A CHARACTERIZATION OF ALMOST ALL
MINIMAL NOT NEARLY PLANAR GRAPHS

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Abstract

In this dissertation, we study nearly planar graphs, that is, graphs that are edgeless or have an edge whose deletion results in a planar graph. We show that all but finitely many graphs that are not nearly planar and do not contain one particular graph have a well-understood structure based on large Möbius ladders.
Chapter 1
Introduction

We shall assume the reader is familiar with the basic terms and definitions and notation of graph theory. Any unexplained terminology used here will follow [Die05] and [Oxl11]. For the readers convenience, we introduce the following terminology from [Bol98], [Big93], [Die05], and [Oxl11]. The vertex set of a graph $G$ is referred to as $V(G)$, its edge set as $E(G)$. The degree of a vertex $v$ is the number $|E(v)|$ of edges at $v$. If all the vertices of a graph $G$ are pairwise adjacent, then $G$ is complete. A complete graph on $n$ vertices is a $K_n$. A complete bipartite graph, $G := (V_1 \cup V_2, E)$, is a bipartite graph such that for any two vertices, $v_1 \in V_1$ and $v_2 \in V_2$, $v_1v_2$ is an edge in $G$. The complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$, is denoted $K_{m,n}$. When $G = (V, E)$ and $G' = (V', E')$ are two graphs, if $G' \subseteq G$ and $G'$ contains all the edges $xy \in E$ with $x, y \in V'$, then $G'$ is an induced subgraph of $G$. A graph $G$ is called $k$-connected (for $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. The greatest integer $k$ such that $G$ is $k$-connected is the connectivity of $G$. An embedding of a graph $G$ into a surface $M$, $G \subset M$, is a realization of a homeomorphic image of $G$ as a subspace of $M$. A graph is called planar if it can be embedded in the plane. A graph $G$ is an edge-transitive graph if, for every pair of edges $e_1$ and $e_2$, $G$ has an automorphism that maps $e_1$ to $e_2$.

This dissertation is focused on nearly planar graphs, that is, graphs $G$ such that $G$ is edgeless or $G \setminus e$ is planar for some edge $e$. The ultimate goal of this project is to fully describe the class of nearly planar graphs by listing all minimal graphs that are not nearly planar. We need to discuss the relations of graphs, such as
minors and topological minors, in order to explain the notion of minimality. All graphs considered in this dissertation are finite, loopless, and may have *multiple edges*, that is, several edges joining the same vertices. Edges are *parallel* to each other if they join the same vertices. If a graph $G$ is not planar but $G \setminus e$ is, then $e$ is a *planarizing edge* of $G$.

Let $e$ be an edge, whose endpoints are $x$ and $y$, of a graph $G$. By $G/e$, we denote the graph obtained from $G$ by *contracting* the edge $e$ into a new vertex $v_e$, which becomes adjacent to all the former neighbors of $x$ and of $y$. A graph $H$ is a *minor* of a graph $G$ if it can be obtained from $G$ by a sequence of operation, each of which is one of the following.

(1) deleting an edge;

(2) deleting an isolated vertex; and

(3) contracting an edge.

A graph $H$ is a *topological minor* of a graph $G$ if it can be obtained from $G$ by a sequence of operation each of which is one of the following.

(1) deleting an edge;

(2) deleting an isolated vertex; and

(3) contracting an edge incident with a vertex of degree two.

We say that a graph $H$ is a *subdivision* of a graph $G$ if there exists a map $\eta : (V(G) \cup E(G)) \to (V(H) \cup E(H))$ with the following properties when $v, w \in V(G)$ and $e, f \in E(G)$;

(1) $\eta(v)$ is a vertex of $H$, and if $v, w$ are distinct then $\eta(v), \eta(w)$ are distinct.
(2) if \( e \) has ends \( v, w \), then \( \eta(e) \) is a path of \( H \) with ends \( \eta(v), \eta(w) \), and otherwise disjoint from \( \eta(V(G)) \), and

(3) if \( e, f \) are distinct, then \( \eta(e) \) and \( \eta(f) \) are edge-disjoint, and if they have a vertex in common, then this vertex is an end of both.

Following [DOTV11], we call such a map \( \eta \) a homeomorphic embedding of \( G \) into \( H \).

The class of nearly planar graphs is not closed under the taking of minors, because we find an example of a graph \( G \) that is nearly planar, but some minor of \( G \) is not nearly planar. This is illustrated in Figure 1.1.

![Figure 1.1: This is an example of nearly planar graphs whose minor is not nearly planar.](image)

However, it is easy to show that the class of nearly planar graphs is closed under the taking of topological minors using Kuratowski’s theorem in [Kur30] and [Kur83].

**Theorem 1.1.** [Kuratowski’s theorem] A graph is planar if and only if it does not contain a subgraph that is a subdivision of \( K_5 \) or of \( K_{3,3} \).

**Lemma 1.2.** The class of nearly planar graphs is closed under the taking of topological minors.
Proof. Suppose that a graph $H$ is a topological minor of a graph $G$ with a homeomorphic embedding $\eta$. Let $H$ be not nearly planar. Then, for every edge $e$ of $H$, $H \setminus e$ is not planar. Therefore, $H \setminus e$ contains a subgraph that is a subdivision of $K_5$ or of $K_{3,3}$.

For every edge $\hat{e}_0$ of $G$ in $\eta(H)$, there is an edge $\hat{e}$ of $H$ such that $\hat{e}_0$ is in $\eta(\hat{e})$. Then, $H \setminus \hat{e}$ is a topological minor of $G \setminus \hat{e}_0$ and $G \setminus \hat{e}_0$ contains a subgraph that is a subdivision of $K_5$ or of $K_{3,3}$. Therefore, $G \setminus \hat{e}_0$ is not planar.

For every edge $\tilde{e}$ of $(E(G) \setminus E(\eta(H)))$, $G \setminus \tilde{e}$ contains $\eta(H)$ as a subgraph and $G \setminus \tilde{e}$ is not planar. As a result, $G$ is not nearly planar, either. 

We will show some properties about minimal graphs that are not nearly planar under the taking of topological minors.

Lemma 1.3. If a graph contains three edges that are parallel to each other, then the graph is not in the class of minimal graphs that are not nearly planar under the taking of topological minors.

Proof. Let a graph $G$ contain edges $e_1$, $e_2$, and $e_3$ that are parallel to each other. Suppose that $G$ is in the class of minimal graphs that are not nearly planar under the taking of topological minors. By the minimality, $G \setminus e_1$ is nearly planar. There exists an edge $e$ such that $(G \setminus e_1) \setminus e$ is planar. Yet, $G \setminus e$ is also planar and this is in contradiction to the fact that $G$ is in the class of minimal graphs that are not nearly planar under the taking of topological minors.

Lemma 1.4. Suppose that graph $G$ contains a vertex $w$ whose neighbors are only two vertices $u$ and $v$. If there are double edges between $w$ and $u$ and if there is one edge between $w$ and $v$, then $G$ is not in the class of minimal graphs that are not nearly planar under the taking of topological minors.
Proof. Let $e_1$ and $e_2$ be the parallel edges between $w$ and $u$. Suppose that $G$ is in the class of minimal graphs that are not nearly planar under the taking of topological minors. By the minimality, $G \setminus e_1$ is nearly planar. There is an edge $e$ such that $(G \setminus e_1) \setminus e$ is planar. However, $G \setminus e$ is also planar and this is in contradiction to the fact that $G$ is in the class of minimal graphs that are not nearly planar under the taking of topological minors.

![Diagram](image)

**FIGURE 1.2**: Minimal graphs that are not nearly planar under the taking of topological minors.

Taking topological minors, we find that each minimal not nearly planar graph containing double edges gives rise to a trivial infinite sequence of not nearly planar graphs $G$ such that every proper topological minor graph of $G$ is nearly planar, as illustrated in Figure 1.2.

To get a simple list of minimal graphs that are not nearly planar, we define a new relation $\lesssim$ between two graphs and this relation is an extension of topological minors. A graph $H$ is dominated by a graph $G$, denoted by $H \lesssim G$, if $H$ can be obtained from $G$ by a sequence of operation each of which is one of the following.

1. deleting an edge;

2. deleting an isolated vertex;

3. deleting a multiple edge;
(3) contracting an edge incident with a vertex with exactly two neighbors and deleting all resulting loops, if any.

Our ultimate goal of this project is to describe all the members of the class \( \mathcal{M} \) that consists of \( \simeq \)-minimal graphs that are not nearly planar. Using Lemma 1.3 and 1.4, we can say that no graphs in \( \mathcal{M} \)

(1) contain parallel edges more than two or

(2) contain a vertex whose neighbors are two vertices.

In Chapter 2, we show the full lists of graphs in \( \mathcal{M} \) whose connectivity is 0, 1, and 2. In Chapter 3, we show that each sufficiently large 3-connected graph in \( \mathcal{M} \) that does not contain \( K_{3,4} \) as a minor dominates a large Möbius ladder, which is explained in Chapter 3. Finally, in Chapter 4, we show that each such graph consists of a large Möbius ladder and a number of small subgraphs of three types.
Chapter 2
Minimal Not Nearly Planar Graphs Whose Connectivity Is Less Than 3

2.1 Graphs Whose Connectivity Is Less Than 2

Suppose that a graph $G \in \mathcal{M}$ is disconnected. Then, there are some properties about the connected components of $G$.

**Lemma 2.1.** For every disconnected graph in $\mathcal{M}$, none of its components is planar.

*Proof.* Let a graph $G \in \mathcal{M}$ be disconnected. Suppose that $A$ is a connected component of $G$ such that $G|A$ is planar. If $A$ does not have any edges, then $A$ is a single vertex and $G$ is not minimal. Therefore, $A$ has an edge. Let $e_A$ be an edge of $A$. Since $G$ is in $\mathcal{M}$, there is an edge $\tilde{e}$ such that $(G \setminus e_A) \setminus \tilde{e}$ is planar. Since $e_A$ is in $A$, $G \setminus \tilde{e}$ is planar. This contradicts the fact that $G$ is in $\mathcal{M}$. Therefore, $G|A$ is not planar. $\square$

**Lemma 2.2.** For every disconnected graph in $\mathcal{M}$, the number of its components is exactly two.

*Proof.* Let a graph $G \in \mathcal{M}$ be disconnected. Suppose that $G$ contains three connected components $A$, $B$, and $C$. By Lemma 2.1, all of $G|A$, $G|B$, and $G|C$ are not planar. Since $G$ is in $\mathcal{M}$, for every edge $e$ of $G$, there is an edge $\tilde{e}$ of $G$ such that $(G \setminus e) \setminus \tilde{e}$ is planar. However, by Pigeonhole principle, at least one of $G|A$, $G|B$, and $G|C$ must be contained in $(G \setminus e) \setminus \tilde{e}$. This is in contradiction to the fact that $G$ is in $\mathcal{M}$. Therefore, the number of components of $G$ is two. $\square$

The following graphs are made by 0-sum of two copies of $K_{3,3}$ and $K_5$. It is easy to show that each of the following graphs is in $\mathcal{M}$. Therefore, if a graph $G \in \mathcal{M}$ is disconnected, then $G$ is one of the following.
FIGURE 2.1: The complete list of disconnected graphs in $M$.

Suppose that a graph $G \in M$ has connectivity-1. Then, $G$ is 1-sum of connected subgraphs $A$ and $B$.

**Lemma 2.3.** Suppose that a graph $G \in M$ is 1-sum of graphs $A$ and $B$. Then, none of $A$ and $B$ is planar.

**Proof.** The argument follows the same idea of Lemma 2.1. □

Therefore, both of $A$ and $B$ are not planar. As a result, each of $A$ and $B$ contains a subdivision of $K_{3,3}$ or $K_5$ by Theorem 1.1. The following graphs are made by 1-sum of subdivisions of $K_{3,3}$ and $K_5$. It is easy to show that each of the graphs in Figure 2.2 is a member of $M$.

Therefore, if a graph $G \in M$ has connectivity one, then $G$ is 1-sum of two subdivisions of $K_{3,3}$ and $K_5$.

**2.2 Connectivity-2 graphs**

To describe connectivity-2 graphs in $M$, we will introduce some definitions in Chapter 4 of [DO02] for readers. If $G$ is a graph, $E_0$ is a subset of $E(G)$, and $S$ is a set, then define a function $L_G : S \times (V(G) \times V(G)) : e \mapsto (s(e), (u(e), v(e)))$ so that for each $e$ in $E_0$, $u(e)$ and $v(e)$ are the end vertices of $e$, and if $s(e) = s(f)$, then $e = f$. Call $L_G$ a directed labeling of $G$ where $u(e)$ and $v(e)$ are the tail and the head, respectively, of $e$. Assume that $L_H : E(H) \to S \times (V(H) \times V(H)) : e \mapsto (s(e), (u_H(e), v_H(e)))$ and $L_K : E(K) \to S \times (V(K) \times V(K)) : e \mapsto (s(e), (u_K(e), v_K(e)))$ are directed labelings of disjoint graphs $H$ and $K$, re-
FIGURE 2.2: The complete list of connectivity-1 graphs in $\mathcal{M}$
respectively, and there is only one pair, \( h \in E(H) \) and \( k \in E(K) \), of edges such that \( s(h) = s(k) \). Then the edge-sum of \( H \) and \( K \) (with respect to \( L_H \) and \( L_K \)), denoted \((H, L_H) \oplus_2 (K, L_K)\) or, more commonly, \( H \oplus_2 K \) is obtained by first identifying \( h \) and \( k \) head-to-head and tail-to-tail, and then deleting the identified edge. We may sometimes refer to \( H \oplus_2 K \) as the edge-sum of \( H \) and \( K \) along \( h \) and \( k \) when \( L_H \) and \( L_K \) are understood.

**Lemma 2.4.** Every 2-sum of two planar graphs is planar.

**Proof.** Let \( A \) and \( B \) be planar graphs. Suppose that \( e_A \) and \( e_B \) are edges of \( A \) and \( B \), respectively, to be used for \( A \oplus_2 B \). Since every planar graph has an embedding in the 2-dimensional sphere, we find a planar embedding of each of \( A \) and \( B \) such that each of \( e_A \) and \( e_B \) is incident to the outer face using stereographic projection. With these new planar embedding for \( A \) and \( B \), we get a plane graph that is isomorphic to \( A \oplus_2 B \). \( \square \)

Let a graph \( G \in \mathcal{M} \) have connectivity two such that \( G \) becomes disconnected by deleting two vertices \( p \) and \( q \). Then we can show the abstract structure of graph \( G \) as illustrated in Figure 2.3.

![Figure 2.3: Connectivity 2](image)

We can say that \( V(G) \) is a disjoint union of non-empty sets \( V(A_1) \), \( V(A_2) \), and \( \{p, q\} \) such that each induced subgraph by \( V(A_1) \) and \( V(A_2) \) is connected. For each \( i \), let \( \tilde{A}_i \) be the induced graph of \( G \) by \( V(A_i) \cup \{p, q\} \) with an edge \( e_i \) which
is incident with $p$ and $q$. Then, one of $\tilde{A}_1$ and $\tilde{A}_2$ is not planar because of Lemma 2.4. The following theorem proves that both of $\tilde{A}_1$ and $\tilde{A}_2$ are not planar.

**Theorem 2.5.** Let a graph $G \in \mathcal{M}$ have connectivity two and is such that $G$ becomes disconnected by deleting two its vertices $p$ and $q$. Let $A_1$ and $A_2$ be the connected components of $G - \{p, q\}$. For each $i = 1, 2$, if $\tilde{A}_i$ is the subgraph of $G$ induced by $V(A_i) \cup \{p, q\}$ joining $p$ and $q$ by an edge $e_i$, then each $\tilde{A}_i$ is not planar.

**Proof.** Without loss of generality, we may assume that $\tilde{A}_1$ is planar and $\tilde{A}_2$ is not. Graph $\tilde{A}_2$ is nearly planar because $G \in \mathcal{M}$ and $\tilde{A}_2 \subsetneq G$. Then, $\tilde{A}_2$ dominates $K_5$ or $K_{3,3}$. Let $M$ be a subgraph of $\tilde{A}_2$ that is a subdivision of $K_{3,3}$ or of $K_5$. Graph $M$ has an edge $e_0$ such that $\tilde{A}_2 \setminus e_0$ is planar. (If not, since the induced subgraph of $G$ by $V(A_i) \cup \{p, q\}$ has a path $P$ from $p$ to $q$ and an edge $e_*$ that does not belong to $E(P)$, $G \setminus e_*$ is not nearly planar. This contradicts the fact that $G$ is a member of $\mathcal{M}$.) If $e_0 \neq e_2$, then by the previous lemma, $G \setminus e_0$ is planar. This is in contradiction to the fact that $G$ is a member of $\mathcal{M}$ if $e_0 \neq e_2$. If $G$ is a member of $\mathcal{M}$, then edge $e_0$ is edge $e_2$ of $\tilde{A}_2$.

We may assume that edge $e_2$ is the only edge that makes $\tilde{A}_2$ planar by edge deletion. If $G$ contains edge $pq$, namely $e_3$, then each of $\tilde{A}_1$ and $\tilde{A}_2$ has double edges between $p$ and $q$. In addition, since $\tilde{A}_2 \setminus e_2$ is not planar because of $e_3$, $\tilde{A}_2$ is not nearly planar. As $G$ is 2-connected, both of $\tilde{A}_1$ and $\tilde{A}_2$ should be 2-connected. Therefore, there is a cycle $C_0$ containing $e_1$ in $\tilde{A}_1$ and we can say that $(C_0 \setminus e_1) \cup e_3$ is a subgraph of $G$. Then, $\tilde{A}_2$ is dominated in $G$ and $\tilde{A}_2 \setminus e_2$ is not nearly planar. This contradicts the fact that $G \in \mathcal{M}$ since $V(A_1)$ is not empty. Therefore, $G$ does not contain an edge $pq$ and the induced subgraph of $V(A_2) \cup \{p, q\}$ with double edges between $p$ and $q$ is not a proper topological minor of $G$. Suppose that there are edge disjoint paths $P_\alpha$ and $P_\beta$ from $p$ to $q$ in $\tilde{A}_1 \setminus e_1$. By the minimality of $G$,
\( \tilde{A}_1 \setminus e_1 \) has an edge \( e^* \) such that \( e^* \) is not in \( E(P_\alpha) \cup E(P_\beta) \). Then, \( G \setminus e^* \) is not nearly planar and \( G \notin \mathcal{M} \). Therefore, \( \tilde{A}_1 \setminus e_1 \) does not have edge disjoint paths from \( p \) to \( q \). As \( \tilde{A}_1 \setminus e_1 \) is planar and \( \tilde{A}_1 \setminus e_1 \) does not have edge disjoint paths from \( p \) to \( q \), by Jordan curve theorem, there exists an edge \( e_3 \) in \( \tilde{A}_1 \setminus e_1 \) such that every path from \( p \) to \( q \) in \( \tilde{A}_1 \setminus e_1 \) contains \( e_3 \). In this case, \( G \setminus e_3 \) is planar. As a result, both of \( \tilde{A}_1 \) and \( \tilde{A}_2 \) are not planar.

Using 2-sum of copies of \( K_{3,3} \) and \( K_5 \), we find connectivity-2 graphs in \( \mathcal{M} \) which are illustrated in Figure 2.4.

![Figure 2.4: Members of \( \mathcal{M} \) constructed by 2-sum of \( K_{3,3} \) and/or \( K_5 \)](image)

Let \( K^+_{3,3} \) be the graph obtained from \( K_{3,3} \) joining two non-adjacent vertices with new edge \( e^+ \). This graph is needed for 2-sum to describe members of \( \mathcal{M} \) whose connectivity is 2. When we 2-sum \( K^+_{3,3} \) and another graph, we 2-sum them along \( e^+ \).

![Figure 2.5: Graph \( K^+_{3,3} \)](image)

Let \( \mathcal{P} \) be the set of \( K_3 \) and \( K_5 \) and let \( \mathcal{P}^+ \) be the set of \( K_3 \), \( K_5 \), and \( K^+_{3,3} \). There are some members of \( \mathcal{M} \) described by 2-sum of \( K^+_{3,3} \) with one of \( \mathcal{P}^+ \).
Using 2-sum of $K_{3,3}^+$ with a subdivision of one of $\mathcal{P}$, we describe other graphs in $\mathcal{M}$. However, every graph $G$ made by 2-sum of $K_{3,3}^+$ with a subdivision of $K_{3,3}^+$ is not nearly planar but not in $\mathcal{M}$ because $G$ dominates a member of $\mathcal{M}$ as stated in Lemma 2.6.

Let $A$ be a copy of one of $\mathcal{P}$ and let $B$ be a subdivision of one of $\mathcal{P}$ with a homeomorphic embedding $\eta$. Suppose that $e$ is an edge of $\eta^{-1}(B)$ such that $\eta(e)$ is a path whose length is at least 2. When we 2-sum $A$ and $B$ along an edge of $A$ and an edge $\tilde{e}$ of $\eta(e)$, $A \oplus_2 B$ is nearly planar as shown in Figure 2.8. In Figure 2.8, red edges are planarizing edges. After doubling some edges of $\eta(e)$, we describe some members of $\mathcal{M}$ in Figure 2.9 by 2-sum.

In Figure 2.10, we introduce new six graphs each of which is obtained by adding an edge $e^+$, indicated red edge, to a subdivision of one of $\mathcal{P}$. These six graphs, needed to describe more graphs in $\mathcal{M}$ using 2-sum, are $K_{3,3}^{S+}$, $K_{3,3}^{S_1S+}$, $K_{3,3}^{S_2S+}$, $K_{5}^{S+}$, $K_{5}^{S_1S+}$, and $K_{5}^{S_2S+}$. 
Each graph in Figure 2.11 made by 2-summing $K_{3,3}^{S+}$ or $K_{5}^{S+}$ with one of $\mathcal{P}^+$ along $e^+$ is not nearly planar and it is not minimal because, when $e$ is one of the red edges of every graph $G$ in Figure 2.11, $G \setminus e$ is not nearly planar, either.

When we 2-sum $K_{3,3}^{S+}$ and $K_{5}^{S+}$, since both of these two graphs are not edge-transitive, we get two graphs by 2-sum as illustrated in Figure 2.12. Both of graphs $G$ in Figure 2.12 are not members of $\mathcal{M}$ even though both are not nearly planar because, when $e$ is a red edge in Figure 2.12, $G \setminus e$ is not nearly planar, either.
FIGURE 2.10: Graphs made by adding one edge to a subdivision of one of $P$

However, if we subdivide $e^+$ of $K_{3,3}^S$ or $K_5^S$ before 2-sum with one of $P$, using 2-sum, we describe members of $M$ in Figure 2.13.

From Figures 2.11 and 2.12, we know that all of $K_{3,3}^S \bigoplus 2 K_{3,3}^S$, $K_5^S \bigoplus 2 K_{3,3}^S$, and $K_{3,3}^S \bigoplus 2 K_5^S$ are not members of $M$. When we 2-sum two of $K_{3,3}^S$, $K_{3,3}^S$, and $K_5^S$, even though we take a subdivision before 2-summing, the graph made by 2-sum is not in $M$ by the following lemma.

**Lemma 2.6.** Let $\mathcal{K} = \{K_{3,3}^S, K_{3,3}^S, K_{3,3}^S \bigoplus 2 K_{3,3}^S, K_5^S \bigoplus 2 K_{3,3}^S, K_5^S \bigoplus 2 K_{3,3}^S, K_{3,3}^S \bigoplus 2 K_5^S, K_5^S \bigoplus 2 K_5^S\}$. Suppose $A$ is a copy of one of $\mathcal{K}$ and $B$ is a graph obtained by subdividing $e^+$ of one of $\mathcal{K}$. If a graph $G$ is made by 2-sum of $A$ and $B$ along $e^+$ of $A$ and an edge in $B$ from the path that is made by subdivision, then $G$ is not in $M$.

**Proof.** Let $\eta(e^+)$ be a path of $B$ made by the subdivision of $e^+$. Suppose $e_B^+$ is an edge of $\eta(e^+)$ such that graph $G$ is made by 2-sum of $A$ and $B$ along $e^+$ of $A$ and $e_B^+$. Since each of $A \setminus e^+$ and $B \setminus (E(\eta(e^+)))$ dominates $K_{3,3}$ or $K_5$, graph $G \setminus (E(\eta(e^+)) \setminus e_B^+)$ dominates one of graphs in Figures 2.1 or 2.2. Therefore, $G$ is not a member of $M$. \qed
By Figures 2.1, 2.2, and 2.12, we use neither $K_{3,3}^{S1+}$ nor $K_{5}^{S1+}$ for 2-sum to describe members of $\mathcal{M}$ by the following lemma.

**Lemma 2.7.** Every graph made by 2-summing a copy of one of $K_{3,3}^{S1+}$ and $K_{5}^{S1+}$ with a graph dominating one of $\mathcal{P}$ is not a member of $\mathcal{M}$.

**Proof.** Let $\alpha$ and $\beta$ be endpoints of $e^+$ of $K_{3,3}^{S1+}$. Then, $V(K_{3,3}^{S1+}) - (N(\alpha) \cup N(\beta))$ is a vertex set with two elements. Suppose $\{\gamma, \delta\} = V(K_{3,3}^{S1+}) - (N(\alpha) \cup N(\beta))$. As $K_{3,3}^{S1+} \setminus \gamma \delta$ dominates $K_{3,3}$, none of $K_{3,3}^{S1+} \oplus_2 K_{3,3}$, $K_{3,3}^{S1+} \oplus_2 K_5$, $K_{3,3}^{S1+} \oplus_2 K_{3,3}$, $K_{3,3}^{S1+} \oplus_2 K_{3,3}$. 

FIGURE 2.11: Graphs, which are not members of $\mathcal{M}$, made by 2-summing $K_{3,3}^{S1+}$ or $K_{5}^{S1+}$ with one of $\mathcal{P}$

FIGURE 2.12: Graphs, which are not members of $\mathcal{M}$, made by 2-summing $K_{3,3}^{S1+}$ and $K_{5}^{S1+}$
FIGURE 2.13: Members of $\mathcal{M}$ made by 2-sum of a copy of one of $\mathcal{P}$ with a subdivision of one of $K_{3,3}^{S^+}$ and $K_5^{S^+}$.

Let $\epsilon$ and $\zeta$ be endpoints of $e^+$ of $K_5^{SS1+}$. Then, $V(K_5^{SS1+}) - (N(\epsilon) \cup N(\zeta))$ is a vertex set with one element. Let $\theta$ be the only element of $V(K_5^{SS1+}) - (N(\epsilon) \cup N(\zeta))$.

Graph $K_5^{SS1+} - \theta$ is a copy of $K_{3,3}$. By the above argument, none of $K_5^{SS1+} \oplus_2 K_{3,3}$, $K_5^{SS1+} \oplus_2 K_5$, $K_5^{SS1+} \oplus_2 K_{3,3}^+$, $K_5^{SS1+} \oplus_2 K_{3,3}^{S^+}$, and $K_5^{SS1+} \oplus_2 K_5^{S^+}$ is a member of $\mathcal{M}$.

In addition, $K_{3,3}^{SS1+} \oplus_2 K_5^{SS1+}$ is not a member of $\mathcal{M}$, either. \hfill $\Box$
Even though we know Lemma 2.7, if we take a subdivision of $e^+$ of $K_{3,3}^{SS1+}$ or of $K_5^{SS1+}$ before 2-sum, then we can describe some members of $\mathcal{M}$ as shown in Figure 2.14.

Since we know that each of $K_{3,3}^S$, $K_{3,3}^{SS1+}$, $K_5^S$, and $K_5^{SS1+}$ dominates a subdivision of $K_{3,3}$ containing $e^+$, we use none of these four for 2-sum with one of $K_{3,3}^{SS2+}$ and $K_5^{SS2+}$. We deal each member of $\mathcal{A}$ with one of $K_{3,3}^{SS2+}$ and $K_5^{SS2+}$ for 2-sum to describe members of $\mathcal{M}$. We notice that none of $K_{3,3}^{SS2+} \bigoplus_2 K_{3,3}^+$, $K_5^{SS2+} \bigoplus_2 K_{3,3}^+$, and $K_{3,3}^{SS2+} \bigoplus_2 K_5^{SS2+}$ is a member of $\mathcal{M}$ because it has some red edge $e$ in Figure 2.15 such that none of $(K_{3,3}^{SS2+} \bigoplus_2 K_{3,3}^+) \setminus e$, $(K_5^{SS2+} \bigoplus_2 K_{3,3}^+) \setminus e$, and $(K_{3,3}^{SS2+} \bigoplus_2 K_5^{SS2+}) \setminus e$ is nearly planar.
However, we describe members of $\mathcal{M}$ using 2-sum of $K_{3,3}^{SS2+}$ with one of $\mathcal{P}$. In addition, subdividing $e^+$ of $K_{3,3}^{SS2+}$, we describe members of $\mathcal{M}$ by 2-summing the subdivided graph with one of $\mathcal{P}$. These members of $\mathcal{M}$ are shown in Figure 2.16.

While $K_{3,3}^{SS2+} \oplus_2 K_{3,3}$ and $K_{3,3}^{SS2+} \oplus_2 K_5$ are members of $\mathcal{M}$, $K_5^{SS2+} \oplus_2 K_{3,3}$ and $K_5^{SS2+} \oplus_2 K_5$ are not members of $\mathcal{M}$ because, for every red edge $e$ in Figure 2.17, both of $(K_5^{SS2+} \oplus_2 K_{3,3}) \setminus e$ and $(K_5^{SS2+} \oplus_2 K_5) \setminus e$ are not nearly planar.

Similar to the previous examples, if we subdivide $e^+$ of $K_5^{SS2+}$ before 2-sum with one of $\mathcal{P}$, then we find members of $\mathcal{M}$, which are illustrated in Figure 2.18.
Fifty-seven graphs in Figures 2.4, 2.6, 2.7, 2.9, 2.13, 2.14, 2.16, and 2.18 are the complete list of connectivity-2 graphs in $\mathcal{M}$. 
Chapter 3
Minimal 3-connected not nearly planar graphs

In this chapter, we will use a result in [DOTV11] to describe 3-connected graphs in $\mathcal{M}$. We begin by introducing some terminology from [DOTV11].

Let $k$ be an integer greater than two. The 2$^k$-spoke alternating double wheel, denoted by $A_k$, has vertices $v_0, v'_0, v_1, v_2, \ldots, v_{2k}$, where $v_1, v_2, \ldots, v_{2k}$ form a cycle in this order, $v_0$ is adjacent to $v_1, v_3, \ldots, v_{2k-1}$, and $v'_0$ is adjacent to $v_2, v_4, \ldots, v_{2k}$. The vertices $v_0$ and $v'_0$ will be called the hubs of $A_k$. The edges incident to a hub are called spokes. The cycle that consists of $v_1, v_2, \ldots, v_{2k}$ is called the rim of $A_k$. As $A_k$ is 3-connected and planar, it has combinatorially unique plane embedding. We define $B_k$ to be the graph obtained from $A_k$ by adding an edge joining its hubs, which is called an axle.

The $k$-rung Möbius ladder $M_k$ has vertices $x_1, x_2, x_3, \ldots, x_k, y_1, y_2, \ldots, y_k$, which form a cycle of length $2k$ in the order listed. Edges of $M_k$ are edges of the above cycle of length $2k$ and $x_iy_i$ for each $i$.

A graph $G$ is $t$-shallow if, for every separation $(A, B)$ of order at most three, one of $G|A$ and $G|B$ has fewer than $t$ vertices and can be drawn in a disk with $A \cap B$ drawn on the boundary of the disk. For example, every 4-connected graph is $k$-shallow for any integer $k \geq 4$ and almost 4-connected graph is 5-shallow.

Theorem 3.1. Suppose that graph $G$ is not $K_{3,4}$. If $G$ has connectivity three and is such that, for some separation $(A, B)$ of order three, neither $G|A$ nor $G|B$ can be drawn in a disk without crossing with $A \cap B$ drawn on the boundary of the disk, then each of $G|A$ and $G|B$ contains $K_{2,3}$ as a topological minor and $G$ contains $K_{3,4}$ as a subgraph or one of the following graphs.
Proof. Since $K_{2,3}$ is a subgraph of $K_5$ and of $K_{3,3}$, we may assume that neither $G|A$ nor $G|B$ dominates $K_5$ or $K_{3,3}$.

Since $G|(B - A)$ is connected, if we contract every edge of $G|(B - A)$, then $G|(B_A)$ becomes a single vertex and $G|B$ becomes $K_{1,3}$ as illustrated in Figure 3.2. Because $G|A$ cannot be drawn in a disk without crossing with $A \cap B$ drawn on the boundary of the disk, $G|A$ with $K_{1,3}$, as shown in Figure 3.2, contains $K_{3,3}$ as a topological minor. Therefore, $G|A$ contains $K_{2,3}$ as a topological minor. Using the same argument, we know that $G|B$ contains $K_{2,3}$ as a topological minor.

Both of $G|A$ and $G|B$ contain one of the following as a topological minor even though $G|A$ or $G|B$ contains $K_5$ or $K_{3,3}$ as a topological minor because $G$ has connectivity three. Combining two graphs from Figure 3.3, we can get the result.

\[\square\]
3.1 Graphs that are $t$-shallow

If $G$ has connectivity three and does not contain $K_{3,4}$ as a minor, then for any separation $(A, B)$ of order three, one of $G|A$ and $G|B$ can be drawn in a disk with $A \cap B$ drawn on the boundary of the disk. For a fixed separation $(A, B)$ of order three, let $V(A \cap B)$ be $\{x, y, z\}$. Without loss of generality, we may assume that $G|B$ can be drawn in a disk. Now, we will describe planar part $G|B$ using the terminology $m_{x,y}$ and $n_z$ in [BORS13]. We define $m_{x,y}$ as the maximum number of edge disjoint paths from $x$ to $y$ without $z$ in $G|B$. Let $n_z$ be the maximum number of edge disjoint paths from $z$ to the union of edge disjoint paths from $x$ to $y$ without $z$ in $G|B$. (We will assume that if there exists an edge between two vertices of $\{x, y, z\}$, then the edge is in $G|A$.) Since $G$ is 3-connected, $m_{x,y} \geq 1$ and $n_z \geq 1$. Now, we will use the results in [BORS13].

In this chapter, if $x$, $y$, and $z$ are all of degree-one vertices of a copy of $K_{1,3}$, then we will call it as a rooted $K_{1,3}$. Suppose that $m_{x,y} > 1$, $m_{y,z} > 1$, $m_{z,x} > 1$, $n_x > 1$, $n_y > 1$, and $n_z > 1$. Then, for some $1 \leq i \leq 20$, $G|B$ contains graph $Y_i$ in Figures 3.4 and 3.5 as a topological minor. From the graphs in Figures 3.4 and 3.5, white points mean $\{x, y, z\}$. In Figure 3.5, the colored edge is a redundant edge for the graphs in $\mathcal{M}$ as stated in Lemma 3.2.
FIGURE 3.4: Possible structures of planar part of a separation of order 3 of graphs in $\mathcal{M}$ (Part 1)
Lemma 3.2. Suppose that $G \in \mathcal{M}$ and it has a separation $(A, B)$ of order three such that $G|B$ can be drawn in a disk with $A \cap B$ drawn on the boundary of the disk. Then $G|B$ cannot contain $Y_{20}$ in Figure 3.5 as a topological minor.

Proof. Suppose that $G|B$ contains $Y_{20}$ as a topological minor. Let the subdivision of the red edge in $Y_{20}$ contain $\alpha$ in $G|B$. Since $G \in \mathcal{M}$, we can find an edge $\beta$ such that $(G \setminus \alpha) \setminus \beta$ is planar. If $\beta$ is in $G|B$, then $\{(G|B) \setminus \alpha\} \setminus \beta$ contains a rooted $K_{1,3}$ as a topological minor, $(G \setminus \alpha) \setminus \beta$ is not planar. Therefore, $\beta$ is in $G|A$. Since $(G|B) \setminus \alpha$ contains a rooted $K_{1,3}$ as a topological minor, $(G|A) \setminus \beta$ can be drawn in a disk with $A \cap B$ drawn on the boundary of the disk. This means $G \setminus \beta$ is planar and $G$ is nearly planar. This is in contradiction to the fact that $G \in \mathcal{M}$. □

![Diagram](image)

FIGURE 3.5: If the planar part of a separation of order 3 of graph $G$ is $Y_{20}$, then $G$ is not in $\mathcal{M}$.

Lemma 3.3. Suppose that $G \in \mathcal{M}$ and it has a separation $(A, B)$ of order three such that $G|B$ can be drawn in a disk with $A \cap B$ drawn on the boundary of the disk. If $G|B$ dominates $Y_i$ with $1 \leq i < 20$, then $G|B$ is homeomorphic to $Y_i$.

Proof. Let $i \in \{k|1 \leq k < 20\}$ and suppose $Y_i \not\preceq G|B$ and $G \in \mathcal{M}$. Then, for any edge $e$ in $G|A$, $(G|A) \setminus e$ cannot be drawn in a disk with $A \cap B$ drawn on the boundary of the disk because $G \in \mathcal{M}$. Since $Y_i \not\preceq G|B$, we can find an edge $\gamma$ in a bridge of $Y_i$ in $G|B$. As $G \in \mathcal{M}$, we can find an edge $e$ such that $(G \setminus \gamma) \setminus e$ is planar. If $e$ is in $G|A$, because $(G|A) \setminus e$ cannot be drawn in a disk with $A \cap B$
on the boundary of the disk, this contradicts the fact that $G \in \mathcal{M}$. If $e$ is in $G|B$, then we can find a rooted $K_{1,3}$ in $\{(G|B) \setminus \gamma \} \setminus e$. This is in contradiction to the fact that $G \in \mathcal{M}$. Therefore, if $G \in \mathcal{M}$, $G|B$ is homeomorphic to $Y_i$. 

Suppose that $m_{x,y} = 1$, $m_{y,z} > 1$, $m_{z,x} > 1$, $n_x > 1$, $n_y > 1$, and $n_z > 1$. Then, for some $21 \leq i \leq 23$, $G|B$ contains graph $Y_i$ in Figure 3.6 as a topological minor. From the graphs in Figure 3.6, white points mean $\{x, y, z\}$. If $G|B$ has an edge $e$ such that $(G|B) \setminus e$ does not contain the rooted $K_{1,3}$, then we colored edge $e$ of graphs in figures of this section. (If the number of edges satisfying this condition is not one, then we use distinct colors for each edge.)

![Figure 3.6: Possible structures of planar part of a separation of order 3 of graphs in $\mathcal{M}$ (Part 2)](image)

**Lemma 3.4.** Suppose that $G \in \mathcal{M}$ and it has a separation $(A, B)$ of order three such that $G|B$ can be drawn in a disk with $A \cap B$ on the boundary of the disk. If $G|B$ does not dominate $Y_i$ for $1 \leq i \leq 20$ but dominates $Y_j$ for some $21 \leq j \leq 23$, then $G|B$ is homeomorphic to $Y_j$.

**Proof.** Assume that there exists some $j$ with $21 \leq j \leq 23$ such that $Y_j \preceq (G|B)$ with $G \in \mathcal{M}$. Without loss of generality, we may assume that $m_{x,y} = 1$ for $G|B$. $G|B$ has the abstract structure in Figure 3.7.

Let $G'$ be made by replacing $G|B$ with $Y_j$ for some $21 \leq j \leq 23$. Then, $G'$ is nearly planar. Let $e$ be the red edge in $Y_j$. Then, $e$ is the only planarizing edge of
FIGURE 3.7: Abstract structures of planar part of a separation of order 3 (Part 1)

$G'$. From every planar embedding of $G' \setminus e$, we can notice that, if $e_{sub}$ is an edge of $G|B$ in the path made by the subdivision of $e$, then $G \setminus e_{sub}$ is planar. This contradicts the fact that $G \in \mathcal{M}$. As a result, $G|B$ is homeomorphic to $Y_j$. □

Suppose that $m_{x,y} = 1$, $m_{y,z} > 1$, $m_{z,x} = 1$, $n_x = 1$, $n_y > 1$, and $n_z > 1$. Then, for some $24 \leq i \leq 28$, $G|B$ contains graph $Y_i$ in Figure 3.8 as a topological minor. In graphs in Figure 3.8, white points mean $\{x, y, z\}$.

FIGURE 3.8: Possible structures of planar part of a separation of order 3 of graphs in $\mathcal{M}$ (Part 3)

**Lemma 3.5.** Suppose that $G \in \mathcal{M}$ and it has a separation $(A, B)$ of order three such that $G|B$ can be drawn in a disk with $A \cap B$ on the boundary of the disk. If
\(G|B\) does not dominate \(Y_i\) for \(1 \leq i \leq 23\) but dominates \(Y_j\) for some \(24 \leq j \leq 28\), then \(G|B\) is homeomorphic to \(Y_j\).

**Proof.** Assume that there exists some \(j\) with \(24 \leq j \leq 28\) such that \(Y_j \preceq (G|B)\) with \(G \in \mathcal{M}\). Without loss of generality, we may assume that \(n_x = 1\), \(m_{z,x} = 1\), and \(m_{x,y} = 1\) for \(G|B\). \(G|B\) has the abstract structure in Figure 3.9.

![Figure 3.9: Abstract structures of planar part of a separation of order 3 (Part 2)](image)

Let \(G'\) be made by replacing \(G|B\) with \(Y_j\) for some \(24 \leq j \leq 28\). Then, \(G'\) is nearly planar. Let \(e\) be the red edge in \(Y_j\). Then, \(e\) is the only planarizing edge of \(G'\). From every planar embedding of \(G' \setminus e\), we can notice that, if \(e_{sub}\) is an edge of \(G|B\) in the path made by the subdivision of \(e\), then \(G \setminus e_{sub}\) is planar. This is in contradiction to the fact that \(G \in \mathcal{M}\). As a result, \(G|B\) is homeomorphic to \(Y_j\).

Suppose that \(m_{x,y} = 1\), \(m_{y,z} = 1\), \(m_{z,x} = 1\), \(n_x > 1\), \(n_y = 1\), and \(n_z = 1\). Then, for some \(29 \leq i \leq 30\), \(G|B\) contains \(Y_i\) in Figure 3.10 as a topological minor. In graphs of Figure 3.10, white points mean \(\{x,y,z\}\).

**Lemma 3.6.** Suppose that \(G \in \mathcal{M}\) and it has a separation \((A, B)\) of order three such that \(G|B\) can be drawn in a disk with \(A \cap B\) on the boundary of the disk. If \(G|B\) does not dominate \(Y_i\) for \(1 \leq i \leq 28\) but dominates \(Y_j\) for some \(29 \leq j \leq 30\), then \(G|B\) is homeomorphic to \(Y_j\).
Proof. Assume that there exists some $j$ with $29 \leq j \leq 30$ such that $Y_j \not\lesssim (G|B)$ with $G \in \mathcal{M}$. Without loss of generality, we may assume that $n_x > 1$, $n_y = 1$, and $n_z = 1$ for $G|B$. $G|B$ has the abstract structure in Figure 3.11.

Let $G'$ be made by replacing $G|B$ with $Y_j$ for some $29 \leq j \leq 30$. Then, $G'$ is nearly planar. Since $G \in \mathcal{M}$ and $G|A$ cannot be drawn in a disk with $A \cap B$ on the boundary, a planarizing edge of $G'$ must be the red edge or the blue edge of $Y_j$. Without loss of generality, we may assume that the red edge is a planarizing edge of $G'$. Let $e$ be the red edge in $Y_j$. From every planar embedding of $G' \setminus e$, we can notice that, if $e_{sub}$ is an edge of $G|B$ in the path made by the subdivision of $e$, then $G \setminus e_{sub}$ is planar. This contradicts the fact that $G \in \mathcal{M}$. Therefore, $Y_j$ is homeomorphic to $G|B$. \hfill $\Box$
Suppose that \( m_{x,y} = m_{y,z} = m_{z,x} = n_x = n_y = n_z = 1 \). Then, \( G|B \) contains graph \( Y_{31} \) in Figure 3.12 as a topological minor. In graph \( Y_{31} \), white points mean \( \{x, y, z\} \).

![Figure 3.12: Possible structures of planar part of a separation of order 3 of graphs in \( \mathcal{M} \) (Part 5)](image)

**Lemma 3.7.** Suppose that \( G \in \mathcal{M} \) and it has a separation \((A, B)\) of order three such that \( G|B \) can be drawn in a disk with \( A \cap B \) on the boundary of the disk. If \( G|B \) does not dominate \( Y_i \) for \( 1 \leq i \leq 30 \) but dominates \( Y_{31} \), then \( G|B \) is homeomorphic to \( Y_{31} \).

**Proof.** Assume that \( Y_{31} \nless (G|B) \) with \( G \in \mathcal{M} \). Then, \( n_x = 1, n_y = 1, \) and \( n_z = 1 \) for \( G|B \). \( G|B \) has the abstract structure in Figure 3.13.

![Figure 3.13: Abstract structures of planar part of a separation of order 4 (Part 4)](image)

Let \( G' \) be constructed by replacing \( G|B \) with \( Y_{31} \). Then, \( G' \) is nearly planar. Since \( G \in \mathcal{M} \) and \( G|A \) cannot be drawn in a disk with \( A \cap B \) on the boundary, a planarizing edge of \( G' \) must be an edge of \( Y_{31} \). Without loss of generality, we may
assume that the red edge is a planarizing edge of \( G' \). Let \( e \) be the red edge in \( Y_{31} \).

From every planar embedding of \( G' \setminus e \), we can notice that, if \( e_{\text{sub}} \) is an edge of \( G|B \) in the path made by the subdivision of \( e \), then \( G \setminus e_{\text{sub}} \) is planar. This is in contradiction to the fact that \( G \in \mathcal{M} \). Therefore, \( Y_{31} \) is homeomorphic to \( G|B \). \( \square \)

In the previous lemmas, we checked all of the possible structures of \( G|B \) which can be drawn in a disk with \( A \cap B \) on the boundary. Therefore, we get the following theorem.

**Theorem 3.8.** If \( G \in \mathcal{M} \) and \( G \) does not contain \( K_{3,4} \) as a minor, then \( G \) is 10-shallow.

By the following theorem in [DOTV11], every large non-planar almost 4-connected graph contains \( B_k, M_k, K_{4,k} \) and \( K'_{4,k} \) as a topological minor.

**Theorem 3.9.** For every two integer \( k, t \geq 4 \), there is an integer \( N \) such that every almost 4-connected \( t \)-shallow non-planar graph with at least \( N \) vertices contains a subgraph isomorphic to a subdivision of one of \( B_k, M_k, K_{4,k}, \) and \( K'_{4,k} \).

Using this theorem, we can get the following corollary.

**Corollary 3.10.** For every \( k \), there is an integer \( N \) such that if a graph in \( \mathcal{M} \) whose connectivity is three has at least \( N \) vertices but it does not contain \( K_{3,4} \) as a minor, then it contains a subgraph isomorphic to a subdivision of one of \( B_k \) or \( M_k \).

**Proof.** Suppose that \( G \) satisfies all of the given conditions. Since \( K_{3,4} \) is not a minor of \( G \), \( G \) contains neither \( K_{4,k} \) nor \( K'_{4,k} \) as a topological minor and \( G \) has a separation \((A, B)\) of order three such that \( G|B \) can be drawn in a disk with \( A \cap B \) on the boundary of the disk. As \( G \) has connectivity three, \( G \) is 10-shallow. We can replace \( G|B \) as \( K_{1,3} \). When we do this action repeatedly, we can get an
almost 4-connected graph $G'$ that is dominated in $G$. Using the previous theorem, $G'$ dominates one of $B_k, M_k, K_{4,k}$, and $K'_{4,k}$. Therefore, $G$ dominates one of $B_k$ or $M_k$.

In this paper, let $N(20)$ be an integer in the above corollary when $k = 20$. Integer $N(20)$ is used for the main theorem of this paper.

### 3.2 The Graphs Containing $K_{4,4}$ as a Minor

From Theorem 3.9, if we do not have the condition that $K_{3,4}$ is not a minor of every graph in $\mathcal{M}$, it is enough to check $B_k, M_k, K_{4,k}$, and $K'_{4,k}$ for infinitely many graphs in $\mathcal{M}$. In this subsection, we show that every graph containing $K_{4,k}$ or $K'_{4,k}$ as a topological minor is not in $\mathcal{M}$.

**Theorem 3.11.** Every graph dominating $K_{4,4}$ or $K'_{4,4}$ is not in $\mathcal{M}$.

**Proof.** Since $K_{3,4} \in \mathcal{M}$, every graph $G$ containing $K_{4,k}$ for $k \geq 4$ as a topological minor is not in $\mathcal{M}$. Let us check $K'_{4,4}$ which is obtained from $K_{4,4}$ by splitting each of the 4 vertices in one set of the bipartition of $K_{4,4}$ in the same way as illustrated in Figure 3.14.

![Figure 3.14: Graph $K'_{4,4}$](image)

Let $\alpha$ be an old edge which exists before split of vertices when we get $K'_{4,4}$ from $K_{4,4}$. If $K'_{4,4}$ is in $\mathcal{M}$, then $K'_{4,4} \setminus \alpha$ should be nearly planar. $K'_{4,4} \setminus \alpha$ has the following $K'_{3,4}$ as a subgraph which is nearly planar.

We can pick an edge $\beta$ in Figure 3.15 such that $K'_{3,4} \setminus \beta$ is planar. If $K'_{4,4} \setminus \alpha \setminus \beta$ is not planar, then $K'_{4,4}$ is not a member of $\mathcal{M}$ because of the symmetry of $K'_{3,4}$.
which is related to the selection of $\beta$. $(K'_{4,4}\setminus \alpha) \setminus \beta$ has a subgraph in Figure 3.16 which is a subdivision of $K_{3,3}$. Therefore, $K'_{4,4}$ is not in $\mathcal{M}$.

Therefore, for $n \geq 4$, every graph dominating $K_{4,k}$ or $K'_{4,k}$ cannot be in $\mathcal{M}$.

### 3.3 Alternating double wheel with axle $B_k$

In this section, we will consider the near planarity of a graph dominating a subdivision of $B_k$. When $k$ is three, we find a member of $\mathcal{M}$ that contains graph $B_3$ as a subgraph. It is easy to show that the graph in Figure 3.17, dominating $B_3$, is in $\mathcal{M}$.

![FIGURE 3.17: A member of $\mathcal{M}$ dominating $B_3$](image)
We want to show that a graph that dominates $B_k$ for $k \geq 6$ can not be in $\mathcal{M}$. In this section, if $H$ is a subgraph of $G \in \mathcal{M}$ and if $H$ is a subdivision of $B_k$, we will call a subdivision of edge between $v_i$ and $v_{i+1}$ of $B_k$ in $H$ as $v_i$-rim-path. In this paper, consecutive spokes mean two edges from one hub to $v_i$ and to $v_{i+1}$.

To investigate $B_k$ for $k \geq 6$, we want to show a definition and a theorem from [DOTV11].

**Definition 3.12.** If there exists a homeomorphic embedding $\mu : C \hookrightarrow G$, then a $\mu$-path in $G$ is a path in $G$ with both ends are in $\mu(C)$ and otherwise disjoint from it.

**Theorem 3.13.** Let $k \geq 4$ be an integer, let $H$ be a non-planar graph, and let $\eta : A_{2k+1} \hookrightarrow H$ be a homeomorphic embedding. Then one of the following holds.

(i) There exist a homeomorphic embedding $\eta' : A_k \hookrightarrow H$ and an $\eta'$-path $P$ in $H$ such that $\eta'$ maps the hubs of $A_k$ to the same pair of vertices $\eta$ maps the hubs of $A_{2k+1}$ to, and the ends of $P$ are the images of the hubs of $A_k$ under $\eta'$.

(ii) There exist a homeomorphic embedding $\eta' : A_{2k+1} \hookrightarrow H$ and a separation $(A, B)$ of $H$ of order at most three such that $|\eta'(V(A_{2k+1})) \cap A - B| \leq 1$ and $H|A$ cannot be embedded in a disk with $A \cap B$ embedded in the boundary of the disk.

In the first case of the previous theorem with $B_{2k+1}$ of $H \in \mathcal{M}$, we can make the following lemma.

**Lemma 3.14.** For $k \geq 4$, suppose that $G$ is not nearly planar with a homeomorphic embedding $\eta : B_{2k+1} \hookrightarrow G$ and that there exist a homeomorphic embedding $\eta' : A_k \hookrightarrow G$ and an $\eta'$-path $P$ in $G$ such that $\eta'$ maps the hubs of $A_k$ to the same
pair of vertices \( \eta \) maps the hubs of \( B_{2k+1} \) to, and the ends of \( P \) are the images of the hubs of \( A_k \) under \( \eta' \).

If the image of the rim of \( A_k \) under \( \eta' \) and the image of the rim of \( B_{2k+1} \) under \( \eta \) are edge disjoint, then \( G \notin \mathcal{M} \).

Proof. Suppose \( G \in \mathcal{M} \). Let \( e \) be an edge in the image of a spoke in \( B_{2k+1} \) under \( \eta \) that contains no edges of \( \eta'(A_k) \). Since \( \eta(B_{2k+1}) \setminus e \) dominates \( B_{2k} \), we need to delete a rim edge of \( \eta(B_{2k+1}) \) or an edge which belongs to the axle of \( B_{2k+1} \) from \( G \setminus e \) because \( G \in \mathcal{M} \). If we delete a rim edge of \( \eta(B_{2k+1}) \), then it is still not planar because of \( \eta'(A_k) \) with the axle. If we delete the axle of \( B_{2k+1} \), we can find a path between two hubs that contains no edges of \( \eta'(A_k) \) using the spokes and the rim of \( \eta(B_{2k+1}) \setminus e \). Using this path and \( \eta'(A_k) \), there exists \( B_k \) which is not planar. This contradicts the fact that \( G \in \mathcal{M} \). \qed

The following lemmas are needed for the coming theorem.

Lemma 3.15. No graphs in \( \mathcal{M} \) dominate one of the graphs in Figure 3.18.

![Figure 3.18](image)

FIGURE 3.18: No graphs in \( \mathcal{M} \) dominate one of the above graphs.

Proof. Let us call edge \( u_9u_{10} \) and path \( u_9u_{12}u_{10} \) (in \( G_3 \), path \( u_9u_{11}u_{10} \)) from Figure 3.18 as two axles. Suppose that there exists a graph \( G \in \mathcal{M} \) dominating \( G_i \) of the above graphs for some \( 1 \leq i \leq 3 \). Let \( H \) be a subgraph of \( G \) such that \( H \) is a subdivision of \( G_i \). If we delete an edge \( e_0 \) of \( G \) from the subdivision of \( u_8u_{10} \),...
then all of planarizing edges $\alpha_0$ of $G \setminus e_0$ must be in the subdivision of the path $u_2u_3u_4u_5u_6$ (in $G_2$, path $u_2u_3u_1u_4u_5u_6$). If we delete an edge $e_1$ of $G$ from the subdivision of $u_1u_9$, then every planarizing edge of $G \setminus e_1$ is also in the subdivision of the path $u_2u_3u_4u_5u_6$ (in $G_2$, path $u_2u_3u_1u_4u_5u_6$) and there are no paths from the inner vertices of the subdivision of $u_8u_{10}$ to the inner vertices of the subdivision of axles because $G \in M$. Since $G \in M$, $G \setminus \alpha_0$ is not planar and $(G \setminus \alpha_0) \setminus e_0$ is planar. Therefore, there exists a minimal set $B$ of bridges of $H$ such that $(\bigcup_{B \in B} B \cup H) \setminus \alpha_0$ is not planar.

If there is a path $P$ from an inner vertex $\beta$ of the subdivision of $u_2u_3$ to an inner vertex of the subdivision of $u_8u_{10}$, then $H \cup P$ has the unique planarizing edge $u_3\beta$. As $(H \cup P) \setminus e_1$ has the same unique planarizing edge $u_3\beta$, there exists a set $B_1$ of bridges. Using the similar argument from the above with $G \setminus e_0$, there are no paths from the inner vertices of the subdivision of $u_1u_9$ to the inner vertices of the subdivision of axles. When we delete an edge of $G$ in the subdivision of $u_9u_{10}$, we notice that $G \notin M$ because the unique planarizing edge of $H \cup P$ is $u_3\beta$ and $G$ contains $B_1$.

Otherwise, when we delete an edge of $G$ in the subdivision of $u_9u_{10}$, we notice that $G \notin M$ because every planarizing edge of $G \setminus u_9u_{10}$ is in the path $u_2u_3u_4u_5u_6$ (in $G_2$, path $u_2u_3u_1u_4u_5u_6$) and because $G$ contains $B$.

Lemma 3.16. No graphs in $M$ dominate one of the graphs in Figure 3.19.

Proof. Let us call one of the graphs in Figure 3.19 as $H$. Since $H \setminus u_5u_9$ is not nearly planar, No graphs in $M$ dominate $H$.

Lemma 3.17. No graphs in $M$ dominate one of the graphs in Figure 3.20.
FIGURE 3.19: No graphs in $\mathcal{M}$ dominate one of the above graphs.

FIGURE 3.20: No graphs in $\mathcal{M}$ dominate one of the above graphs.

Proof. Let us call one of the graphs in Figure 3.20 as $H_0$ and suppose that $G \in \mathcal{M}$ such that $G$ has a subgraph $H$ that is a subdivision of $H_0$ with $\mu : H_0 \hookrightarrow G$. We know that the planarizing edges of each of $H_0$ and $H_0 \setminus u_5u_9$ are $u_8u_{12}$ and $u_2u_{14}$.

(or only $u_8u_{12}$.) Let $e_1$ be an edge in $\mu(u_5u_9)$ and $e_2$ be an edge in $\mu(u_8u_{12})$ such that, by symmetry, $(G \setminus e_1) \setminus e_2$ is planar. Since $G \in \mathcal{M}$, there exists a minimal set $\mathcal{B}$ of bridges of $H$ in $G$ such that $\left( \bigcup_{B \in \mathcal{B}} B \cup H \right) \setminus e_2$ is not planar. We can find a spoke $s_0$ in $H_0$ such that there are no attachments of bridges of $\mathcal{B}$ among the inner vertices of $\mu(s_0)$. (If not, $G$ has a path from an inner vertex of the subdivision of $u_5u_9$ to an inner vertex of the subdivision of each spoke and $G \notin \mathcal{M}$.) Let $e$ be an edge in $\mu(s_0)$. Then, $G \setminus e$ is not nearly planar because of $\mathcal{B}$. \qed
Lemma 3.18. Suppose that, for a graph $G$, there exists a homeomorphic embedding $\eta : B_9 \hookrightarrow G$. Let $e_0$ be the edge of $B_9$ joining the two hubs. Let $\eta_0$ be the restriction of $\eta$ to $A_9$ and $J$ be the union of $\eta_0(A_9)$ and all $\eta_0$-bridges except the one that includes $\eta(e_0)$. If $J$ is planar, then $G \not\in M$.

Proof. For a contradiction, assume $G \in M$. Let $B$ be the $\eta_0$-bridge that includes $\eta(e_0)$. If there is a bridge of $\eta(B_9)$ that contains a path from an inner vertex of $\eta(e_0)$ to $\eta_0(A_9)$, then one of the graphs in Figure 3.21 is dominated by $G$.

![FIGURE 3.21](image)

FIGURE 3.21: No graphs in $M$ dominate one of the above graphs.

Suppose that there is a homeomorphic embedding $\eta_1 : G_1 \hookrightarrow G$. We know set 
\{u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_{14}u_{16}, u_{15}u_{16}\} is the set of planarizing edges of each of $G_1$ and $G_1 \setminus u_{11}u_{15}$. Let $e$ be an edge in $\eta_1(u_{11}u_{15})$ whose endpoint is $\eta_1(u_{15})$. Since $G \in M$, for some edge $\alpha \in \{u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_{14}u_{16}, u_{15}u_{16}\}$, there exists an edge $e_1$ in $\eta_1(\alpha)$ such that $(G \setminus e) \setminus e_1$ is planar. Let $C_{11}$ be a closed path $u_1u_2u_{15}u_7u_6u_{14}u_1$ in $G_1$ and $C_{12}$ be a closed curve $u_9u_{15}u_{13}u_{12}u_{11}u_9u_0$ in $G_1$. In every planar embedding of $(G \setminus e) \setminus e_1$, $\eta_1(C_{11})$ and $\eta_1(C_{12})$ generate three regions $R_{11}$, $R_{12}$, and $R_{13}$ such that the boundary of $R_{11}$ is only $\eta_1(C_{11})$ and that the boundary of $R_{12}$ is only $\eta_1(C_{12})$ and that the boundary of $R_{13}$ is $\eta_1(C_{11})$ and
\(\eta_1(C_{12})\). Inside of \(R_{13}\), there are two paths \(\eta_1(u_1u_{13})\) and \(\eta_1(u_7u_8u_9)\) from \(\eta_1(C_{11})\) to \(\eta_1(C_{12})\). Since we can find a path from \(\eta_1(u_{14})\) to \(\eta_1(u_{15})\) using some edges of \(J\) and at least one edge of \(B \setminus e_1\) in \(((G \setminus e) \setminus e_1) \setminus (\eta_1(C_{11}) \cup \eta_1(C_{12}))\), because of the previous two paths \(\eta_1(u_1u_{13})\) and \(\eta_1(u_7u_8u_9)\), \(B \setminus e_1\) is inside of \(R_{11}\) in every planar embedding of \((G \setminus e) \setminus e_1\) by Jordan curve theorem. We know that \((G \setminus e) \setminus e_1\) is isomorphic to \((J \cup B) \setminus e_1\). Therefore, \(G \setminus e_1\) is isomorphic to \((J \cup B) \setminus e_1\). In every planar embedding of \(J\), \(\eta_1(u_{11}u_{15})\) is inside of \(R_{12}\). Because of the previous argument, in every planar embedding of \((J \cup B) \setminus e_1\), since \(B \setminus e_1\) is inside of \(R_{11}\), we can add edge \(e\) to \((J \cup B) \setminus e_1\) inside of \(R_{12}\). As a result, \(G \setminus e_1\) is nearly planar. This is in contradiction to the fact that \(G \in \mathcal{M}\).

Suppose that there is a homeomorphic embedding \(\eta_2 : G_2 \hookrightarrow G\). We know set \(\{u_2u_3, u_3u_5, u_5u_6, u_6u_7, u_{14}u_{16}\}\) is the set of planarizing edges of each of \(G_2\) and \(G_2 \setminus u_{11}u_{15}\). Let \(e\) be an edge in \(\eta_2(u_{11}u_{15})\) whose endpoint is \(\eta_2(u_{15})\). Since \(G \in \mathcal{M}\), for some edge \(\beta \in \{u_2u_3, u_3u_5, u_5u_6, u_6u_7, u_{14}u_{16}\}\), there exists an edge \(e_2\) in \(\eta_2(\beta)\) such that \((G \setminus e) \setminus e_2\) is planar. Let \(C_{21}\) be a closed path \(u_1u_2u_{15}u_7u_8u_{14}u_1\) in \(G_2\) and \(C_{22}\) be a closed curve \(u_9u_{15}u_{13}u_{12}u_{11}u_{10}u_9\) in \(G_2\). In every planar embedding of \((G \setminus e) \setminus e_2, \eta_2(C_{21})\) and \(\eta_2(C_{22})\) generate three regions \(R_{21}, R_{22},\) and \(R_{23}\) such that the boundary of \(R_{21}\) is only \(\eta_2(C_{21})\) and that the boundary of \(R_{22}\) is only \(\eta_2(C_{22})\) and that the boundary of \(R_{23}\) is \(\eta_2(C_{21})\) and \(\eta_2(C_{22})\). Inside of \(R_{23}\), there are two paths \(\eta_2(u_1u_{13})\) and \(\eta_2(u_8u_9)\) from \(\eta_2(C_{21})\) to \(\eta_2(C_{22})\). Since we can find a path from \(\eta_2(u_{14})\) to \(\eta_2(u_{15})\) using some edges of \(J\) and at least one edge of \(B \setminus e_2\) in \(((G \setminus e) \setminus e_2) \setminus (\eta_2(C_{21}) \cup \eta_2(C_{22}))\), because of the previous two paths \(\eta_2(u_1u_{13})\) and \(\eta_2(u_8u_9)\), \(B \setminus e_2\) is inside of \(R_{21}\) in every planar embedding of \((G \setminus e) \setminus e_2\) by Jordan curve theorem. We know that \((G \setminus e) \setminus e_2\) is isomorphic to \((J \cup B) \setminus e_2\). Therefore, \(G \setminus e_2\) is isomorphic to \((J \cup B) \setminus e_2\). In every planar embedding of \(J, \eta_2(u_{11}u_{15})\) is inside of \(R_{22}\). Because of the previous argument, in every planar embedding of


\((J \cup B) \setminus e) \setminus e_2, \text{ since } B \setminus e_2 \text{ is inside of } R_{21}, \text{ we can add edge } e \text{ to } ((J \cup B) \setminus e) \setminus e_2 \text{ inside of } R_{22}. \text{ As a result, } G \setminus e_2 \text{ is nearly planar. This contradicts the fact that } G \in \mathcal{M}.

Therefore, two hubs are the only attachments of \(B\). Let \(\alpha\) and \(\beta\) be two hubs of \(B_9\). Suppose that an edge \(e\) is in \(B \setminus \eta(\alpha\beta)\). Then, there exists an edge \(e_3\) such that \((G \setminus e) \setminus e_3\) is planar as \(G \in \mathcal{M}\).

1. If \(e_3\) is a rim edge, then \(G \setminus e_3\) is not planar because of \(B\). This means that, for every rim edge \(\hat{e}\), \(G \setminus \hat{e}\) is not planar because of \(B\) since the attachments of \(B\) are only \(\alpha\) and \(\beta\). In addition, for every edge \(\bar{e}\) of \(B\), \(G \setminus \bar{e}\) is not planar because \(G \in \mathcal{M}\). When we pick an edge \(\tilde{e}\) from an \(\eta\)-image of a spoke, \(G \setminus \tilde{e}\) is not nearly planar because of \(B\). This is in contradiction to the fact that \(G \in \mathcal{M}\).

2. If \(e_3\) cannot be a rim edge, then, \(e_3\) is in \(\eta(e_0)\). Since, for every edge \(\hat{e}\) of \(B\), \(G \setminus \hat{e}\) is not planar because \(G \in \mathcal{M}\). When we pick an edge \(\bar{e}\) from an \(\eta\)-image of a spoke, \(G \setminus \bar{e}\) is not nearly planar because of \(B\). This contradicts the fact that \(G \in \mathcal{M}\).

As a result, \(G \not\in \mathcal{M}\).

Using the argument of a theorem in [DOTV11] with these previous results, we can say that if \(B_k \not\succeq G\) for \(k \geq 9\), then \(G \not\in \mathcal{M}\).

**Theorem 3.19.** No graphs in \(\mathcal{M}\) dominate \(B_9\).

**Proof.** Let \(G \in \mathcal{M}\), and suppose for a contradiction that \(G \succeq B_9\). Let \(\eta : B_9 \hookrightarrow G\) be a homeomorphic embedding, and let \(\eta_0\) be the restriction of \(\eta\) to \(A_9\). Let \(e_0\) be the edge of \(B_9\) joining the two hubs. From Theorem 3.13 applied to \(A_9, G,\) and \(\eta_0\), we deduce that (i) or (ii) of Theorem 3.13 holds.
If (i) holds, then suppose that $J$ be the union of $\eta_0(A_0)$ and all $\eta_0$-bridges except the one that includes $\eta(e_0)$. Because of Lemma 3.18, $J$ is not planar. By (i), $J$ contains a $\eta_0$-path $P$ between two hubs. $P$ is an axle and $J$ does not contain $\eta(e_0)$. Therefore, we conclude that $G$ dominates the graph $K_0$ obtained from $B_4$ by adding an edge $e_1$ parallel to $e_0$ under $\preceq$. (In $G$, $\eta(e_1)$ is $P$.) Let $K$ be a subgraph of $G$ such that $K$ is a subdivision of $K_0$ with a homeomorphic embedding $\tilde{\mu}$. For each rim edge $\alpha$ of $K$, we can find a minimal set $B_\alpha$ of Bridges of $K$ in $G$ such that \((\bigcup_{C \in B_\alpha} C \cup K) \setminus \alpha\) is not planar because $G \in \mathcal{M}$. In addition, the inner vertices of neither $\tilde{\mu}(e_0)$ nor $\tilde{\mu}(e_1)$ contain any attachment of $B_\alpha$ because of Lemma 3.15.

For some spoke $s_0$ in $K_0$, let $s$ be an edge of $K$ in $\tilde{\mu}(s_0)$ such that one endpoint of $s$ must be on the rim. Then, there exists a rim edge $\alpha_s$ such that \((G \setminus s) \setminus \alpha_s\) is planar but $G \setminus \alpha_s$ is not planar. In this case, we have two possibilities about the set $B_{\alpha_s}$.

1. Since \((G \setminus \alpha_s) \setminus s\) is planar, for a planar embedding of \((G \setminus \alpha_s) \setminus s\), there exists a face $F$ such that two endpoints of $s$ and all attachments of every bridge in $B_{\alpha_s}$ are in the boundary of $F$. In this case, after the planar embedding of \((G \setminus \alpha_s) \setminus s\), when we draw $s$, $s$ is crossing one of $B_{\alpha_s}$.

2. There exists a path $P_1$ is a bridge of $K$ such that one of endpoints of $P_1$ is an endpoint of $s$ and that \((K \cup P_1) \setminus \alpha_s\) is not planar but \((K \cup P_1) \setminus \alpha_s\) is not planar. In this case, $P_1$ is the only bridge of $B_{\alpha_s}$. After a planar embedding of $K \setminus \alpha_s$, when draw $P_1$ over the embedding, the $P_1$ is crossing a spoke or a rim.

When we delete an edge $e_3$ in the subdivision of $e_0$, we notice that $G \notin \mathcal{M}$. If we delete every rim edge $\alpha$ from $G \setminus e_3$, because of $B_\alpha$, \((G \setminus e_3) \setminus \alpha\) is not planar. If we delete an edge $e_2$ of $K$ in $\tilde{\mu}(e_1)$, because of $B_{\alpha_s}$, \((G \setminus e_3) \setminus e_2\) is not planar. (In
case (1), s is still crossing $B_{\alpha s}$. In case (2), $P_1$ is still crossing a spoke or a rim.)
Because of the contradiction, (i) does not hold.

If (ii) holds, there exist a homeomorphic embedding $\eta' : A_{2k+1} \rightarrow H$ and a separation $(A, B)$ of $G$ of order at most three such that $|\eta'(V(A_{2k+1})) \cap A - B| \leq 1$ and $G|A$ cannot be embedded in a disk with $A \cap B$ embedded in the boundary of the disk. From $|\eta'(V(A_{2k+1})) \cap A - B| \leq 1$, if $|\eta'(V(A_{2k+1})) \cap A - B| = 1$, then $G$ has a graph in Lemma 3.16 or in Lemma 3.17 under $\preceq$ because none of hubs can be in $A \setminus B$. Suppose that $|\eta'(V(A_{2k+1})) \cap A - B| = 0$. Let $e$ be an edge in $\eta$-image of a spoke edge of $B_9$ in $G$. Then, there exist an edge $\tilde{e}$ such that $(G \setminus e) \setminus \tilde{e}$ is planar. Since $\tilde{e}$ is in $\eta$-image of a rim edge or of an axle, edges in $A$ are preserved in $(G \setminus e) \setminus \tilde{e}$. For any planar embedding of $(G \setminus e) \setminus \tilde{e}$, we can find graph $\tilde{G}$ in Figure 3.22 as a part of $(G \setminus e) \setminus \tilde{e}$ under $\preceq$ with a homeomorphic embedding $\xi$ such that vertices $\xi(a)$ and $\xi(b)$ are in $V(A)$ and vertices $\xi(x)$, $\xi(y)$, and $\xi(z)$ are in $V(A \cap B)$.

For some edge $e$ in $\xi(ay)$, there exists an edge $r$ of $\mu(B_9)$ in $B$ such that $(G \setminus e) \setminus r$ is planar. Since $G \in \mathcal{M}$, $G \setminus r$ is not planar. Therefore, there exists a minimal set

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**FIGURE 3.22:** A structure of graph in case (ii)
\( \mathcal{B} \) of bridges of \( \mu(B_9) \cup \tilde{G} \) in \( G \) such that \( \left( \bigcup_{Y \in \mathcal{B}} Y \cup (\mu(B_9) \cup \tilde{G}) \right) \setminus r \) is not planar. In this case, all of attachments of each bridge of \( \mathcal{B} \) are only in \( A \) or only in \( B \). Therefore, every attachment of each bridge of \( \mathcal{B} \) is the inside area made by closed path \( \xi(axbza) \). We can find a spoke \( s_0 \) in \( B_9 \) such that there are no attachments of bridges of \( \mathcal{B} \) among the inner vertices of \( \mu(s_0) \). Let \( s \) be an edge in \( \mu(s_0) \). Then, \( G \setminus s \) is not nearly planar because of \( \mathcal{B} \). This is in contradiction to the fact that \( G \in \mathcal{M} \). By the contradiction, no graphs in \( \mathcal{M} \) dominate \( B_9 \). \( \Box \)
Chapter 4
The $k$-rung Möbius ladder

The $k$-rung Möbius ladder, denoted by $M_k$, has vertices $v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k$, where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_k$ form paths in the order listed, and $v_i$ is adjacent to $u_i$ for $i = 1, 2, \ldots, k$ with edges between $v_1$ and $u_k$ and between $u_1$ and $v_k$. We will call a rung for each $v_i u_i$ edge and call consecutive rungs for edges $v_i u_i$ and $v_{i+1} u_{i+1}$. It is easy to show the following graphs are in $\mathcal{M}$ and dominate $M_3$ or $M_4$.

![Diagram showing members of $\mathcal{M}$ containing $M_3$ or $M_4$.]

Guoli Ding proved the following lemma.

**Lemma 4.1.** Suppose $G$ dominates $H$ which is a subdivision of $M_4$. If there exists a bridge $B$ of $H$ in $G$ such that all attachments of $B$ are inner vertices of subdivisions of two rungs which are not consecutive, then $G$ dominates a subdivision of Petersen graph.

**Proof.** $G$ dominates the graph in Figure 4.2 which is drawn in the projective plane.
The graph in Figure 4.2 is isomorphic to the graph in Figure 4.3.

The above lemma says if a graph $G \in \mathcal{M}$ contains a path from an inner vertex of the subdivision of a rung to that of another rung, then two rungs must be consecutive. In this chapter, we will select a homeomorphic embedding of $M_k$ with the smallest number of bridges. In the graph of Figure 4.4, if the red subgraph means the bridges of a homeomorphic embedding of $M_k$, then this embedding is not what we want because we can reduce the number of bridges.

With a homeomorphic embedding of $M_k$ with the smallest number of bridges, the following red bridges cannot be shown in a separation of the order 4. (Each black path belongs to the homeomorphic embedding of $M_k$.)
FIGURE 4.5: Not allowed sets of bridges for the smallest number of bridges of a homeomorphic embedding of $M_k$

In a homeomorphic embedding of $M_k$ with the smallest number of bridges, the above graphs must be shown in Figure 4.6, respectively. Each red subgraph means bridges and each black subgraph belongs to the homeomorphic embedding of $M_k$.

The following lemmas are trivial but are used for every argument in this chapter.

**Lemma 4.2.** Suppose that $G$ is in $\mathcal{M}$ and $G$ has a subgraph $M$ which is a subdivision of $M_k$ for $k \geq 6$ with a homeomorphic embedding $\eta$. For every rung edge $e$ in $M$, there exists an edge $e_0$ in $M_k$ with $e \in E(\eta(e_0))$ and there is a rim edge $\tilde{e}_0$ in $M_k$ such that, for some edge $\tilde{e}$ in $E(\eta(\tilde{e}_0))$, $(G \setminus e) \setminus \tilde{e}$ is planar.

*Proof.* Since $M_k \setminus e_0$ is not planar, we need a rim edge $\tilde{e}_0$ such that $M_k \setminus \{e_0, \tilde{e}_0\}$ is planar. \hfill \Box

**Lemma 4.3.** Suppose that $G$ dominates $M_{20}$ with a homeomorphic embedding $\eta$. Let $e_3$ be an edge in $\eta(u_3v_3)$. Let $B_3$ be a set of bridges of $\eta(M_{20})$ in $G$ such that every attachment of each member of $B_3$ is on $\eta(u_1u_2u_3u_4u_5) \cup \eta(v_1v_2v_3v_4v_5) \cup \eta(u_2v_2) \cup$...
FIGURE 4.6: Change of bridges by replacement of homeomorphic embeddings of $M_k$.

$\eta(u_3v_3) \cup \eta(u_4v_4)$. For every edge $\hat{e}$ in $\eta(u_5u_6 \ldots u_{19}u_{20}v_1) \cup \eta(v_5v_6 \ldots v_{19}v_{20}u_1)$, if

$$\left( \bigcup_{B \in B_3} B \cup \eta(M_{20}) \right) \setminus \hat{e}$$

is not planar, then $G$ is not in $\mathcal{M}$.

\textbf{Proof.} If subgraph $\bigcup_{B \in B_3} B \cup \eta(M_{20})$ of $G$ does not have any planarizing edges, then for every edge $e_{13}$ in $\eta(u_{13}v_{13})$, another subgraph $\left( \bigcup_{B \in B_3} B \cup \eta(M_{20}) \right) \setminus e$ of $G$ has no planarizing edges, either and graph $G$ is not in $\mathcal{M}$. (Since attachments of every member of $B_3$ are on $\eta(u_1u_2u_3u_4u_5) \cup \eta(v_1v_2v_3v_4v_5) \cup \eta(u_2v_2) \cup \eta(u_3v_3) \cup \eta(u_4v_4)$, path $\eta(u_{13}v_{13})$ is redundant.)

Therefore, if $G$ is in $\mathcal{M}$, then subgraph $\bigcup_{B \in B_3} B \cup \eta(M_{20})$ has planarizing edges on $\eta(u_1u_2u_3u_4u_5) \cup \eta(v_1v_2v_3v_4v_5)$. For some edge $e_{13}$ in $\eta(u_{13}v_{13})$, we can find an edge $\tilde{e}_{13}$ in $\eta(u_1u_2u_3u_4u_5) \cup \eta(v_1v_2v_3v_4v_5)$ such that $(G \setminus e_{13}) \setminus \tilde{e}_{13}$ is planar. Since $G$ is in $\mathcal{M}$, there is a minimal set $B_{13}$ of bridges of $\bigcup_{B \in B_3} B \cup \eta(M_{20})$ in $G$ such that

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for every edge $\tilde{e}$ in $\eta(u_1u_2u_3u_4u_5) \cup \eta(v_1v_2v_3v_4v_5)$,

$$\left( \bigcup_{B_3 \in B_3} B_3 \right) \cup \left( \bigcup_{B_{13} \in B_{13}} B_{13} \right) \cup \eta(M_{20}) \setminus \tilde{e} \text{ is not planar}$$

and

$$\left\{ \left( \bigcup_{B_3 \in B_3} B_3 \right) \cup \left( \bigcup_{B_{13} \in B_{13}} B_{13} \right) \cup \eta(M_{20}) \setminus \tilde{e} \right\} \setminus e_{13} \text{ is planar.}$$

Let $e_{16}$ be an edge in $\eta(u_{16}v_{16})$. Then, graph $G \setminus e_{16}$ is not nearly planar because of $\bigcup_{B_{13} \in B_{13}} B_{13}$. This is in contradiction to the fact that $G$ is in $\mathcal{M}$.

With these lemmas, we will focus on a bridge containing a path from the image of a rim to the image of a rung or another rim.

**Lemma 4.4.** The planarizing edges of the graph in Figure 4.7 are only $u_1u_2$ and $u_6v_1$.

![FIGURE 4.7: A subdivision of $M_k$ with an edge from rim to rung (Part 1)](image)

**Proof.** Let the above graph be $H$. We can get a planar embedding of each of $H \setminus u_1u_2$ and $H \setminus u_6v_1$ as shown in Figure 4.8. It is easy to show that none of rung

![FIGURE 4.8: Planar embeddings of $H \setminus u_1u_2$ and $H \setminus u_6v_1$](image)
edges can be planarizing edges of $H$ because of the structure of Möbius ladder. If we delete a rim edge other than $u_1u_2$ and $u_6v_1$ from $H$, it has a subdivision of $K_{3,3}$ one of whose nodes is vertex $\alpha$ as illustrated in Figure 4.9. (A node means a vertex whose degree is greater than 2.) Therefore, the planarizing edges of graph $H$ are

$H \setminus u_1v_6$

$H \setminus v_1v_2$

FIGURE 4.9: Subgraphs of $H$ containing $K_{3,3}$ subdivisions

only $u_1u_2$ and $u_6v_1$. \hfill \Box

**Lemma 4.5.** The planarizing edges of the graph in Figure 4.10 are only $u_1u_2$, $u_6v_1$, and $\beta v_3$.

FIGURE 4.10: A subdivision of $M_k$ with an edge from rim to rung (Part 2)

**Proof.** Let the above graph be $H$. We can get a planar embedding of each of $H \setminus u_1u_2$ and $H \setminus u_6v_1$ using the argument in Lemma 4.4. We can get a planar embedding of $H \setminus \beta v_3$ as shown in Figure 4.11. Using the similar argument of Lemma 4.4, we know that the planarizing edges of graph $H$ are only $u_1u_2$, $u_6v_1$, and $\beta v_3$. \hfill \Box
Theorem 4.6. If graph $G$ dominates one of the graphs in Figure 4.12, then $G$ is not in $\mathcal{M}$.

Proof. Since these two graphs are satisfying the condition of Lemma 4.3 by Lemma 4.4 and 4.5, graph $G$ is not in $\mathcal{M}$. \qed

In the following lemmas, we want to show that if a graph $G \in \mathcal{M}$ dominates a Möbius ladder with a path from an inner vertex of the subdivision of a rung to a vertex of the subdivision of a rim edge, then the rung and the rim edge are adjacent in the Möbius ladder.

For the convenience, when there is a homeomorphic embedding $\eta : M_k \hookrightarrow G$, if there exist inner vertices $\alpha$ and $\beta$ in $\eta(v_i v_{i+1})$ in the order of $\eta(v_i)\alpha\beta\eta(v_{i+1})$ in $\eta(M_k)$, then we will pick rational numbers $q$ and $r$ with $i < q < r < i + 1$ and rename $\alpha$ as $v_q$ and $\beta$ as $v_r$. If there are inner vertices in $\eta(u_i u_{i+1})$, then we will use the same argument in the above. If there exist inner vertices $\gamma$ and $\delta$ in $\eta(u_k v_1)$
in the order of $\eta(u_k)\alpha\beta\eta(v_1)$ in $\eta(M_k)$, then we will pick rational numbers $q$ and $r$ with $0 < q < r < 1$ and rename $\alpha$ as $v_q$ and $\beta$ as $v_r$. If there are inner vertices in $\eta(v_ku_1)$, then we will use the same argument.

**Lemma 4.7.** Suppose $M_{20} \not\subseteq G$ with a homeomorphic embedding $\eta : M_{20} \hookrightarrow G$ and $\alpha$ is an inner vertex of $\eta(v_1u_1)$. If a bridge of $\eta(M_{20})$ in $G$ has a path $P$ from $\alpha$ to $v_i$ for some $3 \leq i \leq 19$, then $G \notin M$.

**Proof.** Using the symmetry, we may assume that $3 \leq i \leq 11$. We can construct a subdivision of $K_{3,3}$ in $G$ such that $\alpha$ is a node. By Lemma 4.4, every homeomorphic embedding of $K_{3,3}$ with a node $\alpha$ must contain edges $v_1u_{20}$ and $u_1u_2$ and the only planarizing edges of $P \cup \eta(M_{20})$ are edges of $\eta(v_1u_{20})$ and of $\eta(u_1u_2)$.

Suppose that $G \in M$. Let $e$ be an edge of $\eta(v_{10}u_{10})$. Since $G \in M$, there exists an edge $\tilde{e}$ such that $(G \setminus e) \setminus \tilde{e}$ is planar. By the previous argument, $\tilde{e}$ is in $\eta(v_1u_{20})$ or in $\eta(u_1u_2)$. As $G \in M$, $G \setminus \tilde{e}$ is not planar. Therefore, there exists a minimal set $B$ of bridges of $\eta(M_{20})$ in $G$ such that $\left(\bigcup_{B \in B} B \cup \eta(M_{20})\right) \setminus \tilde{e}$ is not planar but $\left(\bigcup_{B \in B} B \cup \eta(M_{20})\right) \setminus \tilde{e} \setminus e$ is planar. There are two possible cases about $B$.

1. A closed path $\eta(v_9v_{10}v_{11}u_{11}u_{10}u_9v_9)$, path $\eta(v_{10}u_{10})$, and $B$ contains a subgraph $K$ that is a subdivision of $K_4$ such that graph $K$ with a path $\eta(v_1v_{12} \ldots v_{20}u_1u_2 \ldots u_9)$ or with a path $\eta(u_{11}u_{12}u_{13} \ldots u_{20}v_1 \ldots v_9)$ becomes a subdivision of $K_{3,3}$.

2. Set $B$ contains a path from an inner vertex of $\eta(v_{10}u_{10})$ to a vertex $v_1$ with $i < 9$ or $11 < i \leq 20$. Then, we can find a subdivision of $K_{3,3}$ using $B$ and this subdivision of $K_4$ such that the inner vertex of $\eta(v_{10}v_{30})$ becomes a node of the subdivision of $K_{3,3}$.
(3) Set $B$ contains a path from an inner vertex of $\eta(v_{10}u_{10})$ to a vertex $u_i$ with $i < 9$ or $11 < i \leq 20$. Then, we can find a subdivision of $K_{3,3}$ using $B$ and this subdivision of $K_4$ such that the inner vertex of $\eta(v_{10}v_{30})$ becomes a node of the subdivision of $K_{3,3}$.

If we want to destroy a subdivision of $K_{3,3}$ made by $B$ by one edge deletion, the possible edges for the deletion in Case (1) are in $\eta(v_8v_9v_{10}v_{11}v_{12})$, in $\eta(v_9u_9)$, in $\eta(v_{10}u_{10})$, or in $\eta(v_{11}u_{11})$ and the possible edges for the deletion in Case (2) are in $\eta(v_8v_9v_{10}v_{11}v_{12})$ or in $\eta(u_8u_9u_{10}u_{11}u_{12})$ by Lemma 4.4 and 4.5. Let $\hat{e}$ be an edge of $\eta(v_{15}u_{15})$. Then $G \setminus \hat{e}$ dominates $M_{19}$ under relation $\prec$ and $G \setminus \hat{e}$ is not nearly planar because of $P$ and $B$. This contradicts the fact that $G \in \mathcal{M}$. □

Theorem 4.6 and Lemma 4.7 show that if a graph $G \in \mathcal{M}$ dominates a Möbius ladder with a path from an inner vertex of the subdivision of a rung to a vertex of the subdivision of a rim edge, then the rung and the rim edge are adjacent in the Möbius ladder.

Now, we want to talk about a path from a rim to another rim. The following lemmas show the prohibited rim-to-rim paths for graphs in $\mathcal{M}$.

**Lemma 4.8.** If graph $G$ is in $\mathcal{M}$, then graph $G$ dominates none of the graphs in Figure 4.13.

<table>
<thead>
<tr>
<th>Planarizing edge set</th>
<th>Planarizing edge set</th>
<th>Planarizing edge set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_S = {u_8v_1, v_1v_2, v_2v_3, u_1u_2, u_2u_3, u_3u_4}$</td>
<td>$E_S = {u_8v_1, v_1v_2, v_2\gamma, \gamma v_3, u_1u_2, u_2u_3}$</td>
<td>$E_S = {v_1v_2, v_2\gamma, \gamma v_3, u_1\delta, \delta u_2, u_2u_3}$</td>
</tr>
</tbody>
</table>

**FIGURE 4.13:** Every graph in $\mathcal{M}$ does not dominate any of the above graphs (Part 2)
Proof. Let $S_1$ be the left graph among the above graphs. In $S_1$, edges $v_2u_2$ and $v_3u_1$ with the closed path $v_1v_2v_3u_3u_2u_1v_1$ become a subdivision of $K_4$. Therefore, $S_1$ has a subdivision of $K_{3,3}$ made by this subdivision of $K_4$ with a path $u_3u_4\ldots u_8v_1$. As a result, the set of planarizing edges of $S_1$ is \{u_8v_1, v_1v_2, v_2v_3, u_1u_2, u_2u_3, u_3u_4\}.

Let $S_2$ be the middle graph among the above graphs. Since $S_2$ is made by splitting vertex $v_3$ from $S_1$, in $S_2$, edges $v_2u_2$ and $u_1\gamma$ with the closed path $v_1v_2v_3u_3u_2u_1v_1$ form a subdivision of $K_4$. We can see a subdivision of $K_{3,3}$ made by this subdivision of $K_4$ with path $u_3u_4\ldots u_8v_1$ or with path $v_3v_4u_4u_5\ldots u_8v_1$. Therefore, the set of planarizing edges of $S_2$ is \{u_8v_1, v_1v_2, v_2\gamma, \gamma v_3, u_1u_2, u_2u_3\}.

Let $S_3$ be the right graph among the above graphs. As $S_3$ is made splitting vertex $u_1$ from $S_2$, using the similar argument from the above, we know that the set of planarizing edges of $S_3$ is \{v_1v_2, v_2\gamma, v_3, u_1\delta, \delta u_2, u_2u_3\}.

Let $S$ be one of the above three graphs and suppose that $G \in \mathcal{M}$ dominates $S$. Then, there exists a subgraph $S_0$ of $G$ such that $S_0$ is a subdivision of $S$ with a homeomorphic embedding $\eta$. Let $E_S$ be the set of planarizing edges of $S$. This means that for every $e \in E_S$, $S \setminus e$ is planar. We know that $v_3u_5 \notin E_S$ and $v_7u_7 \notin E_S$. Since the set of planarizing edges of $S \setminus v_5u_5$ is $E_S$, by the previous lemmas, there is a minimal set $B$ of bridges of $S_0$ in $G$ such that members of set $B$ and the closed path $\eta(v_4v_5v_6u_6u_5u_4v_4)$ with path $\eta(v_5u_5)$ dominate a subdivision of $K_4$ that is a subgraph of a subdivision of $K_{3,3}$ in $G \setminus e$ for every edge $e$ in set $E_S$. However, as the set of planarizing edges of $S \setminus v_7u_7$ is also $E_S$, $G$ cannot be in $\mathcal{M}$ because of $B$. \hfill \Box

**Theorem 4.9.** If graph $G \succeq M_{20}$ dominates one of the graphs in Figure 4.14, then $G$ is not in $\mathcal{M}$

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Proof. By Lemma 4.1, 4.6, and 4.8, if $G$ dominates one of the above with $G \succeq M_{20}$, then $G$ is not in $\mathcal{M}$. \hfill \square

Lemma 4.10. Suppose $M_{20} \preceq G$ with a homeomorphic embedding $\eta : M_{20} \hookrightarrow G$. Graph $G$ contains vertices $v_\alpha$ and $u_\beta$ with $1 \leq \alpha < 2$ and $3 < \beta < 12$. If a bridge of $\eta(M_{20})$ in $G$ has a path $P$ from $v_\alpha$ to $u_\beta$, then $G \not\in \mathcal{M}$.

Proof. Suppose that $G \in \mathcal{M}$. The set of planarizing edges of $P \cup \eta(M_{20})$ is a subset of $E(\eta(v_1v_2\ldots v_{12}) \cup \eta(v_{20}u_1u_2\ldots u_{11}u_{12}))$ by the similar argument of Lemma 4.8. Let $e$ be an edge in $E(\eta(v_{14}u_{14})$. Then, the set of planarizing edges of $(P \cup \eta(M_{20})) \setminus e$ is still a subset of $E(\eta(v_1v_2\ldots v_{12}) \cup \eta(v_{20}u_1u_2\ldots u_{11}u_{12}))$. Since $G \in \mathcal{M}$, there exists a minimal set $\mathcal{B}$ of bridges of $P \cup \eta(M_{20})$ such that members of set $\mathcal{B}$ and the closed path $\eta(v_{13}v_{14}v_{15}u_{15}u_{14}v_{13})$ with path $\eta(v_{14}u_{14})$ dominate a subdivision of $K_4$ that is a subgraph of a subdivision of $K_{3,3}$ in $G \setminus \hat{e}$ for every edge $\hat{e}$ in edge set $E(\eta(v_1v_2\ldots v_{12}) \cup \eta(v_{20}u_1u_2\ldots u_{11}u_{12}))$. Let $\hat{e}$ be an edge in $E(\eta(v_2u_2))$. Then, the set of planarizing edges of $(P \cup \eta(M_{20})) \setminus \hat{e}$ is still a subset of $E(\eta(v_1v_2\ldots v_{12}) \cup \eta(v_{20}u_1u_2\ldots u_{11}u_{12}))$. Because of $\mathcal{B}$, $G \not\in \mathcal{M}$. This is in contradiction to the fact that $G \in \mathcal{M}$. \hfill \square

The previous lemma and the following lemma explain that if a graph $G$ dominates $M_k$ with $k \geq 20$ with a homeomorphic embedding $\eta$ and if a bridge of $\eta(M_k)$
FIGURE 4.15: Every graph in $\mathcal{M}$ does not dominate the above graph (Part 3) in $G$ contains a rim-to-rim path $P$, then the distance of two endpoints of $P$ in the $\eta$-image of the rim of $M_k$ must be small.

**Lemma 4.11.** If a graph dominates the graph in Figure 4.15, then the graph is not in $\mathcal{M}$.

*Proof.* Let us call the above graph as $H$. We know that because edge $\alpha \beta$, the set $A$ of planarizing edges of $H$ is $\{u_1 \alpha, \beta u_7, u_7 u_8, u_8 v_1, v_1 v_2, v_6 v_7, v_7 v_8, v_8 u_1\}$. Suppose that a graph $G \in \mathcal{M}$ dominates $H$ with a homeomorphic embedding $\eta$. Let edge $e$ be in $\eta(u_4 v_4)$. We know that for every edge $\xi$ in $A$, $G \setminus \xi$ is not planar. Therefore, there exists a minimal set $B$ of bridges of $\eta(H)$ in $G$ such that $\left( \bigcup_{B \in B} B \cup \eta(H) \right) \setminus \xi$ is not planar for every $\xi$ in $A$ and that $\left( \left( \bigcup_{B \in B} B \cup \eta(H) \right) \setminus \xi \right) \setminus e$ is planar for every $\xi$ in $A$. By the minimality of $B$, every attachment of each member of $B$ is on $\eta(u_2 u_3 u_4 u_5 u_6) \cup \eta(v_2 v_3 v_4 v_5 v_6) \cup \eta(u_3 v_3) \cup \eta(u_4 v_4) \cup \eta(u_5 v_5)$. Since the set of planarizing edges of $H \setminus u_8 v_8$ is still $A$, because of $B$, $G$ is not in $\mathcal{M}$. This contradicts the fact that $G \in \mathcal{M}$. \(\Box\)

**Lemma 4.12.** None of graphs in $\mathcal{M}$ dominates the graph in Figure 4.16.

FIGURE 4.16: Every graph in $\mathcal{M}$ does not dominate the above graph (Part 4)
Proof. Let the above graph be $H$. Suppose that graph $G \in \mathcal{M}$ dominates $H$ with homeomorphic embedding $\eta$. Let edge $e$ be in $\eta(u_2v_2)$ such that $\eta(u_2)$ is an endpoint of $e$. Since graph $G$ is in $\mathcal{M}$, there is another edge $\hat{e}$ such that $(G \setminus e) \setminus \hat{e}$ is planar such that $G \setminus \hat{e}$ is not planar.

If there is a bridge of $\eta(H)$ in $G$ such that it contains a path $P$ from $u_q$ to $u_r$ with $1 < q < 3$ and $3 < r \leq 11$, then $P \cup \eta(H)$ dominates a forbidden graph in Lemma 4.8 or 4.10 and $G$ is not in $\mathcal{M}$. For example, the graphs in Figure 4.17 are isomorphic to each other. (In these graphs, vertex $\alpha$ means $u_q$ and vertex $\beta$ means $u_r$.) We notice that they dominate a graph in Figure 4.13 and this is in contradiction to the fact that $G$ is in $\mathcal{M}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.17}
\caption{Two isomorphic graphs that are not dominated by graphs in $\mathcal{M}$}
\end{figure}

The second case is coming from the fact that edge $u_1u_2$, $u_2u_3$, $u_1u_3$, $v_1v_2$, $v_2v_3$, $u_1v_1$, $u_2v_2$, and $u_3v_3$ form a subdivision of $K_4$. By the argument of Kuratowski in [Kur30], it is possible that $G$ contains a bridge of $\eta(H)$ containing a path from an inner vertex of $\eta(u_1u_3)$ to an inner vertex of $\eta(u_2v_2)$. Then, $G$ dominates the graph in Figure 4.18. Let us call the following graph as $\tilde{H}$. Since $\tilde{H} \setminus u_4v_4$ is not nearly planar, $\tilde{H}$ is neither nearly planar nor in $\mathcal{M}$. This contradicts the fact that $G$ is in $\mathcal{M}$.

We need to check the third case that a bridge contains a path that can replace path $\eta(v_1v_2)$ or path $\eta(v_2v_3)$ containing $\hat{e}$ to preserve the Möbius structure. In other words, if $\hat{e}$ is in $\eta(v_1v_2)$, then a bridge contains a path from a vertex of $\eta(u_2v_2)$ except $\eta(u_2)$ to a vertex of $\eta(u_1v_1)$ except $\eta(u_1)$. If $\hat{e}$ is in $\eta(v_2v_3)$, then a
FIGURE 4.18: A not nearly planar graph which is not a member of $M$

bridge contains a path from a vertex of $\eta(u_2v_2)$ except $\eta(u_2)$ to a vertex of $\eta(u_3v_3)$ except $\eta(u_3)$. By the symmetry, we may assume that $\tilde{e}$ is in $\eta(v_1v_2)$. Then, by the argument, there is a path between a vertex of $\eta(u_1v_1)$ to $\eta(u_2v_2)$. Since $\eta(u_2)$ is an endpoint of $e$, $(G \setminus \tilde{e}) \setminus e$ is not planar because $(G \setminus \tilde{e}) \setminus e$ dominates a Möbius ladder structure. (Remember that $e$ contains $\eta(u_2)$ as its endpoint.) Therefore, the first case does not occur if $G$ is in $M$.

FIGURE 4.19: Every graph in $M$ does not dominate the above graph (Part 4-1)

Let us check the last case. By Theorem 4.9, a path in a bridge $B_1$ of $\eta(H)$ in $G$ from an inner vertex of $\eta(u_2v_2)$ must have its other endpoint from vertices of the closed path $\eta(u_1u_2u_3v_3v_2v_1u_1)$. By the symmetry, we may assume that a bridge contains a path from an inner vertex $\eta(u_2v_2)$ to a vertex of $\eta(u_2u_1v_1v_2)$ as illustrated in Figure 4.20.

We know that the closed path $u_1u_2v_2v_1u_1$ and the path from an inner vertex $\eta(u_2v_2)$ to a vertex of $\eta(u_2u_1v_1v_2)$ form $\theta$-graph in [Kur30]. Since $G \setminus \tilde{e}$ is
FIGURE 4.20: Every graph in $\mathcal{M}$ does not dominate the above graph (Part 4-2)

not planar, we need more bridge $B_2$ of $H$ in $G$ such that this bridge with the previous $\theta$-graph makes a subdivision of $K_4$. Therefore, all of the attachment of $B_1$ and $B_2$ are on the closed path $\eta(u_1 u_2 v_2 v_1 u_1)$. Then, the possible planarizing edges of $\eta(H) \cup B_1 \cup B_2$ are in $\eta(v_1 v_2)$. Let $\epsilon$ be an edge in $\eta(u_9 v_9)$. Since $G \in \mathcal{M}$, there exists an edge $\tilde{\epsilon}$ such that $G \setminus \epsilon \setminus \tilde{\epsilon}$ is planar. We can notice that $\tilde{\epsilon}$ is in $\eta(v_1 v_2)$ by the previous argument. Therefore, there exists a minimal set $\mathcal{B}$ of bridges of $H$ such that $(\bigcup_{B \in \mathcal{B}} B \cup \eta(H) \cup B_1 \cup B_2) \setminus \tilde{\epsilon}$ is not planar but $\left\{ \left( \bigcup_{B \in \mathcal{B}} B \cup \eta(H) \cup B_1 \cup B_2 \right) \setminus \tilde{\epsilon} \right\} \setminus \epsilon$ is planar. If we select an edge $\xi$ from $\eta(u_{14} v_{14})$, then $G \setminus \xi$ is not nearly planar because of $B_1$, $B_2$, and $\mathcal{B}$. This is in contradiction to the fact that $G \in \mathcal{M}$.

Lemma 4.13. Suppose $M_{20} \not\preceq G$ with a homeomorphic embedding $\eta : M_{20} \hookrightarrow G$. Graph $G$ contains vertices $u_{\alpha}$ and $u_{\beta}$ with $1 \leq \alpha < 2$ and $3 < \beta \leq 6$ If a bridge of $\eta(M_{20})$ in $G$ contains a path $P$ from $u_{\alpha}$ to $u_{\beta}$, then $G \notin \mathcal{M}$.

Proof. The proof is very similar to that of Lemma 4.12. Let $H$ be a graph made by $\eta(M_{20})$ and $P$. To get a contradiction, let $G$ be in $\mathcal{M}$.
If there is a bridge of $H$ in $G$ such that it contains a path $Q$ from $u_q$ to $u_r$ with an integer $n$ satisfying

(1) $\alpha < q < \beta$,

(2) $\beta < r \leq 11$, and

(3) $q < n < r$,

then $Q \cup H$ dominates a forbidden graph in Lemma 4.8 or 4.10 and $G$ is not in $\mathcal{M}$. This contradicts the fact that $G$ is in $\mathcal{M}$.

Let $m$ be an integer with $\alpha < m < \beta$. If a bridge of $H$ in $G$ contains a path $Q$ from an inner vertex of $P$ to an inner vertex of $\eta(u_m v_m)$. As the argument in Lemma 4.12, $H \cup Q$ is neither nearly planar nor in $\mathcal{M}$. Therefore, this is in contradiction to the fact that $G$ is in $\mathcal{M}$.

Let $n$ be an integer with $\alpha < n < \beta$ and $e$ be an edge of $\eta(u_n v_n)$ such that $u_n$ is one of the endpoints of $e$. Suppose that we can find edge $\tilde{e}$ such that $(G \setminus e) \setminus \tilde{e}$ is planar and $\tilde{e}$ is contained in path $\eta(v_{n-1} v_n)$ or path $\eta(v_n v_{n+1})$. If a bridge contains a path that can replace path $\eta(v_{n-1} v_n)$ or path $\eta(v_n v_{n+1})$ containing $\tilde{e}$ to preserve the Möbius structure like the third case of Lemma 4.12, then $(G \setminus e) \setminus \tilde{e}$ is not planar. This is a contradiction of the property of $\tilde{e}$.

Let $r$ be an integer with $\alpha < r - 1 < r < \beta$ and $e$ be an edge contained in $\eta(u_{r-1} v_{r-1})$ or in $\eta(u_r v_r)$. Since $G$ is in $\mathcal{M}$, there is an edge $\tilde{e}$ such that $(G \setminus e) \setminus \tilde{e}$ is planar. If there is a set $\mathcal{B}$ of bridges of $H$ such that every attachment of every member of $\mathcal{B}$ is on path $\eta(u_{r-1} u_r v_r v_{r-1} u_{r-1})$, then $G$ is not in $\mathcal{B}$ because of redundant rung edges of $\eta(M_{20})$ using the argument of Lemma 4.12.

We want to check the first case of each of Lemma 4.12 and 4.13.
Lemma 4.14. If graph $G$ is in $\mathcal{M}$, then none of the graphs in Figure 4.21 is dominated in $G$.

![Figure 4.21: Every graph in $\mathcal{M}$ does not dominate the above graphs (Part 5)](image)

Proof. We notice that the above graphs are isomorphic to the graphs shown in Figure 4.22, respectively.

![Figure 4.22: Isomorphic graphs to graphs in Figure 4.21](image)

By Lemma 4.8, graph $G$ is not in $\mathcal{M}$.

The following lemma is generalization of the above lemma using Lemma 4.11 and Theorem 4.9.

Lemma 4.15. Suppose $M_{20} \preceq G$ with a homeomorphic embedding $\eta : M_{20} \hookrightarrow G$. Graph $G$ contains vertices $u_\alpha$, $u_\beta$, $u_\gamma$, and $u_\delta$ with an integer $n$ such that $1 \leq \alpha < 2 < \beta \leq 6$, $\gamma < \beta < \delta$, and $\gamma < n < \delta$. If each of a path from $u_\alpha$ to $u_\beta$ and a path from $u_\gamma$ to $u_\delta$ belongs to some bridge of $\eta(M_{20})$ in $G$, then $G$ is not in $\mathcal{M}$.

From the argument of the second case of Lemma 4.12, we can get the following lemma.
Lemma 4.16. If a graph is in \( \mathcal{M} \), then it dominates none of the graphs in Figure 4.23.

\[ \begin{array}{ccc}
\text{T}_4 & \text{T}_5 & \text{T}_6 \\
\text{T}_7 & \text{T}_8 & \text{T}_9 \\
\end{array} \]

*FIGURE 4.23: Every graph in \( \mathcal{M} \) does not dominate the above graphs (Part 6)*

Proof. Suppose that graph \( G \) dominates \( T_4 \) or \( T_7 \). Since neither \( T_4 \) nor \( T_7 \) is nearly planar, neither \( T_4 \setminus (u_{12}v_{12}) \) nor \( T_7 \setminus (u_{12}v_{12}) \) is nearly planar. Therefore, \( G \) is not in \( \mathcal{M} \).

Suppose that graph \( G \) dominates \( T_5 \) or \( T_8 \). Since edge \( u_3\alpha \) is the only planarizing edge of both of \( T_5 \) and \( T_8 \), by Lemma 4.3, \( G \) is not in \( \mathcal{M} \).

Suppose that graph \( G \) dominates \( T_6 \) or \( T_9 \). Since \( u_1\beta \) and \( u_3\alpha \) are the only planarizing edges of both of \( T_6 \) and \( T_9 \), by Lemma 4.3, \( G \) is not in \( \mathcal{M} \).

The right picture in Figure 4.24 shows an embedding of \( K_{3,4} \) on the projective plane.

In this paper, we will consider graphs that do not contain \( K_{3,4} \) as a minor. Therefore, we will focus on graphs that do not dominate any of the graphs in Figure 4.25.

To get a graph in \( \mathcal{M} \) from a given Möbius ladder \( M_k \) with \( k \geq 20 \), we are focusing on bridges of a homeomorphic embedding of \( M_k \) and we want to know that the bridges can be used for graphs in \( \mathcal{M} \).
Lemma 4.17. Suppose that a graph $G$ dominates $H_1$ in Figure 4.26 with homeomorphic embedding $\eta$. If $\eta(v_4\alpha)$ is just an edge in $G$, then $G$ is not in $\mathcal{M}$.

Proof. Suppose that $G$ is in $\mathcal{M}$. Since $\eta(v_4\alpha)$ is an edge in $G$, by Lemma 4.7 and 4.10, it is enough to focus on the closed path $\eta(u_4\alpha u_5 v_5 v_4 u_4)$ with $\eta(v_4\alpha)$ as a $\theta$-graph in [Kur30] to focus bridges of $\eta(H_1)$ in $G$. Let $e$ be edge $\eta(v_4\alpha)$. Then, there is an edge $\tilde{e}$ such that $(G \setminus e) \setminus \tilde{e}$ is planar and $G \setminus \tilde{e}$ is not planar. Since every $K_{3,3}$ subdivision in $G \setminus \tilde{e}$ contains $e$, from $\theta$-graph $\eta(v_4\alpha) \cup \eta(u_4\alpha u_5 v_5 v_4 u_4)$, we can find a minimal set $\mathcal{B}$ of bridges of $\eta(H_1)$ in $G$ such that \[ \left( \bigcup_{B \in \mathcal{B}} B \cup \eta(H_1) \right) \setminus \tilde{e} \] is not planar but \[ \left( \bigcup_{B \in \mathcal{B}} B \cup \eta(H_1) \right) \setminus e \] is planar. By Theorem 4.9, we notice that every
attachment of each member of $\mathcal{B}$ is on the closed path $\eta(u_4\alpha u_5 v_5 v_4 u_4)$. If there are planarizing edges of $\bigcup_{B \in \mathcal{B}} B \cup \eta(H_1)$, then they are on $\eta(u_3 u_4 \alpha u_5 u_6) \cup \eta(v_3 v_4 v_5 v_6)$. By Theorem 4.3, graph $G$ is not in $\mathcal{M}$. This is in contradiction to the fact that $G$ is in $\mathcal{M}$.

We can use a similar argument to the one in this lemma for $H_2$ in Figure 4.26.

**Lemma 4.18.** Suppose that a graph $G$ dominates $H_2$ in Figure 4.26 with homeomorphic embedding $\eta$. If $\eta(v_4 u_5)$ is just an edge in $G$, then $G$ is not in $\mathcal{M}$.

**Lemma 4.19.** Graphs in $\mathcal{M}$ dominate none of the graphs in Figure 4.27.

**FIGURE 4.27:** Every graph in $\mathcal{M}$ does not dominate the above graphs (Part 7)

**Proof.** Suppose that $G \in \mathcal{M}$ dominates $H_3$ with a homeomorphic embedding $\eta$. Let $e$ be an edge in $\eta(\beta \gamma)$. There exists an edge $\bar{e}$ such that $(G \setminus e) \setminus \bar{e}$ is planar. Since $G$ is in $\mathcal{M}$, it follows that $G \setminus \bar{e}$ is not planar. We know that every subgraph of $G \setminus \bar{e}$ dominating $K_{3,3}$ contains $e$. Because of Theorem 4.9, it is enough to focus on three $\theta$-graphs made by

1. $\eta(u_3 u_4 \beta \gamma v_4 v_3 u_3) \cup \eta(\beta v_4)$ which is called as $J_1$ in this proof
2. $\eta(u_4 \alpha \gamma v_4 \beta u_4) \cup \eta(\beta \gamma)$ which is called as $J_2$ in this proof
3. $\eta(v_4 \beta \gamma \alpha u_5 v_5 v_4) \cup \eta(\gamma v_4)$ which is called as $J_3$ in this proof

Since $G \setminus \bar{e}$ is not planar, there is a minimal set $\mathcal{B}$ of bridges of $\eta(H_3)$ in $G$ such that $\left( \bigcup_{B \in \mathcal{B}} B \cup \eta(H_3) \right) \setminus \bar{e}$ is not planar but $\left\{ \left( \bigcup_{B \in \mathcal{B}} B \cup \eta(H_3) \right) \setminus e \right\} \setminus \bar{e}$ is planar.
By Theorem 4.9, for each $1 \leq i \leq 3$, if $J_i$ is dominated in a $K_{3,3}$ subdivision of $\left( \bigcup_{B \in \mathcal{B}} B \cup \eta(H_3) \right) \setminus \hat{e}$, then every attachment of each member of $\mathcal{B}$ is on $J_i$. We conclude that every attachment of each member of $\mathcal{B}$ is on $\eta(u_3u_4v_5v_4v_3u_3) \cup \eta(u_4\beta v_4) \cup \eta(\alpha \gamma v_4) \cup \eta(\beta \gamma)$. Then, the set of planarizing edges of $\left( \bigcup_{B \in \mathcal{B}} B \cup \eta(H_3) \right)$ is a subset of $E(\eta(u_3u_4\alpha u_5) \cup \eta(v_3v_4v_5))$ and by Lemma 4.3, $G$ is not in $\mathcal{M}$. We get the fact that $G$ is not in $\mathcal{M}$ and this is in contradiction.

For the right graph of the above, we can use the same argument. Therefore, if $G$ dominates one of the above graphs, then $G$ is not in $\mathcal{M}$.

Suppose that $G$ dominates a Möbius ladder $M_k$ with a homeomorphic embedding $\eta_0$ and $G$ has a bridge of $\eta(M_k)$ such that this bridge has a red path as one of the graphs in Figure 4.28. Since the above graphs are isomorphic to the graphs in Figure 4.29, respectively, for the convenience, we will select another homeomorphic embedding $\eta_1 : M_k \hookrightarrow G$ such that $\eta(M_k)$ is dominated in one of the below graphs not in one of the above.

We want to show that the allowed bridges of a homeomorphic embedding of $M_k$ are the bold lines in the graphs in Figure 4.30.

**Lemma 4.20.** If graph $G$ in $\mathcal{M}$ dominates $M_{20}$, then $G$ dominates none of the graphs in Figure 4.31.
Proof. Suppose that graph $G$ is in $\mathcal{M}$ and graph $X_1$ comes from Figure 4.31. Let $G \supseteq X_1$ with a homeomorphic embedding $\eta$. Suppose that $G$ has a bridge containing a path $P$ from an inner vertex of $\eta(\gamma\delta)$ to a vertex $\varsigma$ of $V(\eta(X_1) \setminus \eta(\alpha\gamma\beta\delta\alpha) \setminus \eta(\gamma\delta))$.

By Theorem 4.9, vertex $\varsigma$ is an inner vertex of path $\eta(\alpha\varsigma\zeta\beta)$. Then, $P \cup \eta(X_1)$ does not have planarizing edges. If $e_0$ is an edge from a subdivision of a rung except $\alpha\varsigma$ and $\beta\zeta$, then $\{P \cup \eta(X_1)\} \setminus e_0$ is not nearly planar and $G$ is not in $\mathcal{M}$. This contradicts the fact that $G$ is in $\mathcal{M}$.

Let $e$ be an edge in $\eta(\gamma\delta)$. Since $G$ is in $\mathcal{M}$, there exists an edge $\tilde{e}$ such that $G \setminus \{e, \tilde{e}\}$ is planar. Suppose that $\tilde{e}$ is not in $\eta(\tau\alpha) \cup \eta(\varepsilon\zeta) \cup \eta(\beta\rho)$. Every planar
embedding of $G\setminus\{e,\tilde{e}\}$ contains the closed path $\eta(\alpha\varepsilon\zeta\beta\gamma\alpha)$. This closed path makes two regions in the planar embedding of $G\setminus\{e,\tilde{e}\}$. Let us call a region containing $\eta(\tau\alpha)$ and $\eta(\beta\rho)$ as the outside region of the closed path $\eta(\alpha\varepsilon\zeta\beta\gamma\alpha)$ in the given planar embedding of $G\setminus\{e,\tilde{e}\}$. Then, we call a region containing $\eta(\alpha\delta\beta) \cup \eta(\gamma\delta)$ as the inside region of the closed path $\eta(\alpha\varepsilon\zeta\beta\gamma\alpha)$ in the given planar embedding of $G\setminus\{e,\tilde{e}\}$. Since $G\setminus\tilde{e}$ is not planar, there is a minimal set $\mathcal{B}$ of bridges of $\eta(X_1)$ in $G$ such that

$$
\left( \bigcup_{B \in \mathcal{B}} B \cup \eta(X_1) \right) \setminus \tilde{e} \text{ is not planar}
$$

and

$$
\left\{ \left( \bigcup_{B \in \mathcal{B}} B \cup \eta(X_1) \right) \setminus \tilde{e} \right\} \setminus e \text{ is planar.}
$$

Let $\hat{e}$ be an edge in $\eta$-image of a rung in the Möbius ladder such that none of endpoints of the rung is in $\{\tau, \alpha, \beta, \rho\}$. Then, because of the property of set $\mathcal{B}$, planarizing edges of $G\setminus\hat{e}$ are in $\eta(\tau\alpha) \cup \eta(\varepsilon\zeta) \cup \eta(\beta\rho)$. By Lemma 4.3, $G$ is not in $\mathcal{M}$. Therefore, $\hat{e}$ is only in $\eta(\tau\alpha) \cup \eta(\varepsilon\zeta) \cup \eta(\beta\rho)$. By Lemma 4.3, $G$ is not in $\mathcal{M}$. This is in contradiction to the fact that $G$ is in $\mathcal{M}$. Therefore, $G$ does not dominate $X_1$.

Using a similar argument, we know that if $G$ is in $\mathcal{M}$, then $G$ does not dominate $X_i$ for $1 \leq i \leq 9$ in Figure 4.31.

\[\square\]

**Lemma 4.21.** Suppose that graph $G \succeq M_{20}$. If $G$ dominates one of the graphs in Figure 4.32, then $G$ is not in $\mathcal{M}$.

**Proof.** Suppose that graph $G \in \mathcal{M}$ dominates $X_{10}$ in Figure 4.32 with a homeomorphic embedding $\eta$. Suppose that $G$ has a bridge containing a path $P$ from an inner vertex of $\eta(\beta\varepsilon)$ to a vertex $\varsigma$ of $V(\eta(X_{10}) \setminus \eta(\alpha\beta\gamma\varepsilon\delta\alpha) \setminus \eta(\beta\varepsilon))$. By Theorem 4.9, vertex $\varsigma$ is an inner vertex of path $\eta(\delta\zeta\lambda\gamma)$. Then, $P \cup \eta(X_{10})$ may have
FIGURE 4.32: Every graph in $\mathcal{M}$ does not dominate the above graphs (Part 9)

planarizing edges on $\eta(\alpha \beta)$. Let $e_0$ be an edge in the $\eta$-image of a rung such that vertex $\rho$ is an endpoint of $e_0$. If $P \cup \eta(X_{10})$ does not have any planarizing edges, then $\{P \cup \eta(X_{10})\} \setminus e_0$ is not nearly planar and $G$ is not in $\mathcal{M}$. If $P \cup \eta(X_{10})$ has planarizing edges on $\eta(\alpha \beta)$, then by Lemma 4.3, $G$ is not in $\mathcal{M}$. This contradicts the fact that $G$ is in $\mathcal{M}$.

Let $e$ be an edge in $\eta(\beta \varepsilon)$. Since $G$ is in $\mathcal{M}$, there is an edge $\tilde{e}$ such that $G \setminus \tilde{e}$ is not planar but $(G \setminus \tilde{e}) \setminus e$ is planar. Every planar embedding of $G \setminus \{e, \tilde{e}\}$ contains the closed path $\eta(\alpha \beta \gamma \varepsilon \delta \alpha)$. This closed path makes two regions in the planar embedding of $G \setminus \{e, \tilde{e}\}$. Let us call a region containing $\eta(\tau \alpha)$ and $\eta(\gamma \rho)$ as the outside region of the closed path $\eta(\alpha \beta \gamma \varepsilon \delta \alpha)$ in the given planar embedding of $G \setminus \{e, \tilde{e}\}$. Then, we call a region containing $\eta(\beta \varepsilon)$ as the inside region of the closed path $\eta(\alpha \beta \gamma \varepsilon \delta \alpha)$ in the given planar embedding of $G \setminus \{e, \tilde{e}\}$. Since $G \setminus \tilde{e}$ is not planar, there is a minimal set $\mathcal{B}$ of bridges of $\eta(X_{10})$ in $G$ such that

$$
\left( \bigcup_{B \in \mathcal{B}} B \cup \eta(X_{10}) \right) \setminus \tilde{e} \text{ is not planar}
$$

and

$$
\left\{ \left( \bigcup_{B \in \mathcal{B}} B \cup \eta(X_{10}) \right) \setminus \tilde{e} \right\} \setminus e \text{ is planar}.
$$

If $\bigcup_{B \in \mathcal{B}} B \cup \eta(X_{10})$ does not have any planarizing edges, then $G$ is not in $\mathcal{M}$ because $G$ contains redundant rungs. If $\bigcup_{B \in \mathcal{B}} B \cup \eta(X_{10})$ has some planarizing edges, then
these planarizing edges are on $\eta(\tau\alpha) \cup \eta(\gamma\rho)$. By Lemma 4.3, $G$ is not in $\mathcal{M}$. As a result, if $G$ is in $\mathcal{M}$, then $G$ does not dominate $X_{10}$.

Using a similar argument, we know that if $G$ is in $\mathcal{M}$, then $G$ does not dominate $X_i$ for $10 \leq i \leq 15$ in Figure 4.32.

**Lemma 4.22.** Suppose that graph $G$ dominates $M_{20}$. If $G$ dominates one of the graphs in Figure 4.33, then $G$ is not in $\mathcal{M}$.

![Figure 4.33](image-url) Every graph in $\mathcal{M}$ does not dominate the above graphs (Part 10) (cont.)
FIGURE 4.33: Every graph in \( \mathcal{M} \) does not dominate the above graphs (Part 10)

**Proof.** Let \( H \) be one of the graphs in Figure 4.33 and \( G \) dominates \( H \) with a homeomorphic embedding \( \eta \). Let \( e \) be an edge of \( \eta \)-image of the red edge of \( H \). Since \( G \in \mathcal{M} \), there exists an edge \( \tilde{e} \) such that \( G \setminus \{e, \tilde{e}\} \) is planar. Using a similar argument to the one of Lemma 4.20 and 4.21, there is a minimal set \( \mathcal{B} \) of bridges of \( \eta(H) \) such that

\[
\left( \bigcup_{B \in \mathcal{B}} B \cup \eta(H) \right) \setminus \tilde{e} \text{ is not planar}
\]

and

\[
\left\{ \left( \bigcup_{B \in \mathcal{B}} B \cup \eta(H) \right) \setminus \tilde{e} \right\} \setminus e \text{ is planar.}
\]

If graph \( \bigcup_{B \in \mathcal{B}} B \cup \eta(H) \) does not have any planarizing edges, then \( G \) is not in \( \mathcal{M} \).

Otherwise, graph \( \bigcup_{B \in \mathcal{B}} B \cup \eta(H) \) satisfies the condition of Lemma 4.3 and \( G \) is not
in \( \mathcal{M} \). This is in contradiction to the fact that \( G \) is in \( \mathcal{M} \). As a result, if \( G \in \mathcal{M} \) dominates \( M_{20} \), then \( G \) does not dominate any of the above graphs.

\[ \square \]

**Theorem 4.23.** If graph \( G \in \mathcal{M} \) dominates Möbius ladder \( M_k \) with \( k \geq 20 \) with a homeomorphic embedding \( \eta \), then each bridge of \( \eta(M_k) \) in \( G \) is one of bridges in Figure 4.30.

**Proof.** By Lemma 4.20, 4.21, and 4.22, this theorem holds.

The following lemmas are showing that if graph \( G \in \mathcal{M} \) dominates \( M_{20} \) with a homeomorphic embedding \( \eta \), then \( G \) has only limited types of bridges of \( \eta(M_{20}) \).

**Lemma 4.24.** If graph \( G \in \mathcal{M} \) dominates \( M_{20} \) under \( \preceq \), then \( G \) dominates none of the graphs in Figure 4.34.

**Proof.** Let \( M \) be one of the above graphs and \( M \) be dominated in graph \( G \in \mathcal{M} \). Since \( M \setminus e \) is not planar, there exists a rim edge \( \tilde{e} \) to make \( M \setminus \{e, \tilde{e}\} \) planar by Lemma 4.2. (In this case, \( \tilde{e} \notin \{g, h\} \) because of another Möbius ladder.) Suppose that \( M_0 \) is a subgraph of \( G \) such that \( M_0 \) is a subdivision of \( M \) under a map \( \eta \). Let \( e_0 \) be an edge of \( M_0 \) which is an edge in the path \( \eta(e) \). Let \( \tilde{e}_0 \) be an edge of \( M_0 \) which is an edge in the path \( \eta(\tilde{e}) \) such that \( G \setminus \{e_0, \tilde{e}_0\} \) is planar. Subgraph \( M_0 \) has a subgraph \( C \) which is the subdivision of the red cycle in \( M \). Graph \( C \) is still a subgraph of \( G \setminus \{e_0, \tilde{e}_0\} \). Since graph \( C \) is a cycle, in any planar embedding \( \Xi \) of \( G \setminus \{e_0, \tilde{e}_0\} \), we can say two open regions \( R_1 \) and \( R_2 \) made by graph \( C \).

![FIGURE 4.34: Every graph in \( \mathcal{M} \) does not dominate the above graphs (Part 11)](image)
(Unfortunately, we cannot say faces.) Without loss of generality, we may assume that the any edges of \((M_0 \setminus \tilde{e}_0) \setminus (E(C) \cup E(\eta(e)))\) should be drawn in \(R_2\) in the planar embedding \(\Xi\) of \(G \setminus \{e_0, \tilde{e}_0\}\). Using this planar embedding \(\Xi\), we can construct a new planar embedding \(\tilde{\Xi}\) of \((M_0 \setminus \tilde{e}_0)\) by erasing the inside of \(R_1\) of \(\Xi\) and add edges of \(\eta(e)\). Since \(G \setminus \tilde{e}_0\) is not planar, there is a minimal set \(\mathcal{B}\) bridges of \(M_0\) in \(G\) such that

\[
\left(\bigcup_{B \in \mathcal{B}} B \cup M_0\right) \setminus \tilde{e}_0 \text{ is not planar}
\]

and

\[
\left\{\left(\bigcup_{B \in \mathcal{B}} B \cup M_0\right) \setminus \tilde{e}_0\right\} \setminus e_0 \text{ is planar}.
\]

As every member of \(\mathcal{B}\) does not contain edge \(e_0\), \(\bigcup_{B \in \mathcal{B}} B\) is a subgraph of \(G \setminus \{e_0, \tilde{e}_0\}\).

If every attachment of every member of \(\mathcal{B}\) is in only closed region \(\overline{R_2}\) of the planar embedding \(\Xi\) of \(G \setminus \{e_0, \tilde{e}_0\}\) and \(G \setminus \tilde{e}\) is not planar, then \(G \setminus \{e_0, \tilde{e}_0\}\) is not planar, either.

If attachments of a member of \(\mathcal{B}\) are in \(R_1\) and \(R_2\) simultaneously under \(\Xi\), then, by Theorem 4.9, subgraph \(\bigcup_{B \in \mathcal{B}} B \cup M_0\) is satisfying the conditions of Lemma 4.3. Therefore, graph \(G\) is not in \(\mathcal{M}\).

\(\mathcal{B}\) must be in drawn in \(R_1\). However, as \(M \setminus f\) is not planar, we will take a rim edge \(\tilde{f}\) to make \(M \setminus \{f, \tilde{f}\}\) planar. We know that \(\tilde{f} \notin \{g, h\}\) because there is another Möbius ladder. Let us select any rim edge \(\tilde{f}\) other than \(g\) and \(h\). We can find a \(W_4\) topological minor \(K\) in \(G \setminus \{f, \tilde{f}\}\) which contains \(e, g, h\) and all of the attachments of \(B\). (Think about the big cycle of \(M_k\).)

By the previous lemmas in this chapter, the graph \(K\) with \(B\) contains a subdivision of \(K_{3,3}\) as a subgraph. Therefore, \(G \setminus \{f, \tilde{f}\}\) cannot be planar. \(G\) cannot be in \(\mathcal{M}\). As a result, the above five graphs cannot be dominated in any graphs in \(\mathcal{M}\). \(\Box\)
Using a similar argument to the one of Lemma 4.24 about a redundant rung, we can get the following lemma.

**Lemma 4.25.** If graphs $G$ dominates $M_{20}$ and one of the following graphs, then $G$ is not in $\mathcal{M}$.

**Lemma 4.26.** If graph $G \in \mathcal{M}$ dominates $M_{20}$, then graph $G$ does not dominate the graph in Figure 4.37.
Proof. Suppose that graph \( G \in \mathcal{M} \) dominates the above. Because of Theorem 4.9 and Lemma 4.3 and 4.24, it is enough to focus on the following six graphs \( \tilde{G} \) in Figure 4.38.

FIGURE 4.38: Not nearly planar graphs none of which is a member of \( \mathcal{M} \) (Part 1)

Since each of the graphs in Figure 4.39 contains a subdivision of \( K_{3,3} \) as a subgraph, graph \( G \setminus f \) is not nearly planar.

FIGURE 4.39: Subgraphs of a graph in Figure 4.38

This contradicts the fact that \( G \in \mathcal{M} \).

By a similar argument of Lemma 4.26, we can get the following lemma.

Lemma 4.27. If graph \( G \in \mathcal{M} \) dominates \( M_{20} \), then graph \( G \) does not dominate the graph in Figure 4.40.

We will investigate other graphs.
Lemma 4.28. If graph $G \in \mathcal{M}$ dominates $M_{20}$, then graph $G$ does not dominate the graphs in Figure 4.41.

Proof. Suppose that $G \in \mathcal{M}$ and $G$ dominates one of the above. By Theorem 4.9 and Lemma 4.3, it is enough to focus on the following graphs $\tilde{G}$ in Figure 4.42.

As each of the graphs in Figure 4.43 contains a subdivision of $K_{3,3}$ as a subgraph, graph $\tilde{G} \setminus f$ is not nearly planar.

This is in contradiction to the fact that $G \in \mathcal{M}$.

Using the similar argument of the previous lemmas, we can get the following.

Lemma 4.29. If graph $G \in \mathcal{M}$ dominates $M_{20}$, then graph $G$ does not dominate the graphs in Figure 4.44.

Then, we can say every graph $G \in \mathcal{M}$ which dominates $M_k$ for some $k \geq 20$ is constructed by graphs in Figure 4.45.
In addition, using the graphs in Figure 4.45, for any \( k \geq 20 \), we can construct a graph \( G \in \mathcal{M} \) dominating \( M_k \) as illustrated in Figure 4.46.

Before stating the theorem, define the class \( \mathcal{N} \) of graphs that can be obtained from \( M_k \), for \( k \geq 20 \), by to get one of the graphs mentioned in Figure 4.45. Be precise, so that there is no doubt in reader’s mind what those graphs are. Then you will have the following theorem.

**Theorem 4.30.** [Main Theorem]

(i) Every graph in \( \mathcal{N} \) is also in \( \mathcal{M} \).

(ii) If a 3-connected graph \( G \in \mathcal{M} \) does not contain \( K_{3,4} \) as a minor with \( |V(G)| \geq N(20) \), then \( G \) is in \( \mathcal{N} \).
FIGURE 4.44: Every graph in $\mathcal{M}$ does not dominate the above graphs (Part 16)
FIGURE 4.45: Fundamental subgraphs to describe every graph $G$ in $\mathcal{M}$ containing $M_k$ for some $k \geq 20$ with $|V(G)| \geq N(20)$

FIGURE 4.46: An example graph in $\mathcal{M}$ made by three fundamental subgraphs on a projective plane
References


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