathcal{T}\mathcal{S} = \mathcal{S}$, by Proposition 6.5.

Now set $D_1^+ := \mathcal{D}\mathcal{T}\mathcal{S} = \mathcal{D}\mathcal{S}$, and call $D_1^+(x)$ the **first positive prolongation** of x . Note that $D_1^+(x)$ coincides with $D^+(x)$, as defined in Chapter 3. Further, D_1^+ is a cluster map, as \mathcal{D} is idempotent. But D_1^+ is not transitive, as shown in [BS, pp. 125] for the case of a single dynamical system.

Next, we set $D_2^+ := \mathcal{D}\mathcal{T}D_1^+$ and call $D_2^+(x)$ the **second prolongation** of x .

In general, having defined D_n^+ , we define $D_{n+1}^+ := \mathcal{D}\mathcal{T}D_n^+$, and call $D_n^+(x)$ to be the **n-th prolongation** of x .

Now, passing to ordinal numbers, we define a prolongation $D_\alpha^+(x)$ of x for any ordinal number α as follows: if α is a successor ordinal, then set $D_\alpha^+ := \mathcal{DT}D_{\alpha-1}^+$, assuming $D_{\alpha-1}^+$ was previously defined. If α is not a successor ordinal, then set $D_\alpha^+ := \mathcal{D}(\cup\{\mathcal{T}D_\beta^+ : \beta < \alpha\})$, assuming D_β^+ defined for every $\beta < \alpha$.

Notice that every D_α^+ defined above is a $c - c$ map. This fact is guaranteed by Lemma 6.10. Also, every $D_\alpha^+(x)$ is closed, by Lemma 6.3.

The following theorem provides an upper bound for the higher prolongations.

Theorem 6.12. *Let Ω be the first uncountable ordinal number. Then*

- (i) $D_\Omega^+ = \cup\{D_\alpha^+ : \alpha < \Omega\}$;
- (ii) D_Ω^+ is a transitive map.

Proof. (i) For simplicity, denote by F the multifunction $\cup\{D_\alpha^+ : \alpha < \Omega\}$. Since Ω is not a successor ordinal, $D_\Omega^+ = \mathcal{D}(\cup\{\mathcal{T}D_\alpha^+ : \alpha < \Omega\})$. For $x \in X$ let $y \in D_\Omega^+(x)$. By Lemma 6.2, there are sequences $\{x_n\}, \{y_n\}$ in X with $x_n \rightarrow x$, $y_n \rightarrow y$, and $y_n \in \cup\{\mathcal{T}D_\alpha^+(x_n) : \alpha < \Omega\}$. So, for every n , there exists some ordinal number α_n , $\alpha_n < \Omega$, such that $y_n \in \mathcal{T}D_{\alpha_n}^+(x_n)$. For the sequence $\{\alpha_n\}$ of countable ordinals, there exists an ordinal β such that for every n we have $\alpha_n < \beta < \beta+1 < \Omega$. By the construction of D_β^+ , we then have $y_n \in \mathcal{T}D_\beta^+(x_n)$ for each n . Again by Lemma 6.2, $y \in \mathcal{DT}D_\beta^+(x) = D_{\beta+1}^+(x)$. It follows that $y \in F(x)$, which shows one inclusion.

Conversely, if $y \in F(x)$, then $y \in D_\alpha^+(x)$ for some $\alpha < \Omega$. But $D_\alpha^+(x) \subset D_\Omega^+(x)$, by the construction of D_Ω^+ . Consequently, $y \in D_\Omega^+(x)$.

(ii) To show the transitivity of D_Ω^+ , it suffices to show that $(D_\Omega^+)^2 = D_\Omega^+$, according to Lemma 6.5. If $z \in (D_\Omega^+)^2(x)$ then there exists $y \in D_\Omega^+(x)$ such that $z \in D_\Omega^+(y)$. Using part (i), there exist ordinals $\alpha_1, \alpha_2 < \Omega$ with $y \in D_{\alpha_1}^+(x)$ and $z \in D_{\alpha_2}^+(y)$. Now if $\beta := \max(\alpha_1, \alpha_2)$ then $y \in D_\beta^+(x)$ and $z \in D_\beta^+(y)$. It follows that $z \in \mathcal{T}D_\beta^+(x) \subset \mathcal{DT}D_\beta^+(x) = D_{\beta+1}^+(x) \subset D_\Omega^+(x)$.

□

Remark 6.13. For any $\alpha > \Omega$, $D_\alpha^+ = D_\Omega^+$.

Proof. As D_Ω^+ is transitive as well as a cluster map, the observation follows inductively from the construction of higher prolongations. □

Definition 6.14. Let X be locally compact and M a compact subset of X . The set M is said to be **stable of order** α , or α -**stable** if $D_\alpha^+(M) = M$. The set M is called **absolutely stable** if it is **stable of order** α for every ordinal α .

Observe that stability of order 1 coincides with stability as defined in Chapter 3. Also, it follows from Remark 6.13 that absolute stability is the same as stability of order Ω , where Ω is the first uncountable ordinal.

Theorem 6.15. If X is locally compact and M is a compact subset of X then M is α -stable if and only if for every neighborhood U of M there exists a neighborhood W of M such that $D_\alpha^+(W) \subset U$.

Lemma 6.16. Let X be locally compact and $M \subset X$ compact. Let $\phi : X \rightarrow \mathbb{R}$ satisfy the condition that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\phi(x) \geq \delta$ whenever $d(x, M) \geq \epsilon$.

Then the sets defined by

$$U_t := \{x \in X : \phi(x) \leq t\}, t > 0$$

form a fundamental system of neighborhoods of M .

Theorem 6.17. Let X be a locally compact metric space and let $M \subset X$ be compact. The following are equivalent:

(i) there exists $\phi : X \rightarrow \mathbb{R}$, continuous in some neighborhood of M , satisfying:

- (a) for every $\epsilon > 0$ there exists a $\delta > 0$ with $\phi(x) \geq \delta$ whenever $d(x, M) \geq \epsilon$;
- (b) for every $\epsilon > 0$ there exists a $\delta > 0$ with $\phi(x) \leq \epsilon$ whenever $d(x, M) \leq \delta$;
- (c) $\phi(gx) \leq \phi(x)$, for all $x \in X$ and $g \in \mathcal{S}$;

(ii) M has a system of absolutely stable compact neighborhoods;

(iii) M is absolutely stable.

Proof. (i) \Rightarrow (ii)

Consider a compact neighborhood U of M so that ϕ is continuous on U and let $m_0 := \min\{\phi(x) : x \in \partial U\}$. Then $m_0 > 0$. By Lemma 6.16, the sets $U_t := \{x \in X : \phi(x) \leq t\}$, $0 < t < m_0$, form a fundamental system of compact neighborhoods of M . Also, from (i)-(c), every U_t is positively invariant.

Define another function on X by

$$\Phi(x) = \phi(x), \text{ if } x \in U_{m_0} \text{ and } \Phi(x) = m_0, \text{ if } x \notin U_{m_0}.$$

The function Φ is continuous and decreases along trajectories. Using Lemma 6.3, we see that Φ satisfies

$$\Phi(y) \leq \Phi(x) \text{ whenever } y \in D_{\Omega}^+(x).$$

Suppose, now, that for some $0 < t < m_0$, $D_{\Omega}^+(U_t) \neq U_t$. Then there exists an $s > t$ such that $D_{\Omega}^+(U_t) \not\subset U_s$. Since D_{Ω}^+ is a $c - c$ map, there exists an $x \in U_t$ and a $y \in D_{\Omega}^+(x) \cup \partial U_s$. We obtain that $\Phi(y) \leq \Phi(x) \leq t$, on one hand, and that $\Phi(y) = s > t$, on the other hand. This contradiction shows that $D_{\Omega}^+(U_t) = U_t$, for every $0 < t < m_0$.

(ii) \Rightarrow (iii) follows from Theorem 6.15

(iii) \Rightarrow (i)

Since D_{Ω}^+ is a $c-c$ map, transitive, and a cluster map, we can use Theorem 6.11 to construct a fundamental system of absolutely stable neighborhoods for M :

$$U_{1/2^n}, \text{ with } U_{1/2^n} \subset \text{Int}(U_{1/2^{n-1}}), n = 0, 1, 2, \dots$$

Using Theorem 6.11 again, we extend this system of absolutely stable compact neighborhoods to one defined over the diadic rationals. So, for every number $r = j/2^n$ ($j = 1, 2, 3, \dots, 2^n$), we construct an absolutely stable compact neighborhood U_r of M , in such a way that:

$$U_{r_1} \subset \text{Int}(U_{r_2}), \text{ if } r_1 < r_2$$

and

$$M = \cap \{U_r : r \text{ diadic rational}\}.$$

For $x \in U_1$, define $\phi(x) := \inf\{r : x \in U_r, r \text{ diadic rational}\}$.

Then ϕ satisfies conditions (a) and (b). To see that ϕ satisfies (c), let $g \in \mathcal{S}$. If $x \in U_r$, by the positive invariance of U_r , $gx \in U_r$. So $\phi(gx) \leq \phi(x)$.

□

Conclusions

The notion of dynamical polysystem originates from the theory of control, in its most widely accepted sense. A dynamical polysystem consists of a family of real dynamical systems, all defined on a metric space. The results in this dissertation are based on the natural idea that a motion in the state space obtained by starting at a point and discretely switching from one dynamical system to another is similar to a motion that is obtained by starting at a point and controlling it with a piece-wise constant control function. In the first chapter we showed how certain dynamical polysystem, namely one for which smooth steering is possible, gives rise to a control system that accepts as inputs regulated functions. The important aspect of this construction is that we can define a continuous-time time-invariant control system without having to solve differential equations. We also showed that this method works for many control systems defined by means of differential equations, namely for those whose dynamics function satisfies a global lipshitz condition. Chances are that such condition can be relaxed and still permit the approach of control systems using dynamical polysystems as discussed above. This is one of our important objectives to be researched in further work.

Another goal for future developments in the direction of dynamical polysystems is motivated by the results on stability and attraction presented in the remaining chapters. Some of the characterizations for various types of stability in dynamical polysystems may generate similar approaches in the field of control systems, given the tight connection that exists between the two mathematical concepts.

Certain aspects of attractors and stable sets can be studied better if Lyapunov functions, and, more importantly, strict Lyapunov functions can be showed to

exist. This is what motivated the ideas presented in Chapter 5, ideas focused on regarding dynamical polysystems through the light of closed relations. We consider closed relations to be a powerful tool in characterizing the existence of Lyapunov functions and they constitute another immediate interest in the attempt to develop a stronger theory for dynamical polysystems.

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Vita

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