

VIRTUAL STRINGS FOR CLOSED CURVES WITH
MULTIPLE COMPONENTS AND
FILAMENTATIONS FOR VIRTUAL LINKS

A Dissertation

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William Schellhorn

B.S. in Mathematics, Mount Mercy College, Cedar Rapids, Iowa, 2000

M.S. in Mathematics, Louisiana State University, Baton Rouge, Louisiana, 2002

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Abstract

The theory of filaments on oriented chord diagrams can be used to detect some non-classical virtual knots. We extend existing filament techniques to virtual links with more than one component and give examples of virtual links that these techniques can detect as non-classical. Given a signed Gauss word w underlying an oriented chord diagram, we describe how to construct a finite sequence of integers $Z(w)$ that encodes all of the filament information for the diagram. We also introduce a square array of integers called a MIN-square that summarizes the filament information about all of the signed Gauss words having a given Gauss word shape.

A Gauss paragraph is a combinatorial formulation of a generic closed curve with multiple components on some surface. A virtual string is a collection of circles with arrows that represent the crossings of such a curve. We use the theory of virtual strings to obtain a combinatorial description of closed curves in the 2-sphere (and therefore 2-dimensional Euclidean space) in terms of Gauss paragraphs and word-wise partitions of their alphabet sets. In addition, we prove that the unordered triple consisting of the Gauss paragraph, the word-wise partition, and a related word-wise partition associated to a closed curve on the 2-sphere is a full homeomorphism invariant of the closed curve. We conclude by introducing a multi-variable polynomial that is a homotopy invariant of virtual strings with multiple circles.

Chapter 1

Filamentations for Virtual Links

1.1 Introduction

Virtual links were first introduced by L.H. Kauffman in 1996. A link, in the classical sense, is a special type of virtual link. Hence any results that hold for virtual links must also apply to classical links. This leads to the study of invariants of virtual links. In particular, we are interested in invariants that have the ability to distinguish virtual links that are not classical links. D. Hrencecin and L.H. Kauffman introduced one such invariant on virtual knots using filamentations on oriented chord diagrams, see [HK02].

In this chapter, we extend the filamentation invariant to virtual links with more than one component. In Section 1.2, we recall the basic definitions of virtual link theory and indicate how chord diagrams characterize virtual links. In Section 1.3, we define filamentations on oriented chord diagrams that represent virtual links. The theorems of Section 1.4 give the central results of this paper. Theorem 1.4.2 shows that the existence of a filamentation is an invariant of oriented chord diagrams. Theorem 1.4.3 and Corollary 1.4.4 then describe how filamentations can be used to distinguish non-classical virtual links. Some additional results and interesting examples are included in Section 1.5.

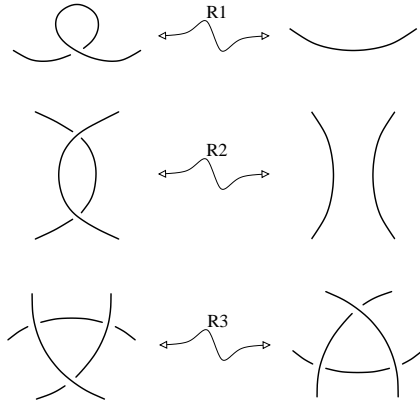


Figure 1.1: The Reidemeister moves.

1.2 Chord Diagrams

A *link* of n components is a subset of S^3 that consists of n disjoint, piecewise linear, simple closed curves, i.e. the image of a piecewise smooth embedding of n copies of S^1 into S^3 . In particular, a link of one component is called a *knot*. A link can be put in general position with respect to the standard projection from \mathbb{R}^3 to \mathbb{R}^2 , so that the only singularities are transversal double points. A *link diagram* for a link is the image of the link in \mathbb{R}^2 enhanced with “over” and “under” crossing information at each double point. Therefore a link diagram is a 4-valent graph in \mathbb{R}^2 with additional structure associated to its vertices. There are three types of *Reidemeister moves* that can be performed on a link diagram, as shown in Figure 1.1. Each Reidemeister move replaces a configuration of arcs and crossings in a disc by a different configuration while leaving the rest of the link diagram unchanged. Two link diagrams are called *equivalent* if there exists a sequence of Reidemeister moves relating them. A link can then be defined in a combinatorial sense as an equivalence class of link diagrams under the Reidemeister moves. This is the approach we will take in this paper.

A *virtual link diagram* is also a 4-valent graph in \mathbb{R}^2 with additional structure associated to its vertices. A vertex in a virtual link diagram can be an over or under crossing as in a link diagram, or it can be a *virtual crossing*. A virtual crossing is represented in a virtual

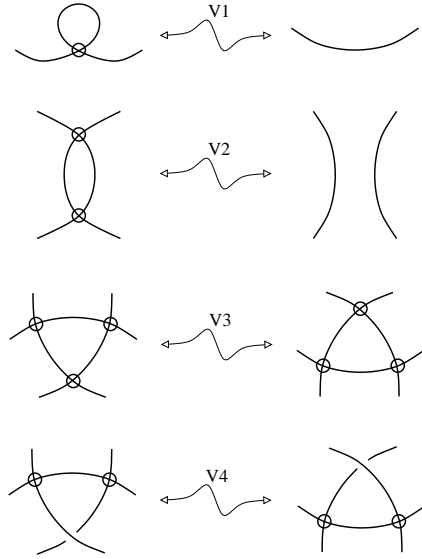


Figure 1.2: The virtual Reidemeister moves.

link diagram as a vertex with a small circle around it. The over and under crossings of a virtual link diagram are called *real crossings* to distinguish them from its virtual crossings. There are four types of *virtual Reidemeister moves* that can be performed on a virtual link diagram, as shown in Figure 1.2. Each virtual Reidemeister move replaces a configuration of arcs and crossings in a disc by a different configuration while leaving the rest of the virtual link diagram unchanged. Two virtual link diagrams are called *equivalent* if there exists a sequence of Reidemeister moves and virtual Reidemeister moves relating them. A *virtual link* can be defined in a combinatorial sense as an equivalence class of virtual link diagrams under the Reidemeister and virtual Reidemeister moves. A detailed introduction to the theory of virtual links is presented in [Kau99]. Notice that a link diagram is simply a virtual link diagram with no virtual crossings. Hence link diagrams are referred to as *classical link diagrams*, and a virtual link is called a *classical link* if it can be represented by a classical link diagram.

Definition 1.2.1. A *flat virtual link diagram* is a 4-valent graph in \mathbb{R}^2 with vertices that may or may not be virtual crossings. A flat virtual link diagram can be obtained from a

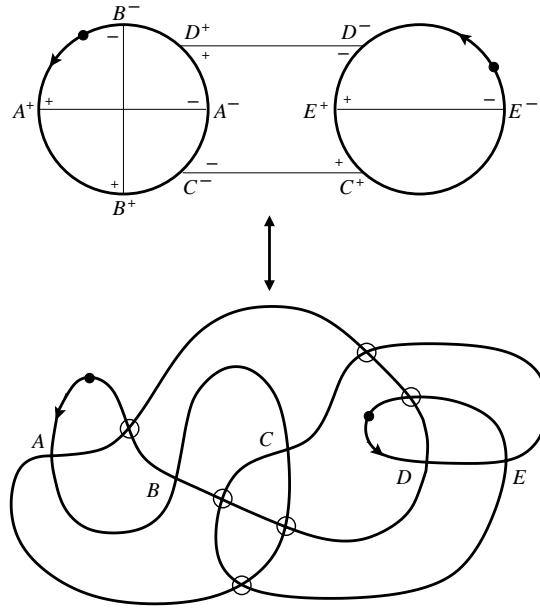


Figure 1.3: An OCD and its related flat virtual link diagram.

virtual link diagram by *flattening* it, that is, stripping the real crossing information from its vertices.

As described in [HK02], any virtual link diagram can be characterized by an *arrow diagram*¹ and any flat virtual link diagram can be characterized by an *oriented chord diagram*. We will refer to arrow diagrams as AD's and oriented chord diagrams as OCD's.

Example 1.2.2. Figure 1.3 shows a flat virtual link diagram and its associated OCD.

Example 1.2.3. Figure 1.4 shows a virtual link diagram and its associated AD.

The flat Reidemeister moves for flat virtual link diagrams are depicted in Figure 1.5. These moves are flattened versions of the Reidemeister moves and virtual Reidemeister moves. Two flat virtual link diagrams are called *equivalent* if there exists a sequence of flat Reidemeister moves relating them. Then a *flat virtual link* can be defined as an equivalence class of flat virtual link diagrams under the flat Reidemeister moves.

¹Such a chord diagram was referred to as a *Gauss diagram* in [GPV99].

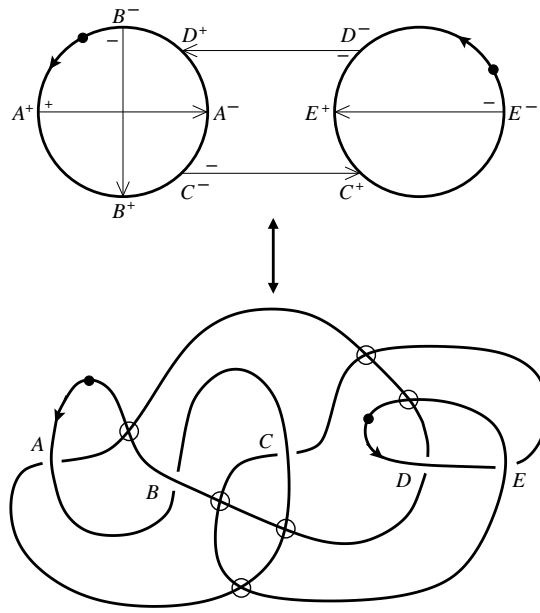


Figure 1.4: An AD and its related virtual link diagram.

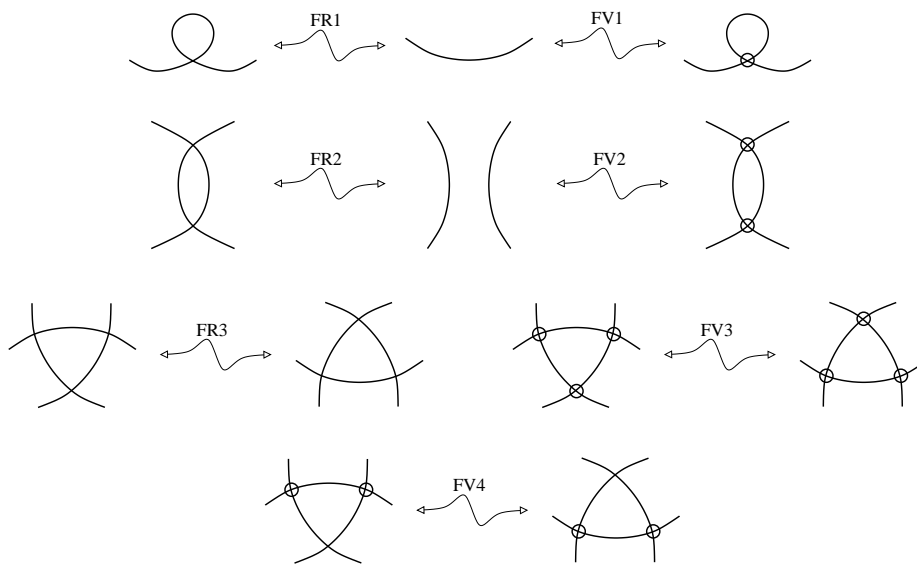


Figure 1.5: The flat Reidemeister moves.

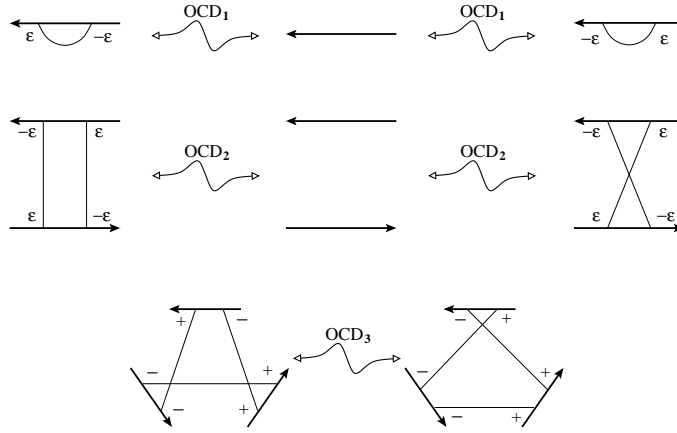


Figure 1.6: The OCD moves.

Definition 1.2.4. Let L be a virtual link, and let D be the flat virtual link diagram that results from flattening a virtual link diagram for L . The *universe* of L is the flat virtual link represented by D .

Example 1.2.5. The universe of the virtual link in Figure 1.4 is represented by the flat virtual link diagram in Figure 1.3.

An equivalence relation can be defined on the set of all OCD's that reflects the flat Reidemeister moves. Notice the flat Reidemeister moves involving virtual crossings do not change the associated OCD since virtual crossings are not represented by chords in the OCD. Figure 1.6 depicts the OCD moves resulting from the flat Reidemeister moves that do not involve virtual crossings, where the move OCD_i results from the move $\text{FR}i$ for $i \in \{1, 2, 3\}$. Note ε can be either $+$ or $-$. The thick arcs depicted are unordered and may be on different circles of the OCD.

1.3 Filamentations

Definition 1.3.1. An *acceptable chord pair* in an oriented chord diagram D is an unordered pair of chords (a, b) such that the positive end of a and the negative end of b are on the same

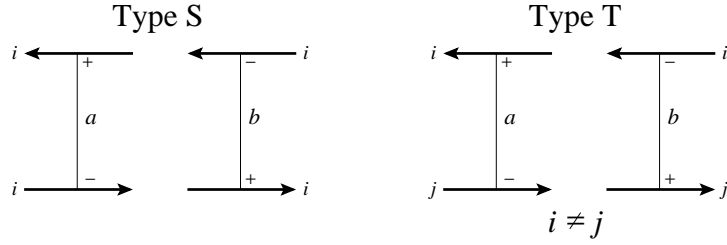


Figure 1.7: The two types of acceptable chord pairs.

circle, and the negative end of a and the positive end of b are on the same circle. We will say the chord pair (a, b) is of *type S* if the endpoints of a and b are all on the same circle of D , and otherwise we will say the chord pair (a, b) is of *type T*. These terms are illustrated in Figure 1.7, where the labels i and j indicate the circle of D on which the chord endpoints reside. Note that it is possible for $a = b$ in an acceptable chord pair of type S, but not in an acceptable chord pair of type T.

Definition 1.3.2. A *pairing* \mathcal{P} for a chord diagram D is a collection of chord pairs such that each chord in D occurs in exactly one pair in the collection and each chord pair is acceptable.

Throughout the remainder of this paper, we will use lowercase letters to denote chords and we will use the corresponding uppercase letters with sign indicators to denote the endpoints of chords.

Example 1.3.3. The pairings $\{(a, a), (b, b), (c, d), (e, e)\}$ and $\{(a, b), (c, d), (e, e)\}$ are the only possible pairings for the OCD in Figure 1.3.

Definition 1.3.4. Let D be an OCD, and let (a, b) be an acceptable chord pair for D . A *filament* α associated with (a, b) is a generic curve between an endpoint A^ε of a and the endpoint $B^{-\varepsilon}$ of b , where $\varepsilon \in \{+, -\}$. Excluding its endpoints, the curve must lie entirely within the interior of a single circle of D and may contain a finite number of transverse self-intersections. A filament is oriented from its negative endpoint to its positive endpoint. The filament between $A^{-\varepsilon}$ and B^ε is called the *dual* of α , and is denoted α' .

If the chords a and b in the above definition are distinct, then the filaments α and α' are called *bifilaments*. Otherwise, the filament α is its own dual and α is called a *monofilament*. Note that a filament α and its dual filament α' reside on different circles when (a, b) is an acceptable chord pair of type T. Therefore filaments coming from type T pairs cannot be monofilaments. We may assume that distinct filaments on an OCD intersect each other transversely. The oriented intersection number $\alpha \overrightarrow{\cap} \beta$ of two filaments α and β and the intersection number $\langle \alpha \rangle$ of a filament α are defined in [HK02].

Definition 1.3.5. Let (a, b) be an acceptable chord pair in a pairing \mathcal{P} , and let α be a filament for (a, b) . The *intersection number* of (a, b) is defined as

$$\langle (a, b) \rangle = \sum_{\beta \in \{\alpha, \alpha'\}} \sum_{\gamma \notin \{\alpha, \alpha'\}} \beta \overrightarrow{\cap} \gamma$$

where the second sum runs over all filaments for the pairs in $\mathcal{P} - \{(a, b)\}$.

Definition 1.3.6. A *filamentation* \mathcal{F} on an oriented chord diagram is a pairing for which the related filaments intersect transversely (if they intersect at all) and the intersection number of each pair is zero.

Example 1.3.7. The OCD in Figure 1.3 does not admit a filamentation.

Example 1.3.8. The OCD in Figure 1.8 admits the filamentation $\{(a, b), (c, d), (e, f)\}$ since

$$\begin{aligned} \langle (a, b) \rangle &= \langle \alpha \rangle + \langle \alpha' \rangle = -1 + 1 = 0, \\ \langle (c, d) \rangle &= \langle \beta \rangle + \langle \beta' \rangle = -1 + 1 = 0, \text{ and} \\ \langle (e, f) \rangle &= \langle \gamma \rangle + \langle \gamma' \rangle = +1 - 1 = 0. \end{aligned}$$

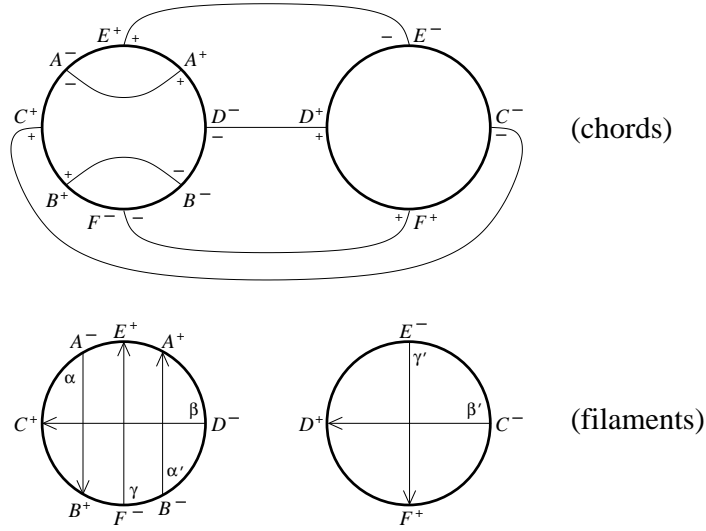


Figure 1.8: An OCD that admits a filamentation.

1.4 Theorems

Lemma 1.4.1. *If a virtual link is classical, then its universe is flat equivalent to the unlink.*

Proof. Let L be a classical link. Any classical link can be unlinked via a finite number of crossing changes. So there exists a link diagram for L and a diagram D_U for the unlink that share the same universe. There is a finite sequence of Reidemeister moves taking D_U to an unlink diagram with no crossings. The corresponding sequence of flat Reidemeister moves will unlink the universe of L . ■

Theorem 1.4.2. *Let D be an OCD for a flat virtual link which admits a filamentation \mathcal{F} , and suppose D' is a chord diagram equivalent to D via OCD moves. Then there exists a filamentation \mathcal{F}' on D' .*

Proof. It suffices to show that a filamentation can be preserved under any single OCD move. This result was proved in [HK02] for cases involving a single circle of an OCD because Hrencecin and Kauffman considered OCD's representing flat virtual knots. Therefore we need only consider cases involving multiple components. We will show how to construct the

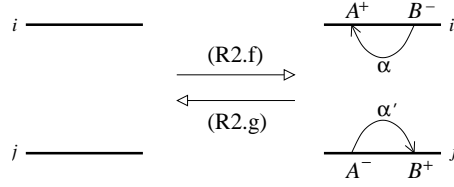


Figure 1.9: Cases (R2.f) and (R2.g) for the OCD_2 move.

desired filamentation \mathcal{F}' from the existing filamentation \mathcal{F} .

In an OCD_1 move, both endpoints of the removed or added chord are on the same circle. Consequently, any acceptable chord pair in \mathcal{F} including the chord must be of type S. So either the filament α for this chord pair is a monofilament, or α and its dual filament α' are on the same circle of D . Hence the OCD_1 move produces no new cases for our consideration.

There are three new cases to consider for the OCD_2 move. The first two, (R2.f) and (R2.g)², are depicted in Figure 1.9. In (R2.f), the induced filamentation \mathcal{F}' is defined by adding the type T pair (a, b) to \mathcal{F} , that is $\mathcal{F}' = \mathcal{F} \cup \{(a, b)\}$. In (R2.g), remove (a, b) from \mathcal{F} by setting $\mathcal{F}' = \mathcal{F} - \{(a, b)\}$. The third case is depicted in Figure 1.10. Here we can define $\mathcal{F}' = (\mathcal{F} - \{(a, x), (b, y)\}) \cup \{(x, y)\}$ since

$$\begin{aligned}
 \langle (x, y) \rangle_{\mathcal{F}'} &= \langle \gamma \rangle + \langle \gamma' \rangle \\
 &= (\langle \alpha \rangle_{\mathcal{F}} + \langle \beta' \rangle_{\mathcal{F}}) + (\langle \beta \rangle_{\mathcal{F}} + \langle \alpha' \rangle_{\mathcal{F}}) \\
 &= (\langle \alpha \rangle_{\mathcal{F}} + \langle \alpha' \rangle_{\mathcal{F}}) + (\langle \beta \rangle_{\mathcal{F}} + \langle \beta' \rangle_{\mathcal{F}}) \\
 &= 0.
 \end{aligned}$$

Now consider the OCD_3 move. As mentioned in [HK02] for knot OCD's, it is not necessary to consider all possible permutations of the circle arcs. The same is true for link OCD's with more than one component by the following argument. We can position the filaments of \mathcal{F} so that none of the filament intersections occur within small neighborhoods of each circle of D .

²The cases (R2.a) through (R2.e) were discussed in [HK02] because they concern a single circle of an OCD. We use the letters “f” and “g” here to be consistent with this notational convention.

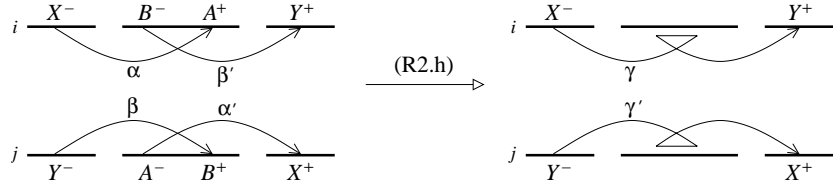


Figure 1.10: Case (R2.h) for the OCD_2 move.

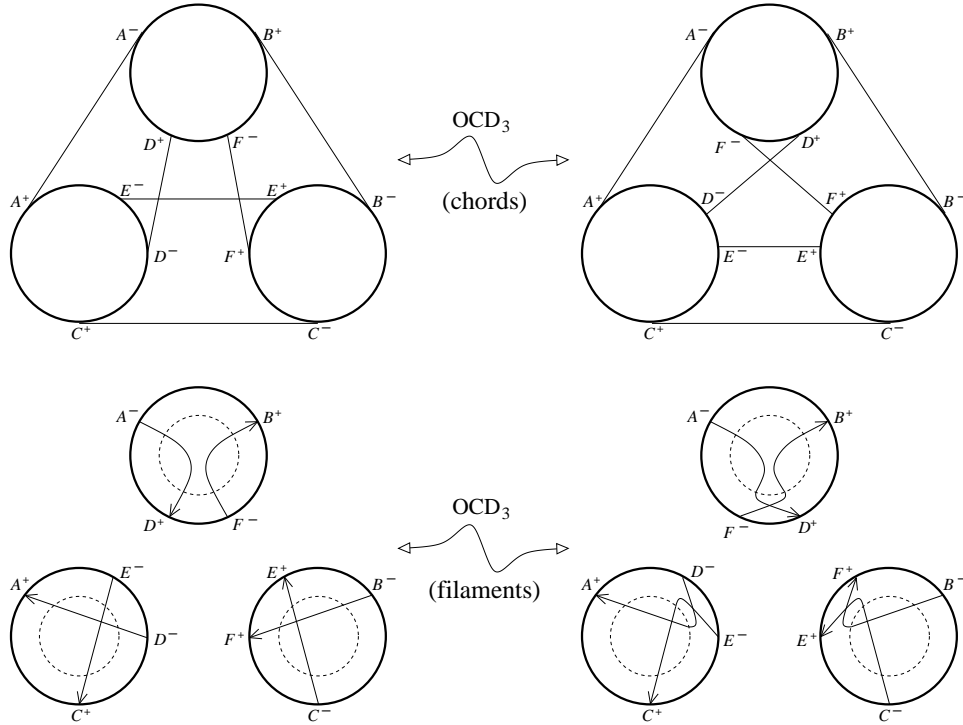


Figure 1.11: The effects of an OCD_3 move.

Then filaments for D' can be arranged in the same way, except within these neighborhoods the two filaments that emanate from each arc cross. These new intersections of the filaments account for the fact that the OCD_3 move simply switches the endpoints on each arc. Refer to Figure 1.11 for an example of the effects of an OCD_3 move. Note that the filaments in the figure do not represent a filamentation.

For any possible configuration of an OCD_3 move, we define $\mathcal{F}' = \mathcal{F}$ and consider the effect of modifying the filaments for D to form the filaments for D' as described above. In general, we have the case (R3) illustrated in Figure 1.12, with the following subcases:

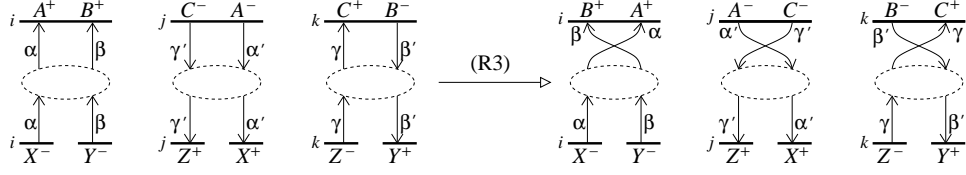


Figure 1.12: General case (R3) for the OCD_3 move.

- (i) $i = j \neq k$,
- (ii) $i \neq j = k$,
- (iii) $i = k \neq j$, and
- (iv) i, j, k are pairwise distinct.

Notice that if the depicted filaments $\alpha, \beta, \gamma, \alpha', \beta'$, and γ' are all distinct, then

$$\langle (a, x) \rangle_{\mathcal{F}'} = \langle (a, x) \rangle_{\mathcal{F}} + 1 - 1 = 0,$$

$$\langle (b, y) \rangle_{\mathcal{F}'} = \langle (b, y) \rangle_{\mathcal{F}} - 1 + 1 = 0, \text{ and}$$

$$\langle (c, z) \rangle_{\mathcal{F}'} = \langle (c, z) \rangle_{\mathcal{F}} - 1 + 1 = 0.$$

Therefore under this condition, \mathcal{F}' is a filamentation because \mathcal{F} is a filamentation. In subcases (ii), (iii), and (iv), the depicted filaments must all be distinct. Therefore to prove the result we now suppose subcase (i) holds and the filaments are not pairwise distinct. If the pairs (a, x) and (b, c) occur in \mathcal{F} , where x is neither b nor c , then we have the situation depicted in Figure 1.13 with

$$\langle (a, x) \rangle_{\mathcal{F}'} = \langle (a, x) \rangle_{\mathcal{F}} + 1 - 1 = 0 \text{ and}$$

$$\langle (b, c) \rangle_{\mathcal{F}'} = \langle (b, c) \rangle_{\mathcal{F}} - 1 + 1 = 0$$

since the self-intersections of β' do not contribute to $\langle \beta' \rangle$.

If the pairs (a, a) and (b, c) occur in \mathcal{F} , then we have the situation depicted in Figure

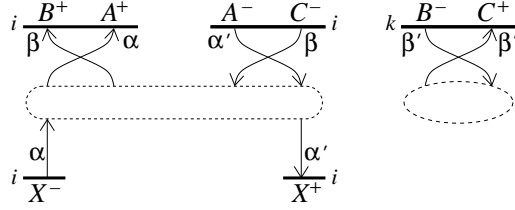


Figure 1.13: The filaments when the pairs (a, x) and (b, c) occur in \mathcal{F} .

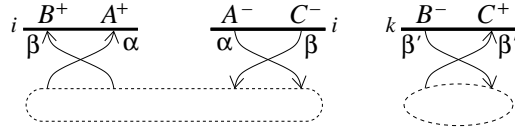


Figure 1.14: The filaments when the pairs (a, a) and (b, c) occur in \mathcal{F} .

1.14 with

$$\langle (a, a) \rangle_{\mathcal{F}'} = \langle (a, a) \rangle_{\mathcal{F}} - 1 + 1 = 0 \text{ and}$$

$$\langle (b, c) \rangle_{\mathcal{F}'} = \langle (b, c) \rangle_{\mathcal{F}} + 1 - 1 = 0$$

since again the self-intersections of β' do not contribute to $\langle \beta' \rangle$.

If the pair (a, a) occurs in \mathcal{F} , but the pair (b, c) does not, then we have the situation depicted in Figure 1.15 with

$$\langle (a, a) \rangle_{\mathcal{F}'} = \langle (a, a) \rangle_{\mathcal{F}} - 1 + 1 = 0,$$

$$\langle (b, y) \rangle_{\mathcal{F}'} = \langle (b, y) \rangle_{\mathcal{F}} - 1 + 1 = 0, \text{ and}$$

$$\langle (c, z) \rangle_{\mathcal{F}'} = \langle (c, z) \rangle_{\mathcal{F}} - 1 + 1 = 0.$$

Thus we have shown how to construct a filamentation \mathcal{F}' on D' whenever there exists a

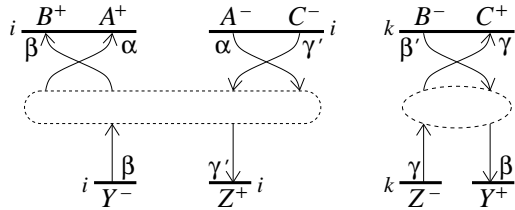


Figure 1.15: The filaments when the pair (a, a) occurs in \mathcal{F} , but the pair (b, c) does not.

filamentation \mathcal{F} on D , as desired. ■

Theorem 1.4.2 gives that the existence of a filamentation is an invariant of OCD's, and hence can be considered an invariant of flat virtual links.

Theorem 1.4.3. *If D is an OCD for which a filamentation does not exist, then the flat virtual link represented by D is non-trivial.*

Proof. If the flat virtual link represented by D is trivial, then there is a sequence of OCD moves that takes D to a trivial OCD with no chords. But such a trivial OCD admits a trivial filamentation, so by Theorem 1.4.2, a filamentation for D can be constructed. ■

Corollary 1.4.4. *If D is an OCD for which a filamentation does not exist, then any virtual link with universe represented by D is non-classical.*

Proof. The flat virtual link represented by D is non-trivial by Theorem 1.4.3. Hence the result follows from Lemma 1.4.1. ■

1.5 Examples

Definition 1.5.1. The *parity* invariant of a flat virtual link or virtual link is the parity of the total number of either real or virtual crossings between distinct components of the link. Notice that the parity of a flat virtual link (respectively virtual link) is the same as the parity of the total number of chords between distinct components of an OCD (respectively AD) for the link.

Example 1.5.2. As mentioned in Example 1.3.7, the OCD in Figure 1.3 does not admit a filamentation, so any virtual link with universe represented by this chord diagram is non-classical. Therefore the virtual link in Figure 1.4 is a non-classical virtual link with two classical components that are both unknotted. The parity invariant does not distinguish the virtual link from the unlink since it has two real crossings between its two components.

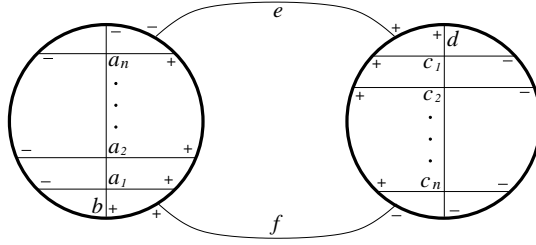


Figure 1.16: The OCD of a non-trivial flat virtual link with two non-trivial components.

The previous example illustrates that the converse of the following proposition is false.

Proposition 1.5.3. *If a flat virtual link L has odd parity and D is an OCD representing L , then D does not admit a filamentation.*

Proof. Since L has odd parity, it follows that the total number of chords between distinct components of D is odd. Then no pairing exists for D since two such chords must be paired together to form an acceptable chord pair. Hence D cannot admit a filamentation. ■

Example 1.5.4. Fix $n \in \mathbb{N}$ and let \mathcal{P} be a pairing for the OCD depicted in Figure 1.16. Then \mathcal{P} must contain the type T pair (e, f) , for neither e nor f can be paired with any other chord. Notice $\langle\langle e, f \rangle\rangle = 2n \neq 0$, so this OCD does not admit a filamentation. Hence any virtual link L with universe represented by this OCD is non-classical. However, the parity invariant is unable to detect this fact. Each component of L is itself a non-classical knot, as shown in [HK02].

Example 1.5.5. The flat virtual link depicted in Figure 1.17 was given in [HK02] as an example of a non-trivial flat virtual link that can be detected by a certain specialization of the Alexander Biquandle. Note that the parity invariant fails at this task. However, since its related OCD does not admit any pairings, the filamentation invariant succeeds in detecting that this flat virtual link is indeed linked.

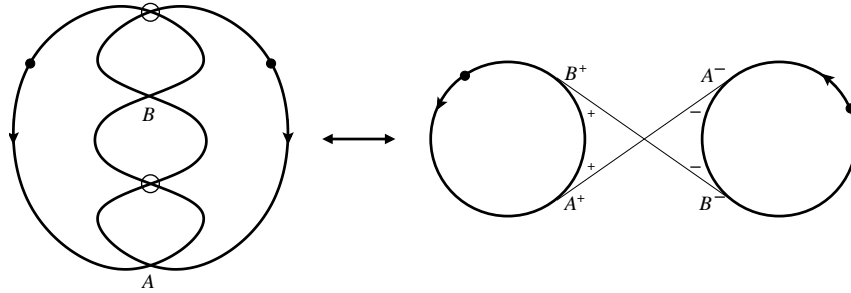


Figure 1.17: A non-trivial flat virtual link with two trivial components and its related OCD.

1.6 Open Questions

- Filament techniques on single-circle diagrams are related to intersection curves on generic immersions of 2-dimensional discs in 3-manifolds. J.S. Carter discussed this relation in a series of papers that were published in the early 1990's, see for example [Car91b]. What are filament techniques on multiple-circle diagrams related to in terms of immersions of surfaces?

Chapter 2

Signed Gauss Words and MIN-Squares

2.1 Introduction

In Chapter 1 we described how an oriented chord diagram (OCD) can be associated to a flat virtual link diagram. In this chapter we will restrict our attention to flat virtual knot diagrams, i.e. flat virtual link diagrams with only one component, and therefore to OCD's with only one circle.

Let K be a flat virtual knot diagram, with its real crossings labeled by the elements in some finite set E . The set E will be called an *alphabet set* and the elements of E will be called its *letters*. If we assign an orientation to K and traverse K according to this orientation, then as we approach a real crossing we will say that the crossing has sign $+1$ if the arc crossing our path is oriented from right to left and sign -1 if this arc is oriented from left to right. Pick a base point on K that is not a crossing of K . A signed Gauss word for K in the alphabet E can be formed in the following way. Start at the chosen base point of K and begin to traverse K according to its orientation. When a real crossing is encountered, record the letter of the crossing and assign an exponent $+1$ or -1 to the letter depending on the sign of the crossing as it is approached. Stop traversing K after returning to the basepoint.

The constructed sequence w of letters with exponents is a *signed Gauss word* underlying K . The entries of a signed Gauss word are called its *symbols*. The *inverse* of a given symbol A^ϵ in a signed word is the symbol $A^{-\epsilon}$. Notice that every letter in E occurs twice in the symbols of w , once with a $+1$ exponent and once with a -1 exponent. Also note that any virtual crossings in K are not indicated in w . Now let D be the OCD underlying K . Then the chosen base point on K corresponds to a base point on D . Start at this base point and transverse the circle of D according to its orientation. When the endpoint of a chord is encountered, record the label of the endpoint. Stop traversing the circle after returning to the basepoint. The entries of the constructed sequence are letters with exponents that are $+$ and $-$ signs. Replacing the $+$ exponents in this sequence with $+1$ and the $-$ exponents with -1 gives the signed Gauss word w . Hence w is considered to be an underlying signed Gauss word of both K and D .

A signed Gauss word underlying a flat virtual knot diagram is not unique. The signed Gauss word obtained from the above procedure depends on the choice of labels for the crossings of the diagram, the choice of a base point, the orientation assigned to the diagram, and the sign convention used on the crossings. Specifically, the effects of these factors on the signed Gauss word are the choice of an alphabet set, the choice of the beginning symbol, writing the word forwards or backwards, and inverting every symbol. The equivalence classes of signed Gauss words under these modifications are often called “signed Gauss words” also, but we will not impose this convention in this chapter.

Example 2.1.1. A flat virtual knot diagram K and its underlying OCD are depicted in Figure 2.1. A signed Gauss word underlying these diagrams is $A^{+1}B^{+1}C^{+1}D^{-1}B^{-1}D^{+1}A^{-1}C^{-1}$, which was constructed by starting at the indicated base points and traversing the diagrams according to the orientations indicated. If the letters B and C on the crossings of K are interchanged, then a signed Gauss word underlying K is $A^{+1}C^{+1}B^{+1}D^{-1}C^{-1}D^{+1}A^{-1}B^{-1}$.

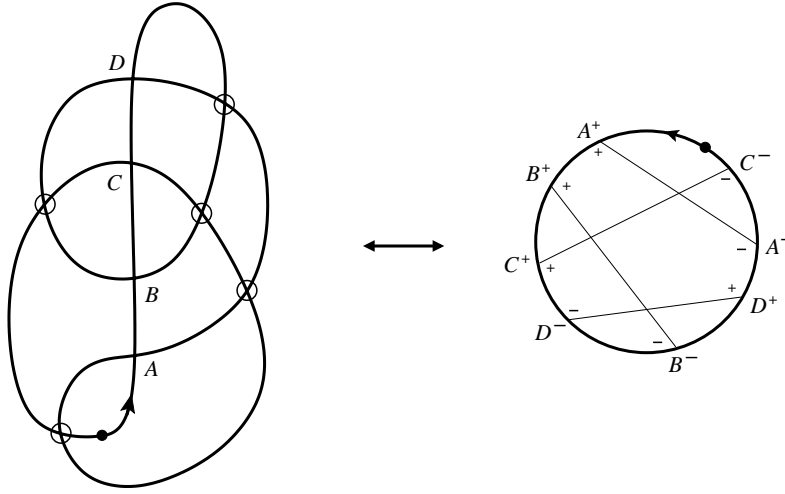


Figure 2.1: A flat virtual knot diagram and its underlying OCD.

Picking a different base point on K gives that $C^{+1}D^{-1}B^{-1}D^{+1}A^{-1}C^{-1}A^{+1}B^{+1}$ also underlies K . If the indicated orientation of K is reversed, then $C^{-1}A^{-1}D^{+1}B^{-1}D^{-1}C^{+1}B^{+1}A^{+1}$ underlies K . If we switch the convention used to determine the sign of a crossing, then $A^{-1}B^{-1}C^{-1}D^{+1}B^{+1}D^{-1}A^{+1}C^{+1}$ would be a signed Gauss word underlying K . Note that all of these signed Gauss words represent the same equivalence class.

2.2 Gauss Word Shapes

A signed Gauss word has an underlying *sign sequence* given by replacing each symbol in the word by the sign of its exponent. Note that if a signed Gauss word has length $2n$ then its underlying sign sequence has length $2n$ with n plus signs and n minus signs. The signed Gauss words underlying a given flat virtual link diagram vary according to the choice of an alphabet set, the choice of the beginning symbol, writing the word forwards or backwards, and the negation of every exponent. Hence we will consider classes of sign sequences under analogous modifications, specifically the choice of the beginning sign, writing the sequence forwards or backwards, and the negation of every sign. The *shape* of a signed Gauss word

is the equivalence class of its underlying sign sequence under these modifications. Observe that all signed Gauss words in the same equivalence class have the same shape by definition, so two signed Gauss words of different shapes cannot be related via the word modifications discussed above.

Example 2.2.1. The signed Gauss word $A^{+1}B^{+1}C^{+1}D^{-1}B^{-1}D^{+1}A^{-1}C^{-1}$ has underlying sign sequence $+++--+-$. The shape of this signed Gauss word can therefore be represented by the sequence $+++--+-$ as well as the sequences $+- - + - - ++$, $--+--++$, and $---++-++$ for example.

The equivalence classes of Gauss word shapes can be counted using a method developed by Hoskins and Street in [HS82]. They mathematically analyze the weaving of “twill” fabric on a loom with a given number of harnesses. The result in their paper that can be applied here shows the total number of equivalence classes of Gauss word shapes of length $2n$ is 1, 2, 3, 7, 13, 35, 85, 257, and 765 for n equal to 1, 2, 3, 4, 5, 6, 7, 8, and 9, respectively.

2.3 MIN-Squares

Definition 2.3.1. Let w be a signed Gauss word of length $2n$, and let S be an oriented circle with $2n$ distinguished points. Label the distinguished points of S with the symbols in w so that w can be obtained by recording in order the symbols encountered while traversing once around S according to its orientation. For each of the n pairs of symbols A^{-1}, A^{+1} in w , attach an arrow to S with its tail on the point labeled A^{-1} and its head on the point labeled A^{+1} . The union of the oriented circle S and its attached arrows is called a *monofilament diagram* for the signed Gauss word w .

Example 2.3.2. Figure 2.2 depicts a monofilament diagram for the signed Gauss word $A^{+1}B^{+1}C^{+1}D^{-1}B^{-1}D^{+1}A^{-1}C^{-1}$.

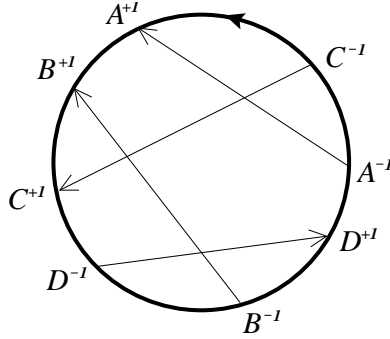


Figure 2.2: A monofilament diagram for the signed Gauss word $A^{+1}B^{+1}C^{+1}D^{-1}B^{-1}D^{+1}A^{-1}C^{-1}$.

Recall from Chapter 1 that a monofilament is the filament associated to an acceptable chord pair (a, a) where a is a chord in an oriented chord diagram. Suppose D is the OCD associated with a given flat virtual knot diagram. Then a monofilament diagram for a signed Gauss word underlying D can easily be obtained from D by replacing each chord of D with an arrow from its negative endpoint to its positive endpoint. Therefore we will consider the arrows of a monofilament diagram for a signed Gauss word to be monofilaments on an OCD that has the word as an underlying signed Gauss word. This convention allows us to use the definitions that we introduced in the previous chapter. For example, the intersection number of a monofilament in a monofilament diagram is the intersection number of the monofilament as defined in Definition 1.3.5.

Definition 2.3.3. Let $s = s_1s_2 \cdots s_{2n}$ be a sequence of signs that represents some Gauss word shape g . Replace each of the $+$ entries in s with different letters from some alphabet set and equip them with $+1$ exponents to obtain a sequence $s' = s'_1s'_2 \cdots s'_{2n}$ of n symbols of the form A^{+1} and n negative signs. For a letter A in a symbol s'_k of s' and an index j of a negative sign in s' , define $\zeta_{s'}(A, j)$ to be the number of $+$ signs minus the number of $-$ signs that appear in the (possibly empty) subsequence $s_{k+1} \cdots s_{j-1}$ of s if $k < j$, and define $\zeta_{s'}(A, j)$ to be the number of $-$ signs minus the number of $+$ signs that appear in the (possibly empty) subsequence $s_{j+1} \cdots s_{k-1}$ of s if $j < k$. A *MIN-square* for the Gauss word shape g is an

array of integers with the following properties: the rows correspond to the n letters with $+1$ exponents in s' , the columns correspond to the n different indices of the negative signs in s' , and the (i, j) -th entry is $\zeta_{s'}(A, j)$ if the letter corresponding to the i -th row is A . Suppose the sets $\{i_1, i_2, \dots, i_n\}$ and $\{j_1, j_2, \dots, j_n\}$ contain the indices of the entries of s that are $+$ and $-$ signs, respectively, with $i_1 < i_2 < \dots < i_n$ and $j_1 < j_2 < \dots < j_n$. Then the *MIN-square obtained from s* is the MIN-square for the Gauss word shape represented by s with its rows arranged in increasing order from top to bottom with respect to the index set $\{i_1, i_2, \dots, i_n\}$ and its columns arranged in increasing order from left to right with respect to the index set $\{j_1, j_2, \dots, j_n\}$.

Example 2.3.4. If $s = + + + - - + - -$, then

$$\begin{vmatrix} +2 & +1 & +1 & 0 \\ +1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -2 \\ +1 & 0 & 0 & -1 \end{vmatrix}$$

is the MIN-square obtained from s for the Gauss word shape represented by s .

Let w be a signed Gauss word, D be a monofilament diagram for w , and M be the MIN-square for the shape of w obtained from the sign sequence s underlying w . Then for each letter A in the alphabet set of w , the intersection number of the monofilament in D associated to A is the (i, j) -th entry of M if the i -th row of M corresponds to A and the j -th row corresponds to the index of the entry in s associated to A^{-1} in w . The term “MIN-square” is used for the array M because M.I.N. is an abbreviation for monofilament intersection numbers.

Example 2.3.5. Recall that a monofilament diagram for the signed Gauss word $w = A^{+1}B^{+1}C^{+1}D^{-1}B^{-1}D^{+1}A^{-1}C^{-1}$ is depicted in Figure 2.2. Then the MIN-square in Ex-

ample 2.3.4, call it M , is the MIN-square for the shape of w obtained from the sign sequence $s = +++--+-$ underlying w . The first, second, third, and fourth rows of M correspond to A, B, C , and D , respectively. The first, second, third, and fourth columns of M correspond to the fourth, fifth, seventh, and eighth entries of s , respectively, because these entries are negative signs. Since the letter A occurs with an exponent of -1 in the seventh symbol of w , it follows that the monofilament associated to A in the monofilament diagram for w should have intersection number equal to the integer in entry $(1, 3)$ of M . We can easily verify that both of these numbers are $+1$ from the monofilament diagram and M . Similarly, the monofilaments associated to B, C , and D in the monofilament diagram have intersection numbers equal to the integers in entries $(2, 2), (3, 4)$, and $(4, 1)$ of M , respectively.

2.4 The Integer Sequences $Z(w)$ for Signed Gauss Words

Let s be a sequence of signs of length $2n$ that represents some Gauss word shape g . Replace each of the $+$ entries in s with different letters from some alphabet set and equip them with $+1$ exponents to obtain a sequence $s' = s'_1 s'_2 \cdots s'_{2n}$ of n symbols of the form A^{+1} and n negative signs. Replacing the $-$ entries in s' with some combination of the inverse symbols gives a signed Gauss word. There are $n!$ signed Gauss words that can be constructed in this way, all of which have shape g . In fact, a representative for every class of signed Gauss words with shape g can be constructed in this way, although each class may be represented more than once. Therefore $n!$ is an obvious upper bound for the number of distinct signed Gauss words of length $2n$ that have a given Gauss word shape. Let M be the MIN-square for g obtained from s . Pick one entry in each of the n rows of M such that the entries all come from distinct columns. A signed Gauss word can be associated with this selection of entries, namely the signed Gauss word that results from replacing the j -th $-$ entry of s' for each entry (i, j) chosen with the symbol composed of the letter corresponding to row i in M

equipped with a -1 exponent.

Let $w = A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_{2n}^{\varepsilon_{2n}}$ be a signed Gauss word in some alphabet set. Every letter B in the alphabet set occurs twice in w , once with a positive exponent and once with a negative exponent, so suppose $A_i^{\varepsilon_i}$ is B^{+1} and $A_j^{\varepsilon_j}$ is B^{-1} . Define $\zeta_w(B)$ to be the number of $+$ exponents minus the number of $-$ exponents that appear in the symbols of the (possibly empty) subsequence $A_{i+1}^{\varepsilon_{i+1}} \cdots A_{j-1}^{\varepsilon_{j-1}}$ of w if $i < j$, and define $\zeta_w(B)$ to be the number of $-$ exponents minus the number of $+$ exponents that appear in the symbols of the (possibly empty) subsequence $A_{j+1}^{\varepsilon_{j+1}} \cdots A_{i-1}^{\varepsilon_{i-1}}$ of w if $j < i$. Suppose $\{i_1, i_2, \dots, i_n\}$ with $i_1 < i_2 < \cdots < i_n$ is the set of indices of the symbols in w that have positive exponents. Let $Z(w)$ denote the integer sequence $(\zeta_w(A_{i_1}), \zeta_w(A_{i_2}), \dots, \zeta_w(A_{i_n}))$. By definition, the sequence $Z(w)$ includes the n integers appearing in the n entries associated to w in the MIN-square obtained from the sign sequence underlying w .

The integer sequence $Z(w)$ for a signed Gauss word w has the property that the sum of its entries is zero. For two letters A, B that occur in w , the symbol B^{ε_2} in w contributes ± 1 to $\zeta_w(A)$ if and only if w has the form $\cdots A^{\varepsilon_1} \cdots B^{\varepsilon_2} \cdots A^{-\varepsilon_1} \cdots B^{-\varepsilon_2} \cdots$. If $\varepsilon_1 = +1$ and $\varepsilon_2 = +1$, then B^{+1} contributes $+1$ to $\zeta_w(A)$ and A^{-1} contributes -1 to $\zeta_w(B)$. If $\varepsilon_1 = +1$ and $\varepsilon_2 = -1$, then B^{-1} contributes -1 to $\zeta_w(A)$ and A^{-1} contributes $+1$ to $\zeta_w(B)$. If $\varepsilon_1 = -1$ and $\varepsilon_2 = +1$, then B^{+1} contributes -1 to $\zeta_w(A)$ and A^{+1} contributes $+1$ to $\zeta_w(B)$. If $\varepsilon_1 = -1$ and $\varepsilon_2 = -1$, then B^{-1} contributes $+1$ to $\zeta_w(A)$ and A^{+1} contributes -1 to $\zeta_w(B)$. Hence the overall contribution to the sum of the entries of $Z(w)$ is zero in each of these four cases.

Let $w = A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_{2n}^{\varepsilon_{2n}}$ be a signed Gauss word in some alphabet set. We want to summarize how the integer sequence $Z(w)$ is affected by the choice of a different alphabet set for w , the choice of a different beginning symbol for w , writing w backwards, and the negation of every exponent in w , but first we need to make some key observations. Suppose the letter B occurs in w in the symbols $A_i^{\varepsilon_i}$ and $A_j^{\varepsilon_j}$ with $\varepsilon_i = +1$, $\varepsilon_j = -1$, and $i < j$. Then

$\zeta_w(B) = \varepsilon_{i+1} + \varepsilon_{i+2} + \cdots + \varepsilon_{j-1}$. Let w' be the signed Gauss word

$$B^{-1}A_{j+1}^{\varepsilon_{j+1}} \cdots A_{2n}^{\varepsilon_{2n}} A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_{i-1}^{\varepsilon_{i-1}} B^{+1} A_{i+1}^{\varepsilon_{i+1}} \cdots A_{j-1}^{\varepsilon_{j-1}}.$$

Then $\zeta_{w'}(B) = -(\varepsilon_{j+1} + \cdots + \varepsilon_{2n} + \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{i-1})$. Since the sum of the integers in $Z(w)$ is zero, it follows that $\zeta_w(B) = \zeta_{w'}(B)$. Notice that we could have began w' with any symbol between B^{+1} and B^{-1} in w and still have $\zeta_w(B) = \zeta_{w'}(B)$. If w'' is the signed Gauss word $A_{2n}^{\varepsilon_{2n}} A_{2n-1}^{\varepsilon_{2n-1}} \cdots A_1^{\varepsilon_1}$, then $\zeta_{w''}(B) = -(\varepsilon_{j-1} + \varepsilon_{j-2} + \cdots + \varepsilon_{i+1}) = -\zeta_w(B)$. If w''' is the signed Gauss word $A_1^{-\varepsilon_1} A_2^{-\varepsilon_2} \cdots A_{2n}^{-\varepsilon_{2n}}$, then $\zeta_{w'''}(B) = -(-\varepsilon_{i+1} - \varepsilon_{i+2} - \cdots - \varepsilon_{j-1}) = \zeta_w(B)$. Similar results can be deduced when $j < i$. Based on these observations, we can conclude that the word modifications may rearrange the order of the integers in $Z(w)$ and possibly negate them all.

Example 2.4.1. As described in Example 2.1.1, the signed Gauss words

$$\begin{aligned} w_1 &= A^{+1}B^{+1}C^{+1}D^{-1}B^{-1}D^{+1}A^{-1}C^{-1}, \\ w_2 &= A^{+1}C^{+1}B^{+1}D^{-1}C^{-1}D^{+1}A^{-1}B^{-1}, \\ w_3 &= C^{+1}D^{-1}B^{-1}D^{+1}A^{-1}C^{-1}A^{+1}B^{+1}, \\ w_4 &= C^{-1}A^{-1}D^{+1}B^{-1}D^{-1}C^{+1}B^{+1}A^{+1}, \text{ and} \\ w_5 &= A^{-1}B^{-1}C^{-1}D^{+1}B^{+1}D^{-1}A^{+1}C^{+1} \end{aligned}$$

all represent the same equivalence class of signed Gauss words. Observe that

$$Z(w_1) = (+1, 0, -2, +1),$$

$$Z(w_2) = (+1, 0, -2, +1),$$

$$Z(w_3) = (-2, +1, +1, 0),$$

$$Z(w_4) = (-1, +2, 0, -1), \text{ and}$$

$$Z(w_5) = (+1, 0, +1, -2).$$

If w_6 is a signed Gauss word such that $Z(w_6) = (+3, +1, -1, -3)$, then we know that w_6 is not in the same equivalence class as w_1 because $Z(w_6)$ cannot be obtained from $Z(w_1)$ by rearranging its order and/or negating all of its entries.

Let M be the MIN-square obtained from a sequence of signs that represents a given Gauss word shape. As mentioned above, if we pick one entry in each of the rows of M such that the entries all come from distinct columns, then we can associate a signed Gauss word w with this selection of entries and the integers appearing in these entries will be the integers in $Z(w)$. Consider the set of integer sequences $Z(w)$ for all the signed Gauss words that can be constructed in this manner from M . If we consider these sequences modulo reordering and the negation of all entries, then the number of inequivalent sequences in this set is a lower bound for the number of equivalence classes of signed Gauss words with the given Gauss word shape.

2.5 Filamentations Revisited

Given a Gauss word shape, consider the collection of all OCD's that have an underlying signed Gauss word of that shape. For every oriented chord diagram D in this collection, a MIN-square M for the given shape encodes the intersection numbers of the monofilaments

on D . In other words, M encodes the intersection numbers of the chord pairs the pairing for D that contains chord pairs of the form (a, a) for each chord a in D . In this section, we will show that M actually contains complete information about the intersection numbers associated to every pairing for D .

Let D be an OCD with a lone circle and let \mathcal{P} be a pairing for D . If a and b are chords of D , then (a, b) is an acceptable chord pair, so we can consider the filaments α and its dual α' associated with (a, b) . Suppose the initial and terminal points of α on D are A^{-1} and B^{+1} , respectively. Then the initial and terminal points of α' on D are B^{-1} and A^{+1} , respectively. Let w be a signed Gauss word underlying D and let MD be a monofilament diagram for w . A diagram showing the filaments associated to the pairs in \mathcal{P} can be constructed from MD in the following way. There will be no intersections of the monofilaments in MD in some neighborhood of its circle. In this neighborhood, if $a \neq b$, construct an H-like union of arcs by introducing a new arc with a minimal number of intersections that has one endpoint on the interior of the short arc emanating from A^{-1} and one endpoint on the interior of the short arc emanating from B^{-1} . We will denote the union of these three arcs by $A \vdash B$ and refer to such a union as an *H-union*. The new arc in $A \vdash B$ will be called its *middle* arc and the arcs emanating from A^{-1} and B^{-1} in $A \vdash B$ will be called its *end* arcs. Now in a small neighborhood of the middle arc of the H-union $A \vdash B$, replace the middle arc with two arcs that cross. These two arcs will have opposite orientations induced by the end arcs of $A \vdash B$. The resulting long arcs from A^{-1} to B^{+1} and B^{-1} to A^{+1} represent α and α' , respectively. If $a = b$, then α is a monofilament and is already represented in MD . Repeat this procedure for every chord pair in \mathcal{P} , and call the resulting diagram FD .

The above construction of FD from MD can be performed in two steps. First, for every chord pair (a, b) in \mathcal{P} with $a \neq b$, in a neighborhood of the circle of MD construct an H-union $A \vdash B$. Call the resulting diagram HD . Second, replace the middle arc of every H-union with two arcs that cross. Notice that the first step can be performed so that there are no

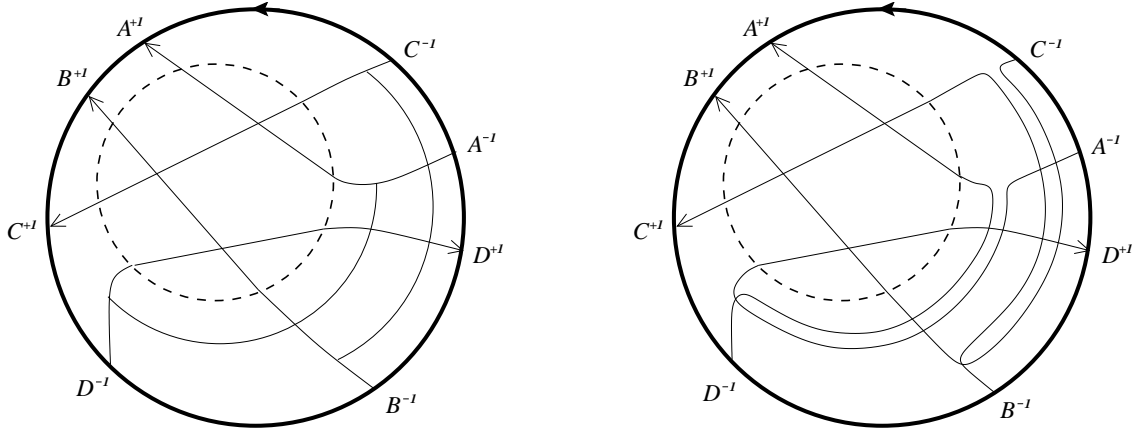


Figure 2.3: A diagram with H-unions and a diagram with the resulting filaments.

intersections between the middle arcs of distinct H-unions in HD .

Example 2.5.1. Figure 2.3 contains a diagram depicting two H-unions $A \vdash D$ and $B \vdash C$ as well as a diagram showing the resulting filaments for the pairing $\{(a, d), (b, c)\}$.

We claim that the intersection number of the chord pair (a, b) in \mathcal{P} can be determined from the intersection numbers of the monofilaments associated to a and b in MD . The filaments α and α' in FD have the same intersections as the monofilaments associated to a and b in MD , respectively, except in some neighborhood N of the circle in FD . We will investigate the intersections in N by considering an analogous neighborhood N' in HD .

If $a = b$, then an intersection of the monofilament α in N' must be between α and the middle arc of an H-union $U \vdash V$. To construct FD , the middle arc of $U \vdash V$ is replaced with two arcs, each of which intersects α exactly once in a neighborhood of the original intersection. Since the two replacement arcs have opposite orientation, it follows that these two intersections in N contribute $+1 - 1 = 0$ to the intersection number of α and to the intersection number of chord pair (u, v) in \mathcal{P} associated to $U \vdash V$.

Now suppose $a \neq b$ so that we can investigate the intersections of the H-union $A \vdash B$ in N' . If $A \vdash B$ intersects a monofilament in N' , then the above argument shows that there are no contributions made to the intersection numbers of any chord pairs in \mathcal{P} . Since there

are no intersections between the middle arcs of distinct H-unions in HD , it follows that an intersection between $A \vdash B$ and another H-union $U \vdash V$ in N' must be an intersection between an end arc of one and the middle arc of the other. If an end arc of $A \vdash B$ intersects the middle arc of $U \vdash V$ in N' , then the intersection corresponds to two intersections in N . Specifically, the two arcs that replace the middle arc of $U \vdash V$ each intersect either α or α' exactly once in a neighborhood of the original intersection. Therefore these two intersections in N do not contribute to the intersection number of (a, b) nor to the intersection number of the pair (u, v) in \mathcal{P} associated with $U \vdash V$. If the middle arc of $A \vdash B$ intersects an end arc of $U \vdash V$ in N' , then interchanging the roles of $A \vdash B$ and $U \vdash V$ in the previous argument shows that again there are no contributions made to the intersection numbers of any chord pairs in \mathcal{P} .

We have considered all possible intersections in the neighborhood N that may contribute to the intersection number of the chord pair (a, b) when $a \neq b$ except one. Specifically, we have yet to consider the intersection between α and α' that results from replacing the middle arc of $A \vdash B$ in HD with two arcs. However, if this intersection contributes ε to the intersection number of α , then it contributes $-\varepsilon$ to the intersection number of α' . So this intersection also does not effect the intersection number of (a, b) .

We have proved that none of the intersections in the neighborhood N in FD contribute to the intersection numbers of the pairs in \mathcal{P} . Therefore the intersection number of the chord pair (a, b) in \mathcal{P} is simply the sum of the intersection numbers of the monofilaments associated to a and b in MD . If w is a Gauss word underlying the oriented chord diagram D , then the integers in the sequence $Z(w)$ are the intersection numbers of the monofilaments in MD . Thus the intersection number of the chord pair (a, b) in \mathcal{P} is the sum of the integers $\zeta_w(A)$ and $\zeta_w(B)$ in $Z(w)$. Notice that the integer sequence $Z(w)$ encodes information about every possible pairing for the oriented chord diagram D .

Example 2.5.2. Let w be the signed Gauss word $A^+B^+C^+D^-B^-D^+A^-C^-$ and suppose w underlies the oriented chord diagram D , with each chord of D labeled with the lowercase version of the uppercase letters on its endpoints. Then the rightmost diagram in Figure 2.3 depicts the filaments on D associated to the pairing $\{(a, d), (b, c)\}$ for D . Observe from these filaments that the intersection numbers $\langle(a, d)\rangle$ and $\langle(b, c)\rangle$ are $+2$ and -2 , respectively. Figure 2.2 depicts a monofilament diagram for w . In this diagram, the sum of the intersection numbers of the monofilaments associated to a and d is indeed $+1 + 1 = +2$ and the sum of the intersection numbers of the monofilaments associated to b and c is indeed $0 - 2 = -2$. Notice that $Z(w) = (\zeta_w(A), \zeta_w(B), \zeta_w(C), \zeta_w(D)) = (+1, 0, -2, +1)$ with $\langle(a, d)\rangle = \zeta_w(A) + \zeta_w(D) = +2$ and $\langle(b, c)\rangle = \zeta_w(B) + \zeta_w(C) = -2$ as claimed.

Let w be a signed Gauss word and suppose the integer sequence $Z(w)$ is $(\zeta_1, \zeta_2, \dots, \zeta_n)$ for some $\zeta_i \in \mathbb{Z}$. For two indices i, j of the entries in $Z(w)$, call the pair (ζ_i, ζ_j) an *entry pair* in $Z(w)$ and define the *sum* of the entry pair to be $\zeta_i + \zeta_j$ if $i \neq j$ and just ζ_i if $i = j$. A *pairing* for the integer sequence $Z(w)$ is a collection of entry pairs such that each entry in $Z(w)$ occurs in exactly one pair in the collection. A *filamentation* on $Z(w)$ is a pairing for which the sum of each entry pair is zero.

Theorem 2.5.3. *Let D be an OCD with only one circle and suppose w is a signed Gauss word underlying D . Then D admits a filamentation if and only if $Z(w)$ admits a filamentation.*

Proof. Label the chords of D by the letters a_1, a_2, \dots, a_n , with the endpoints of the chord a_i in D labeled A_i^{+1} and A_i^{-1} . Then w has length $2n$ and we may assume that $Z(w) = (\zeta_w(A_1), \zeta_w(A_2), \dots, \zeta_w(A_n))$. As discussed above, the intersection number of the chord pair (a_i, a_j) in any pairing for D is $\zeta_w(A_i) + \zeta_w(A_j)$ if $a_i \neq a_j$ and just $\zeta_w(A_i)$ if $a_i = a_j$. Therefore the intersection number of a chord pair (a_i, a_j) is zero if and only if the sum of the entry pair $(\zeta_w(A_i), \zeta_w(A_j))$ is zero. ■

Corollary 2.5.4. *Suppose D is an OCD with only one circle and w is a signed Gauss word underlying D . If $Z(w)$ does not admit a filamentation, then the flat virtual knot represented by D is non-trivial. Consequently, any virtual knot with universe represented by D is non-classical.*

Proof. This result follows immediately from theorems 2.5.3 and 1.4.3 and Corollary 1.4.4. ■

2.6 Open Questions

- The MIN-square for a given Gauss word shape summarizes all monofilament information (and therefore all filament information) about every signed Gauss word having the shape. Therefore MIN-squares can be used to distinguish signed Gauss words and determine if flat virtual knots are non-classical. In what other ways can MIN-squares be used?
- What other information about a given signed Gauss word w does the integer sequence $Z(w)$ encode?
- MIN-squares are square arrays of integers, so they can be interpreted as matrices. Can such a matrix structure on MIN-squares be exploited in any sense that is useful?
- Define an equivalence relation on MIN-squares that respects the equivalence relation used to define Gauss word shapes. Explore the induced classes of MIN-squares.
- Develop a formula for the total number of equivalence classes of signed Gauss words of a given length. Is there an efficient algorithm that generates such classes? The algorithm developed by Sawada in [Saw02] for chord diagrams may be helpful.

Chapter 3

Virtual Strings for Closed Curves with Multiple Components

3.1 Introduction

A *word* in a finite alphabet set, i.e. a finite set with elements referred to as letters, is a sequence of letters up to circular permutation and choice of letters. A *Gauss paragraph* in a finite alphabet is a finite set of words in the alphabet such that each letter occurs exactly twice in the words of the set. A Gauss paragraph with only one word is called a *Gauss word* or *Gauss code*. Such words were introduced by C.F. Gauss as a combinatorial formulation of closed curves on \mathbb{R}^2 .

The concept of a virtual string was first introduced by Vladimir Turaev in [Tur03] to describe a single copy of S^1 with distinguished ordered pairs of points, which can be represented as a set of arrows attached to the circle. We extend this definition to allow multiple copies of S^1 . Every virtual string has an underlying Gauss paragraph, with the circles and arrows of the virtual string corresponding to the words and letters of the Gauss paragraph, respectively.

A *closed curve* with $N \in \mathbb{N}$ components on a surface Σ is a generic smooth immersion of N oriented circles into the surface Σ . Every closed curve ρ has an underlying virtual

string, with the components and crossings of the closed curve corresponding to the circles and arrows of the virtual string, respectively. The Gauss paragraph of a closed curve is the Gauss paragraph of its underlying virtual string.

Let ρ be a closed curve on a surface, with crossings labeled by the elements of some finite set E . Pick a base point on each component of ρ that is not a crossing of ρ . Having selected these base points, it is relatively easy to determine the Gauss paragraph p of ρ in the alphabet E . The word of p corresponding to a given component is the sequence of letters formed in the following way: start at the base point of the component and record, in order, the labels of the crossings encountered while traversing the component according to its orientation. A Gauss paragraph p is said to be *realizable* by a closed curve on a surface Σ if there exists some closed curve ρ on the surface Σ whose Gauss paragraph is p .

Many mathematicians have questioned when a given Gauss word is realizable by a closed curve on \mathbb{R}^2 , or equivalently on the 2-sphere S^2 . Four of the most notable studies include [Gau00], [LM76], [Ros76], and [DT83]. In this chapter we address when a Gauss paragraph is realizable by a closed curve on S^2 . We use the theory of virtual strings to obtain a combinatorial description of closed curves on the 2-sphere in terms of Gauss paragraphs and “word-wise partitions”.

This chapter is organized as follows. In Section 3.2, we discuss Gauss paragraphs and define the concept of a word-wise partition. Section 3.3 contains the definitions for several maps that are related to Gauss paragraphs. We define virtual strings on multiple copies of S^1 in Section 3.4, and explain how virtual strings have underlying Gauss paragraphs. In Section 3.5, we discuss closed curves on surfaces, explain how closed curves have underlying virtual strings, and describe how to construct a surface containing a closed curve realizing a given virtual string. In Section 3.6, we observe that a given virtual string may or may not give rise to a word-wise partition, and explain why it is acceptable to restrict our attention to those virtual strings that do give rise to word-wise partitions. We discuss a homological

intersection pairing in Section 3.7 and relate the pairing to the maps defined in Section 3.3. Section 3.8 contains a statement of the main theorem and the proofs of several lemmas. These lemmas are used to prove the main theorem in Section 3.9. Section 3.10 contains some additional results about Gauss paragraphs.

3.2 Gauss Paragraphs

Let us impose the condition that it is not possible to partition the words of a Gauss paragraph into two sets such that the words in one set have no letters in common with the words of the other. This condition assures that a Gauss paragraph is not a “disjoint union” of other paragraphs. A letter in a word w of a Gauss paragraph is called a *single letter* of w if it occurs once in w , and a *double letter* of w if it occurs twice in w . Given a set A , denote its cardinality by $\#(A)$. Then the *length* of a word w of a Gauss paragraph is defined as

$$2(\#\{\text{double letters in } w\}) + \#\{\text{single letters in } w\}.$$

Although a Gauss paragraph is by definition a finite set of words, we will write a Gauss paragraph as a finite sequence of words when it is necessary to reference the words individually. This convention is reasonable because we will not discuss any notion of equivalence amongst Gauss paragraphs.

Throughout the remainder of this section, let p be a Gauss paragraph in an alphabet E .

Definition 3.2.1. A *word-wise partition* P of E with respect to p is a partition of the letters of E that satisfies the following conditions:

- (i) P associates two disjoint, possibly empty subsets A and A' to every word w in p ;

(ii) for each word w in p , the union

$$(A \cap \{\text{double letters of } w\}) \cup (A' \cap \{\text{double letters of } w\})$$

gives a bipartition of the set of double letters of w ; and

(iii) if a word w of p has $2n$ letters in common with another word w' of p , then n of these letters appear in the two sets in P associated to w and n of them appear in the two sets in P associated to w' .

In practice, if p is written as a sequence (v_1, v_2, \dots, v_N) of words, then we will express a word-wise partition P using the notation $(A_1 \cup A'_1, A_2 \cup A'_2, \dots, A_N \cup A'_N)$, where the sets A_n and A'_n are associated to the word v_n . Notice that a word-wise partition is a partition of E such that its sets satisfy the above conditions with respect to the words of p . However, it does not partition the letters in each word of p unless p has only one word, and in this case it is a bipartition of the letters in the word (it is the bipartition Turaev defined in [Tur03]).

When we use the terms “sequence” and “subsequence”, we mean finite sequences of letters that are not considered up to circular permutation. Recall, however, that words are considered up to circular permutation. Given a word w , if we write $w = i \cdots j$ for some sequence of letters $i \cdots j$, we mean that w can be written in the form $i \cdots j$ up to circular permutation. A finite sequence x_1 of letters is called a *subsequence* of a word w if $w = x_1 x_2$ for some sequence x_2 of letters. Consequently, x_2 is also a subsequence of w . The *length* of a sequence is the number of entries in the sequence, and the length of a sequence s will be denoted by $\ell(s)$. We will use the symbol \emptyset to denote an empty sequence and define $\ell(\emptyset) = 0$.

Definition 3.2.2. Let $i \in E$ be a double letter of a word w of p . The two p -sets of i , denoted p_i and p'_i , are the two sets of letters in w defined as follows. Since w is considered up to circular permutation, assume it has the form ix_1ix_2 , with x_1 and x_2 being subsequences

(possibly empty) of the word w . The two p -sets are the set of letters occurring exactly once in x_1 and the set of letters occurring exactly once in x_2 .

Note that $p_i = p'_i$ if p has only one word. Turaev denotes this set by w_i in [Tur03], and uses the concept of interlacing to define w_i . In general, two distinct letters $i, j \in E$ in a word w of a Gauss paragraph are called w -interlaced if w has the form $i \cdots j \cdots i \cdots j \cdots$ up to circular permutation. Then w_i is the set of letters w -interlaced with i .

Definition 3.2.3. A word-wise partition P of E (with respect to p) is *compatible* with p if the following two conditions are satisfied. Suppose $i, j \in E$ are w -interlaced in a word w of p . Then $w = ix_1jx_2ix_3jx_4$ for some (possibly empty) subsequences x_1, x_2, x_3, x_4 of w . The first condition is that for any two such letters i and j ,

$$\begin{aligned} & \#(w_i \cap w_j) + \#\{\text{single letters of } w \text{ in } x_1\} \\ & \equiv \#(w_i \cap w_j) + \#\{\text{single letters of } w \text{ in } x_3\} \pmod{2} \\ & \equiv \begin{cases} 0 \pmod{2} & \text{if } i, j \text{ appear in different subsets of } P; \\ 1 \pmod{2} & \text{if } i, j \text{ appear in the same subset of } P. \end{cases} \end{aligned}$$

Suppose $i, j \in E$ are single letters in a word w of p that appear in the union of the two subsets associated to w in P . Then $w = ix_1jx_2$ for some (possibly empty) subsequences x_1, x_2 of w . The second condition is that for any two such letters i and j ,

$$\begin{aligned} & \ell(x_1) \equiv \ell(x_2) \pmod{2} \\ & \equiv \begin{cases} 0 \pmod{2} & \text{if } i, j \text{ appear in different subsets of } P; \\ 1 \pmod{2} & \text{if } i, j \text{ appear in the same subset of } P. \end{cases} \end{aligned}$$

Example 3.2.4. Consider the Gauss paragraph $p = (127351263488, 4567)$ in the alphabet $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$. The partition $P = (\{1, 2, 7\} \cup \{3, 6, 8\}, \{4\} \cup \{5\})$ of E is a word-

wise partition of E with respect to p . Let w denote the first word of p , and notice that 1, 2, and 3 are double letters of w . Then the p -sets of 1 are $\{2, 3, 5, 7\}$ and $\{2, 3, 4, 6\}$, the p -sets of 2 are $\{1, 3, 5, 7\}$ and $\{1, 3, 4, 6\}$, and the p -sets of 3 are $\{1, 2, 5, 6\}$ and $\{1, 2, 4, 7\}$. Moreover $w_1 = \{2, 3\}$, $w_2 = \{1, 3\}$, and $w_3 = \{1, 2\}$ with

$$\#(w_1 \cap w_2) \equiv \#(w_1 \cap w_3) \equiv \#(w_2 \cap w_3) \equiv 1 \pmod{2}.$$

Therefore since

$$\begin{aligned} \#\{\text{single letters of } w \text{ in } 735\} &\equiv \#\{\text{single letters of } w \text{ in } 63488\} \equiv 0 \pmod{2}, \\ \#\{\text{single letters of } w \text{ in } 27\} &\equiv \#\{\text{single letters of } w \text{ in } 5\} \\ &\equiv \#\{\text{single letters of } w \text{ in } 26\} \equiv \#\{\text{single letters of } w \text{ in } 488\} \equiv 1 \pmod{2}, \\ \#\{\text{single letters of } w \text{ in } 7\} &\equiv \#\{\text{single letters of } w \text{ in } 51\} \\ &\equiv \#\{\text{single letters of } w \text{ in } 6\} \equiv \#\{\text{single letters of } w \text{ in } 4881\} \equiv 1 \pmod{2}, \\ \ell(3512) &\equiv \ell(348812) \equiv 0 \pmod{2}, \text{ and} \\ \ell(67) &\equiv \ell(\emptyset) \equiv 0 \pmod{2}, \end{aligned}$$

it follows that the word-wise partition P is compatible with p .

3.3 Maps Related to Gauss Paragraphs

Throughout this section, let $p = (v_1, v_2, \dots, v_N)$ be a Gauss paragraph in an alphabet set E with word-wise partition $P = (A_1 \cup A'_1, A_2 \cup A'_2, \dots, A_N \cup A'_N)$.

Notation. Given a sequence of letters $i \cdots j$, let $o(i \cdots j)$ denote the letters that occur exactly once between i and j . Given a word w of a Gauss paragraph, let $o(w)$ denote the single letters in w .

Definition 3.3.1. Let $a \cdots b$ and $y \cdots z$ be subsequences of a word v_n of p , where y and z are distinct single letters of v_n . Set

$$\delta_n(a \cdots b, \emptyset) = \delta_n(\emptyset, y \cdots z) = 0.$$

Define $\delta_n(a \cdots b, y \cdots z)$ as follows:

(i) if $v_n = a \cdots y \cdots b \cdots z \cdots$, set it equal to

$$\#(o(a \cdots y) \cap o(y \cdots b)) + \#(o(a \cdots y) \cap o(b \cdots z)) + \#(o(y \cdots b) \cap o(b \cdots z))$$

(ii) if $v_n = a \cdots z \cdots b \cdots y \cdots$, set it equal to

$$\#(o(a \cdots z) \cap o(y \cdots a)) + \#(o(z \cdots b) \cap o(y \cdots a)) + \#(o(z \cdots b) \cap o(a \cdots z))$$

(iii) if $v_n = a \cdots y \cdots z \cdots b \cdots$, set it equal to

$$\#(o(a \cdots y) \cap o(y \cdots z)) + \#(o(z \cdots b) \cap o(y \cdots z))$$

(iv) if $v_n = a \cdots z \cdots y \cdots b \cdots$, set it equal to

$$\begin{aligned} &\#(o(a \cdots z) \cap o(b \cdots a)) + \#(o(z \cdots y) \cap o(y \cdots b)) + \#(o(z \cdots y) \cap o(b \cdots a)) \\ &+ \#(o(z \cdots y) \cap o(a \cdots z)) + \#(o(y \cdots b) \cap o(b \cdots a)) \end{aligned}$$

(v) if $v_n = a \cdots b \cdots y \cdots z \cdots$, set it equal to

$$\#(o(a \cdots b) \cap o(y \cdots z))$$

(vi) if $v_n = a \cdots b \cdots z \cdots y \cdots$, set it equal to

$$\#(o(a \cdots b) \cap o(y \cdots a)) + \#(o(a \cdots b) \cap o(b \cdots z))$$

Although $y \neq z$ by hypothesis, no other pairwise distinct requirements are enforced in this definition. For example, if $a = y$ and $b = z$, then the first, second, third, and sixth cases are the same. The definition of δ_n is consistent under all such circumstances.

Definition 3.3.2. Let ax_1b and yx_2z be two subsequences of v_n , where y and z are distinct single letters of v_n and x_1, x_2 are subsequences of v_n . Set

$$\epsilon_n^P(ax_1b, \emptyset) = \epsilon_n^P(\emptyset, yx_2z) = 0.$$

If $a = b$ is a double letter of v_n , define $\epsilon_n^P(ax_1b, yx_2z)$ to be 1 if an odd number of the following statements is true and 0 otherwise:

- (i) $a = b$ occurs exactly once in x_2
- (ii) y occurs in x_1 and $y \notin A_n \cup A'_n$
- (iii) z occurs in x_1 and $z \in A_n \cup A'_n$

If a and b are distinct single letters of v_n , define $\epsilon_n^P(ax_1b, yx_2z)$ to be 1 if an odd number of the following statements is true and 0 otherwise:

- (i) a occurs in x_2 and $a \in A_n \cup A'_n$
- (ii) b occurs in x_2 and $b \notin A_n \cup A'_n$
- (iii) y occurs in x_1 and $y \notin A_n \cup A'_n$
- (iv) z occurs in x_1 and $z \in A_n \cup A'_n$

Although $y \neq z$ by hypothesis, no other pairwise distinct requirements are enforced in this definition.

Definition 3.3.3. Let $X = (x_1, x_2, \dots, x_N)$ be a sequence consisting of subsequences of the words in p , where x_n is a subsequence of the word v_n and more than one x_n is nonempty. Let X' be the set containing the entries of X that are nonempty subsequences, and define $M = \#(X') \geq 2$. The set X is called a *cyclic sequence* associated to p if a sequence $(x'_1, x'_2, \dots, x'_M)$ including all elements in X' can be constructed such that the first letter of x'_1 is the last letter of x'_M and the first letter of x'_m is the last letter of x'_{m-1} for $2 \leq m \leq M$. Note that the first and last letters of each subsequence in a cyclic sequence are single letters of the same word. Let \mathcal{D}_p denote the collection of all cyclic sequences associated to p .

Suppose $d = (x_1, x_2, \dots, x_N) \in \mathcal{D}_p$. Denote the sequence $x_n \in d$ by $d(n)$. Then $o(d(n))$ is the set of letters in $d(n) = x_n$ that occur exactly once between the first and last letters if it is a nonempty sequence, and the empty set otherwise. We will now define a collection of maps that involve \mathcal{D}_p . These maps will eventually be related to a homological intersection form.

Definition 3.3.4. Suppose i, j are distinct single letters in a word v_n of p . Define $\gamma_n^P(i, j)$ to be 0 if exactly one of the letters i, j appears in $A_n \cup A'_n$, and 1 if both or neither of these letters appears in $A_n \cup A'_n$.

Definition 3.3.5. The map $W : p \times \mathcal{D}_p \rightarrow \mathbb{N} \cup \{0\}$ is defined by

$$W(v_n, d) = \sum_{k \neq n} \#(o(v_n) \cap o(d(k)))$$

if $d(n)$ is an empty sequence, and

$$W(v_n, d) = \#(o(ix_1j) \cap o(jx_2i)) + \gamma_n^P(i, j) + \sum_{k \neq n} \#(o(v_n) \cap o(d(k)))$$

if $d(n)$ is subsequence ix_1j of $v_n = ix_1jx_2$ where i, j are distinct single letters of v_n and x_1, x_2 are subsequences of v_n .

Definition 3.3.6. Suppose $i \in E$ is a double letter in word v_n of p . Then v_n can be written as ix_1ix_2 and ix_2ix_1 for some subsequences x_1 and x_2 , but we may assume the p -sets p_i and p'_i contain the letters that occur exactly once in x_1 and x_2 , respectively. Then for $d \in \mathcal{D}_p$, set

$$Q_n(p_i, d) = \delta_n(ix_1i, d(n)) + \epsilon_n^P(ix_1i, d(n)) + \sum_{k \neq n} \#(p_i \cap o(d(k)))$$

and

$$Q_n(p'_i, d) = \delta_n(ix_2i, d(n)) + \epsilon_n^P(ix_2i, d(n)) + \sum_{k \neq n} \#(p'_i \cap o(d(k))).$$

Definition 3.3.7. Define a map $D_n : \mathcal{D}_p \times \mathcal{D}_p \rightarrow \mathbb{N} \cup \{0\}$ as follows. For $d_1, d_2 \in \mathcal{D}_p$, define

$$D_n(d_1, d_2) = \delta_n(d_1(n), d_2(n)) + \epsilon_n^P(d_1(n), d_2(n)) + \sum_{k \neq n} \#(o(d_1(n)) \cap o(d_2(k))).$$

Definition 3.3.8. The word-wise partition P is *compatible* with \mathcal{D}_p if the following conditions are satisfied:

- (i) for every word v_n of p , $W(v_n, d) \equiv 0 \pmod{2}$ for all $d \in \mathcal{D}_p$;
- (ii) if $i \in E$ is a double letter of word v_n of p , then $Q_n(p_i, d) \equiv Q_n(p'_i, d) \equiv 0 \pmod{2}$ for all $d \in \mathcal{D}_p$; and
- (iii) $\sum_{n=1}^N D_n(d_1, d_2) \equiv 0 \pmod{2}$ for all $d_1, d_2 \in \mathcal{D}_p$.

Example 3.3.9. Consider the Gauss paragraph $p = (v_1, v_2, v_3)$ in the alphabet $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \bar{1}, \bar{2}, \bar{3}\}$, where word $v_1 = 1273512634$, word $v_2 = \bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{3}945678$, and word $v_3 = 89$. The partition $P = (\{1, 2, 7\} \cup \{3, 6\}, \{\bar{1}, \bar{2}, \bar{3}, 5\} \cup \{4, 8\}, \{9\} \cup \emptyset)$ of E is a word-wise partition of E with respect to p , where the symbol \emptyset denotes the empty set. We

claim that P is compatible with \mathcal{D}_p . Although we will not include all of the calculations necessary to justify this claim, we do include several here to illustrate the above definitions. Two cyclic sequences in \mathcal{D}_p are $d_1 = (73512634, 4567, \emptyset)$ and $d_2 = (\emptyset, 945678, 89)$. To verify condition (i) of Definition 3.3.8, we must for example calculate that

$$\begin{aligned} W(v_1, d_2) &= \#(o(v_1) \cap o(945678)) + \#(o(v_1) \cap o(89)) = 4 + 0 \equiv 0 \pmod{2} \text{ and} \\ W(v_1, d_1) &= \#(o(73512634) \cap o(4127)) + \gamma_1^P(7, 4) + \#(o(v_1) \cap o(4567)) + \#(o(v_1) \cap \emptyset) \\ &= 2 + 0 + 2 + 0 \equiv 0 \pmod{2}. \end{aligned}$$

If we assume that $p_3 = \{5, 1, 2, 6\}$ and $p'_3 = \{4, 1, 2, 7\}$, then to verify condition (ii) of Definition 3.3.8, we must for example calculate that

$$\begin{aligned} Q_1(p_3, d_1) &= \delta_1(351263, 73512634) + \epsilon_1^P(351263, 73512634) + \#(p_3 \cap o(4567)) + \#(p_3 \cap \emptyset) \\ &= \#(o(351263) \cap o(73)) + \#(o(351263) \cap o(34)) + 0 + 2 + 0 \equiv 0 \pmod{2} \text{ and} \\ Q_1(p'_3, d_1) &= \delta_1(341273, 73512634) + \epsilon_1^P(341273, 73512634) + \#(p'_3 \cap o(4567)) + \#(p'_3 \cap \emptyset) \\ &= \#(o(34) \cap o(351263)) + \#(o(4127) \cap o(73)) + \#(o(4127) \cap o(351263)) \\ &\quad + \#(o(4127) \cap o(34)) + \#(o(73) \cap o(351263)) + 0 + 0 + 0 \equiv 0 \pmod{2}. \end{aligned}$$

To verify condition (iii) of Definition 3.3.8, we must for example calculate that

$$\begin{aligned} D_1(d_1, d_2) &= \delta_1(73512634, \emptyset) + \epsilon_1^P(73512634, \emptyset) + \#(o(d_1(1)) \cap o(945678)) \\ &\quad + \#(o(d_1(1)) \cap o(89)) = 0 + 0 + 2 + 0 = 2, \\ D_2(d_1, d_2) &= \delta_2(4567, 945678) + \epsilon_2^P(4567, 945678) + \#(o(d_1(2)) \cap \emptyset) + \#(o(d_1(2)) \cap o(89)) \\ &= \#(o(4567) \cap o(94)) + \#(o(4567) \cap o(78)) + 0 + 0 + 0 = 0, \text{ and} \\ D_3(d_1, d_2) &= \delta_3(\emptyset, 89) + \epsilon_3^P(\emptyset, 89) + \#(o(d_1(3)) \cap \emptyset) + \#(o(d_1(3)) \cap o(945678)) = 0, \end{aligned}$$

and observe that $D_1(d_1, d_2) + D_2(d_1, d_2) + D_3(d_1, d_2) = 2 \equiv 0 \pmod{2}$.

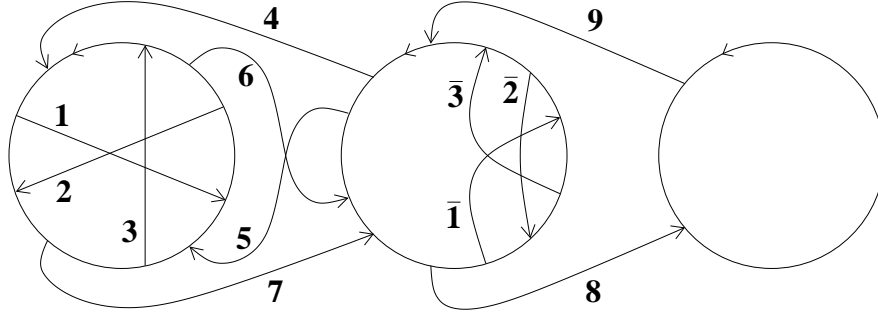


Figure 3.1: A virtual string with three core circles and twelve arrows.

3.4 Virtual Strings

Turaev introduced the term “virtual string” in [Tur03] to describe a single copy of S^1 with distinguished ordered pairs of points, which can be represented as a set of arrows attached to the circle. We extend this definition to allow multiple copies of S^1 . Consider a collection of N copies of S^1 with a distinguished set of points on each copy, such that the total number of distinguished points amongst all N copies is even. A *virtual string* α with N components is such a collection with the distinguished points partitioned into ordered pairs. The copies of S^1 in α are called the *core circles* of α and the ordered pairs are called the *arrows* of α . For an arrow (a, b) of α , the endpoints a and b are called its *tail* and *head*, respectively. We will impose the condition that it is not possible to partition the core circles of a virtual string into two sets such that the circles in one set have no arrows in common with the circles of the other. This condition assures that a virtual string is not a “disjoint union” of other virtual strings.

Example 3.4.1. Figure 3.1 depicts a virtual string with three core circles and twelve arrows.

Let S be a core circle of a virtual string α . Two distinguished points a, b on S separate S into two arcs, namely ab and ba . Suppose $e = (a, b)$ is an arrow of α . A different arrow $f = (c, d)$ on S is said to *link* e if one endpoint of f lies on the interior $(ab)^\circ$ of the arc ab and the other lies on the interior $(ba)^\circ$ of the arc ba . The arrow f links e *positively* if the arrow

endpoints lie in the cyclic order a, d, b, c around S , and the arrow f links e *negatively* if the arrow endpoints lie in the cyclic order a, c, b, d around S . Note that if f links e positively then e links f negatively.

Two virtual strings are *homeomorphic* if there is an orientation-preserving homeomorphism of their core circles such that the arrows of the first string are mapped onto the arrows of the second string. The homeomorphism classes of virtual strings will also be called virtual strings.

Every virtual string α has an underlying Gauss paragraph $p = p_\alpha$. Pick any alphabet set E with $\#(E)$ equal to the number of arrows in α , and label each arrow with a different letter in E . Select a base point on each core circle of α that is not an endpoint of any arrow. The word of p associated to the core circle S of α is obtained as follows: starting at the base point of S , traverse S in the positive direction and record the label of an arrow each time one of its endpoints is encountered. The resulting word will be well-defined up to circular permutations (and of course the choice of arrow labels). The Gauss paragraph underlying α is the set of words obtained in this way, with one word associated to every core circle.

Example 3.4.2. The arrows of the virtual string in Figure 3.1 are labeled with letters from the alphabet set $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \bar{1}, \bar{2}, \bar{3}\}$. The Gauss paragraph underlying the virtual string is $(1273512634, \bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{3}945678, 89)$.

3.5 Closed Curves on Surfaces

Recall that a *surface* is a smooth oriented 2-dimensional manifold. Also recall that a smooth map from a collection of oriented circles $\mathbb{I}S^1$ into a surface Σ is called an *immersion* if its differential is nonzero at all points of the circle. For an immersion $\rho : \mathbb{I}S^1 \rightarrow \Sigma$, a point $x \in \Sigma$ with $\#(\rho^{-1}(x)) = 2$ is called a *double point* or *crossing* of ρ . The immersion ρ is called *generic* if $\#(\rho^{-1}(x)) \leq 2$ for all $x \in \Sigma$, it has a finite set of double points, and all its

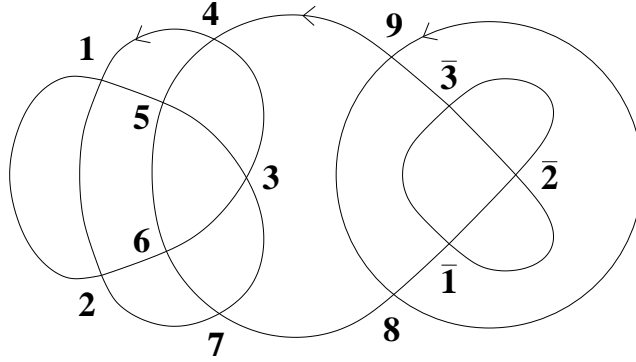


Figure 3.2: A closed curve with three components and twelve crossings that realizes the virtual string in Figure 3.1.

double points are transverse intersections of two branches. A generic smooth immersion of N oriented circles into a surface Σ is called a *closed curve* with N components on the surface Σ .

Every closed curve $\rho : \mathbb{R}S^1 \rightarrow \Sigma$ has an underlying virtual string $\alpha = \alpha_\rho$. The core circles of α are the copies of S^1 in the domain of ρ . The arrows of α are all ordered pairs (a, b) of distinguished points such that $\rho(a) = \rho(b)$ and the pair (a positive tangent vector of ρ at a , a positive tangent vector of ρ at b) is a positive basis in the tangent space of $\rho(a)$. A virtual string is said to be *realized* by a closed curve $\rho : \mathbb{R}S^1 \rightarrow \Sigma$ if it is homeomorphic to α_ρ . The Gauss paragraph of a closed curve is the Gauss paragraph of its underlying virtual string.

Example 3.5.1. The virtual string in Figure 3.1 is realized by the closed curve in Figure 3.2, which can be considered as a closed curve in either \mathbb{R}^2 or S^2 . The Gauss paragraph of the closed curve is $(1273512634, \bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{3}945678, 89)$, so this Gauss paragraph is realizable by a closed curve in S^2 .

Every virtual string admits a canonical realization by a closed curve on a surface. Turaev showed this result for virtual strings with one core circle in [Tur03]. He used a well-known construction of surfaces from four-valent graphs, see for example [Car91a]. We will describe the same construction for virtual strings with multiple core circles. Let α be a virtual string

with more than one core circle. Transform α into a 1-dimensional CW-complex $\Gamma = \Gamma_\alpha$ by identifying the head and tail of each arrow in α . A thickening of Γ gives a surface Σ_α in the following manner. The 0-cells of Γ are 4-valent vertices. A vertex $v \in \Gamma$ results from an arrow (a, b) , with points a and b on some core circles of α . Call these core circles S_a and S_b , and note that possibly $S_a = S_b$. A neighborhood of a point x in a core circle is an oriented arc, which x splits into one incoming arc and one outgoing arc with respect to x . The incoming and outgoing arcs in neighborhoods of a in S_a and b in S_b can be identified with the four arcs in a neighborhood of v in Γ . Therefore this neighborhood of v can be embedded in the unit 2-disc $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ so that the image of v is the origin, the images of the incoming arcs at a and b are the intervals $0 \times [-1, 0]$ and $[-1, 0] \times 0$ respectively, and the images of the outgoing arcs at a and b are the intervals $0 \times [0, 1]$ and $[0, 1] \times 0$ respectively. By repeating this procedure for all vertices of Γ , the vertices can be thickened to 2-discs endowed with counterclockwise orientation. A 1-cell of Γ either connects two different vertices or forms a loop at a single vertex. The 1-cells of Γ can be thickened to ribbons, with the thickening uniquely determined by the condition that the orientations of the 2-discs extend to their unions with the ribbons. Thickening Γ in the way just described gives an embedding of Γ onto a surface Σ_α . Notice that Σ_α is a compact, connected, oriented surface with boundary. A closed curve $\rho_\alpha : \mathbb{H}S^1 \rightarrow \Sigma_\alpha$ realizing α is obtained by composing the natural projection $\mathbb{H}S^1 \rightarrow \Gamma$ with the inclusion $\Gamma \hookrightarrow \Sigma_\alpha$.

The surface Σ_α constructed above is the surface of minimal genus containing a closed curve realizing α . Suppose $\rho : \mathbb{H}S^1 \hookrightarrow \Sigma$ is a generic closed curve realizing α on some surface Σ . Then a regular neighborhood of $\rho(\mathbb{H}S^1)$ in Σ is homeomorphic to Σ_α . This homeomorphism can be chosen to transform ρ into ρ_α , that is composing ρ_α with an orientation-preserving embedding of Σ_α into Σ results in the curve ρ . Gluing 2-discs to all the components of the boundary of Σ_α gives a closed surface of minimal genus that contains a curve realizing α .

Table 3.1: Summary of notational conventions

Gauss paragraph p	Virtual string α	Closed curve ρ
word v_n	core circle S_n	component $\rho(S'_n)$
letter i in v_n	arrow $e_i = (a_i, b_i)$	crossing i at $\rho(a_i) = \rho(b_i)$
p -sets p_i and p'_i	arcs $a_i b_i$ and $b_i a_i$	two loops in $\rho(S'_n)$ based at $\rho(a_i) = \rho(b_i)$
sequence $d \in \mathcal{D}_p$	sequence of arcs C_d	loop $\rho(C'_d)$ with segments $\rho(C'_d(n))$

Notation. Let $\rho : \coprod_{n=1}^N S'_n \rightarrow \Sigma$ be a closed curve with $N \in \mathbb{N}$ components on some surface Σ , where each S'_n is a copy of S^1 . Then ρ has an underlying virtual string α with core circles S_1, S_2, \dots, S_N , where the core circle S_n corresponds to the circle S'_n in the domain of ρ . Let $p = (v_1, v_2, \dots, v_N)$ be the underlying Gauss paragraph of α , with word v_n corresponding to core circle S_n . A cyclic sequence $d = (x_1, x_2, \dots, x_N) \in \mathcal{D}_p$ corresponds to a sequence $C_d = (c_1, c_2, \dots, c_N)$ of arcs in α such that the arc $c_n \subset S_n$ corresponds to the sequence x_n in d . Moreover, C_d corresponds to a sequence $C'_d = (c'_1, c'_2, \dots, c'_N)$ of arcs on the components S'_n in the domain of ρ such that the arc $c'_n \subset S'_n$ corresponds to the arc $c_n \subset S_n$. Let $C_d(n)$ denote the arc c_n of C_d and $C'_d(n)$ denote the arc c'_n of C'_d . In addition, let $\rho(C'_d)$ denote the loop on Σ that is the union $\bigcup_{n=1}^N \rho(C'_d(n))$. Table 3.1 displays the correspondences and notation discussed thus far.

3.6 Observations and Restrictions

Every virtual string with an even number of arrow endpoints on each of its core circles naturally gives rise to a word-wise partition. Let α be such a virtual string, and suppose S is a core circle of α . Denote the set of arrows of α with tails on S by $arr_t(S)$. For $e = (a, b)$ and $f = (c, d)$ in $arr_t(S)$, define $q(e, f) \in \mathbb{Z}$ to be the number of arrowheads lying on the semi-open arc $ac - \{a\} \subset S$ minus the number of arrowtails lying on $ac - \{a\}$. Set $q(e, f) = 0$ when $e = f$. Use q to define an equivalence relation on $arr_t(S)$ by defining two arrows

$e = (a, b)$ and $f = (c, d)$ to be *equivalent* if $q(e, f) \equiv 0 \pmod{2}$. Then the arrows e and f are equivalent if either $e = f$ or the number of arrow endpoints lying on the interior of the arc $ac \subset S$ is odd. Notice that there are at most two equivalence classes on $arr_t(S)$ that result from this relation, and these classes partition $arr_t(S)$ into two subsets. Pick any alphabet set E with $\#(E)$ equal to the number of arrows in α , and label each arrow with a different letter in E . Suppose $E' \subset E$ contains the letters that label the arrows in $arr_t(S)$. Then the bipartition of $arr_t(S)$ induces a bipartition of the set E' . Applying the same procedure on all core circles of α creates a word-wise partition of E . Thus α naturally gives rise to a pair (the underlying Gauss word p of α , a word-wise partition of E with respect to p).

If α is the underlying virtual string of a closed curve ρ on some surface, then the above construction gives a partition of the set of labels (namely E) on the double points of ρ . Thus ρ gives rise to the pair (the underlying Gauss word p of α , a word-wise partition of E with respect to p) as well. Note that each component of ρ has an even number of intersections between it and the other components because we assumed each core circle of α had an even number of arrow endpoints.

An important observation should be made here. Let e be an arrow of α with tail on one core circle, say S_1 , and head on a different core circle, say S_2 . Then e corresponds to a letter $i \in E$ that appears in one of the sets associated to S_1 in the word-wise partition, and in neither of the sets associated to S_2 . Moreover, the sign of crossing i of ρ is $+1$ with respect to the component of ρ corresponding to S_2 and -1 with respect to the component of ρ corresponding to S_1 .

A pair (a Gauss paragraph p in an alphabet E , a word-wise partition P of E) is said to be *realizable* by a closed curve on a surface if there exists a closed curve on the surface that gives rise to the pair. The surface of particular interest to us is S^2 , but observe that a pair (a Gauss paragraph p in an alphabet E , a word-wise partition P of E) is realizable by a closed curve on S^2 if and only if the pair is realizable by a closed curve on \mathbb{R}^2 .

Example 3.6.1. The virtual string in Figure 3.1 has an even number of arrow endpoints on each of its core circles. If a pairing $q(e, f)$ is defined on its arrows as described above, then the induced word-wise partition of the alphabet $\{1, 2, 3, 4, 5, 6, 7, 8, 9, \bar{1}, \bar{2}, \bar{3}\}$ is $P = (\{1, 2, 7\} \cup \{3, 6\}, \{\bar{1}, \bar{2}, \bar{3}, 5\} \cup \{4, 8\}, \{9\} \cup \emptyset)$. Since the closed curve in Figure 3.2 has Gauss paragraph $p = (1273512634, \bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{3}945678, 89)$ and realizes the virtual string in Figure 3.1, it follows that the pair (p, P) is realizable by a closed curve on S^2 .

If a virtual string has a core circle with an odd number of arrow endpoints on it, then the pairing $q(e, f)$ above is not well-defined on that core circle. Therefore such virtual strings do not give rise to word-wise partitions in any sense that we will discuss in this chapter. It is well known that a closed curve on S^2 cannot have an odd number of intersections between any two distinct components, and therefore each component of a closed curve on S^2 cannot have an odd number of intersections between it and all the other components. Consequently, the underlying virtual string of a closed curve on S^2 cannot have a core circle with an odd number of arrow endpoints on it.

Since Gauss paragraphs that can be realized by closed curves on S^2 are our primary interest, it is reasonable to now restrict our attention to virtual strings with an even number of arrow endpoints on each core circle and closed curves for which each component has an even number of crossings between it and all the other components.

3.7 Homological Intersection Form

In this section, we relate a homological intersection pairing to the maps described in Section 3.3. We develop formulas that help prove the main theorem.

Let Σ be an oriented surface, and let $H_1(\Sigma)$ denote its first integral homology group $H_1(\Sigma; \mathbb{Z})$. Let $\rho : \coprod_{n=1}^N S'_n \rightarrow \Sigma$ be a closed curve with $N \in \mathbb{N}$ components, where each S'_n is a copy of S^1 . Then ρ has an underlying virtual string α , so suppose the Gauss paragraph

underlying α is $p = (v_1, v_2, \dots, v_N)$. Label the N core circles of α by S_1, S_2, \dots, S_N , with core circle S_n corresponding to circle S'_n and word v_n . A distinguished point z on S_n corresponds to a point on the circle S'_n , and for convenience this point on S'_n will also be called z .

If $e = (a, b)$ is an arrow on core circle S_n of α , then $\rho(a) = \rho(b)$. Therefore the images $\rho(ab)$ and $\rho(ba)$ of the arcs ab and ba of S'_n , respectively, are loops in Σ . To stay consistent with Turaev's convention in [Tur03], we will use the notation $[e]$ and $[e]^*$ to denote the homology classes $[\rho(ab)]$ and $[\rho(ba)]$ of $H_1(\Sigma)$, respectively. The image under ρ of each circle S'_n is also a loop in Σ , so let $[S_n]$ denote the homology class $[\rho(S'_n)] \in H_1(\Sigma)$. Let $(x'_1, x'_2, \dots, x'_M)$ with $M \geq 2$ be a sequence of arcs on the circles S'_n that satisfies the following properties: at most one arc exists on each S'_n , the initial point of x'_1 and the terminal point of x'_M have the same image under ρ , and the initial point of x'_m and the terminal point of x'_{m-1} have the same image under ρ for $2 \leq m \leq M$. Then the union $\bigcup_{m=1}^M \rho(x'_m)$ constitutes a loop in Σ . The sequence of arcs $(x'_1, x'_2, \dots, x'_M)$ corresponds to sequence of arcs on the core circles S_n , which in turn corresponds to a sequence $d = (x_1, x_2, \dots, x_N)$ with x_n being a subsequence (possibly empty) of the word v_n of p . Since more than one x_n is nonempty, it follows that $d \in \mathcal{D}_p$. Therefore let $[C_d]$ denote the homology class of the union $\bigcup_{m=1}^M \rho(x'_m) = \bigcup_{n=1}^N \rho(C'_d(n)) = \rho(C'_d)$ in $H_1(\Sigma)$. Notice that if a loop in ρ has lone segments on multiple components and the orientations of these segments agree, then the loop can be expressed as such a union for some cyclic sequence $d \in \mathcal{D}_p$.

Example 3.7.1. Consider the closed curve depicted in Figure 3.3. Each of the dashed loops in the figure represents one of the types of homology classes described above; specifically the loops K_1, K_2 , and K_3 represent homology classes of the types $[S_n], [e]^*$, and $[C_d]$, respectively. The virtual string underlying the closed curve is depicted in Figure 3.4, where the dashed arcs correspond to the loops represented in Figure 3.3. In particular, the arcs κ_1 and κ_2 correspond to the loops K_1 and K_2 , respectively, and the two arcs κ_{31} and κ_{32} correspond to

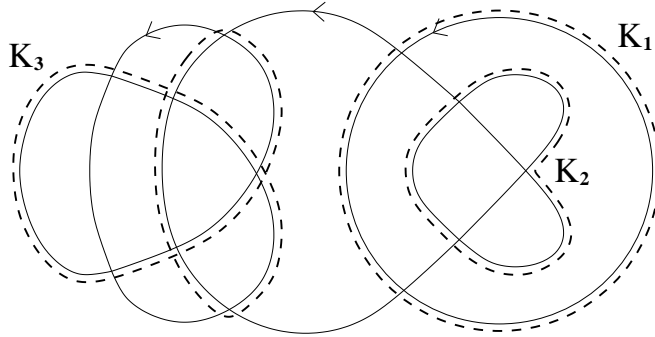


Figure 3.3: Representatives of three types of homology classes.

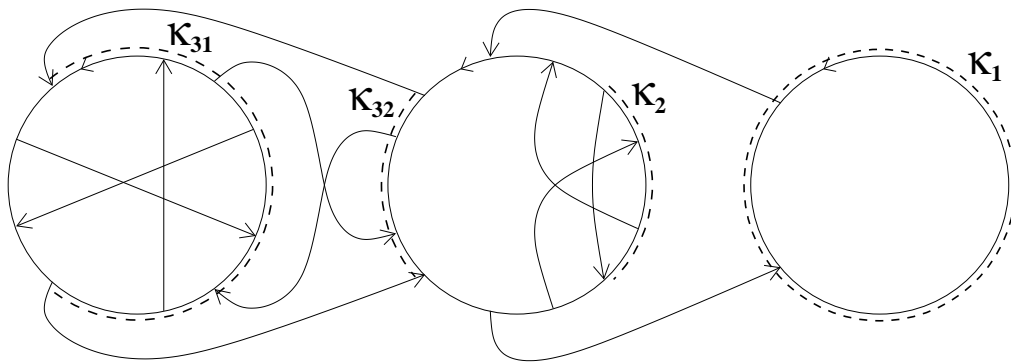


Figure 3.4: The arcs that correspond to the representatives in Figure 3.3.

the two segments of the loop K_3 on different components of the closed curve.

In accordance with the restrictions discussed at the end of Section 3.6, assume that each component of ρ has an even number of crossings between it and all the other components. Then α has an even number of arrow endpoints on each core circle. To this point, we have not specified an alphabet set for p . Suppose p is a paragraph in the alphabet set E , and suppose α gives rise to the word-wise partition $P = (A_1 \cup A'_1, A_2 \cup A'_2, \dots, A_N \cup A'_N)$ of E . Let $e_i = (a_i, b_i)$ denote the arrow in virtual string α that corresponds to the letter $i \in E$. Label a crossing of the closed curve ρ with the letter i if the crossing corresponds to the arrow e_i . Under these notational conventions, we may assume that p_i and p'_i contain the letters corresponding to the arrows in α with exactly one endpoint on $(a_i b_i)^\circ$ and $(b_i a_i)^\circ$, respectively.

The orientation of the surface Σ determines a homological intersection form $B : H_1(\Sigma) \times H_1(\Sigma) \rightarrow \mathbb{Z}$. In the propositions and theorems that follow, we develop formulas that will help determine the parities of the intersection numbers for the homology classes discussed above.

Notation. The set of arrows $arr_{i,j}(\alpha)$ consists of the arrows with tail on S_i and head on S_j . For $e = (a, b) \in arr_{i,i}(\alpha)$, let $n(e) \in \mathbb{Z}$ denote

$$\#\{f \in arr_{i,i}(\alpha) \mid f \text{ links } e \text{ positively}\} - \#\{f \in arr_{i,i}(\alpha) \mid f \text{ links } e \text{ negatively}\}.$$

Let $n_{i,j}(e)$ denote

$$\#\{f \in arr_{j,i}(\alpha) \mid \text{head of } f \text{ in } (ab)^\circ\} - \#\{f \in arr_{i,j}(\alpha) \mid \text{tail of } f \text{ in } (ab)^\circ\},$$

and let $n_{i,j}^*(e)$ denote

$$\#\{f \in arr_{j,i}(\alpha) \mid \text{head of } f \text{ in } (ba)^\circ\} - \#\{f \in arr_{i,j}(\alpha) \mid \text{tail of } f \text{ in } (ba)^\circ\}.$$

Notice that $n_{i,i}(e) = n(e)$ and $n_{i,i}^*(e) = -n(e)$.

Proposition 3.7.2. *If $e = (a, b) \in arr_{i,i}(\alpha)$ and $i \neq j$, then*

$$B([e], [S_i]) = B([e], [e]^*) = n(e)$$

$$B([e], [S_j]) = n_{i,j}(e)$$

$$B([e]^*, [S_j]) = n_{i,j}^*(e)$$

and

$$B([e], [S_j]) \equiv \#(p_i \cap o(v_j)) \pmod{2}$$

$$B([e]^*, [S_j]) \equiv \#(p'_i \cap o(v_j)) \pmod{2}.$$

Proof. The proof of the first claim appears in [Tur03], but we include it here for completeness. The loops $\rho(ab)$ and $\rho(ba)$ intersect transversely, except at their common origin $\rho(a) = \rho(b)$. However, a small deformation makes these loops disjoint in a neighborhood of $\rho(a) = \rho(b)$. Notice that there is a bijective correspondence between the transversal intersections and the arrows of $arr_{i,i}(\alpha)$ linked with e . The sign of a transversal intersection with respect to $\rho(ab)$ is $+1$ when its corresponding arrow links e positively, and -1 when its corresponding arrow links e negatively. Hence $B([e], [e]^*) = n(e)$. Since $[S_i] = [e] + [e]^*$ in $H_1(\Sigma)$, it follows that

$$B([e], [S_i]) = B([e], [S_i]) - B([e], [e]) = B([e], [e]^*) = n(e).$$

Any intersections of the loops $\rho(ab)$ and $\rho(S'_j)$ are transversal intersections. There is a bijective correspondence between these intersections and the union of the sets $\{f \in arr_{j,i}(\alpha) \mid \text{head of } f \text{ in } (ab)^\circ\}$ and $\{f \in arr_{i,j}(\alpha) \mid \text{tail of } f \text{ in } (ab)^\circ\}$. The sign of an intersection with respect to $\rho(ab)$ is $+1$ when its corresponding arrow is in the first set, and -1 when its corresponding arrow is in the second set. Hence $B([e], [S_j]) = n_{i,j}(e)$. Moreover, the arrows in the union of these two sets correspond to the letters that the sets p_i and $o(v_j)$ have in common. Hence $B([e], [S_j]) \equiv \#(p_i \cap o(v_j)) \pmod{2}$. The claims about $[e]^*$ follow from similar arguments. ■

Proposition 3.7.3. *If $i \in E$ is a double letter of the word v_j in p , then*

$$\begin{aligned} \#(p_i) &\equiv \sum_{k=1}^N n_{j,k}(e_i) \pmod{2} \\ \#(p'_i) &\equiv \sum_{k=1}^N n_{j,k}^*(e_i) \pmod{2}. \end{aligned}$$

Proof. We have assumed that the letters in p_i correspond to the arrows in α with exactly one endpoint on $(a_i b_i)^\circ$. The other endpoint of such an arrow is either on $(b_i a_i)^\circ$ or on another

component of α . For $k \neq j$, $n_{j,k}(e_i)$ is equivalent to

$$\#\{f \in \text{arr}_{k,j}(\alpha) \mid \text{head of } f \text{ in } (a_i b_i)^\circ\} + \#\{f \in \text{arr}_{j,k}(\alpha) \mid \text{tail of } f \text{ in } (a_i b_i)^\circ\}$$

modulo two, and this sum is equal to the number of arrows between S_j and S_k with one endpoint on $(a_i b_i)^\circ$. Notice $n_{j,j}(e_i) = n(e_i)$ is equivalent to

$$\#\{f \in \text{arr}_{j,j}(\alpha) \mid f \text{ links } e \text{ positively}\} + \#\{f \in \text{arr}_{j,j}(\alpha) \mid f \text{ links } e \text{ negatively}\}$$

modulo two, with this sum equal to the number of arrows in $\text{arr}_{j,j}(\alpha)$ with one endpoint on $(a_i b_i)^\circ$ and the other on $(b_i a_i)^\circ$. Hence $\#(p_i) \equiv \sum_{k=1}^N n_{j,k}(e_i)$. The claim about $\#(p'_i)$ can be proved in a similar manner. ■

Proposition 3.7.4. *For distinct core circles S_i and S_j of α ,*

$$B([S_i], [S_j]) = \#(\text{arr}_{j,i}(\alpha)) - \#(\text{arr}_{i,j}(\alpha)).$$

The number of letters that the words v_i and v_j of p have in common is equal to $\#(\text{arr}_{j,i}(\alpha)) + \#(\text{arr}_{i,j}(\alpha))$. Therefore $B([S_i], [S_j])$ is equivalent modulo two to the number of letters that the words v_i and v_j of p have in common.

Proof. Any intersections of the loops $\rho(S'_i)$ and $\rho(S'_j)$ are transverse intersections. There is a bijective correspondence between these transverse intersections and the set of arrows between S_i and S_j in α , that is the disjoint union $\text{arr}_{j,i}(\alpha) \cup \text{arr}_{i,j}(\alpha)$. The sign of an intersection with respect to $\rho(S'_i)$ is +1 when its corresponding arrow is in $\text{arr}_{j,i}(\alpha)$, and -1 when its corresponding arrow is in $\text{arr}_{i,j}(\alpha)$. Hence $B([S_i], [S_j]) = \#(\text{arr}_{j,i}(\alpha)) - \#(\text{arr}_{i,j}(\alpha))$. The arrows in the disjoint union $\text{arr}_{j,i}(\alpha) \cup \text{arr}_{i,j}(\alpha)$ correspond to the letters that v_i and v_j have in common, so the total number of such letters is $\#(\text{arr}_{j,i}(\alpha)) + \#(\text{arr}_{i,j}(\alpha))$. ■

Proposition 3.7.5. *If $i, j \in E$ are double letters of a word w of p that are not w -interlaced and $\#(w_i) \equiv \#(w_j) \equiv 0 \pmod{2}$, then $\#(w_i \cap w_j) \equiv B([e_i], [e_j]) \pmod{2}$.*

Proof. The proof of this result is included as part the proof of Theorem 5.3.1 in [Tur03]. The same arguments apply because the intersections that contribute to $B([e_i], [e_j])$ are self-intersections of a single component of ρ . ■

Proposition 3.7.6. *Suppose $i \in E$ is a double letter of the word v_m of p and $j \in E$ is a double letter of word the v_n of p , where $m \neq n$. Then*

$$\begin{aligned} \#(p_i \cap p_j) &\equiv B([e_i], [e_j]) \pmod{2} \\ \#(p_i \cap p'_j) &\equiv B([e_i], [e_j]^*) \pmod{2} \\ \#(p'_i \cap p_j) &\equiv B([e_i]^*, [e_j]) \pmod{2} \\ \#(p'_i \cap p'_j) &\equiv B([e_i]^*, [e_j]^*) \pmod{2} \end{aligned}$$

Proof. Any intersections of the loops $\rho(a_i b_i)$ and $\rho(a_j b_j)$ are transverse intersections. There is a bijective correspondence between these transverse intersections and the set of arrows between S_m and S_n in α with one endpoint on $(a_i b_i)^\circ$ and the other on $(a_j b_j)^\circ$. Such arrows correspond to the letters of p appearing in $p_i \cap p_j$. Hence $\#(p_i \cap p_j) \equiv B([e_i], [e_j]) \pmod{2}$. The other three claims follow from similar arguments. ■

Theorem 3.7.7. *For all $1 \leq n \leq N$ and $d \in \mathcal{D}_p$,*

$$B([S_n], [C_d]) \equiv W(v_n, d) \pmod{2}.$$

Proof. If $d(n)$ is an empty sequence, then the intersections of $\rho(S'_n)$ and $\rho(C'_d)$ are all transversal intersections. These intersections are in bijective correspondence with the arrows of α with one endpoint on S_n and the other on any arc $C_d(k)$, with $k \neq n$ since $d(n) = \emptyset$. If an arrow of α has one endpoint on S_n and the other on $C_d(k)$, then it corresponds

to a letter of p that v_n and $d(k)$ have in common. Hence $B([S_n], [C_d])$ is equivalent to $\sum_{k \neq n} \#(o(v_n) \cap o(d(k))) = W(v_n, d)$ modulo two in this case.

Now assume $d(n)$ is a nonempty sequence with first letter i and last letter j . Then $d(n)$ corresponds to an arc $C'_d(n)$ on S'_n . A neighborhood of the segment $\rho(C'_d(n))$ in Σ is depicted in Figure 3.5, where the thickened horizontal line segment represents the segment of $\rho(S'_n)$ between crossings i and j . The vertical lines depict neighborhoods of crossings i and j in some components $\rho(S'_t)$ and $\rho(S'_u)$, respectively, and their ends are labeled with the letters i_1, i_2 and j_1, j_2 to represent the letters flanking i and j in the words v_t and v_u of p , respectively.

An intersection of the loops $\rho(S'_n)$ and $\rho(C'_d)$ is either a transversal intersection involving $\rho(S'_n)$ and a segment of $\rho(C'_d)$ not on $\rho(S'_n)$ or an intersection resulting from the segment $\rho(C'_d(n))$ on $\rho(S'_n)$. As in the above discussion, the transversal intersections correspond to letters of p that v_n and any $d(k)$ for $k \neq n$ have in common, so there are a total of $\sum_{k \neq n} \#(o(v_n) \cap o(d(k)))$ such intersections. The other intersections require further consideration.

If the points i, i_1, i_2 lie in the cyclic order $i_1 i i_2$ (respectively $i_2 i i_1$) on S'_t , then the sign of the crossing i with respect to $\rho(S'_n)$ is -1 (respectively $+1$) and $i \in A_n \cup A'_n$ (respectively $i \in A_t \cup A'_t$). Similarly, if the points j, j_1, j_2 lie in the cyclic order $j_1 j j_2$ (respectively $j_2 j j_1$) on S'_u , then the sign of the crossing j with respect to $\rho(S'_n)$ is -1 (respectively $+1$) and $j \in A_n \cup A'_n$ (respectively $j \in A_u \cup A'_u$). The possible cyclic orders of the points i, i_1, i_2 and j, j_1, j_2 on S'_t and S'_u , respectively, yield the following four cases:

- (1) $i_1 i i_2$ on S'_t and $j_1 j j_2$ on S'_u ;
- (2) $i_1 i i_2$ on S'_t and $j_2 j j_1$ on S'_u ;
- (3) $i_2 i i_1$ on S'_t and $j_1 j j_2$ on S'_u ;

(4) $i_2i_1i_1$ on S'_t and $j_2j_1j_1$ on S'_u .

Consider the loop σ in Σ that is the “push off” of $\rho(S'_n)$ to the right in a neighborhood of $\rho(S'_n)$. In particular, for the above four cases, consider a neighborhood of the segment $\rho(C'_d(n))$, as depicted in Figure 3.5. The dashed lines in the figure represent σ and the thickened lines represent a portion of the loop $\rho(C'_d)$. The intersections resulting from segment $\rho(C'_d(n))$ on $\rho(S'_n)$ are transversal intersections of σ and $\rho(C'_d)$ in this neighborhood. In every case, the number of intersections on the interior of the segment $\rho(C'_d(n))$ is the same. In a neighborhood of a self-intersection of $\rho(S'_n) \cap \rho(C'_d(n))$, there are two crossings involving σ and the interior of $\rho(C'_d(n))$. These two crossings have opposite signs, so they do not contribute to $B([S_n], [C_d])$. Therefore we only consider those intersections of σ and $\rho(C'_d(n))$ that do not result from self-intersections of $\rho(S'_n) \cap \rho(C'_d(n))$. Such intersections bijectively correspond to letters that the subsequences x_1 and x_2 of $v_n = ix_1jx_2$ have in common, and there are $\#(o(ix_1j) \cap o(jx_2i))$ such letters. Notice that there may be additional transverse intersections of σ and $\rho(C'_d)$ in each of the four cases, resulting from the portions of the remaining segments of $\rho(C'_d)$ depicted in Figure 3.5. Specifically, there are 1, 0, 2, and 1 additional intersections in cases (1), (2), (3), and (4), respectively, and these totals are equivalent to $\gamma_n^P(i, j)$ modulo two in each case. Hence $B([S_n], [C_d])$ is equivalent to

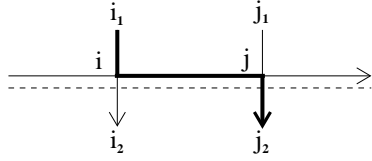
$$\#(o(ix_1j) \cap o(jx_2i)) + \gamma_n^P(i, j) + \sum_{k \neq n} \#(o(v_n) \cap o(d(k))) = W(v_n, d)$$

modulo two. ■

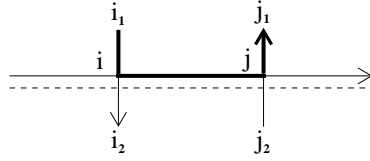
Theorem 3.7.8. *If a word v_n of p has a double letter $i \in E$, then*

$$B([e_i], [C_d]) \equiv Q_n(p_i, d) \pmod{2}$$

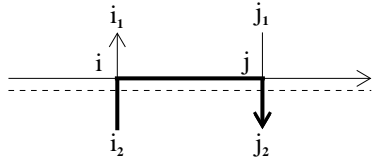
Case (1)



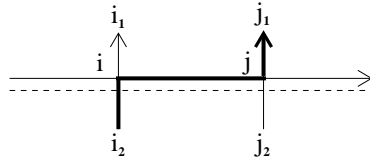
Case (2)



Case (3)



Case (4)

Figure 3.5: Neighborhood of the segment $\rho(C'_d(n))$.

and

$$B([e_i]^*, [C_d]) \equiv Q_n(p'_i, d) \pmod{2}.$$

Proof. Since i is a double letter of the word v_n , it follows that $v_n = ix_1ix_2$ for some subsequences x_1, x_2 . Recall that we have assumed p_i and p'_i contain the letters corresponding to the arrows in α with exactly one endpoint on $(a_ib_i)^\circ$ and $(b_ia_i)^\circ$, respectively. The letters in p_i then correspond to the letters that occur exactly once in either x_1 or x_2 , but we may assume it is x_1 . Then the letters in p'_i correspond to the letters that occur exactly once in x_2 .

If $d(n)$ is an empty sequence, then the intersections of the loops $\rho(a_ib_i)$ and $\rho(C'_d)$ are all transversal intersections. These intersections are in bijective correspondence with the arrows of α with one endpoint on $(a_ib_i)^\circ$ and the other on any arc $C_d(k)$, with $k \neq n$ since $d(n) = \emptyset$. If an arrow of α has one endpoint on $(a_ib_i)^\circ$ and the other on $C_d(k)$, then it corresponds to a letter of p that p_i and $o(d(k))$ have in common. Therefore $B([e_i], [C_d]) \equiv \sum_{k \neq n} \#(p_i \cap o(d(k))) \pmod{2}$. Hence $B([e_i], [C_d]) \equiv Q_n(p_i, d) \pmod{2}$ in this case since $\delta_n(ix_1i, d(n)) = \delta_n(ix_1i, \emptyset) = 0$ and $\epsilon_n^P(ix_1i, d(n)) = \epsilon_n^P(ix_1i, \emptyset) = 0$.

Now suppose $d(n) = yxz$ for some single letters y, z of v_n and subsequence x of v_n . The subsequence $d(n)$ of v_n corresponds to an arc $C'_d(n) \subset S'_n$ with initial point y and terminal point z . The subsequence ix_1i of v_n corresponds to the arc $a_i b_i \subset S'_n$ with initial point a_i and terminal point b_i . Notice that the points a_i, b_i, y, z must be pairwise distinct since they are the four endpoints of two different arrows in α .

An intersection of $\rho(a_i b_i)$ and $\rho(C'_d)$ is either a transversal intersection involving $\rho(a_i b_i)$ and a segment of $\rho(C'_d)$ not on $\rho(S'_n)$ or an intersection resulting from the segment $\rho(C'_d(n))$ on $\rho(S'_n)$. As in the above discussion, the transversal intersections correspond to letters of p that p_i and $o(d(k))$ for $k \neq n$ have in common, so there are a total of $\sum_{k \neq n} \#(p_i \cap o(d(k)))$ such intersections.

Now consider the intersections resulting from the segment $\rho(C'_d(n))$ on $\rho(S'_n)$. The six possible cyclic orders of the points a_i, b_i, y, z on S'_n are:

- (1) a_i, y, b_i, z ;
- (2) a_i, z, b_i, y ;
- (3) a_i, y, z, b_i ;
- (4) a_i, z, y, b_i ;
- (5) a_i, b_i, y, z ;
- (6) a_i, b_i, z, y .

Consider the loop σ in Σ that is the “push off” of $\rho(a_i b_i)$ to the right in a neighborhood of $\rho(S'_n)$. The intersections resulting from the segment $\rho(C'_d(n))$ are transversal intersections of σ and $\rho(C'_d)$ in this neighborhood. In particular, for the above six cases, consider a neighborhood of $\rho(S'_n)$ in Σ as depicted in Figure 3.6. The dashed loops in the figure represent σ and the thickened lines represent the segment $\rho(C'_d(n))$. The segments of $\rho(C'_d)$

immediately before and after $\rho(C'_d(n))$ are not indicated because they depend on the signs of crossings y and z with respect to $\rho(S'_n)$. If the sign of crossing y is -1 (respectively +1), then $y \in A_n \cup A'_n$ (respectively $y \notin A_n \cup A'_n$). If the sign of crossing z is -1 (respectively +1), then $z \in A_n \cup A'_n$ (respectively $z \notin A_n \cup A'_n$).

In each of the six cases, there may be transversal intersections of σ and $\rho(C'_d)$ in neighborhoods of crossings i , y , and z . In a neighborhood of crossing i , there are two intersections of σ and $\rho(C'_d(n))$ in Cases (4) and (6), and there is one intersection in Cases (1) and (2). In a neighborhood of crossing y , there is an intersection of σ and the segment of $\rho(C'_d)$ immediately before $\rho(C'_d(n))$ in Cases (1), (3), and (4) if and only if the sign of crossing y with respect to $\rho(S'_n)$ is +1. In a neighborhood of crossing z , there is an intersection of σ and the segment of $\rho(C'_d)$ immediately after $\rho(C'_d(n))$ in Cases (2), (3), and (4) if and only if the sign of crossing z with respect to $\rho(S'_n)$ is -1. Observe that there are no intersections of σ and $\rho(C'_d)$ in these three neighborhoods otherwise. Since p_i contains the letters that occur exactly once in the subsequence x_1 of v_n , it follows that the total number of transversal intersections of σ and $\rho(C'_d)$ in these neighborhoods of crossings i , y , and z is given by $\epsilon_n^P(ix_1i, d(n))$.

The only remaining transversal intersections of σ and $\rho(C'_d)$ left to consider are intersections of σ and $\rho(C'_d(n))$ that do not occur in the three crossing neighborhoods discussed above. In a neighborhood of a self-intersection of $\rho(a_i b_i) \cap \rho(C'_d(n))$, there are two crossings involving σ and the interior of $\rho(C'_d(n))$. These two crossings have opposite signs, so they do not contribute to $B([e_i], [C_d])$. Therefore we only consider those intersections of σ and $\rho(C'_d(n))$ that do not result from self-intersections of $\rho(a_i b_i) \cap \rho(C'_d(n))$. Such intersections bijectively correspond to arrows in $arr_{n,n}(\alpha)$ with one endpoint on arc $a_i b_i$ and the other on $C_d(n)$, provided both endpoints are not on $a_i b_i \cap C_d(n)$. In the following arguments, let the labels s_1, s_2, s_3, s_4 denote subsequences of the word v_n and assume the endpoints a_i and b_i correspond to the first and second instances of i , respectively. In Case (1), if $v_n = i s_1 y s_2 i s_3 z s_4$, then these arrows correspond to letters that s_1 and s_2 have in common, s_1 and s_3 have in

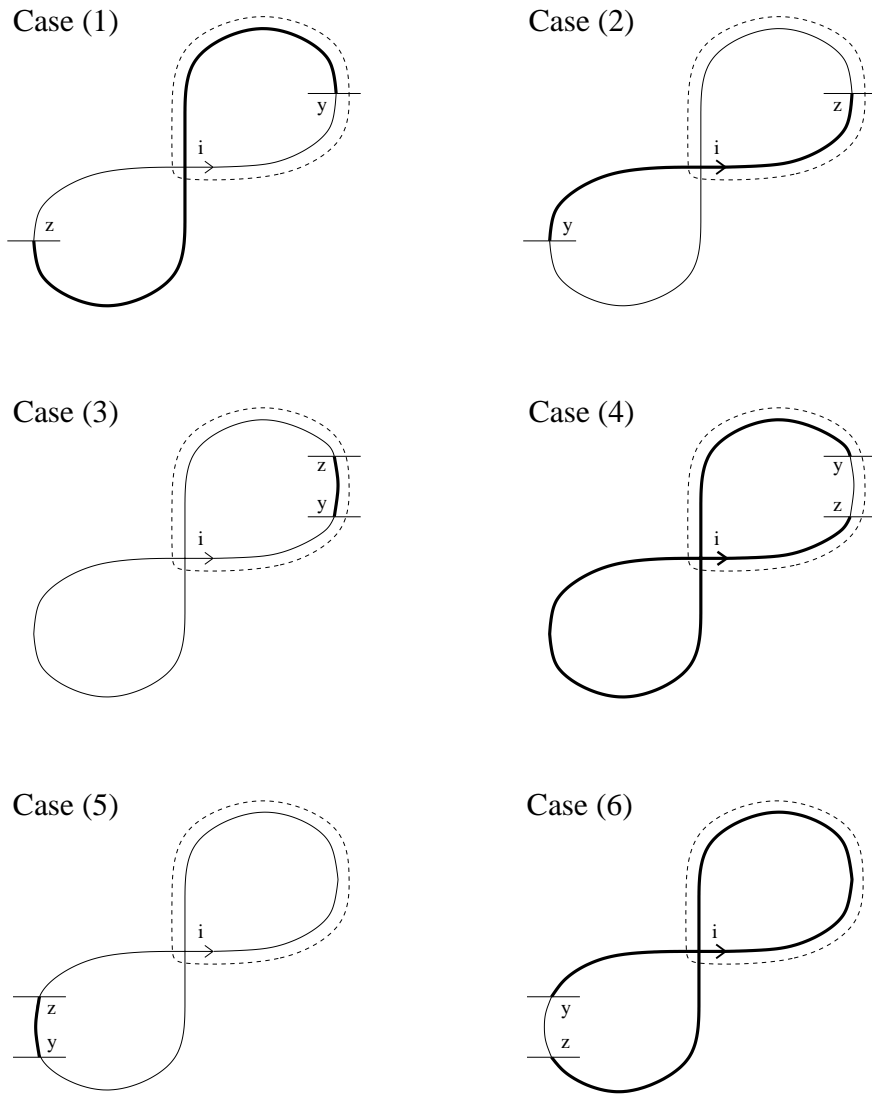


Figure 3.6: Neighborhood of the loop $\rho(S'_n)$.

common, or s_2 and s_3 have in common. In Case (2), if $v_n = i s_1 z s_2 i s_3 y s_4$, then these arrows correspond to letters that s_1 and s_4 have in common, s_2 and s_4 have in common, or s_2 and s_1 have in common. In Case (3), if $v_n = i s_1 y s_2 z s_3 i s_4$, then these arrows correspond to letters that subsequences s_1 and s_2 have in common or letters that s_3 and s_2 have in common. In Case (4), if $v_n = i s_1 z s_2 y s_3 i s_4$, then these arrows correspond to letters that s_1 and s_4 have in common, s_2 and s_3 have in common, s_2 and s_4 have in common, s_2 and s_1 have in common, or s_3 and s_4 have in common. In Case (5), if $v_n = i s_1 i s_2 y s_3 z s_4$, then these arrows correspond to letters that subsequences s_1 and s_3 have in common. In Case (6), if $v_n = i s_1 i s_2 z s_3 y s_4$, then these arrows correspond to letters that s_1 and s_4 have in common or s_1 and s_2 have in common. Hence the total number of such arrows is $\delta_n(ix_1i, d(n))$ in each of the six cases.

Having considered all possible intersections that contribute to $B([e_i], [C_d])$, we conclude that $B([e_i], [C_d])$ is equivalent to

$$\delta_n(ix_1i, d(n)) + \epsilon_n^P(ix_1i, d(n)) + \sum_{k \neq n} \#(p_i \cap o(d(k))) = Q_n(p_i, d)$$

modulo two. The claim $B([e_i]^*, [C_d]) \equiv Q_n(p'_i, d) \pmod{2}$ can be proven with similar arguments. ■

Theorem 3.7.9. *If $d_1, d_2 \in \mathcal{D}$, then*

$$B([C_{d_1}], [C_{d_2}]) \equiv \sum_{n=1}^N D_n(d_1, d_2) \pmod{2}.$$

Proof. If at least one of $d_1(n)$ and $d_2(n)$ is empty for every $1 \leq n \leq N$, then all intersections of $\rho(C'_{d_1})$ and $\rho(C'_{d_2})$ are transversal intersections since these loops have segments on distinct components of ρ . Suppose $d_1(n) = acb$ for some single letters a, b of v_n and subsequence c of v_n . The subsequence $d_1(n)$ of v_n corresponds to an arc $C'_{d_1}(n)$ in S'_n with initial point

a and terminal point b . The transversal intersections of the arc $\rho(C'_{d_1}(n))$ and $\rho(C'_{d_2})$ are in bijective correspondence with the arrows of α with one endpoint on arc $C_{d_1}(n)$ and the other on any arc $C_{d_2}(k)$, with $k \neq n$ since $d_2(n) = \emptyset$. If an arrow of α has one endpoint on $C_{d_1}(n)$ and the other on $C_{d_2}(k)$, then it corresponds to a letter of p that $d_1(n)$ and $d_2(k)$ have in common. Therefore the total number of intersections of $\rho(C'_{d_1}(n))$ and $\rho(C'_{d_2})$ is $\sum_{k \neq n} \#(o(d_1(n)) \cap o(d_2(k)))$, with $\#(o(d_1(n)) \cap o(d_2(k))) = 0$ for any k with $d_2(k) = \emptyset$. Hence $B([C_{d_1}], [C_{d_2}]) \equiv \sum_{n=1}^N D_n(d_1, d_2) \pmod{2}$ since $\delta_n(d_1(n), d_2(n)) = \epsilon_n^P(d_1(n), d_2(n)) = \sum_{k \neq n} \#(o(d_1(n)) \cap o(d_2(k))) = 0$ when $d_1(n)$ is an empty sequence and $\delta_n(d_1(n), d_2(n)) = \epsilon_n^P(d_1(n), d_2(n)) = 0$ when $d_2(n)$ is an empty sequence.

Now remove the hypothesis that at least one of $d_1(n)$ and $d_2(n)$ is empty for every $1 \leq n \leq N$. Again suppose $d_1(n) = acb$ for some single letters a, b of v_n and subsequence c of v_n . If $d_2(n) = \emptyset$, then the total number of intersections of $\rho(C'_{d_1}(n))$ and $\rho(C'_{d_2})$ is $\sum_{k \neq n} \#(o(d_1(n)) \cap o(d_2(k))) = D_n(d_1, d_2)$ as shown above since $\delta_n(d_1(n), \emptyset) = \epsilon_n^P(d_1(n), \emptyset) = 0$.

Now suppose $d_2(n) = yxz$ for some single letters y, z of v_n and subsequence x of v_n . For any $k \neq n$, the intersections of the segments $\rho(C'_{d_1}(n))$ and $\rho(C'_{d_2}(k))$ are transversal intersections. These intersections are in bijective correspondence with the arrows of α with one endpoint on arc $C_{d_1}(n)$ and the other on arc $C_{d_2}(k)$. Such an arrow corresponds to a letter of p that $d_1(n)$ and $d_2(k)$ have in common. Therefore the total number of intersections of $\rho(C'_{d_1}(n))$ and the segments of $\rho(C'_{d_2})$ not on $\rho(S'_n)$ is $\sum_{k \neq n} \#(o(d_1(n)) \cap o(d_2(k)))$.

The intersections of $\rho(C'_{d_1})$ and $\rho(C'_{d_2})$ on $\rho(S'_n)$ may also contribute to $B([C_{d_1}], [C_{d_2}])$. Consider the loop σ in Σ that is the “push off” of $\rho(C'_{d_1})$ to the right in a neighborhood of ρ . The intersections of $\rho(C'_{d_1})$ and $\rho(C'_{d_2})$ on $\rho(S'_n)$ are transversal intersections of σ and $\rho(C'_{d_2})$ in a neighborhood of $\rho(S'_n)$.

The six possible cyclic orders of the points a, b, y, z on S'_n are:

- (1) a, y, b, z ;
- (2) a, z, b, y ;
- (3) a, y, z, b ;
- (4) a, z, y, b ;
- (5) a, b, y, z ;
- (6) a, b, z, y ,

with the possibility that a, y, z and b, y, z are not pairwise distinct where appropriate (but a, b and y, z are distinct). For these six cases, consider a neighborhood of the loop $\rho(S'_n)$ in Σ as depicted in Figure 3.7. The dashed segments in the figure represent segments of σ and the thickened lines represent the segment $\rho(C'_{d_2}(n))$. The segments of σ immediately before and after $\rho(C'_{d_1}(n))$ are not indicated because they depend on the signs of crossings a and b with respect to $\rho(S'_n)$. The segments of $\rho(C'_{d_2}(n))$ immediately before and after $\rho(C'_{d_2}(n))$ are also not indicated because they depend on the signs of crossings y and z with respect to $\rho(S'_n)$. If the sign of one of the crossings is -1 (respectively +1), then the letter it is labeled by appears in $A_n \cup A'_n$ (respectively does not appear in $A_n \cup A'_n$).

In each of the six cases, there may be transversal intersections of σ and $\rho(C'_{d_2}(n))$ in neighborhoods of crossings $a, b, y,$ and z . If the sign of crossing a with respect to $\rho(S'_n)$ is -1 and a, y, z are pairwise distinct, then there is one intersection of σ and $\rho(C'_{d_2}(n))$ in a neighborhood of crossing a in cases (2), (4), and (6). It is possible for $a = y$ in Cases (1), (2), (3), and (6), and if so there are no intersections of σ and $\rho(C'_{d_2}(n))$ in a neighborhood of crossing $a = y$. It is possible for $a = z$ in Cases (1), (2), (4), and (5), and if so there are two intersections of σ and $\rho(C'_{d_2}(n))$ in a neighborhood of crossing $a = z$ when the sign of crossing $a = z$ is -1 and none when the sign of crossing $a = z$ is +1. Observe that there are no intersections of σ and $\rho(C'_{d_2}(n))$ in a neighborhood of crossing a otherwise.

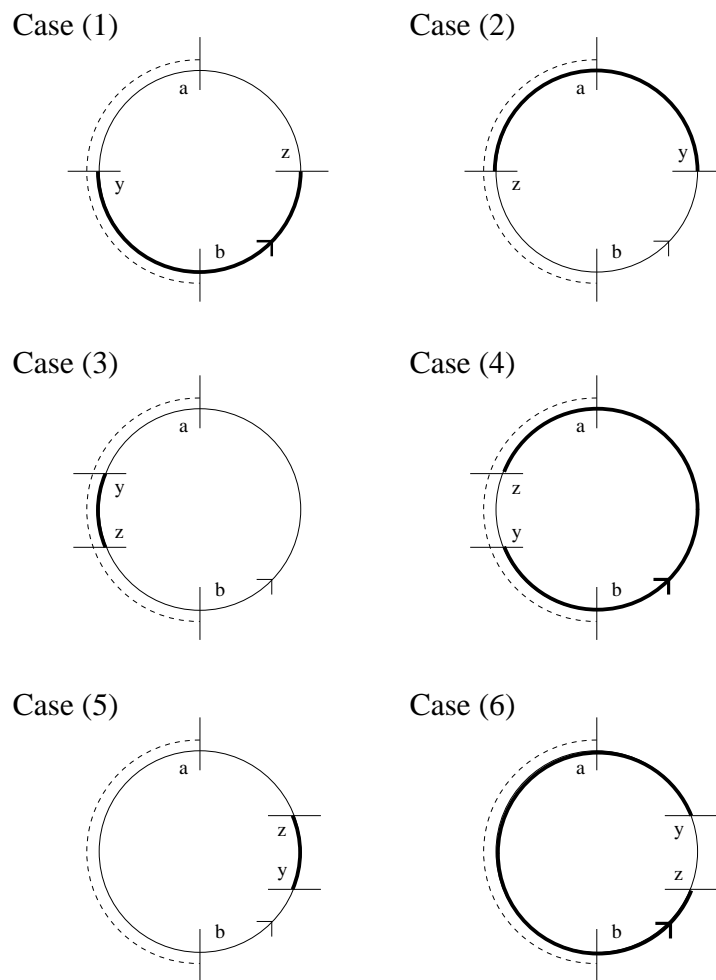


Figure 3.7: Neighborhood of the loop $\rho(S'_n)$.

If the sign of crossing b with respect to $\rho(S'_n)$ is $+1$ and b, y, z are pairwise distinct, then there is one intersection of σ and $\rho(C'_{d_2}(n))$ in a neighborhood of crossing b in cases (1), (4), and (6). It is possible for $b = y$ in Cases (1), (2), (4), and (5), and if so there are two intersections of σ and $\rho(C'_{d_2})$ in a neighborhood of crossing $b = y$ when the sign of crossing $b = y$ is $+1$ and none when the sign of crossing $b = y$ is -1 . It is possible for $b = z$ in Cases (1), (2), (3), and (6), and if so there are no intersections of σ and $\rho(C'_{d_2})$ in a neighborhood of crossing $b = z$. Observe that there are no intersections of σ and $\rho(C'_{d_2})$ in a neighborhood of crossing b otherwise.

Assume a, b, y, z are pairwise distinct. If the sign of crossing y with respect to $\rho(S'_n)$ is $+1$, then there is one intersection of σ and $\rho(C'_{d_2})$ in a neighborhood of crossing y in cases (1), (3), and (4). If the sign of crossing z with respect to $\rho(S'_n)$ is -1 , then there is one intersection of σ and $\rho(C'_{d_2})$ in a neighborhood of crossing z in cases (2), (3), and (4). Observe that there are no intersections of σ and $\rho(C'_{d_2})$ in these neighborhoods of crossings y and z otherwise when a, b, y, z are pairwise distinct.

Combining the intersection information in the previous three paragraphs, notice that the total number of transversal intersections of σ and $\rho(C'_{d_2})$ in neighborhoods of crossings $a, b, y,$ and z is given by $\epsilon_n^P(d_1(n), d_2(n))$.

The only remaining transversal intersections of σ and $\rho(C'_{d_2})$ in a neighborhood of $\rho(S'_n)$ in Σ to consider are intersections of σ and $\rho(C'_a(n))$ that do not occur in the four crossing neighborhoods discussed above. In a neighborhood of a self-intersection of $\rho(C'_{d_1}(n)) \cap \rho(C'_{d_2}(n))$, there are two crossings involving σ and the interior of $\rho(C'_{d_2}(n))$. These two crossings have opposite signs, so they do not contribute to $B([C_{d_1}], [C_{d_2}])$. Therefore we only consider those intersections of σ and $\rho(C'_{d_2}(n))$ that do not result from self-intersections of $\rho(C'_{d_1}(n)) \cap \rho(C'_{d_2}(n))$. Such intersections bijectively correspond to arrows in $arr_{n,n}(\alpha)$ with one endpoint on arc $C_{d_1}(n)$ and the other on $C_{d_2}(n)$, provided both endpoints are not on the arc $C_{d_1}(n) \cap C_{d_2}(n)$. In the following arguments, let the labels s_1, s_2, s_3, s_4 denote

subsequences of the word v_n . In Case (1), if $v_n = a s_1 y s_2 b s_3 z s_4$, then these arrows correspond to letters that s_1 and s_2 have in common, s_1 and s_3 have in common, or s_2 and s_3 have in common. In Case (2), if $v_n = a s_1 z s_2 b s_3 y s_4$, then these arrows correspond to letters that s_1 and s_4 have in common, s_2 and s_4 have in common, or s_2 and s_1 have in common. In Case (3), if $v_n = a s_1 y s_2 z s_3 b s_4$, then these arrows correspond to letters that subsequences s_1 and s_2 have in common or letters that s_3 and s_2 have in common. In Case (4), if $v_n = a s_1 z s_2 y s_3 b s_4$, then these arrows correspond to letters that s_1 and s_4 have in common, s_2 and s_3 have in common, s_2 and s_4 have in common, s_2 and s_1 have in common, or s_3 and s_4 have in common. In Case (5), if $v_n = a s_1 b s_2 y s_3 z s_4$, then these arrows correspond to letters that subsequences s_1 and s_3 have in common. In Case (6), if $v_n = a s_1 b s_2 z s_3 y s_4$, then these arrows correspond to letters that s_1 and s_4 have in common or s_1 and s_2 have in common. Hence the total number of such arrows is $\delta_n(d_1(n), d_2(n))$ in each of the six cases.

Having considered all possible intersections of segment $\rho(C'_{d_1}(n))$ and $\rho(C'_{d_2})$, we conclude that there are a total of

$$\delta_n(d_1(n), d_2(n)) + \epsilon_n^P(d_1(n), d_2(n)) + \sum_{k \neq n} \#(o(d_1(n)) \cap o(d_2(k)))$$

such intersections. Thus $B([C_{d_1}], [C_{d_2}]) \equiv \sum_{n=1}^N D_n(d_1, d_2) \pmod{2}$. ■

3.8 Statement of the Main Theorem

The Main Theorem. *A pair (a Gauss paragraph p in an alphabet E , a word-wise partition P of E) is realizable by a closed curve on S^2 if and only if the following seven conditions are satisfied:*

(i) if $i \in E$ is a double letter of a word w of p , then

$$\#(w_i) \equiv \#(p_i) \equiv \#(p'_i) \equiv 0 \pmod{2};$$

(ii) if $i \in E$ is a double letter of a word of p that is not the word w , then

$$\#(p_i \cap o(w)) \equiv \#(p'_i \cap o(w)) \equiv 0 \pmod{2};$$

(iii) any two words in p have an even number of letters in common;

(iv) if $i, j \in E$ are both double letters of the word w of p and are not w -interlaced, then

$$\#(w_i \cap w_j) \equiv 0 \pmod{2};$$

(v) P is compatible with p ;

(vi) if $i \in E$ is a double letter of a word of p and $j \in E$ is a double letter of a different word of p , then

$$\#(p_i \cap p_j) \equiv \#(p_i \cap p'_j) \equiv \#(p'_i \cap p_j) \equiv \#(p'_i \cap p'_j) \equiv 0 \pmod{2}; \text{ and}$$

(vii) P is compatible with \mathcal{D}_p .

Two corollaries of this theorem are Theorem 5.3.1 of Turaev's paper [Tur03] and the central theorem of Rosenstiehl's paper [Ros76]. Condition (i) is analogous to Gauss' famous necessary condition for closed curves with a single component in [Gau00], and Condition (iv) is analogous to the second condition of Rosenstiehl's central theorem in [Ros76]. The theorem above will be proved in the next section. The following lemmas will be used in the proof.

Lemma 3.8.1. *Suppose any two words of a Gauss paragraph p have an even number of letters in common. Then every word of p has even length.*

Proof. Recall that the length of a word w of p is

$$2(\#\{\text{double letters in } w\}) + \#\{\text{single letters in } w\}.$$

Therefore the length of w is equivalent modulo two to the number of single letters of w . But the number of single letters in w is the sum of the numbers of letters that w has in common with each other word of p . Since each term of this sum is even, it follows that w has even length. ■

Lemma 3.8.2. *If a pair (a Gauss paragraph p in an alphabet E , a word-wise partition P of E) satisfies conditions (i), (iii), and (v) of the main theorem, then a virtual string α can be constructed such that α has underlying Gauss word p and induced word-wise partition P .*

Proof. The three conditions allow us to construct a virtual string α from (p, P) in the following manner. Suppose $p = (v_1, v_2, \dots, v_N)$ and $P = (A_1 \cup A'_1, A_2 \cup A'_2, \dots, A_N \cup A'_N)$. Each word v_n of p can be written in the form $z_1 z_2 \cdots z_{2M_n}$ for some $M_n \in \mathbb{N}$ since each word of p has even length by Condition (iii) and Lemma 3.8.1. Consider a copy R_n of \mathbb{R} with distinguished points $\{1, 2, \dots, 2M_n\}$. The core circle S_n of α corresponding to v_n will be the union of R_n and a point at infinity with orientation extending the right-handed orientation on R_n . A distinguished point a of this core circle will correspond to the endpoint of an arrow in α for the letter $z_a \in E$. If word v_n has a double letter, then $z_a = z_b$ for some indices a and b . Note $a - b \equiv 1 \pmod{2}$ by Condition (i), so one of the indices a, b is odd and the other is even. A consequence of Condition (v) is that for single letters z_x, z_y of v_n in $A_n \cup A'_n$, we have $x - y \equiv 0 \pmod{2}$ if z_x, z_y appear in the same set of this union and $x - y \equiv 1 \pmod{2}$ if z_x, z_y appear in different sets of this union. Therefore the single letters of A_n are all in

either even or odd positions in $z_1 z_2 \cdots z_{2M_n}$, and the single letters of A'_n all have the other position parity.

Any two distinct words v_j, v_k of p have an even number $2m$ (with $m \in \mathbb{N} \cup \{0\}$) of letters of common by Condition (iii), m of which appear in $A_j \cup A'_j$ and m of which appear in $A_k \cup A'_k$ by the definition of word-wise partition. If a single letter of both v_j and v_k is an element of $A_j \cup A'_j$, then attach an arrow to the points in R_j and R_k corresponding to the letter, with the arrow's tail on R_j and its head on R_k . If a single letter of both v_j and v_k is not an element of $A_j \cup A'_j$, then attach an arrow to the points in R_j and R_k corresponding to the letter, but with the arrow's head on R_j and its tail on R_k . Note these arrow assignments are consistent since if a single letter appears in $A_j \cup A'_j$ then it does not appear in $A_k \cup A'_k$ by the definition of word-wise partition. Using the method just described, all arrows between distinct copies of \mathbb{R} can be attached.

If a word $v_n = z_1 z_2 \cdots z_{2M_n}$ of p has a double letter, then attach arrows on R_n using the following method. As noted above, the single letters of v_n in A_n all have the same position parity in $z_1 z_2 \cdots z_{2M_n}$ and the single letters of v_n in A'_n all have the other position parity by Condition (v). Since the words of p are considered up to circular permutation, assume for convenience that the single letters in A_n are all in odd positions and the single letters in A'_n are all in even positions. Let $i \in E$ be a double letter of v_n , and suppose z_a, z_b are the two occurrences of i in v_n . Recall that one of the indices a, b is odd and the other is even by Condition (i), and that i is an element of either A_n or A'_n by the definition of word-wise partition. Without loss of generality, we may assume a is odd and b is even. If $i \in A_n$, then attach an arrow to R_n with its tail on point a and its head on point b . If $i \in A'_n$, then attach an arrow to R_n with its head on point a and its tail on point b . Note that when a letter in E appears in A_n , the tail of its corresponding arrow is on an odd point of R_n , and when a letter in E appears in A'_n , the tail of its corresponding arrow is on an even point of R_n . Using the method just described, all arrows with tail and head on the same copy of \mathbb{R} can

be attached.

Now to construct the virtual string α . The core circle of α corresponding to v_n is the union of R_n and a point at infinity with orientation extending the right-handed orientation on R_n . It follows from the above methods of assigning arrows that the underlying Gauss paragraph of α is p and the induced word-wise partition is P . ■

Lemma 3.8.3. *Suppose a pair (a Gauss paragraph p in an alphabet E , a word-wise partition P of E) satisfies conditions (i), (iii), and (v) of the main theorem. Let α be a virtual string constructed from (p, P) as in Lemma 3.8.2, so that the Gauss paragraph of α is p and the induced word-wise partition is P . Let ρ_α be the closed curve that is the canonical realization of α on the surface Σ_α as described in Section 3.5. Then $H_1(\Sigma_\alpha; \mathbb{Z})$ is a nontrivial free abelian group with a system of free generators consisting of three types of generators: $[S_n]$ for $1 \leq n \leq N$, $[e_i]$ for all letters $i \in E$ that are double letters of the words in p , and $[C_d]$ for the cyclic sequences $d \in \mathcal{D}_p$ associated to a subset of the single letters of the words in p that will be described below.*

Proof. A graph G_p can be associated to the Gauss paragraph p in the following way: the words of p correspond to the vertices of G_p , the letters in E correspond to the edges of G_p , and the endpoints of the edge of G_p corresponding to a given letter are the vertices corresponding to the words in which the letter occurs. The graph G_p is connected since by definition p is not a disjoint union of other paragraphs. Label each edge of G_p with the letter in E to which it corresponds. Let \widehat{T} be the set of letters that label the edges of some maximal spanning tree of G_p . Notice that no double letters of the words of p appear in \widehat{T} since the edges labeled by double letters are loops on the vertices of G_p .

Although the virtual string α is an abstract combinatorial object, we can easily associate a trivalent directed graph G_α to α . Notice that the endpoints of the arrows of α divide the core circles of α into oriented arcs. The edges of G_α correspond to these arcs and the arrows

of α , with the edges directed in ways that agree with the orientation of the arcs and a “tail to head” orientation on the arrows. The vertices of G_α correspond to the arrow endpoints of α . If an edge of G_α corresponds to an arrow of α , label it with the letter of E associated to the arrow. The subgraph of G_α that consists of the edges and vertices corresponding to the core circle S_n of α will also be referred to as S_n . Note that G_α is not a tree since each S_n is a cycle in G_α . Observe that the graph G_p is isomorphic to the graph that results from G_α after contracting all S_n to points and stripping directions from the remaining edges.

Pick an edge in each S_n and let \overline{S}_n be the collection S_n excluding the selected edge. Let T be the maximal spanning tree of G_α that is the union of all the \overline{S}_n for $1 \leq n \leq N$ and the set of edges in G_α labeled with the letters in \widehat{T} . The labeled edges of G_α not in T correspond to either arrows in α with both endpoints on the same core circle or arrows between different core circles whose labels do not appear in \widehat{T} . Therefore the labeled edges of G_α not in T correspond to the letters in E that are either double letters of some word of p or single letters of words that do not appear in \widehat{T} , in other words the letters in $E \setminus \widehat{T}$. There are N remaining unlabeled edges of G_α not in T , namely one edge in each set $S_n \setminus \overline{S}_n$.

In general, if X is a connected graph that is not a tree, then the first integral homology group of X is a nontrivial free abelian group. Given a maximal spanning tree of X , the first integral homology group of X has a system of free generators that corresponds bijectively to the collection of edges of X not in the tree. Moreover, these generators can be calculated explicitly in a manner that we will utilize later. These results are well known, see for example [Mun00]. Recall G_α is a connected graph that is not a tree and T is a maximal spanning tree of G_α , so $H_1(G_\alpha) = H_1(G_\alpha; \mathbb{Z})$ has a system of free generators in bijective correspondence with the collection of $\#(E \setminus \widehat{T}) + N$ edges of G_α not in T . Since ρ is a deformation retract of both G_α and $\Sigma = \Sigma_\alpha$, it follows that $H_1(\Sigma) = H_1(\Sigma; \mathbb{Z})$ is isomorphic to $H_1(G_\alpha)$. Therefore we can describe a system of free generators for $H_1(\Sigma)$ with $\#(E \setminus \widehat{T}) + N$ generators by investigating $H_1(G_\alpha)$.

As discussed above, there are three types of edges in G_α that are not in T . The first type consists of unlabeled edges, with one in each set $S_n \setminus \overline{S}_n$. Add the edge in $S_n \setminus \overline{S}_n$ to T and “prune” the resulting graph to make a cycle graph, i.e. repeatedly remove all pendant vertices until only a cycle remains. The resulting cycle graph is S_n itself. The generator of $H_1(G_\alpha)$ represented by S_n corresponds to $[S_n] \in H_1(\Sigma)$, so $[S_n]$ is in the system of free generators for $H_1(\Sigma)$. The second type consists of edges labeled with a double letter $i \in E$ of a word v_n in p . Such an edge e has both endpoints in S_n . Add edge e to T and prune the resulting graph to a cycle graph. The resulting cycle graph corresponds to the arrow e_i of α with endpoints a_i, b_i on S_n and either arc $a_i b_i$ or arc $b_i a_i$ on S_n . Therefore the generator of $H_1(G_\alpha)$ represented by the cycle graph corresponds to either $[e_i]$ or $[e_i]^*$ in $H_1(\Sigma)$. We may assume the corresponding generator is $[e_i]$ since $[S_n]$ is also in the system of generators and $[e_i] + [e_i]^* = [S_n]$ in $H_1(\Sigma)$.

The third type of edges to consider are edges labeled with a single letter of two words of p . Such an edge e has endpoints in distinct sets S_m and S_n . Add edge e to T and prune the resulting graph to a cycle graph C , with orientation induced by the edge e . For any of the subgraphs S_k in G_α , the intersection $C \cap S_k$ is either empty or a connected subgraph of S_k . At least two such intersections must be subgraphs because e has an endpoint in S_m and an endpoint in S_n , so suppose there are $M \geq 2$ total. Then C consists of one connected subgraph in M of the S_k and M edges that connect these subgraphs to make a cycle. Therefore C corresponds to a set of M arrows between M core circles of α and a single arc on each of these M core circles. Let $C(k)$ denote the arc of core circle S_k associated to C if such an arc exists, and let $C'(k)$ denote the arc in the domain of $\rho = \rho_\alpha$ corresponding to $C(k)$. This collection of M arcs is mapped to a loop c on Σ that has segments on M different components of ρ . The generator of $H_1(G_\alpha)$ represented by C then corresponds to the equivalence class $[c] \in H_1(\Sigma)$. The orientation of cycle graph C induces an orientation on c . However, on a given segment $\rho(C'(k))$ of c , the induced orientation may not agree with the orientation

of the component $\rho(S'_k)$. Let \bar{c} be the loop on Σ that results from removing the segment $\rho(C'(k))$ from c and replacing it with the segment $\rho(S'_k \setminus (C'(k))^\circ)$. Then $[S_k] + [c] = [\bar{c}]$ in $H_1(\Sigma)$, so we may assume the generator corresponding to C is $[\bar{c}]$. Repeating this procedure for each segment $\rho(C'(k))$ of $\bigcup_{C'(k) \text{ exists}} \rho(C'(k))$ whose induced orientation does not agree with $\rho(S'_k)$, we may conclude that the generator corresponding to C is $[C_d]$ for some $d \in \mathcal{D}_p$.

We have now displayed a system of free generators for $H_1(\Sigma)$ with $\#(E \setminus \widehat{T}) + N$ generators, consisting of three kinds of generators: $[S_n]$ for $1 \leq n \leq N$, $[e_i]$ for all letters $i \in E$ that are double letters of words in p , and $[C_d]$ for sequences $d \in \mathcal{D}_p$ associated to the letters $i \in E \setminus \widehat{T}$ that are single letters of words in p . ■

3.9 Proof of the Main Theorem

Now to finally prove the main theorem.

Proof of the main theorem. Suppose the given pair (p, P) is realizable by a closed curve $\rho : \prod_{n=1}^N S'_n \rightarrow S^2$, where each S'_n is a copy of S^1 . Let α be the virtual string underlying ρ , and let S_n denote the core circle of α corresponding to S'_n . Write p as a sequence (v_1, v_2, \dots, v_N) , where word v_n corresponds to the core circle S_n . Let $e_i = (a_i, b_i)$ for $i \in E$ be a labeling of the arrows of α . If a word w of p has a double letter i , then assume the p -sets p_i and p'_i contain the letters that occur exactly once in the subsequences of w corresponding to the interiors of arcs $a_i b_i$ and $b_i a_i$, respectively, of the core circle of α associated to w . Any two cycles in S^2 have intersection number zero, so the homological intersection form B takes only the value zero in this case. We need to show the seven conditions are satisfied.

For Condition (i), notice that two letters $i, j \in E$ are w -interlaced in word $w = v_m$ of p if and only if the arrows e_i, e_j are linked on S_m . So $\#(w_i)$ is equal to the number of arrows in $arr_{m,m}(\alpha)$ that are linked with e_i . Therefore $\#(w_i) \equiv n(e_i) \pmod{2}$. Proposition 3.7.2 implies $n(e_i) = B([e_i], [S_m]) = 0$, so $\#(w_i) \equiv n(e_i) \equiv 0 \pmod{2}$. Then $\#(p_i) \equiv \sum_{k=1}^N n_{m,k}(e_i) \equiv$

$\sum_{k \neq m} n_{m,k}(e_i) \pmod{2}$ and $\#(p'_i) \equiv \sum_{k=1}^N n_{m,k}^*(e_i) \equiv \sum_{k \neq m} n_{m,k}^*(e_i) \pmod{2}$ by Proposition 3.7.3. But $n_{m,k}(e_i) = B([e_i], [S_k]) = 0$ and $n_{m,k}^*(e_i) = B([e_i]^*, [S_k]) = 0$ when $k \neq m$ by Proposition 3.7.2. Hence $\#(p_i) \equiv \#(p'_i) \equiv 0 \pmod{2}$.

For Condition (ii), suppose $i \in E$ is a double letter of a word of p and let v_n be a different word of p . Then $\#(p_i \cap o(v_n)) \equiv B([e_i], [S_n]) \equiv 0 \pmod{2}$ and $\#(p'_i \cap o(w)) \equiv B([e_i]^*, [S_n]) \equiv 0 \pmod{2}$ by Proposition 3.7.2.

For Condition (iii), suppose p has at least two words. If v_m and v_n are distinct words of p , then $B([S_m], [S_n])$ and the number of letters that these two words have in common are equivalent modulo two by Proposition 3.7.4. Since $B([S_m], [S_n]) = 0$, it follows that the number of letters that v_m and v_n have in common is an even number.

For Condition (iv), if $i, j \in E$ are both double letters of the word w of p and are not w -interlaced, then Proposition 3.7.5 implies $\#(w_i \cap w_j) \equiv B([e_i], [e_j]) \equiv 0 \pmod{2}$ since we have already shown that Condition (i) is satisfied.

For Condition (v), we must show that P satisfies the two conditions in the definition of compatibility. If $i, j \in E$ are w -interlaced in a word w of p , then $w = ix_1jx_2ix_3jx_4$ for some subsequences x_1, x_2, x_3, x_4 of w . By permuting i and j if necessary, we may assume the arrowtails a_i and a_j correspond to the first occurrences of i and j , respectively, and the arrowheads b_i and b_j correspond to the second occurrences of i and j , respectively. If a letter $k \in p_i$, then k either appears in p'_i or occurs in another word of p . The letter k appears in p'_i if and only if $k \in w_i$, so

$$\begin{aligned} \#(p_i) &= \#(w_i) + \#\{\text{single letters of } w \text{ in } x_1jx_2\} \\ &= \#(w_i) + \#\{\text{single letters of } w \text{ in } x_1\} + \#\{\text{single letters of } w \text{ in } x_2\}. \end{aligned}$$

It follows that the number of single letters in x_1 and the number of single letters in x_2 have the same parity since we have already shown that Condition (i) is satisfied. Similar

arguments for p'_i, p_j , and p'_j give that the numbers of single letters in x_1, x_2, x_3 , and x_4 all have the same parity. Therefore $\#(w_i \cap w_j) + \#\{\text{single letters of } w \text{ in } x_1\} \equiv \#(w_i \cap w_j) + \#\{\text{single letters of } w \text{ in } x_3\} \pmod{2}$. To justify Formula 5.3.3 in the proof of Theorem 5.3.1 in [Tur03], Turaev showed that

$$(3.1) \quad B([e_i], [e_j]) \equiv q(e_i, e_j) + \#\{\text{single letters of } w \text{ in } x_1\} + \#(w_i \cap w_j) + 1 \pmod{2}$$

when $\#(w_i) \equiv 0 \pmod{2}$, where q is the pairing defined in Section 3.6. Since $B([e_i], [e_j]) = 0$ and Condition (i) is satisfied, it follows that

$$\#\{\text{single letters of } w \text{ in } x_1\} + \#(w_i \cap w_j) \equiv 0 \pmod{2}$$

if and only if $q(e_i, e_j) \equiv 1 \pmod{2}$, i.e. i and j belong to different subsets of P . Hence P satisfies the first condition in the definition of compatibility.

To prove that P satisfies the second condition in the definition of compatibility, suppose $i, j \in E$ are single letters in a word w of p that appear in the union of the two subsets associated to w in P . Then $w = ix_1jx_2$ for some subsequences x_1, x_2 of w . The arrowtails of e_i and e_j are both on the same core circle of the virtual string, specifically the core circle corresponding to word w . We have already shown that Condition (iii) is satisfied, so the length of w is even by Lemma 3.8.1. Consequently $\ell(x_1) \equiv \ell(x_2) \pmod{2}$. If i and j are in the same subset of P , then $q(e_i, e_j) \equiv \ell(x_1) + 1 \equiv 0 \pmod{2}$ and $q(e_j, e_i) \equiv \ell(x_2) + 1 \equiv 0 \pmod{2}$, so $\ell(x_1) \equiv \ell(x_2) \equiv 1 \pmod{2}$. If i and j are in different subsets of P , then $q(e_i, e_j) \equiv \ell(x_1) + 1 \equiv 1 \pmod{2}$ and $q(e_j, e_i) \equiv \ell(x_2) + 1 \equiv 1 \pmod{2}$, so $\ell(x_1) \equiv \ell(x_2) \equiv 0 \pmod{2}$. Thus P is compatible with p .

Condition (vi) is satisfied because otherwise B takes non-zero values by Proposition 3.7.6. Theorems 3.7.7, 3.7.8, and 3.7.9 imply Condition (vii) since B takes only the value zero.

Now suppose the pair (p, P) satisfies all seven conditions. Let α be a virtual string constructed from (p, P) as in Lemma 3.8.2, so that the Gauss word of α is p and the induced word-wise partition is P . Write p as a sequence (v_1, v_2, \dots, v_N) of words for some $N \in \mathbb{N}$ and let S_1, S_2, \dots, S_N denote the core circles of α , where word v_n corresponds to the core circle S_n . Let $e_i = (a_i, b_i)$ for $i \in E$ be a labeling of the arrows of α . If a word w of p has a double letter i , then assume the p -sets p_i and p'_i contain the letters that occur exactly once in the subsequences of w corresponding to the interiors of arcs $a_i b_i$ and $b_i a_i$, respectively, of the core circle of α associated to w . Using the virtual string α , we will show that the pair (p, P) is realizable by a closed curve on S^2 .

If the surface $\Sigma = \Sigma_\alpha$ constructed in Section 3.5 is a 2-disc with holes, then Σ embeds in S^2 , so the canonical realization of α in Σ gives a realization of α in S^2 . If the intersection form $B : H_1(\Sigma) \times H_1(\Sigma) \rightarrow \mathbb{Z}$ takes only even values, then Σ is a disc with holes by the classification of compact surfaces. Lemma 3.8.3 implies that $H_1(\Sigma)$ is a nontrivial free abelian group with a system of free generators consisting of three types of generators: $[S_n]$ for $1 \leq n \leq N$, $[e_i]$ for all letters $i \in E$ that are double letters of words of p , and $[C_d]$ for the cyclic sequences $d \in \mathcal{D}_p$ associated to a subset of the single letters of the words in p as described in the proof of the lemma. To show that B takes only even values on $H_1(\Sigma)$, it suffices to show B takes only even values on these generators.

For three cases, Turaev's arguments in the proof of Theorem 5.3.1 in [Tur03] can be generalized as follows. If $i \in E$ is a double letter of the word $w = v_m$ of p , then $B([e_i], [S_m]) = n(e_i)$ by Proposition 3.7.2. But $n(e_i) \equiv \#(w_i) \pmod{2}$, so $B([e_i], [S_m])$ is even by Condition (i). Suppose $j \in E$ is another double letter of word w . If the arrows e_i and e_j are not linked, then $B([e_i], [e_j]) \equiv \#(w_i \cap w_j) \equiv 0 \pmod{2}$ by Proposition 3.7.5 and conditions (i) and (iv). If e_i and e_j are linked, then first assume that their endpoints lie in the cyclic order a_i, a_j, b_i, b_j around S_m , with the subsequence x of w corresponding to the interior of the arc $a_i a_j$ on S_m . If i, j appear in the same subset of P , then $q(e_i, e_j) \equiv 0 \pmod{2}$ by the construction of α

and $\#(w_i \cap w_j) + \#\{\text{single letters in } x\} \equiv 1 \pmod{2}$ by Condition (v). Therefore $B([e_i], [e_j])$ is even by Formula (3.1) and Condition (i). If i, j appear in different subsets of P , then $q(e_i, e_j) \equiv 1 \pmod{2}$ by the construction of α and $\#(w_i \cap w_j) + \#\{\text{single letters in } x\} \equiv 0 \pmod{2}$ by Condition (v). Therefore $B([e_i], [e_j])$ is even by Formula (3.1) and Condition (i). The case where the arrows e_i and e_j are linked and their endpoints are assumed to lie in the cyclic order a_i, b_j, b_i, a_j around S_m follows from the previous case and the skew-symmetry of B .

We will now consider all other cases. If $i \in E$ is a double letter of the word v_m of p and $j \in E$ is a double letter of a different word v_n of p , then $B([e_i], [e_j]) \equiv \#(p_i \cap p_j) \equiv 0 \pmod{2}$ by Proposition 3.7.6 and Condition (vi). Moreover, $B([e_i], [S_n]) \equiv \#(p_i \cap o(v_n)) \equiv 0 \pmod{2}$ by Proposition 3.7.2 and Condition (ii). Proposition 3.7.4 implies $B([S_m], [S_n])$ is equivalent modulo two to the number of letters that the words v_m and v_n of p have in common, so $B([S_m], [S_n]) \equiv 0 \pmod{2}$ by Condition (iii). Since P is compatible with \mathcal{D}_p by Condition (vii), it follows from theorems 3.7.7, 3.7.8, and 3.7.9 that

$$B([S_n], [C_{d_1}]) \equiv B([e_i], [C_{d_1}]) \equiv B([C_{d_1}], [C_{d_2}]) \equiv 0 \pmod{2}$$

for all $1 \leq n \leq N$, double letters $i \in E$ of the words in p , and $d_1, d_2 \in \mathcal{D}_p$.

We have proved that B takes only even values on $H_1(\Sigma)$, which implies Σ is a disc with holes. Then Σ embeds in S^2 , giving a realization of α in S^2 induced by the canonical realization of α in Σ . Thus the pair (p, P) is realizable by a closed curve on S^2 . ■

3.10 Additional Results about Gauss Paragraphs

Throughout this section, let p be a Gauss paragraph in an alphabet E .

Lemma 3.10.1. *Suppose $i, j \in E$ are w -interlaced in a word w of p . Write w in the*

form $ix_1jx_2ix_3jx_4$ for some subsequences x_1, x_2, x_3, x_4 of w , and assume $p_i = o(ix_1jx_2i)$, $p'_i = o(ix_3jx_4i)$, $p_j = o(jx_2ix_3j)$, and $p'_j = o(jx_4ix_1j)$. Then

$$\begin{aligned}\#(p_i \cap p_j) &= \#(w_i \cap w_j) + \text{out}(x_2) \\ \#(p_i \cap p'_j) &= \#(w_i \cap w_j) + \text{out}(x_1) \\ \#(p'_i \cap p_j) &= \#(w_i \cap w_j) + \text{out}(x_3) \\ \#(p'_i \cap p'_j) &= \#(w_i \cap w_j) + \text{out}(x_4),\end{aligned}$$

where $\text{out}(x_k) \in \mathbb{N} \cup \{0\}$ denotes the number of single letters of w in x_k , i.e. the number of letters that the subsequence x_k has in common with the other words of p .

Proof. Note $\#(p_i \cap p_j)$ is the number of letters that occur exactly once in both p_i and p_j . There are three types of such letters: letters in both x_1 and x_3 , letters in both x_2 and x_4 , and letters in $\text{out}(x_2)$. A letter appears in both x_1 and x_3 or both x_2 and x_4 if and only if it appears in $\#(w_i \cap w_j)$. Hence $\#(p_i \cap p_j) = \#(w_i \cap w_j) + \text{out}(x_2)$. The remaining three claims can be proved similarly. \blacksquare

Note that when p is a one-word Gauss paragraph, the above lemma implies $\#(w_i \cap w_j) = \#(p_i \cap p_j) = \#(p_i \cap p'_j) = \#(p'_i \cap p_j) = \#(p'_i \cap p'_j)$ since $\text{out}(x_2) = \text{out}(x_1) = \text{out}(x_3) = \text{out}(x_4) = 0$.

Lemma 3.10.2. *Suppose $i \in E$ occurs twice in a word w of p . Write w in the form ix_1ix_2 for some subsequences x_1, x_2 of w , and assume $p_i = o(ix_1i)$ and $p'_i = o(ix_2i)$. Then*

$$\begin{aligned}\#(p_i) &= \#(w_i) + \text{out}(x_1) \\ \#(p'_i) &= \#(w_i) + \text{out}(x_2)\end{aligned}$$

Proof. If $j \in p_i$, then j either appears in p'_i or occurs in another word of p . The letter j appears in p'_i if and only if $j \in w_i$. The letter j occurs in another word of p if and only if

$j \in \text{out}(x_1)$. The equation involving p'_i follows from similar arguments. ■

Theorem 3.10.3. *Suppose $i, j \in E$ are w -interlaced in a word w of p . Write w in the form $ix_1jx_2ix_3jx_4$ for some subsequences x_1, x_2, x_3, x_4 of w , and assume $p_i = o(ix_1jx_2i)$ and $p'_i = o(ix_3jx_4i)$. Then the following are equivalent:*

(i) $\#(p_i \cap p_j) \equiv \#(p_i \cap p'_j) \equiv \#(p'_i \cap p_j) \equiv \#(p'_i \cap p'_j) \pmod{2}$;

(ii) $\text{out}(x_1) \equiv \text{out}(x_2) \equiv \text{out}(x_3) \equiv \text{out}(x_4) \pmod{2}$;

(iii) $\text{out}(x_1jx_2) \equiv \text{out}(x_2ix_3) \equiv \text{out}(x_3jx_4) \equiv \text{out}(x_4ix_1) \equiv 0 \pmod{2}$;

(iv) $\#(p_i) \equiv \#(p'_i) \equiv \#(w_i) \pmod{2}$ and $\#(p_j) \equiv \#(p'_j) \equiv \#(w_j) \pmod{2}$.

Proof. Statements (i) and (ii) are equivalent via Lemma 3.10.1. Statements (ii) and (iii) are equivalent since i and j are double letters of the word w . Statements (iii) and (iv) are equivalent via Lemma 3.10.2. ■

Proposition 3.10.4. *Suppose $i \in E$ is a double letter of a word w in p and $j \in E$ is a double letter of a different word w' in p . If*

$$\#(p_i \cap p_j) \equiv \#(p_i \cap p'_j) \equiv \#(p'_i \cap p_j) \equiv \#(p'_i \cap p'_j) \pmod{2},$$

then p_i and p'_i each have an even number of letters in common with word w' , and hence the words w and w' have an even number of letters in common.

Proof. Note $\#(p_i \cap p_j) + \#(p_i \cap p'_j) \equiv \#(p'_i \cap p_j) + \#(p'_i \cap p'_j) \equiv 0 \pmod{2}$ by hypothesis. The first and second sums are equal to the number of letters that p_i and p'_i , respectively, have in common with word w' . If both p_i and p'_i have an even number of letters in common with w' , then w must have an even number of letters in common with w' . ■

3.11 Open Questions

- It would be interesting to write a computer program that implements an algorithm for testing whether a given pair (a Gauss paragraph, a word-wise partition) satisfies the conditions of the main theorem in this chapter. If a given word of the Gauss paragraph has a double letter, then some of the conditions on that word are consequences of the others. Incorporating the detection of such relationships into the algorithm would obviously make it more efficient.
- Any virtual string with an even number of endpoints on each core circle induces an underlying word-wise partition as described in Section 3.6. What are the repercussions of this fact for closed curves on surfaces other than S^2 ?

Chapter 4

Invariants and Virtual Strings

4.1 Introduction

In Chapter 3 we displayed that a virtual string has an underlying Gauss paragraph in some alphabet set. We also introduced the concept of a word-wise partition of an alphabet set with respect to a given Gauss paragraph, and then showed how a virtual string with an even number of arrow endpoints on each of its core circles naturally induces a word-wise partition of the alphabet set. Let us again restrict our attention to such virtual strings in this chapter.

Notation. Given two arrow endpoints y, z on a core circle of a virtual string, let $\tau(yz)$ denote the number of arrow endpoints on the interior of the arc yz . Note $\tau(yz) \equiv \tau(zy) \pmod{2}$ since there are an even number of arrow endpoints on the core circle.

4.2 A Homeomorphism Invariant of Closed Curves

Definition 4.2.1. Let α be a virtual string and suppose S is a core circle of α . Let $e = (a, b)$ and $f = (c, d)$ be arrows in α with each arrow having at least one endpoint on S . Define e to be α -equivalent to f with respect to S if one of the following conditions is satisfied:

- (i) $e = f$;
- (ii) all of the endpoints a, b, c, d are on S and $\tau(ac) \equiv 1 \pmod{2}$;
- (iii) both endpoints of e and only one endpoint of f , call it z , are on S and $\tau(az) \equiv 1 \pmod{2}$;
- (iv) only one endpoint of e , call it y , and both endpoints of f are on S and $\tau(yz) \equiv 1 \pmod{2}$;
- (v) only one endpoint of e , call it y , and only one endpoint of f , call it z , are on S and $\tau(yz) \equiv 1 \pmod{2}$.

This notion of α -equivalence is an equivalence relation because α has an even number of arrow endpoints on each of its core circles.

Let α be a virtual string and suppose S is a core circle of α . Label the arrows of α with the letters in some set E . Let p be the Gauss paragraph in E underlying α and let w be the word of p that corresponds to S . The relation of α -equivalence on the arrows with at least one endpoint on S has at most two equivalence classes. These classes partition this set of arrows into two subsets, and these subsets correspond to a bipartition of the letters in E that occur in the word w . Suppose α induces the bipartition $Z \cup Z'$ of the letters in w . The set Z can be decomposed into three subsets $Z = Z_d \cup Z_t \cup Z_h$, where Z_d, Z_t , and Z_h contain double letters of w , single letters of w for which the corresponding arrows of α have tails on S , and single letters of w for which the corresponding arrows of α have heads on S , respectively. The set Z' can be decomposed into three subsets Z'_d, Z'_t, Z'_h in an analogous manner.

Lemma 4.2.2. *Let α be a virtual string with underlying Gauss paragraph p and induced word-wise partition P . Then the two subsets associated to a word of p by P are the sets $Z_d \cup Z_t$ and $Z'_d \cup Z'_t$ constructed above.*

Proof. Let w be a word of p and let S be the core circle of α that corresponds to w . A letter appears in the two subsets associated to w in P if and only if the arrow in α corresponding to the letter has its tail on S , that is if and only if the letter appears in $(Z_d \cup Z_t) \cup (Z'_d \cup Z'_t)$. Suppose the tails a and c of two arrows e and f , respectively, are on S . Then e, f are α -equivalent if and only if $\tau(ac) \equiv 1 \pmod{2}$, that is if and only if e and f are equivalent in the sense of the arrow equivalence defined in Section 3.6. ■

Definition 4.2.3. Let α be a virtual string with underlying Gauss paragraph p in an alphabet E and induced word-wise partition P of E . We can define a related word-wise partition \overline{P} of E in the following manner. Let w be a word in p . The α -equivalence relation induces a bipartition $Z \cup Z'$ of the letters in w with $Z = Z_d \cup Z_t \cup Z_h$ and $Z' = Z'_d \cup Z'_t \cup Z'_h$ as discussed above. The two subsets associated to w in \overline{P} are defined to be $Z_d \cup Z'_h$ and $Z'_d \cup Z_h$. The fact that \overline{P} is indeed a word-wise partition is verified in the proof of the next lemma. Observe that P and \overline{P} are the same word-wise partition when p is a Gauss word, i.e. a one-word Gauss paragraph.

Lemma 4.2.4. *The collection \overline{P} of subsets of the alphabet E just defined is a word-wise partition of E .*

Proof. We must show that \overline{P} is a partition of E that satisfies the three properties of a word-wise partition. By definition, \overline{P} associates two disjoint subsets to every word in the Gauss paragraph p . If $i \in E$, then the head of the arrow corresponding to i lies on some core circle of α . The letter i appears in exactly one subset in \overline{P} , specifically in exactly one of the two subsets in \overline{P} associated to the word of p corresponding to this core circle. Hence \overline{P} is a partition of E . Let v and w be two different words in p . The α -equivalence relation induces bipartitions $Y \cup Y'$ and $Z \cup Z'$ of the letters in v and w , respectively, with $Y = Y_d \cup Y_t \cup Y_h$, $Y' = Y'_d \cup Y'_t \cup Y'_h$, $Z = Z_d \cup Z_t \cup Z_h$, and $Z' = Z'_d \cup Z'_t \cup Z'_h$ using the conventions established above. Any double letter of w appears in exactly one of the sets

Z_d, Z'_d because P is a word-wise partition. Hence $Z_d \cup Z'_d$ is a bipartition of the set of double letters in w . Suppose the words v and w have an even number $2n$ of letters in common. Since P is a word-wise partition, it follows from Lemma 4.2.2 that n of these letters appear in $Z_t \cup Z'_t$ and n of them appear in $Y_t \cup Y'_t$. Let $i \in E$ be a letter that v and w have in common. If $i \in Z_t \cup Z'_t$, then $i \notin Y_t \cup Y'_t$, so i must appear in $Y_h \cup Y'_h$. If $i \in Y_t \cup Y'_t$, then $i \notin Z_t \cup Z'_t$, so i must appear in $Z_h \cup Z'_h$. Therefore n of the letters that v and w have in common appear in $(Y_d \cup Y'_h) \cup (Y'_d \cup Y_h)$ and n of them appear in $(Z_d \cup Z'_h) \cup (Z'_d \cup Z_h)$. Thus \overline{P} is indeed a word-wise partition of E . \blacksquare

Definition 4.2.5. Given a virtual string α , its *inverse string* $\overline{\alpha}$ is the virtual string obtained by interchanging the heads and tails of all the arrows of α . Notice that a virtual string and its inverse string have the same underlying Gauss paragraph.

Theorem 4.2.6. *Suppose a virtual string α has underlying Gauss paragraph p in an alphabet E and induced word-wise partition P of E . If any two words of p have an even number of letters in common and $\tau(ab) \equiv \tau(ba) \equiv 0 \pmod{2}$ for any arrow (a, b) with both endpoints on the same core circle of α , then the inverse string $\overline{\alpha}$ induces the word-wise partition \overline{P} of E .*

Proof. For any word w of p , the α -equivalence relation induces a bipartition $Z \cup Z'$ of the letters in w with $Z = Z_d \cup Z_t \cup Z_h$ and $Z' = Z'_d \cup Z'_t \cup Z'_h$ as constructed before. The two subsets associated to w in P are $Z_d \cup Z_t$ and $Z'_d \cup Z'_t$ by Lemma 4.2.2. Let \overline{A} and \overline{A}' be the two subsets associated to w in the word-wise partition induced by $\overline{\alpha}$.

Let S be the core circle of α that corresponds to w , and suppose (a, b) and (c, d) are arrows in α with all of the endpoints a, b, c, d on S . There are six possible cyclic orders of the points a, b, c, d around S : (1) a, c, b, d ; (2) a, d, b, c ; (3) a, c, d, b ; (4) a, d, c, b ; (5) a, b, c, d ; and (6) a, b, d, c . We need to show that $\tau(ac) \equiv \tau(bd) \pmod{2}$ in each of these six cases. For the arcs $ab, ba, cd, dc \subset S$, we have that $\tau(ab) \equiv \tau(ba) \equiv \tau(cd) \equiv \tau(dc) \equiv 0 \pmod{2}$ by

hypothesis. In case (1),

$$\begin{aligned}
\tau(ac) + \tau(bd) &\equiv \tau(ac) + \tau(bd) + 2\tau(cb) + 2 \\
&\equiv (\tau(ac) + \tau(cb) + 1) + (\tau(cb) + \tau(bd) + 1) \\
&\equiv \tau(ab) + \tau(cd) \equiv 0 \pmod{2}.
\end{aligned}$$

In case (2),

$$\begin{aligned}
\tau(ac) + \tau(bd) &\equiv (\tau(ab) + \tau(bc) + 1) + (\tau(ba) + \tau(ad) + 1) \\
&\equiv \tau(bc) + \tau(ad) \\
&\equiv \tau(bc) + \tau(ad) + 2\tau(db) + 2 \\
&\equiv (\tau(ad) + \tau(db) + 1) + (\tau(db) + \tau(bc) + 1) \\
&\equiv \tau(ab) + \tau(dc) \equiv 0 \pmod{2}.
\end{aligned}$$

In case (3),

$$\begin{aligned}
\tau(ac) + \tau(bd) &\equiv \tau(ac) + (\tau(ba) + \tau(ac) + \tau(cd) + 2) \\
&\equiv \tau(ba) + \tau(cd) \equiv 0 \pmod{2}.
\end{aligned}$$

In case (4),

$$\begin{aligned}
\tau(ac) + \tau(bd) &\equiv (\tau(ad) + \tau(dc) + 1) + (\tau(ba) + \tau(ad) + 1) \\
&\equiv \tau(dc) + \tau(ba) \equiv 0 \pmod{2}.
\end{aligned}$$

In case (5),

$$\begin{aligned}
\tau(ac) + \tau(bd) &\equiv (\tau(ab) + \tau(bc) + 1) + (\tau(bc) + \tau(cd) + 1) \\
&\equiv \tau(ab) + \tau(cd) \equiv 0 \pmod{2}.
\end{aligned}$$

In case (6),

$$\begin{aligned}
\tau(ac) + \tau(bd) &\equiv (\tau(ab) + \tau(bd) + \tau(dc) + 2) + \tau(bd) \\
&\equiv \tau(ab) + \tau(dc) \equiv 0 \pmod{2}.
\end{aligned}$$

Therefore, in the sense of the arrow equivalence defined in Section 3.6, two arrows (a, b) and

(c, d) with endpoints on the same core circle of α are equivalent if and only if the reversed arrows (b, a) and (d, c) of $\bar{\alpha}$ are equivalent. Since the arrows (a, b) and (c, d) correspond to double letters in w , it follows that the subsets associated to w by P and the subsets associated to w by \bar{P} agree on the double letters of w . Hence we may assume that $\bar{A} = Z_d \cup Y'$ and $\bar{A}' = Z'_d \cup Y$, where Y', Y are some sets of single letters of w .

If $i \in Z_t \cup Z'_t$, then $i \notin Y' \cup Y$ because i corresponds to an arrow of $\bar{\alpha}$ with its head on S but not its tail. However, if $i \in Z_h \cup Z'_h$, then $i \in Y' \cup Y$ because i corresponds to an arrow of $\bar{\alpha}$ with its tail on S but not its head. Consequently, $Y' \cup Y = Z'_h \cup Z_h$.

Suppose $i, j \in Z'_h \cup Z_h$ with the arrows $e = (a, b)$ and $f = (c, d)$ of α corresponding to i and j , respectively. Then b and d lie on S with e and f being α -equivalent if and only if $\tau(bd) \equiv 1 \pmod{2}$ on S , which in this case happens if and only if the reversed arrows (b, a) and (d, c) in $\bar{\alpha}$ are equivalent. Therefore i, j are in the same subset of $Z'_h \cup Z_h$ if and only if they are in the same subset of $Y' \cup Y$.

If the word w has no double letters, then the two subsets associated to w by \bar{P} are simply $\bar{A} = Y'$ and $\bar{A}' = Y$. In this particular case, it does not matter whether we have $Y' = Z'_h$ and $Y = Z_h$ or we have $Y' = Z_h$ and $Y = Z'_h$ because the two sets associated to a word in a word-wise partition are unordered anyway. We will now show that $Y' = Z'_h$ and $Y = Z_h$ when w has at least one double letter. Suppose $i \in Z_d$ and $j \in Z'_h$ with i and j corresponding to the arrows $e = (a, b)$ and $f = (c, d)$ in α , respectively. Then the endpoints a, b, d (but not c) are on the core circle S of α corresponding to w , with the point d on the interior of one of the arcs $ab, ba \subset S$. Since e and f are not α -equivalent, it follows that $\tau(ad) \equiv \tau(da) \equiv 0 \pmod{2}$. Note $\tau(ab) \equiv \tau(ba) \equiv 0 \pmod{2}$ by hypothesis. Therefore $\tau(ab) \equiv \tau(ad) + \tau(db) + 1 \equiv \tau(db) + 1 \equiv 0 \pmod{2}$ when d is on the interior of ab and $\tau(ba) \equiv \tau(bd) + \tau(da) + 1 \equiv \tau(bd) + 1 \equiv 0 \pmod{2}$ when d is on the interior of ba . Hence in either case the reversed arrows (b, a) and (d, c) in $\bar{\alpha}$ are equivalent, which implies $Y' = Z'_h$. Similar arguments give that $Y = Z_h$. Thus $\bar{A} = Z_d \cup Z'_h$ and $\bar{A}' = Z'_d \cup Z_h$ as desired. ■

Example 4.2.7. As discussed in Example 3.6.1, the virtual string α in Figure 3.1 has an underlying Gauss paragraph in the alphabet $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \bar{1}, \bar{2}, \bar{3}\}$ that contains the word $w = 1273512634$. The α -equivalence relation induces the bipartition $Z \cup Z' = \{1, 2, 5, 7\} \cup \{3, 4, 6\}$ of the letters in w with $Z = Z_d \cup Z_t \cup Z_h = \{1, 2\} \cup \{7\} \cup \{5\}$ and $Z' = Z'_d \cup Z'_t \cup Z'_h = \{3\} \cup \{6\} \cup \{4\}$. The two subsets of E associated to w in the word-wise partition P induced by α are $Z_d \cup Z_t = \{1, 2, 7\}$ and $Z'_d \cup Z'_t = \{3, 6\}$. Therefore the inverse string $\bar{\alpha}$ induces the word-wise partition \bar{P} with $Z_d \cup Z'_h = \{1, 2, 4\}$ and $Z'_d \cup Z_h = \{3, 5\}$ being the two subsets of E associated to w .

Let Σ be a surface and suppose $\rho, \rho' : \mathbb{H}S^1 \rightarrow \Sigma$ are two closed curves on Σ . Then the curves ρ and ρ' are called *homeomorphic* if there exists a homeomorphism $\psi : \Sigma \rightarrow \Sigma$ such that $\rho = \psi \circ \rho'$. Note that ψ does not necessarily preserve the orientation of Σ .

Theorem 4.2.8. *Two closed curves on S^2 are homeomorphic if and only if they give rise to pairs (p, P_1) and (p, P_2) consisting of a Gauss paragraph p and word-wise partitions P_1, P_2 with $P_1 = P_2$ or $\bar{P}_1 = P_2$.*

Proof. If two closed curves on S^2 are related via a homeomorphism that preserves the orientation of S^2 , then they clearly give rise to the same Gauss paragraph and word-wise partition. Therefore we must investigate how the Gauss paragraph and word-wise partition underlying a closed curve ρ on S^2 behave when the orientation of S^2 is reversed. The Gauss paragraph of a closed curve on any surface does not depend on the orientation of the surface by definition. If α is the virtual string underlying ρ , then the virtual string underlying ρ after the orientation of S^2 is reversed is the inverse string $\bar{\alpha}$. If α induces the word-wise partition P , then $\bar{\alpha}$ induces the word-wise partition \bar{P} by Theorem 4.2.6 and the main theorem of Chapter 3.

Suppose that two closed curves ρ, ρ' on S^2 give rise to the same pair (a Gauss paragraph p in an alphabet E , a word-wise partition P of E). Furthermore, suppose ρ and ρ' have

underlying virtual strings α and α' , respectively. Both α and α' have the same underlying Gauss paragraph, so they coincide except for possibly the orientation of their arrows.

First suppose that α and α' each have only one core circle S . If there is an arrow (a, b) in α such that (b, a) is an arrow in α' , then let (c, d) be another arrow of α and assume (c, d) is also an arrow in α' . If c lies on the interior of the arc $ab \subset S$ in α , then $\tau(ac) + \tau(bc) + 1 \equiv \tau(ac) + \tau(cb) + 1 \equiv \tau(ab) \equiv 0 \pmod{2}$ by condition (i) of the main theorem of Chapter 3. If c lies on the interior of the arc $ba \subset S$ in α , then $\tau(ac) \equiv \tau(ab) + \tau(bc) + 1 \equiv \tau(bc) + 1 \pmod{2}$. Therefore in both cases $\tau(ac)$ and $\tau(bc)$ have different parity. So (a, b) and (c, d) are equivalent in α if and only if (b, a) and (c, d) are not equivalent in α' . This is a contradiction because α and α' induce the same word-wise partition of E . So (d, c) must be an arrow of α' . Hence α' is homeomorphic to either α or its inverse $\bar{\alpha}$ when ρ, ρ' are closed curves with one component that give rise to the same pair (p, P) .

Next suppose α and α' each have at least two core circles. The orientations of the arrows between different core circles in α and α' must agree because the letter in E corresponding to such an arrow only appears in the subsets associated to one of the words corresponding to the core circles, specifically the word corresponding to the core circle on which the tail of the arrow lies. So α and α' coincide except for possibly the orientations of the arrows with both endpoints on the same core circle. Let S be a core circle of α . Since α underlies a closed curve on S^2 , it follows that there is an arrow (c, d) in α with c lying on S . We will show that (c, d) determines the orientation of all the arrows of α with both endpoints on S . Suppose e is an arrow with both of its endpoints a, b on S . If c lies on the arc $ab \subset S$, then we can deduce from arguments similar to those above that $\tau(ac)$ and $\tau(bc)$ have different parity. Let i and j be the letters in E corresponding to the arrows e and (c, d) , respectively. If i, j are in the same subset of P , it follows that $e = (a, b)$ if and only if $\tau(ac)$ is odd. If i, j are in different subsets of P , it follows that $e = (a, b)$ if and only if $\tau(ac)$ is even. The orientations of the arrows between distinct core circles agree in α and α' , and therefore the

orientations of the other arrows in α and α' must also agree. Hence α' is homeomorphic to α when ρ, ρ' are closed curves with multiple components that give rise to the same pair (p, P) .

Now suppose that the two closed curves ρ and ρ' on S^2 give rise to the pairs (p, P) and (p, \overline{P}) , respectively, where p is a Gauss paragraph in an alphabet E and P is a word-wise partition of E . Furthermore, suppose ρ and ρ' have underlying virtual strings α and α' , respectively. Both α and α' have the same underlying Gauss paragraph, so they coincide except for possibly the orientation of their arrows. If α and α' each have only one core circle, then P and \overline{P} are the same word-wise partition because p has only one word and it has no single letters. Therefore α' is homeomorphic to either α or its inverse $\overline{\alpha}$ in this case by previous arguments.

Suppose α and α' each have at least two core circles. Let w be a word of p and let S be the core circle of α that corresponds to w . The two subsets associated to w in P are the sets $Z_d \cup Z_t$ and $Z'_d \cup Z'_t$ described in Lemma 4.2.2. Then by definition the two subsets associated to w in \overline{P} are $Z_d \cup Z'_h$ and $Z'_d \cup Z_h$. If (c, d) is an arrow of α with only its head d on S , then the single letter j of w corresponding to (c, d) appears in $Z_h \cup Z'_h$. Since j appears in the two sets associated to w by \overline{P} , it follows that the tail of the arrow in α' corresponding to j must lie on S . So this arrow in α' is (d, c) . Suppose (a, b) is an arrow of α with both endpoints on S . We may assume that the letter $i \in E$ corresponding to (a, b) appears in Z_d rather than Z'_d . The union $Z_h \cup Z'_h$ is non-empty because ρ is a closed curve on S^2 , so let $j \in Z_h \cup Z'_h$. Then j corresponds to an arrow (c, d) of α with its head d on S , so (d, c) is an arrow of α' . If $j \in Z_h$, then $\tau(ad) \equiv 1 \pmod{2}$ by α -equivalence. So (b, a) is an arrow of α' in this case because otherwise the arrows $(a, b), (d, c)$ of α' would be equivalent although i, j appear in different subsets of \overline{P} . If $j \in Z'_h$, then $\tau(ad) \equiv 0 \pmod{2}$ by α -equivalence. So (b, a) is an arrow of α' in this case because otherwise the arrows $(a, b), (d, c)$ of α' would not be equivalent although i, j appear in the same subset of \overline{P} . Hence α' is homeomorphic to $\overline{\alpha}$ when ρ and ρ' are closed curves with multiple components that give rise to the pairs (p, P)

and (p, \overline{P}) , respectively.

We have shown that if a closed curve ρ on S^2 gives rise to the pair (a Gauss paragraph p in an alphabet E , a word-wise partition P of E) and a closed curve ρ' on S^2 gives rise to the pair (p, P) or the pair (p, \overline{P}) , then the virtual string α' underlying ρ' is homeomorphic to either the virtual string α underlying ρ or its inverse string $\overline{\alpha}$. If α' is homeomorphic to $\overline{\alpha}$, then compose ρ' with an orientation-reversing self-homeomorphism of S^2 so that we may assume that α and α' are indeed homeomorphic. Recall from Chapter 3 that ρ is a composition of the canonical realization $\rho_\alpha : \mathbb{H}S^1 \rightarrow \Sigma_\alpha$ with an embedding $\Sigma_\alpha \hookrightarrow S^2$. The closed curve ρ' is an analogous composition. Any homeomorphism $\alpha \rightarrow \alpha'$ extends to a homeomorphism $\phi : \Sigma_\alpha \rightarrow \Sigma_{\alpha'}$ that transforms ρ_α into $\rho'_{\alpha'}$. All the components in the complements of Σ_α in S^2 and $\Sigma_{\alpha'}$ in S^2 are discs, and any homeomorphism of circles extends to a homeomorphism of the discs bounded by these circles. Thus ϕ extends to a homeomorphism $S^2 \rightarrow S^2$ transforming ρ into ρ' . ■

Corollary 4.2.9. *(Theorem 5.3.3 in [Tur03]) Two single-component closed curves on S^2 are homeomorphic if and only if they give rise to the same pair (a Gauss word, a word-wise partition of the alphabet with respect to the Gauss word).*

Theorem 4.2.8 implies that the unordered triple consisting of the Gauss paragraph, the word-wise partition P , and the word-wise partition \overline{P} associated to a closed curve on S^2 is a full homeomorphism invariant of the closed curve. Corollary 4.2.9 implies that the pair consisting of the Gauss word and the word-wise partition associated to a closed curve on S^2 with a single component is a full homeomorphism invariant.

4.3 A Homotopy Invariant of Virtual Strings

A homotopy relation is defined on virtual strings with one core circle in [Tur03] that interprets the usual homotopy of closed curves on surfaces in terms of such virtual strings. We can

easily generalize this homotopy relation to virtual strings with any number of core circles. Specifically, two virtual strings are *homotopic* if they can be related by some finite sequence of homeomorphisms, the *homotopy moves* $(a)_s$, $(b)_s$, $(c)_s$ that will be defined shortly, and the inverses of these homotopy moves. Then two homotopic closed curves on a surface will have underlying virtual strings that are homotopic.

Let α be a virtual string. Suppose x and y are distinct points on the same core circle S of α such that the arc $xy \subset S$ does not contain any arrow endpoints. The move $(a)_s$ adds the arrow (x, y) to α . Suppose two given arcs of the core circles of α are disjoint and neither contains any arrow endpoints (note these two arcs may be on the same core circle or on different core circles). The move $(b)_s$ adds two arrows (x, y) and (y', x') to α , where x, x' are the endpoints of one of the given arcs and y, y' are the endpoints of the other arc. Since the endpoints x, x' and y, y' here can be chosen arbitrarily, it follows that this move has four forms. Now suppose α has three arrows $(x', y), (y', z), (z', x)$ such that the arcs xx', yy', zz' are disjoint from each other and none of them contain any arrow endpoints (again these arcs may be on the same core circle or on different core circles). The move $(c)_s$ replaces these three arrows with the three arrows $(x, y'), (y, z'), (z, x')$.

Let α be a virtual string, and label the core circles of α with S_1, S_2, \dots, S_M for a minimal $M \in \mathbb{N}$. The set of arrows $arr_{i,j}(\alpha)$ consists of the arrows with tail on S_i and head on S_j . For $e = (a, b) \in arr_{i,i}(\alpha)$, let $n(e) \in \mathbb{Z}$ denote

$$\#\{f \in arr_{i,i}(\alpha) \mid f \text{ links } e \text{ positively}\} - \#\{f \in arr_{i,i}(\alpha) \mid f \text{ links } e \text{ negatively}\}.$$

For every integer $k \geq 1$, let $U_k^m(\alpha) \in \mathbb{Z}$ denote

$$\#\{e \in arr_{m,m}(\alpha) \mid n(e) = k\} - \#\{e \in arr_{m,m}(\alpha) \mid n(e) = -k\}.$$

For integers m and n , let $U_{m,n}(\alpha) \in \mathbb{Z}$ denote $\#(arr_{n,m}(\alpha)) - \#(arr_{m,n}(\alpha))$.

We claim that all of the differences U_k^m and $U_{m,n}$ for α are preserved under the homotopy moves. If e_0 is the arrow added to α by an $(a)_s$ move, then $e_0 \in arr_{i,i}(\alpha)$ for some $1 \leq i \leq M$ and e_0 does not link any arrows in $arr_{i,i}(\alpha)$. Hence $(a)_s$ moves preserve U_k^m and $U_{m,n}$ for all integers k, m, n . If e_1 and e_2 are the two arrows added to α by a $(b)_s$ move, then either $e_1, e_2 \in arr_{i,i}(\alpha)$ for some i or $e_1 \in arr_{i,j}(\alpha)$ and $e_2 \in arr_{j,i}(\alpha)$ for some integers i, j . In the first case, $n(e_1) = -n(e_2)$ and $n(f)$ is unaffected for all other $f \in arr_{i,i}(\alpha)$. In the second case, the arrows e_1 and e_2 contribute $-1 + 1 = 0$ to $U_{i,j}$ and $+1 - 1 = 0$ to $U_{j,i}$. Hence $(b)_s$ moves preserve U_k^m and $U_{m,n}$ for all integers k, m, n .

Now consider a $(c)_s$ move. Each arrow $e = (x', y), (y', z), (z', x)$ involved before the move results in an arrow $e' = (x, y'), (y, z'), (z, x')$, respectively, after the move. First, suppose the arcs xx', yy', zz' are all on the same core circle S of α . For $e = (x', y)$ and $e' = (x, y')$, note that the endpoints z and z' either lie on the arc xy of S or the arc yx of S . If $z, z' \in xy$, then the arrows (y', z) and (z', x) contribute $+1$ and -1 to $n(e)$, respectively, and the arrows (y, z') and (z, x') each contribute 0 to $n(e')$. If $z, z' \in yx$, then the arrows (y', z) and (z', x) each contribute 0 to $n(e)$ and the arrows (y, z') and (z, x') contribute -1 and $+1$ to $n(e')$, respectively. Whether $z, z' \in xy$ or $z, z' \in yx$, all other arrows make the same contributions to $n(e)$ and $n(e')$, so $n(e) = n(e')$ in either case. Similar arguments show that $n(e) = n(e')$ for the other two possibilities for e and e' . For all other arrows $f \in arr_{i,i}(\alpha)$, the number $n(f)$ is clearly also preserved. Hence, when the arcs xx', yy', zz' are all on the same core circle, a $(c)_s$ move preserves U_k^m and $U_{m,n}$ for all integers k, m, n . Second, suppose the arcs xx', yy', zz' are all on different core circles of α . An arrow e has its tail and head on the same core circles as the tail and head of the arrow e' , respectively, so these two arrows make the same ± 1 contribution to some $U_{i,j}$ and the same ∓ 1 contribution to $U_{j,i}$. Hence, when the arcs xx', yy', zz' are all on different core circles, a $(c)_s$ move preserves U_k^m and $U_{m,n}$ for all integers k, m, n . Third, suppose two of the three arcs, say xx' and yy' , are on the same core

circle S_i and the last arc is on a different core circle S_j of α . For $e = (x', y)$ and $e' = (x, y')$, all the arrows in $arr_{i,i}(\alpha)$ make the same contributions to $n(e)$ and $n(e')$. For all other arrows $f \in arr_{i,i}(\alpha)$, the number $n(f)$ is clearly also preserved. The arrows (y', z) and (z', x) contribute $-1 + 1 = 0$ to $U_{i,j}$ and $+1 - 1 = 0$ to $U_{j,i}$, as do the arrows (y, z') and (z, x') . Hence, when two of the three arcs xx', yy', zz' are on the same core circle and the last arc is on a different core circle, a $(c)_s$ move preserves U_k^m and $U_{m,n}$ for all integers k, m, n . Thus we have justified our claim that all of the differences U_k^m and $U_{m,n}$ for α are preserved under the homotopy moves.

Definition 4.3.1. Let α be a virtual string with its core circles labeled with S_1, S_2, \dots, S_M for a minimal $M \in \mathbb{N}$. Define the polynomial $U(\alpha)$ in M variables t_1, t_2, \dots, t_M by

$$U(\alpha) = \sum_{m=1}^M \sum_{k \geq 1} U_k^m(\alpha) t_m^k + \sum_{m=1}^M \sum_{n=m+1}^M |U_{m,n}(\alpha)| t_m t_n.$$

Since all of the differences U_k^m and $U_{m,n}$ for α are preserved under the homotopy moves, it follows that the U -polynomial $U(\alpha)$ is a homotopy invariant of the virtual string α .

Example 4.3.2. Let α_1 be the virtual string in Figure 4.1, and label the left and right core circles depicted with S_1 and S_2 , respectively. Then

$$\begin{aligned} U(\alpha_1) &= U_1^1 t_1 + U_2^1 t_1^2 + U_1^2 t_2 + U_2^2 t_2^2 + U_3^2 t_2^3 + |U_{1,2}| t_1 t_2 \\ &= (1 - 3) t_1 + (1 - 0) t_1^2 + (3 - 0) t_2 + (0 - 0) t_2^2 + (0 - 1) t_2^3 + |2 - 0| t_1 t_2 \\ &= -2t_1 + t_1^2 + 3t_2 - t_2^3 + 2t_1 t_2 \end{aligned}$$

Let α_2 be the virtual string in Figure 3.1, and label the core circles of α_2 with S_1, S_2, S_3 as they appear from left to right in the figure. Then

$$U(\alpha_2) = |U_{1,2}| t_1 t_2 + |U_{1,3}| t_1 t_3 + |U_{2,3}| t_2 t_3 = |2 - 2| t_1 t_2 + |0 - 0| t_1 t_3 + |1 - 1| t_2 t_3 = 0.$$

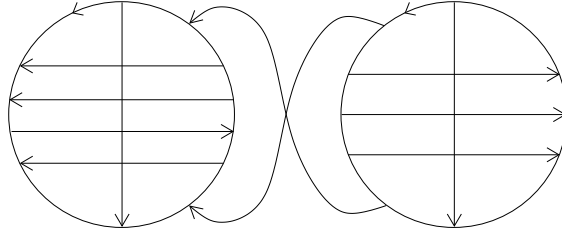


Figure 4.1: A virtual string with two core circles and eleven arrows.

Turaev defined a polynomial $u(\alpha)$ for virtual strings α with only one core circle in [Tur03]. The U -polynomial we have defined extends this polynomial to virtual strings with any number of components, i.e. $U(\alpha) = u(\alpha)$ when the virtual string α has only one core circle. The U -polynomial of a virtual string α also has the following properties:

- (i) the free term of $U(\alpha)$ is always 0 by definition;
- (ii) the degree of every two-variable term in $U(\alpha)$ is two by definition;
- (iii) if α has only one core circle, then $U(\alpha)$ has no two-variable terms;
- (iv) if α has only one core circle, then the degree of $U(\alpha)$ is less than the number of arrows in α ;
- (v) if α has more than one core circle, then the degree of $U(\alpha)$ is less than or equal to $\max\{m - 1, 2\}$, where m is the maximal number of arrows in α with both endpoints on a given core circle;
- (vi) $U(\alpha) = 0$ when α is homotopic to a trivial virtual string, i.e. a virtual string with no arrows;
- (vii) $U(\alpha) = 0$ when α is realizable by a closed curve on S^2 because every closed curve on S^2 is homotopically trivial.

4.4 Open Questions

- Do U -polynomials have any other interesting properties that were not mentioned in the previous section?
- What other information does the U -polynomial of a virtual string encode? Turaev proves several neat results about his u -polynomial in [Tur03], so it is reasonable to expect that the U -polynomial is equally intriguing.
- Any virtual string with an even number of endpoints on each core circle induces an underlying word-wise partition as described in Section 3.6. Is classification using the unordered triple (a Gauss paragraph p , a word-wise partition P , the related word-wise partition \overline{P}) of this chapter still valid for curves on any surface with such underlying virtual strings?

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Vita

William Joseph Schellhorn was born on October 27, 1977, in Cedar Rapids, Iowa. He attended Mount Mercy College in Cedar Rapids for his undergraduate studies and graduated *summa cum laude* with honors distinction in May 2000, earning a Bachelor of Science degree with majors in mathematics and computer science. In August 2000 he came to Louisiana State University to pursue graduate studies in mathematics. He earned a Master of Science degree in mathematics from Louisiana State University in May 2002. He conducted his dissertation research under the direction of professor Richard A. Litherland and is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2005.