ORBIT STRUCTURE ON THE SILOV BOUNDARY OF A TUBE DOMAIN
AND THE PLANCHEREL DECOMPOSITION OF A CAUSALLY
COMPACT SYMMETRIC SPACE,
WITH EMPHASIS ON THE RANK ONE CASE

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# Table of Contents

Acknowledgements ........................................................................................................ ii

Abstract ....................................................................................................................... v

1 Introduction ................................................................................................................ 1
  1.1 Cartan’s Contribution ........................................................................................ 2
  1.2 The Boundary of a Domain ............................................................................ 3
  1.3 Origin of the Problems .................................................................................. 4

2 Harmonic Analysis and Complex Geometry Related to Tube Domains ............... 6
  2.1 Introduction ..................................................................................................... 6
  2.2 Convex Cones and Domains of Tube Type .................................................... 6
  2.3 Function Spaces Associated to a Tube Domain ............................................. 8
  2.4 The Bounded Realization .......................................................................... 12
  2.5 The Unbounded Realization ..................................................................... 14
  2.6 Geometric Connection between the Realizations ....................................... 15

3 Geometry of Compactly Causal Symmetric Spaces .............................................. 18
  3.1 Introduction ..................................................................................................... 18
  3.2 Structure Theory for Reductive Symmetric Spaces ....................................... 18
  3.3 Causal Structures and Orientations ............................................................... 21
  3.4 The Causal Compactification of a Compactly Causal Symmetric Space .......... 25
  3.5 Causal Structure on the Silov Boundary ....................................................... 27
  3.6 Existence of Lowest Weight Representations and Spherical Vectors .......... 27
  3.7 The Function $\varpi_m$ ................................................................................. 29
  3.8 Identification of $L^2$-Spaces .................................................................... 32

4 The Finer Details in the Geometric Realization of $SO_e(2, n)/SO_e(1, n)$ .......... 33
  4.1 Introduction ..................................................................................................... 33
  4.2 Details on the Structure Theory ................................................................. 33
  4.3 A priori Group-Actions .............................................................................. 34
  4.4 Explicit Model for the Silov Boundary ......................................................... 39
  4.5 A Group-Action on the Silov Boundary ....................................................... 41
  4.6 Action of the Non-Trivial Weyl Group Element .......................................... 46
  4.7 The Realization of the Hyperboloids in the Silov Boundary ......................... 49
Abstract

We construct a $G$-equivariant causal embedding of a causally compact symmetric space $G/H$ as an open dense subset of the Silov boundary $S$ of the unbounded realization of an explicitly given Hermitian symmetric space $G_1/K_1$ of tube type. Then $S$ is an Euclidean space that is open and dense in the flag manifold $G_1/P'$, where $P'$ is a certain parabolic subgroup of $G_1$. The regular representation of $G$ on $L^2(G/H)$ is thus realized in $L^2(S)$, and we use abelian harmonic analysis in the study thereof. In particular, the holomorphic discrete series of $G/H$ is being realized in function spaces on the boundary via the Euclidean Fourier transform on the boundary.

Let $P' = L_1 N_1$ denote the Langlands decomposition of $P'$. The Levi factor $L_1$ of $P'$ then acts on the Silov boundary $S$, and the orbits $\mathcal{O}$ can be characterized completely. For $G/H$ of rank one we associate to each orbit $\mathcal{O}$ the irreducible representation

$$L^2_{\mathcal{O}_i} := \{ f \in L^2(S, dx) \mid \text{supp} \hat{f} \subset \overline{\mathcal{O}_i} \}$$

of $G_1$ and show that the representation of $G_1$ on $L^2(S)$ decompose as an orthogonal direct sum of these representations.

We show that by restriction to $G$ of the representations $L^2_{\mathcal{O}_i}$, we thus obtain the Plancherel decomposition of $L^2(G/H)$, in the sense of Delorme, and van den Ban and Schlichtkrull.
Chapter 1

Introduction

The present work describes an intimate three-way connection between the complex geometry of bounded homogeneous domains in $\mathbb{C}^n$ and their unbounded realizations, function spaces of holomorphic functions, and certain unitary representations of Lie groups. The close interconnection allows for a more geometric approach to harmonic analysis of a certain class of reductive symmetric spaces, as we will demonstrate in the course of this dissertation.

For the sake of simplicity and motivation, however, we will focus on the geometry in the present chapter and merely indicate the relation to Lie groups.

Classical harmonic analysis is, to a large extent, the theory of Poisson and Cauchy integrals, Hardy spaces and the boundary value behavior of holomorphic and harmonic functions. In the one-variable theory, the geometry plays a subordinate role in the sense that the geometry of the unit disc and upper half-plane is simple enough to not require a separate study.

In the several-variable theory, many new problems arise that make the resulting theory distinctly different from the one-variable theory. In some sense, the subject of classical harmonic analysis in several variables was born in 1907 when Poincaré ([Poi07]) discovered that the Riemann mapping theorem failed to extend to higher dimensions\(^1\). Thus arose the very difficult problem of classifying, up to holomorphic equivalence, the simply-connected domains in $\mathbb{C}^n$.

Regarding function theory, a less ambitious classification problem is already sufficiently interesting: The natural domains of interest are the \emph{domains of holomorphy}\(^2\). Even this more restricted classification problem turned out to be very difficult, so additional constraints on the domain were imposed, and the motivation came from a closer look at the unit disc.

In addition to being a simply connected domain of holomorphy in $\mathbb{C}$, the unit disc is also homogeneous, has a metric, and is symmetric. More precisely

\textbf{Homogeneity}: The automorphism group of the unit disc acts transitively on the disc;

\textbf{Metric property}: The Poincaré metric;

\textbf{Symmetry}: Let $\sigma$ denote the automorphism $\sigma : z \mapsto -z$. Then $\sigma$ has as its only fixed point the origin, and $\sigma^2 = \text{id}$. Let $a$ be any point on the disc and let $g$ denote the automorphism

---

\(^1\)The counterexample of Poincaré was to prove that the Cartesian product of two discs is not holomorphically equivalent to the unit ball in $\mathbb{C}^2$

\(^2\)Domains on which there are holomorphic functions that cannot be extended analytically into a larger domain.
mapping $a$ to 0. Then $\sigma_a = g^{-1}\sigma g$ is an automorphism that fixes the point $a$ and none other, and $\sigma_a^2 = \text{id}$.

The homogeneity and the Poincaré metric are both crucial in the role which the disc plays in the theory of automorphic forms and in the theory of Riemann surfaces, but the symmetry property will turn out to be equally important for what we will do.

1.1 Cartan’s Contribution

The first systematic study of the bounded homogeneous domains was initiated by Henri Cartan in [Car35b]. One of the most important initial results was that the automorphism group of such a domain is a Lie group, thus demonstrating a close connection to differential geometry. These domains (the bounded symmetric domains) are sometimes called Cartan domains after Élie Cartan (who introduced them in [Car35a]).

The fact that every bounded domain has a natural Riemannian metric, the Bergman metric, in presence of the symmetry, now turns a Cartan domain into a Riemannian symmetric space (introduced in the late twenties by É. Cartan). Cartan was then able to classify the bounded symmetric domains case-by-case (as opposed to using the structure theory of semisimple Lie groups which was still in its infancy at that time). In order to state Cartan’s classification, let us first note that if $D_1$ and $D_2$ are symmetric domains, so is their product $D_1 \times D_2$. A domain is irreducible if it is not the Cartesian product of lower-dimensional domains.

There are four classes of irreducible domains, each containing infinitely many members (the “classical domains”), and then there are two isolated domains (the “exceptional domains”), of dimension 16 and 27, respectively. Let $M_{m,n}$ denote the set of $m \times n$ complex matrices, and let $Z^*$ denote the conjugate transpose of a matrix $Z$. If $A \in M_n = M_{n,n}$, the notation “$A > 0$” signifies that $A$ is a positive definite matrix.

Theorem (Classification). Every irreducible classical domain is holomorphically equivalent to one of the domains on the following list:

(I) $\mathcal{A}_n = \{Z \in M_{m,n} \mid I - Z^*Z > 0\}, m \geq n \geq 1$;

(II) $\mathcal{B}_n = \{Z \in M_n \mid Z \text{ is symmetric, and } I - Z^*Z > 0\}$;

(III) $\mathcal{C}_n = \{Z \in M_n \mid Z \text{ is skew symmetric, and } I - Z^*Z > 0\}$;

(IV) $\mathcal{D}_n = \left\{z \in \mathbb{C}^n \left| \sum_{k=1}^n |z_k|^2 - 2 \sum_{k=1}^n |z_k|^2 + 1 > 0, \text{and } 1 - \sum_{k=1}^n |z_k|^2 > 0 \right\}, n \geq 5$.

The domains $\mathcal{D}_n$ are called Lie spheres or Lie balls. The restriction on dimensions is made to avoid incidental double occurrences of domains on the list due to low-dimensional domains from different classes being holomorphically equivalent.

The presence of two additional (exceptional) domains is irksome and is one of the reasons why a general approach based on structure theory of Riemannian symmetric spaces and Lie groups is indispensable. The structure theory works for all the domains - classical or not - at the same time.
(a) The $n \times n$ real symmetric matrices that are positive definite;
(b) The $n \times n$ complex Hermitian matrices that are positive definite;
(c) The $2n \times 2n$ complex Hermitian matrices that are positive definite and subject to the condition $Jz = -J^t z J$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$;
(d) The Lorentz cone in $\mathbb{R}^n$,
\[ \{ x \in \mathbb{R}^n | x_1 > (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \} . \]
(e) The set of $3 \times 3$ matrices whose entries are Cayley numbers and which are Hermitian and positive definite in a suitable sense.

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Figure 1.1: List of homogeneous self-dual cones

The new idea, which eventually led to the solution of Cartan’s classification problem, was to investigate Siegel domain, a term introduced by Pjateckii-Šapiro to describe domains that appeared in Siegel’s paper [Sie43]. In the same way that bounded symmetric domains served to generalize the unit disc, so Siegel domains are generalizations of the upper half-plane (hence sometimes called generalized upper half-planes). Every Cartan domain has a realization as a Siegel domain. The corresponding mapping is called the Cayley transform, to emphasize the analogy with the fact in classical geometry that the unit disc is holomorphically equivalent to the upper half-plane via the map
\[ z \mapsto i \frac{1 - z}{1 + z}, \]
called the Cayley transform.

For example, one shows that the half-plane realization of the domains labelled as (iv) in Cartan’s list (the Lie balls) is the tube domain $\mathbb{R}^n + iC$ over the Lorentz cone $C$ mentioned in part (d) of Figure 1.1. Similar half-plane realizations exist for the other Cartan domains, and we return to the matter in the following chapter.

1.2 The Boundary of a Domain

Cartan domains are diffeomorphic to simply connected, connected bounded open subspaces of $\mathbb{C}^n$, although bounded, classical domains are not compact in $\mathbb{C}^n$. They are therefore diffeomorphic (not holomorphically) to $\mathbb{R}^{2n}$ and are thus $2n$-dimensional manifolds without boundary.

Realizing an abstract Cartan domain as a topological subspace of $\mathbb{C}^n$, it does have a topological boundary, however. For example, the topological boundary of the open disc in $\mathbb{C}$ is the circle, and the topological boundary of $D_4$ in $\mathbb{R}^8$ is diffeomorphic to the standard seven-sphere $S^7$.

In order to give an abstract definition of the boundary of a Cartan domain (as a homogeneous space of Lie groups), we use the Borel embedding. The observation is that, writing $D = G/K$, the group $G$ embeds into its complexified group $G_\mathbb{C}$, $K$ into $K_\mathbb{C}$ and $G/K$ into the compact quotient $G_\mathbb{C}/K_\mathbb{C}$. One then observes that $G_\mathbb{C}/K_\mathbb{C}$ is isomorphic to the quotient $G_\mathbb{C}/K_\mathbb{C}$,
where $G_c$ is a real compact form of $G$. An irreducible non-compact Hermitian symmetric space therefore embeds into an irreducible compact Hermitian symmetric space.

For example, the non-compact space $D_1 = SO(1, 2)/SO(2)$ embeds into the compact space $S^2 = SO(3)/SO(2)$. In the same way, the eight-dimensional Lie ball $D4 = SO(4, 2)/SO(4) \times SO(2)$ is identified with a subspace of the compact Grassmanian $SO(6)/SO(4) \times SO(2)$.

The topological boundary of the Cartan domain embedded in its compact dual is called the \textit{weak} boundary of the domain. One can easily see that $K$ acts on the weak boundary, but \textit{not} always transitively. The weak boundary, in general, will therefore be stratified under the action of $K$. However, one of the strata consists of a single orbit, the \textit{Silov boundary} $\partial_s D$ of the domain $D$. When represented as a bounded domain in $\mathbb{C}^n$, the Silov boundary can be given an analytic definition: Let $\mathcal{A}$ be a set of non-constant holomorphic functions in a bounded domain $D$ of $\mathbb{C}^n$ that are continuous on $\overline{D}$. Then $\partial_s D \subset \partial D$ is the smallest closed subset of the (topological) boundary $\partial D$ so that every $f \in \mathcal{A}$ attains its maximum on $\partial_s D$. In the case of $D_n$, one gets $\partial_s D_n = S^{n-1} \times Z \times S^1$, which is a higher-dimensional analogue of the Klein bottle.

Since Cartan domains are $2n$-dimensional manifolds that can be represented as open subsets of $\mathbb{C}^n$, they admit a global complex chart coming from such an embedding. For instance, there is a global chart sending the ‘abstract’ domain $D_1 = SL(2, \mathbb{R})/U(1)$ to the open unit disc in the complex plane. The Cayley transform is a holomorphic mapping that sends the unit disc to the upper half-plane, and its Silov boundary - the unit circle - to the real line. This transformation is singular on the boundary, to the Cayley transformation does not give a global chart; one point is mapped to infinity.

The disc is called a bounded realization of $D_1$ and the upper half-plane an unbounded realization. In general, there exists a higher-dimensional analogue of the Cayley transform for other Cartan domains, and we can see it as a bi-holomorphic change of charts mapping a bounded realization to an unbounded one (or the unbounded to the bounded) which is singular at the boundary.

### 1.3 Origin of the Problems

The first indication that orbits on the Silov boundary had a connection to the decomposition of a representation seems to have appeared in [KV79], where the authors studied the metaplectic representation of the symplectic group $G = Sp(n, \mathbb{R})$. With $P = LN$ a maximal parabolic subgroup of $G$, and $N$ being the set of symmetric $n \times n$ matrices, it was proved that the representation of $G$ on $L^2(N)$ decomposed into irreducible sub-representations associated to the orbits of $L$ on $N$.

It was recently noticed that some of the ideas we use in a realization of the holomorphic discrete series representations associated to $G/H$ could be interpreted as a natural generalization of results from the papers [VR73] and [VR76]. In these papers, the authors consider a Hermitian Lie group $G$ such that its holomorphic discrete series is nonempty, and construct a realization of these representations in function spaces of limits of holomorphic functions on the Silov boundary of a tube domain via the Euclidean Fourier transform on the boundary. They also discuss the important issue of analytic continuation of the representations.

The motivation for some of our results (which we describe in a moment) is provided mainly
by three papers. In [ÓØ99], the authors construct a causal compactification of symmetric spaces of Cayley type, i.e., a realization of $G/H$ as an open dense subset of the Silov boundary of the bounded symmetric domain $G \times G/K \times K$. It is suggested that the method could be useful in harmonic analysis on these spaces. The work in [Bet97] generalize the results of [ÓØ99] to apply to compactly causal symmetric spaces in general; it is shown that most of these spaces $G/H$ can be realized as a dense open subset of the Silov boundary of a certain bounded symmetric domain $G_1/K_1$. The $G$-orbit structure on the boundary is described but is not present in the English translation [Bet03]. It is suggested in [Óla98] that the $G$-orbit structure on the Silov boundary could be important in harmonic analysis.

Finally we mention [BÓ01], where the results in [Bet97] are applied to realize the regular representation of $G$ on $L^2(G/H)$ on the $L^2$-space of the Silov boundary of a bounded symmetric domain. The theory of generalized Hardy spaces, as discussed in [HÓØ91], is related to the classical Hardy space on the Silov boundary. There is no mention of the orbit structure on the Silov boundary, and the Plancherel decomposition is not being brought into focus, either.

The implications of the work at hand may therefore be stated briefly as follows:

1. We combine ideas from [KV79], the existence of a causal compactification from [Bet97], and the realization of the regular representation in [BÓ01] to show that the regular representation associated to a compactly causal symmetric space has an orthogonal decomposition into sub-representations according to the orbit structure on the Silov boundary of a certain tube type domain. We do not study $G$-orbits but rather $L$-orbits, thus obtaining a finer decomposition. We present the details for the rank one case in the main text and sketch the situation for arbitrary rank in Section 7.3.

2. We generalize the results from [VR73] to explicitly realize the holomorphic discrete series representations of compactly causal symmetric spaces in function spaces on the Silov boundary of a tube domain (as suggested in the introduction to [Óla91]), using the Euclidean Fourier transform on the boundary. Similar results, for the bounded realization, appeared already in [ÓØ88b]. We describe the intertwining operator implementing this particular realization.

3. We show that when $G/H$ is rank one ($G/H$ is locally isomorphic to $SO_c(2, n)/SO_c(1, n)$), the decomposition subordinate to the orbit structure on the Silov boundary is the Plancherel decomposition. The paper [BÓ01] served as the primary source of influence.

An important part of the work done in the initial stages of this project was suggested to me by professor Ólafsson during the seminar talks I gave on the Plancherel formula for $\mathbb{R}^+SO(1, n) \ltimes \mathbb{R}^{n+1}$ in Fall 2002. Some ideas I use in Chapter 6 are already present in [FÓ03], albeit in a different context and with no obvious connection to the present setup.
Chapter 2

Harmonic Analysis and Complex Geometry Related to Tube Domains

2.1 Introduction

In this chapter we study the geometry of domains of tube type and the Silov boundary of such a domain. We cite a Paley-Wiener type characterization of the Hardy space on a tube domain in terms of the limit functions on the Silov boundary. We adopt a geometric point of view, where Lie groups are introduced as automorphism groups of the tube domains, thus avoiding a lengthy discussion of abstract structure theory.

Most of the results are present in the literature in one form or the other, and standard references include [Wol72], [SW71] (an old classic), [Vág79] (for a detailed historical overview) and [FK94] (a modern exposition, cast in the language of Jordan algebras). An encyclopedic treatment of the algebraic theory of symmetric domains is given in [Sat80].

2.2 Convex Cones and Domains of Tube Type

In the following, $V$ denotes a finite-dimensional real vector space and $V^C := V + iV$ its complexification, with the inner product $(\cdot, \cdot)$ on $V$ extended to a Hermitian inner product on $V^C$. We have in mind some specific choices for $V$ (see Figure 2.1 on 9) but prefer to work in full generality at this point.

**Definition 2.2.1.** An open subset $\Omega$ of $V$ is an open cone if all elements $\lambda x$ with $\lambda > 0$ and $x \in \Omega$ belong to $\Omega$. The cone $\Omega$ is proper if $\Omega \cap (-\Omega) = \{0\}$. The open dual cone of $\Omega$ is the set

$$\Omega^* = \{y \in V \mid (x, y) > 0 \text{ for all } x \in \Omega\}.$$ 

The cone $\Omega$ is self-dual if $\Omega = \Omega^*$.

**Definition 2.2.2.** The automorphism group of an open convex cone $\Omega$ is the group

$$G(\Omega) = \{g \in GL(V) \mid g\Omega = \Omega\}.$$ 

The cone $\Omega$ is homogeneous if $G(\Omega)$ acts transitively on $\Omega$. 
Observe that an element $g$ in $GL(V)$ belongs to $G(\Omega)$ if and only if $g\Omega = \overline{\Omega}$, implying that $G(\Omega)$ is a closed subgroup of $GL(V)$ and therefore a Lie group. Define the adjoint of $g \in G(\Omega)$ by $(gx, y) = (x, g^*y)$. Then $G(\Omega^*) = G(\Omega)^*$ if and only if $\Omega$ is self-dual (see [FK94], p.20).

**Definition 2.2.3.** Let $D$ be a domain in $V^C$. A holomorphic automorphism of $D$ is a bijective holomorphic mapping $D \to D$ with holomorphic inverse\(^1\). The group of all holomorphic automorphisms is denoted by $Aut(D)$.

A domain $D$ in $V^C$ is homogeneous if $Aut(D)$ acts transitively on $D$. The domain $D$ is symmetric if $D$ is homogeneous and if there exists a point $z_0$ in $D$ and an element $s$ in $Aut(D)$ such that $s^2 = id$ and $z_0$ is an isolated fixed point of $s$.

**Definition 2.2.4.** A domain $D$ in $V^C$ is a domain of tube type (or a tube domain for short) if there exists a proper homogeneous convex cone $\Omega$ in $V$ such that $D$ is biholomorphic to the domain $T_\Omega := V + i\Omega$.

**Lemma 2.2.5.** Let $\Omega$ be a proper convex homogeneous cone in a real vector space $V$. Then the tube domain $T_\Omega$ over $\Omega$ in $V^C$ is homogeneous.

**Proof.** For $g$ in $G(\Omega)$ and $a$ in $V$, the mapping $z \mapsto gv + a$ is a holomorphic automorphism of $T_\Omega$, and the group of all such transformations acts transitively on $T_\Omega$. \hfill \blacksquare

**Lemma 2.2.6.** Let $\Omega$ be a proper convex symmetric cone in $V$. Then $T_\Omega$ is a symmetric domain in $T_\Omega$.

**Remark.** The result is typically proved using Jordan algebra techniques: $V$ is viewed as a simple Euclidean Jordan algebra, and $\Omega$ is the associated symmetric cone. For Jordan algebras there is a notion of inverse, and one then proves that the map $z \mapsto -z^{-1}$ is an involutive holomorphic automorphism of $T_\Omega$ having $ie$ as its unique fixed points. Hence $T_\Omega$ is a symmetric domain. We refer to Theorem X.1.1 in [FK94] for details.

Write $G(T_\Omega) = Aut(T_\Omega)$. An element $g$ in $G(\Omega)$ acts on $T_\Omega$ by $z \mapsto gz$, so we can identify $G(\Omega)$ with a subgroup of $G(T_\Omega)$. For $v$ in $V$, the translation $\tau_v : z \mapsto z + v$ is a holomorphic automorphism of $T_\Omega$ and the group of all real translations $\tau_v$ form an abelian subgroup $N$ of $G(T_\Omega)$ that is isomorphic to $V$. The map $j : z \mapsto -z^{-1}$ belongs to $G(T_\Omega)$ and we set $N^- = j \circ N \circ j^{-1}$. Then $N^-$ is the abelian subgroup of $G(T_\Omega)$, consisting of all maps $z \mapsto (z^{-1} - v)^{-1}$, $v \in V$.

**Proposition 2.2.7.**

1. An affine linear transformation $z \mapsto Az + b$ is in $G(T_\Omega)$ if and only if $A$ belongs to $G(\Omega)$ and $b$ belongs to $V$.

2. Let $G(T_\Omega)_{ie}$ denote the subgroup of automorphisms in $G(T_\Omega)$ that fix the point $ie$. Then $G(T_\Omega) = NG(\Omega)G(T_\Omega)_{ie}$.

**Proof.** The first statement is Proposition X.5.4 in [FK94], and the second statement is Proposition X.5.5. \hfill \blacksquare

\(^1\)The inverse will, in fact, always be holomorphic.
In particular, the group of all affine linear automorphisms of $T_\Omega$ is $NG(\Omega)$, the semi-direct product of $N$ and $G(\Omega)$.

**Remark.** The map $j : z \mapsto -z^{-1}$ has an interpretation in the language of structure theory of Lie groups, as we will mention a little later. It is a “geometric version” of the action of a non-trivial element of the so-called Weyl group. We will explain its meaning when $G = Sp(n, \mathbb{R})$. Then $G = G(T_\Omega)$, where $\Omega$ is the cone of positive definite symmetric $n \times n$ matrices and $T_\Omega$ the 'Siegel upper half-plane' of complex symmetric $n \times n$ matrices $Z$ with $\text{Im}Z$ positive definite.

The group $G$ is generated by the parabolic subgroup

$$P = \left\{ \begin{pmatrix} a & 0 \\ 0 & t a^{-1} \end{pmatrix} \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} \mid x \in \text{Sym}(n, \mathbb{R}), a \in GL(n, \mathbb{R}) \right\}$$

and the Weyl group element

$$w = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

It can be seen that $G$ acts on $T_\Omega$ by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Z = (aZ + b)(cZ + D)^{-1}$$

so in particular the element $w$ acts on an element $Z \in T_\Omega$ through the formula

$$\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \cdot Z = (0 + 1_n)(-1_nZ + 0) = -Z^{-1}.$$

From the list of automorphism groups $G(T_\Omega)$ (Figure 2.1 on page 9), the only case in which this analogy between the map $j$ and the action of the Weyl group element $w$ is not the same, is when $G = SO(2, n)$ (or its connected component). This is exactly the case for which we will be able to study the Plancherel decomposition in later chapters, so we will have to make explicit computations in the ‘structure theory’-model. In many ways, working with $G = SO(2, n)$ involves the worst possible geometric framework.

**Theorem 2.2.8.** The subgroups $G(\Omega)$ and $N$, together with the element $j$, generate $G(T_\Omega)$.

For a proof we refer to [FK94], p.207-208.

Finally we list in Figure 2.1 on page 9 the Lie algebra $g(T_\Omega)$ of the automorphism groups $G(T_\Omega)$ of a tube domain $T_\Omega$ together with the Lie algebra $g$ of $G(\Omega)$, the compact real form $u$ of $g$, and the Lie algebra $\mathfrak{k}$ of a maximal compact subgroup $K$ of $G(T_\Omega)$.

### 2.3 Function Spaces Associated to a Tube Domain

Let $A(T_\Omega)$ denote of functions that are bounded and continuous on $\overline{T_\Omega}$ and holomorphic on $T_\Omega$. 
### Definition 2.3.1.

A Silov boundary for $T_\Omega$ (or Silov boundary, for short) is a closed subset $B$ of the topological boundary $\partial T_\Omega$ of $T_\Omega$ that is minimal with respect to the property that

$$\forall f \in A(T_\Omega) : \sup_{z \in T_\Omega} |f(z)| = \sup_{z \in B} |f(z)|.$$

We need the following basic result in order to prove the existence and uniqueness of the Silov boundary.

### Lemma 2.3.2.

Let $\varphi$ be a positive integrable function on $\Omega^*$ with integral 1, and let

$$\psi(z) = \int_{\Omega^*} e^{i(z,u)} \varphi(u) \, du.$$

Then $\psi$ is continuous on $T_\Omega$, $\psi(0) = 1$, and $|\psi(z)| < 1$ for all non-zero $z$ in $T_\Omega$.

### Theorem 2.3.3.

The real vector space $V$ is the unique Silov boundary of $T_\Omega$, and we denote it by $\partial_s T_\Omega$.

**Proof.** It is easy to see that if $f$ belongs to $A(T_\Omega)$, then $|f(z)| \leq \sup_{x \in V} |f(x)|$ for all $z \in T_\Omega$. Now let $B$ be a closed subset of $\partial T_\Omega$ such that $\sup_{z \in T_\Omega} |f(z)| = \sup_{z \in \mathbb{F}} |f(z)|$ for all $f$ in $A(T_\Omega)$, and define $\psi_v$ by $\psi_v(z) = \psi(z - v)$ for $v \in V$. Then $|\psi_v(z)| < \psi_v(v)$ for all $z$ in $T_\Omega \setminus \{v\}$, showing that $v$ belongs to $B$. Hence $V \subset B$. ■

### Remark.

The Silov boundary $\partial_s T_\Omega$ of $T_\Omega$ is an abelian group related to the Lie structure of the semi-simple automorphism group of the domain $T_\Omega$. At present, we simply need to observe that $\partial_s T_\Omega$ is an Euclidean vector space and thus allows for classical Fourier analysis.

The Silov boundary of a Cartan domain is not a group but merely a homogeneous space. Analysis on the Silov boundary of a Cartan domain thus becomes much more complicated than analysis on the Silov boundary of a tube domain.

Let $f$ be a function defined on $T_\Omega$ and let $t \in \Omega$. Then we define $f_t : \partial_s T_\Omega \to \mathbb{C}$ by $f_t(z) = f(z + it)$. For $0 < p < \infty$, the Hardy space $H^p(T_\Omega)$ is defined as the set of holomorphic functions $f$ on $T_\Omega$ such that

$$\sup_{t \in \Omega} \int_{\partial_s T_\Omega} |f_t|^p \, d\beta < \infty.$$

Note that the supremum in the defining inequality is taken over a set of dimension half the real dimension of the domain.
The $H^2$-spaces for tube domains were first introduced and studied by Bochner in [Boc44] and marks the beginning of harmonic analysis on symmetric domains. Let $T_\Omega$ denote such a domain. Bochner then showed that a function $f$ belongs to $H^2(T_\Omega)$ if and only if
\[
f(z) = \int_{\Omega^*} e^{2\pi i (\lambda, z)} g(\lambda) \, d\lambda
\]
for some $g \in L^2(\Omega^*)$. From this result it follows that $f$ has a boundary value function $f^*$ on $\partial_s T_\Omega$ such that
\[
\int_{\partial_s T_\Omega} |f^*(x) - f(x + iy)|^2 \, dx \rightarrow 0
\]
as $y$ tends to 0 in $\Omega$. The $L^2$-Fourier transform of $f^*$ is equal to $g(\lambda)$ for $\lambda \in \Omega^*$ and zero almost everywhere outside $\Omega^*$. The representation formula for $f$ can therefore be written as
\[
f(z) = \int_{\Omega^*} e^{2\pi i (\lambda, z)} g(\lambda) \, d\lambda = \int_{\partial_s T_\Omega} e^{2\pi i (\lambda, z)} \hat{f}^*(\lambda) \, d\lambda,
\]
thus characterizing $H^2$-functions by means of the Fourier transform of their boundary functions.

Classically, a function $f$ in $L^2(T)$ (with $T$ denoting the torus) is the boundary value of an element in $H^2(U)$ (with $U$ denoting the upper half-plane) if and only if its Fourier coefficients
\[
\hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \, d\theta
\]
vanish for $n < 0$. The analogous result for bounded symmetric domains is much more difficult but was settled in Schmid’s paper [Sch69]. For the upper half-plane the characterization of $H^2$ is given in a well-known theorem by Paley and Wiener, stating that $f \in L^2(\mathbb{R})$ is the boundary value of a function in $H^2(U)$ if and only if its Fourier transform $\hat{f}(\xi)$ vanishes for almost all $\xi < 0$. As already indicated, the result was extended to tube domains by Bochner in the following way: Use Bochner’s representation formula for $H^2$-functions,
\[
f_t(x) = \int_{\Omega^*} e^{2\pi i (x, \lambda)} e^{-2\pi i (\lambda, t)} \hat{f}^*(\lambda) \, d\lambda \quad , \quad t \in \Omega,
\]
and let $t$ tend to zero. Then we obtain the exact analogue of the Paley-Wiener description of $H^2$:

**Theorem 2.3.4.** A function $f \in L^2(\partial_s T_\Omega)$ belongs to $H^2(\partial_s T_\Omega)$ if and only if its Fourier transform vanishes almost everywhere outside $\Omega^*$.

**Corollary 2.3.5.** For a function $F$ in $H^2(T_\Omega)$ and a vector $y$ in $\Omega$, we write $F_y(x) = F(x + iy)$. Then
\[
\lim_{\Omega \ni y \rightarrow 0} F_y = F_0
\]
exists in $L^2(V)$ and the map $F \rightarrow F_0$ is an isometric embedding of $H^2(T_\Omega)$ into $L^2(V)$. 

10
Proof. We define \( F_0(x) = (2\pi)^{-\frac{n}{2}} \int_{\Omega^*} e^{i(x|u)} f(u) \, du \). Then
\[
\|F_y - F_0\|^2 = \int_{\Omega^*} \left| e^{-(y|u)} - 1 \right|^2 |f(u)|^2 \, du.
\]

Recall the geometric transformation \( j : z \mapsto -z^{-1} \) of \( T_\Omega \). Then \( j \) gives rise to a unitary mapping \( V_\nu \) in \( H^2_\nu(T_\Omega) \). Let \( L_\nu \) denote the (weighted) Laplace transform defined by
\[
L_\nu f(y) = \int_{\Omega} e^{-i(x,y)} f(x) \Delta(x) \nu - \frac{n}{r} \, dx \quad \text{for} \quad f \in D(\Omega).
\]
We then define the generalized Hankel transform \( U_\nu \) by \( U_\nu = L_{\nu}^{-1} \circ V_\nu \circ L_\nu \) that acts on the space \( L^2_\nu(\Omega) \). Without being too precise regarding the additional notation, we mention the following result that is proved in [FK94], p.341-342.

Theorem 2.3.6. Assume \( \nu > d(r - 1) + 1 \).

1. The transform \( V_\nu : F \mapsto G, \)
\[
G(z) = \Delta \left( \frac{z}{I} \right)^{-\nu} F(-z^{-1})
\]
is an involutive automorphism of \( H^2_\nu(T_\Omega) \).

2. The transform \( U_\nu = L_{\nu}^{-1} \circ V_\nu \circ L_\nu \) is an involutive automorphism of \( L^2_\nu(\Omega) \) with kernel \( H_\nu(u,v) \): For compactly supported functions \( f \) in \( L^2_\nu(\Omega) \) we have the formula
\[
U_\nu f(u) = \int_{\Omega} H_\nu(u,v) f(v) \Delta(v)^{\nu - \frac{n}{r}} \, dv.
\]
The kernel function \( H_\nu \) has the invariance property that \( H_\nu(gu,v) = H_\nu(u,g^*v) \) for \( g \in G \), and
\[
H_\nu(u,e) = \frac{1}{I_\Omega(\nu)} J_\nu(u),
\]
where \( J_\nu \) is the Bessel function \( J_\nu(z) = {}_0F_1(\nu; -z) \).

Finally we mention that (as part of the folk-lore), with \( T_\Omega \) being the unbounded realization of \( G/K \), the Lie group \( G \) has a unitary action on the Hardy space \( H^2(T_\Omega) \), defined by
\[
\lambda(g) f(x) = j(g^{-1},x)^{\frac{1}{2}} f(g^{-1} \cdot x).
\]
Here the action \( \cdot \) refers to the action of \( G \) on \( T_\Omega \) as an automorphism group, and \( j \) is the determinant of the holomorphic action \( x \mapsto g \cdot x \). See [FK94], Chapter X, for further details.
2.4 The Bounded Realization

Let $M$ be a Hermitian symmetric space of non-compact type, $G^0$ its connected group of isometries, and let $K$ denote the isotropy group. Then $M = G^0/K$, $G^0$ is a semi-simple Lie group, and $K$ is a maximal compact subgroup. Let $\mathfrak{g}^0$ denote the Lie algebra of $G^0$, $\mathfrak{g}^C$ its complexification, and $G^C$ the adjoint group of $\mathfrak{g}^C$. Furthermore, let $\mathfrak{t}$ denote the Lie algebra of $K$, $\mathfrak{t}^C$ its complexification and $K^C$ the analytic subgroup of $G^C$ corresponding to $\mathfrak{t}^C$. Finally, let $\mathfrak{g}$ denote the compact form of $\mathfrak{g}^C$ such that the involution of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$ leaves $\mathfrak{g}^0$ invariant. Then $\mathfrak{g}^0 = \mathfrak{t} \oplus \mathfrak{p}^0$ is the decomposition of $\mathfrak{g}^0$ into $(+1)$-, respectively, $(-1)$-eigenspaces. Additionally, $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ where $\mathfrak{p} := \mathfrak{p}^0$.

Roots of $\mathfrak{g}^C$ that are also roots of $\mathfrak{t}^C$ will be called compact roots, and non-compact otherwise.

Now choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{t}$. Then $\mathfrak{h}^C$ is a Cartan subalgebra of $\mathfrak{t}^C$. Given a root $\alpha$, choose elements $H_\alpha, E_\alpha \in \mathfrak{g}^C$ in the usual way. Let $\mathfrak{z}$ denote the center of $\mathfrak{t}$ and choose an element $Z \in \mathfrak{z}$ so that $[Z, E_{\pm \alpha}] = \mp i E_\alpha$ for every non-compact positive root $\alpha$. For $\alpha$ a non-compact positive root, let $X_0^\alpha = E_\alpha + E_{-\alpha}$ and $Y_0^\alpha = -i(E_\alpha - E_{-\alpha})$. Then $\mathfrak{p}^0$ is spanned by all such elements $X_0^\alpha$ and $Y_0^\alpha$. Let $J = \text{ad}(Z)|_{\mathfrak{p}^0}$ (defines the complex structure).

**Definition 2.4.1.** Two roots $\alpha$ and $\beta$ are strongly orthogonal if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root.

Let $\Delta$ denote a maximal set of strongly orthogonal non-compact positive roots such that $\Delta^0 := \text{span}_\mathbb{R}\{X_0^\alpha \mid \alpha \in \Delta\}$ is a maximal abelian subalgebra in $\mathfrak{p}^0$. Additionally, $JX_0^\alpha = [Z, Y_0^\alpha] = Y_0^\alpha, HY_0^\alpha = [Z, Y_0^\alpha] = -X_0^\alpha, \text{and } [X_0^\alpha, Y_0^\alpha] = 2iH_\alpha$ for all $\alpha \in \Delta$. The set of non-compact roots in $\Delta$ is denoted by $\Delta^+_K$, and the set of compact roots in $\Delta$ by $\Delta_k$.

Let $\mathfrak{p}^+ = \text{span}_\mathbb{C}\{E_\alpha \mid \alpha \in \Delta^+_K\}$ and $\mathfrak{p}^- = \text{span}_\mathbb{C}\{E_{-\alpha} \mid \alpha \in \Delta^+_K\}$, and let $P^+$ resp. $P^-$ denote the corresponding analytic subgroups of $G^C$. The semi-direct product $K^C \cdot P^+$ is the normalizer of $P^+$ in $G^C$, so we may identify the compact dual $M^* = G/K$ of the Hermitian symmetric space $M = G^0/K$ with $G^C/K^C \cdot P^+$ through the natural inclusion $G \hookrightarrow G^C$.

Let $\mathfrak{k}$ be the identity coset in $G^C/K^C P^+$. The orbit $G^0(\mathfrak{x})$ is then a holomorphic embedding of $M$ as an open set in $M^*$, called the Borel embedding of $M$. Define $\zeta : \mathfrak{p}^- \to M^*$ by $\zeta(X) = \exp(X) \cdot (\mathfrak{x})$. Then $\zeta$ is an injective holomorphic mapping onto a dense open subset of $M^*$, and $D := \zeta^{-1}(G^0(\mathfrak{x}))$ can be shown to be a bounded starlike domain in $\mathfrak{p}^-$. The set $D$ is called the Harish-Chandra realization, or the bounded realization, of $M$ in $\mathfrak{p}^-$. Note that the $\zeta$-equivariant action of $G^0$ on $D$ is the action of the connected group of holomorphic automorphisms of $D$.

**Definition 2.4.2.** Let $M$ be a Hermitian symmetric space of non-compact type. Then $M$ is of tube type if it is holomorphically equivalent to the tube domain over a self-dual homogeneous cone.

Let $D_b$ denote the realization of $G/K$ as a bounded symmetric domain in $\mathfrak{p}^+ \subset \mathbb{C}^N$ for some integer $N$. For $g \in P^+ K^C P^-$ we denote by $k_C(g)$ the $K_C$-part of $g$ and by $p^+(g)$ the $P^+$-part of $g$. In that notation, $D_b = \{p^+(g), \mid g \in G\}$.

For $(g, z) \in G_C \times D$ such that $g \exp z$ belongs to $P^+ K_C P^-$ we define $g(z) \in p^+$ and $J(g, z) \in K_C$ by $\exp g(z) = p^+(g \exp z)$ and $J(g, z) = k_C(g \exp z)$. Whenever $g \exp z$ is in $P^+ K_C P^-$ we say that the holomorphic action $g(z)$ is defined.
Example: For concrete groups, the action of $G$ on the bounded realization $D_b$ is simple to describe. For $G = SU(n, n)$ and $G_C = SL(2n, \mathbb{C})$, say, the decomposition of an open subset of $G_C$ as $P^+ \times K_C \times P^-$ is

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
1 & BD^{-1} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
A - BD^{-1}C & 0 \\
0 & D
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
D^{-1}C & 1
\end{pmatrix},
$$

whenever $D$ is nonsingular. With $\omega \in D_b$ of the form $\omega = \begin{pmatrix} 1 & Z \\
0 & 1 \end{pmatrix}$, one has $g\omega = \begin{pmatrix} A & AZ + B \\
C & CZ + D \end{pmatrix}$, the $P^+$-component of which is

$$
\begin{pmatrix}
1 & (AZ + B)(CZ + D)^{-1} \\
0 & 1
\end{pmatrix}.
$$

The action is therefore described by

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \cdot \begin{pmatrix}
1 & Z \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & (AZ + B)(CZ + D)^{-1} \\
0 & 1
\end{pmatrix}.
$$

It turns out that for the groups $G$ in Cartan’s list of Cartan domains, $G$ always acts by fractional linear transformations. See [Hel01], Chapter X.6.

The $K_C$-valued function $J$ is called the canonical automorphy factor of $G$. Both $g(z)$ and $J(g, z)$ are defined on an open subset of $G_C \times p^+$ containing $G \times D$ and $K_C \times p^+$, and both are holomorphic in $g \in G_C$ and $z \in p^+$, whenever defined.

**Lemma 2.4.3.** The canonical automorphy factor $J$ satisfies the following relations:

$$
J(g, o) = k_C(g) \text{ for all } g \in P^+K_CP^-,
$$

$$
J(k, z) = 1 \text{ for all } k \in K_C \text{ and } z \in p^+.
$$

Let $g, g' \in G_C$ and $z \in p^+$. If $g'(z)$ and $g(g'(z))$ are defined, then so is $(gg')(z)$ and

$$
J(gg', z) = J(g, g'(z))J(g', z) \text{ (cocycle property)}.
$$

The $K_C$-valued function $K$ defined on an open subset of $p^+ \times p^+$ (containing $D \times D$) is called the canonical kernel function of $G_C$. Clearly, $K(z, w)$ is holomorphic in $z$ and anti-holomorphic in $w$, whenever defined.

**Lemma 2.4.4.** For $g \in G_C$, the Jacobian (a linear map) of the holomorphic action $z \mapsto g(z)$ (whenever defined) is given by

$$
\text{Jac}(z \mapsto g(z)) = \text{Ad}_{p^+}(J(g, z)),
$$

where $\text{ad}_{p^+}$ is the restriction of $\text{ad}$ to $p^+$.

For a (holomorphic) character $\chi : K_C \to \mathbb{C}$, we define the canonical automorphy factor of type $\chi$ and the canonical kernel function of type $\chi$ by

$$
\chi_j(g, z) = \chi(J(g, z)) \text{ and } k_{\chi}(z, w) = \chi(K(z, w)) \text{ respectively.}
$$

Since $\chi(k) = \overline{\chi(k)}^{-1}$, it follows at once that $k_{\chi}(w, z) = \overline{k_{\chi}(z, w)}$ and $k_{\chi}(g(z), \overline{g}(w)) = j_{\chi}(g, z)k_{\chi}(z, w) j_{\chi}(\overline{g}, w)$. The character $\chi_1$ defined by

$$
\chi_1(k) = \text{det}(\text{ad}_{p^+}(k)) \quad , \quad k \in K_C
$$

is particularly important, since the associated automorphy factor $j_{\chi_1}$ is the jacobian of the holomorphic transformation $z \mapsto g(z)$ in the usual sense.
Example: $SU(1, 1)$ Let $g = su(1, 1)$. For $g = (\frac{\alpha \beta}{\overline{\beta} \overline{\alpha}}) \in G$, one has
\[
J(g, z) = \begin{pmatrix} (\overline{\beta} z + \overline{\alpha})^{-1} & 0 \\ 0 & \overline{\beta} z + \overline{\alpha} \end{pmatrix}, \quad K(z, w) = \begin{pmatrix} 1 - z \overline{w} & 0 \\ 0 & (1 - z \overline{w})^{-1} \end{pmatrix}
\]
and $\chi_1((\frac{\alpha}{0} \frac{0}{\alpha^{-1}})) = \alpha^2, \alpha \in \mathbb{C}$. Hence $j_{\chi_1}(g, z) = (\overline{\beta} z + \overline{\alpha})^{-1}$ and $k_{\chi_1}(z, w) = (1 - z \overline{w})^2$.

Theorem 2.4.5 (Bott-Koranyi). The orbit $K(\sum_{\alpha \in \Delta} E_{-\alpha})$ is the unique $K$-orbit on $\overline{D}$ that is also a $G^0$-orbit; thus $\tilde{S} = K(\sum_{\alpha \in \Delta} E_{-\alpha})$ is the Silov boundary of the bounded domain $D$.

Let $g^0_{\alpha}$ denote the subalgebra of $g^0$ spanned by $\{iH_\alpha, X^0_\alpha, Y^0_\alpha \mid \alpha \in \Delta \}$ and let $g^C_{\alpha}$ denote the complexification of $g^0_{\alpha}$. Then $[g^C_{\alpha}, g^C_{\beta}] = 0$ for $\alpha \neq \beta$.

Put $X_\alpha = iX^0_\alpha \in g$ and $c_\alpha = \exp \frac{\pi}{4} X_\alpha \in G$ (the partial Cayley transforms). Furthermore, let $X = iX^0 = \sum_{\alpha \in \Delta} X_\alpha \in g$ and define the (full) Cayley transform $c$ of $G$ as
\[
c = \exp \frac{\pi}{4} X = \prod_{\alpha \in \Delta} c_\alpha.
\]

Lemma 2.4.6. In the notation above,
\[
c(x) = \zeta \left( i \sum_{\alpha \in \Delta} E_{-\alpha} \right) \in \tilde{S}.
\]

Let $L$ denote the isotropy subgroup of $G^0$ in $c(x)$. Then it follows from the Lemma that $\tilde{S} = K(c(x)) \approx K/L$, a compact symmetric space.

Example: Let $M$ be the open unit ball in $\mathbb{C}^n$. Then $M^*$ is the complex projective space $\mathbb{P}^n(\mathbb{C})$, equipped with the Fubini-Study metric, $\tilde{S}$ is the full topological boundary $S^{2n-1}$ of $M$ in $M^*$, $K(c^2(x))$ is the polar hyperplane $\mathbb{P}^{n-1}(\mathbb{C})$ to $x$ in $M^*$, and the fibering
\[
\tilde{S} \longrightarrow K(c^2(x))
\]
is the usual circle bundle $S^{2n-1} \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$.

We can also express the Silov boundary $\tilde{S}$ of the bounded domain $D$ as a coset space of the connected group of analytic automorphisms: Let $B$ denote the isotropy subgroup of $G^0$ at $c(x)$. Then $\tilde{S} = G^0(c(x)) \approx G^0/B$, and a careful study of the Lie algebra of $B$ reveals that $B$ is a parabolic subgroup of $G^0$ (see section 5 in [KW65] for the details).

### 2.5 The Unbounded Realization

Let $\nu$ and $\nu^0$ denote the involutions of $g^C$ with respect to $g$ and $g^0$, respectively, and let $\langle , \rangle$ be the Killing form. Then $\langle U, V \rangle_\nu := -\langle U, \nu V \rangle$ defines a positive definite Hermitian form on $g^C$. If $f$ is a linear transformation on $g^C$, let $f^*$ denote its adjoint with respect to $\langle , \rangle_\nu$. Then $\text{ad}(V)^* = -\text{ad}(\nu V)$ for every $V \in g^C$, and $\text{ad}(K)$ acts on $p^-$ by unitary transformations with respect to $\langle , \rangle_\nu$. The restriction of $\langle , \rangle_\nu$ to $n^-$ is a real positive definite bilinear form.

Let $o$ be the zero element of $p^-$ (the base point of $D$), and let $o^C$ be the image of $o$ under the map $\zeta^{-1} c \zeta$. Then $o^C = \zeta^{-1}(c(x)) = \sum_{\alpha \in \Delta} E_{-\alpha}$. 

Proposition 2.5.1 (Proposition 6.2, [KW65]). Let \( c = -iK^*(o^c) \). Then \( c \) is a cone in \( \pi^- \) that is self-dual with respect to the restriction to \( \pi^- \) of the positive definite form \( \langle \cdot, \cdot \rangle_v \).

When \( M \) is tube type, we have \( ic = K^*/L \), so \( ic \) (or \( c \)) is just the non-compact dual of the Silov boundary \( \hat{S} \) which, in this case, is a compact symmetric space.

Theorem 2.5.2 (Theorem 6.8, [KW65]). The Cayley transformed domain \( D^c \) is equal to the domain \( \{ E \mid \text{Im} E \in c \} \).

Theorem 2.5.3 (Theorem 6.9, [KW65]). Put \( \hat{S}^c = c(\hat{S}) \). Then
\[
\hat{S}^c \cap \zeta(p^-) = \zeta(\{ E \mid \text{Im} E = 0 \}),
\]
and \( \hat{S}^c \cap \zeta(p^-) \) is an open dense subset of \( \hat{S}^c \).

In particular, \( \hat{S}^c \) is a real analytic sub-manifold of \( M^* \).

We postpone a discussion of the group action in the unbounded realization until the next section, where we discuss two different points of view.

2.6 Geometric Connection between the Realizations

To discuss the action in the unbounded realization, thus on the Silov boundary, the most direct approach is the following: Let the notation be as above, and let \( g \in G \) and \( \pi \in \hat{\pi} \) be such that \( g \exp \pi \in NMAN \). In that case we say that \( g \) acts on \( \pi \), and the action is defined uniquely by \( (g \cdot \pi)MAN = \exp(g\pi)MAN \), i.e., \( g \cdot \pi \) is the \( N \)-component of \( g \exp \pi \). This is the action we will be analyzing further in later chapters.

There is however, as the section heading suggests, a close connected to the action in the bounded realization. In the spirit of the previous chapter we will therefore include a short discussion (also for the sake of reference). The outcome is a realization of \( G/K \) (in the Hermitian case) as a particularly nice unbounded open subset \( \Omega' \) of \( P^+ \). In the following, let \( b = t \oplus \bigoplus_{\alpha \in \Delta^+} g_{-\alpha} \) and let \( B \) be the associated analytic subgroup of \( G_C \).

We will first discuss the special case where \( G = SU(1,1) \) in order to facilitate an \( su(1,1) \)-calculation. Let \( c \) be the Cayley transform given by
\[
c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},
\]
let \( G' = SL(2, \mathbb{R}) \), and put
\[
\Omega' = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid \text{Im} z > 0 \right\}.
\]
It is then easy to verify directly that \( cGc^{-1} = G' \). In fact, \( cGB = G'cB = \Omega'K_C P^- \), and \( G' \) acts on \( \Omega' \) by the usual action of \( SL(2, \mathbb{R}) \) on the upper half plane (through fractional linear transformations).

**Proof.** From the fact that \( cGc^{-1} = G' \), we have \( cGB = G'cB \). Since \( GB = \Omega K_C P^- \) we also have \( cGB = c\Omega'K_C P^- \). Here
\[
c \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z + i \\ i & iz + 1 \end{pmatrix}
\]
has $P^+$-component given by
\[
\begin{pmatrix}
1 & \frac{z+i}{iz+1} \\
0 & 1
\end{pmatrix}
\]
so $c\mathcal{G}B = \Omega'' K_C P^-$, where
\[
\Omega'' := \left\{ \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \mid w = \frac{z+i}{iz+1}, |z| < 1 \right\}.
\]

Notice that $\Omega''$ is just $\Omega'$ (the mapping from $\Omega$ to $\Omega'$ being the classical Cayley transform). The action of $G'$ on $\Omega'$ is given by $g(w') = P^+$-component of $gw'$, and this action is given by linear fractional transformation, according to the same computations as for the action of $G$ on $\Omega$. ■

Notice the important distinction: The group $SU(1,1)$ acts on the unit disc by fractional linear transformations, but it does not act on the upper half-plane; it is the group $SL(2,\mathbb{R})$ that acts on the upper half-plane by fractional linear transformation. Since $SL(2,\mathbb{R})$ and $SU(1,1)$ are conjugate within $SL(2,\mathbb{C})$ via the Cayley transform, and the unit disc and upper half-plane are holomorphically equivalent, one could – by a slight abuse of language – state that $SU(1,1)$ also acts on the upper half-plane. Defining the action of an element $g \in SU(1,1)$ on a complex number $z$ in the upper half-plane would therefore look something like this:
\[
g \circ z := g \cdot (c^{-1}z)
\]
the · denoting the action of $SU(1,1)$ on the unit disc, or (which amounts to the same thing)
\[
g \circ z := \text{Ad}(c)(g) \cdot
\]
where the · now refers to the action of $\text{Ad}(c)(g) \in SL(2,\mathbb{R})$ on the upper half-plane.

In the general case, let $\gamma_1,\ldots,\gamma_s$ be a maximal set of strongly orthogonal roots and construct, for each $j$, the partial Cayley transform $c_j$ in $G_C$ that behaves for the three-dimensional group corresponding to $\gamma_j$ like the Cayley transform $c$ considered for $SL(2,\mathbb{R})$ (we are getting ready to do an $\mathfrak{sl}(2,\mathbb{R})$-calculation). Let $c = \prod_{j=1}^s c_j$. Then $c$ may be decomposed according to $P^+K_CP^-$. Let $a_0$ be the maximal abelian subspace of $p_0$ constructed elsewhere, and let $A_p = \exp a_0$. Then $cA_pc^{-1} \subset K_C$. Furthermore, for a particular ordering on $a_0^*$, $cN_p c^{-1} \subset P^+K_C$ when $N_p$ is built from the positive restricted roots.

**Proof.** The element $c$ has the triangular decomposition
\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
The element $c_j$ is the (partial) Cayley transform $c_{\gamma_j}$ defined as in (xx) with root vectors normalized so that $[E_{\gamma_j},E_{\gamma_j}] = 2|\gamma_j|^{-2}H_{\gamma_j} = H'_{\gamma_j}$. More precisely, we have in mind setting up for an $\mathfrak{sl}(2,\mathbb{C})$-calculation, so we have in mind the correspondence
\[
E_{\gamma_j} \simeq \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E_{\gamma_j} \simeq \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}.
\]
so that
\[ c_j = \exp \pi \frac{1}{4} (\overline{E_{\gamma j}} - E_{\gamma j}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \]

Then the decomposition for \( c_j \) is
\[ c_j = \exp(-E_{\gamma j}) \exp((\frac{1}{2} \log 2)H'_{\gamma j}) \exp(\overline{E_{\gamma j}}). \]

It can then be seen that the factor \( c_j \) of \( c \) only acts non-trivially on the \( j \)th factor of \( \exp(\sum c_j(E_{\gamma j} + \overline{E_{\gamma j}})) \). A computation in \( \text{sl}(2, \mathbb{C}) \) then reveals that
\[ \text{Ad}(c) \left( \exp(\sum c_j(E_{\gamma j} + \overline{E_{\gamma j}})) \right) = \exp \sum c_j \text{Ad}(c_j)(E_{\gamma j} + \overline{E_{\gamma j}}) = \exp \sum (-c_jH'_{\gamma j}). \]

Now, for the remainder of the proof, define a restricted root \( \beta \) to be positive if \( \beta(E_{\gamma j} + \overline{E_{\gamma j}}) < 0 \) for the first index \( j \) such that \( \beta(E_{\gamma j} + \overline{E_{\gamma j}}) \neq 0 \). If \( X \) is a restricted-root vector for such a \( \beta \) and \( j \) is the distinguished index, then \( [E_{\gamma i} + \overline{E_{\gamma i}}, X] = -c_iX \) for all indices \( i \) such that \( c_1 = \cdots = c_{j-1} = 0 \) and \( c_j > 0 \). For such \( i \)
\[ [H'_{\gamma i}, \text{Ad}(c)X] = -[\text{Ad}(c)(E_{\gamma i} + \overline{E_{\gamma i}}), \text{Ad}(c)X] = c_i \text{Ad}(c)X. \]

Thus \( \text{Ad}(c)X \) is a sum of root vectors for \( \tilde{\beta} \) such that \( \langle \tilde{\beta}, \gamma_i \rangle = c_i \). If \( \tilde{\beta} \) is negative and noncompact, then \( \langle \tilde{\beta}, \gamma_j \rangle \) is negative whenever it first becomes nonzero. Since \( \langle \tilde{\beta}, \gamma_j \rangle > 0 \) by construction, we conclude that \( \tilde{\beta} \) is either compact or positive noncompact. The last conclusion now follows from the fact that \( cGB = (cN_p c^{-1})(cA_p c^{-1})cKB \subset P^+K_C \cdot K_C \cdot P^+K_C \cdot P^- \cdot KB \subset P^+K_C P^- \).

Finally, writing \( G = N_p A_p K \) according to the Iwasawa decomposition, we prove that \( cGB \subset P^+K_C P^- \).

Let \( G' = cGC^{-1} \). Then \( G'cB = \Omega'K_C P^- \) for some open subset \( \Omega' \) of \( P^+ \). The resulting action of \( G' \) on \( \Omega' \) is holomorphic and transitive, and we may identify \( \Omega' \) with \( G/K \).

It follows from the proof, in particular, that \( (cGC^{-1}) \cap P^+ = N \), so taking \( N \)-component or Cayley transforming the action in the bounded realization amounts to the same thing. More precisely, let \( g \in G \) and \( z \in D_u \) be such that \( g \circ z \) is defined. Then
\[ g \circ z = (c^{-1} gc) \cdot (c^{-1} z). \]
Chapter 3

Geometry of Compactly Causal Symmetric Spaces

3.1 Introduction

In this chapter we construct the $G$-equivariant causal realization of a compactly causal symmetric space $G/H$ as an open dense sub-manifold of the Silov boundary of unbounded realization of a certain Hermitian symmetric space $(G_1,K_1,\theta)$. While the construction is closely related to, indeed motivated by the causal compactification of $G/H$ in [ÓØ99] for Cayley type spaces and [Bet97] in general (see also [Bet03]) we will only use the initial setup in these references. Part of the initial setup can be described as follows:

1. Choose a Cayley type symmetric space $(G_1,G^T_1,\tau)$ such that $(G_1,K_1,\theta)$ is of tube type, and where $\theta$ is a Cartan involution commuting with $\tau$.
2. Construct an involution $\sigma$ that commutes with $\tau$ and $\sigma$, with the additional property that the triple $(G,G^T,\tau)$ (with $G := G^T_1$) becomes a causally symmetric triple by restriction of the causal structure from $(G_1,G^T_1,\tau)$.

3.2 Structure Theory for Reductive Symmetric Spaces

We give a short account of the most important definitions regarding the structure theory of a reductive symmetric space, and explain in detail what it means for the spaces $G/H = SO_e(p,q)/SO_e(p-1,q)$. The point about discussing all the hyperboloids is that we have to consider several different hyperboloids simultaneously in a later chapter.

Definition 3.2.1. A symmetric triple is a triple $(G,H,\tau)$, where $G$ is a Lie group and $\tau$ is an involution on $G$ such that $(G^T)_e \subset H \subset G^T$. Here $G^T$ is the subgroup of elements in $G$ being fixed by $\tau$ and $(G^T)_e$ is the connected component of the identity in $G^T$.

A symmetric space is a space $X$ for which there exists a symmetric triple such that $X$ is diffeomorphic to $G/H$. 

18
Definition 3.2.2. Let $G$ be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$, and let $B(X,Y) = \text{tr}(\text{ad}X \text{ad}Y)$ be its Cartan-Killing form. A Cartan involution on $\mathfrak{g}$ is an involutive automorphism $\vartheta$ on $\mathfrak{g}$ such that the bilinear form $B_\vartheta(X,Y) := -B(X, \vartheta(Y))$ is positive definite.

Remark. The presence of the involution $\tau$ makes the structure theory of symmetric spaces much more profound than the structure theory for groups. The structural interplay between $\tau$ and the Cartan involution $\vartheta$ will take center stage later on.

Let $\mathfrak{g}$ be the (real) Lie algebra of $G$, and let $\tau$ denote the involution of $\mathfrak{g}$ obtained from $\tau$ on $G$ by differentiation. Then $\mathfrak{g}$ decomposes into $\tau$-eigenspaces as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. The usual convention is to let $\mathfrak{h}$ denote the $+1$-eigenspace and $\mathfrak{q}$ the $-1$-eigenspace. Then $\mathfrak{h}$ is the Lie algebra of $H$.

Given a semisimple symmetric triple $(G,H,\tau)$, one can always construct a Cartan involution $\vartheta$ on $G$ commuting with $\tau$. The Lie algebra of $G$ has an orthogonal decomposition with respect to $\vartheta$, the Cartan decomposition, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and since $\vartheta$ and $\tau$ commute, it follows that $\mathfrak{h}$ and $\mathfrak{q}$ are $\vartheta$-invariant and $\mathfrak{k}$ and $\mathfrak{p}$ are $\tau$-invariant. Let $K$ be the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$. We thus have a joint decomposition of $\mathfrak{g}$

$$(3.1) \quad \mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}).$$

Furthermore, $\tau \vartheta$ is an involution with respect to which $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \quad \text{where} \quad \mathfrak{g}_+ = \mathfrak{g}^{\tau \vartheta} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}) \quad \text{and} \quad \mathfrak{g}_- = (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}).$$

Let $\mathfrak{a}_q \mathfrak{a}_q$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Since $\mathfrak{g}_+ = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q})$ is the Cartan decomposition of $\mathfrak{g}_+$, and $K \cap H$ is a maximal compact subgroup of $G_+$, one can apply standard theory for noncompact Riemannian symmetric spaces (as introduced by Cartan) to see that $\mathfrak{a}_q$ is unique up to $K \cap H$-conjugacy. Let $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+)$ be the corresponding set of restricted roots, $\Sigma_+(\mathfrak{a}_q, \mathfrak{g}_+)$ a choice of positive roots $\mathfrak{a}_q^+$ the associated positive Weyl chamber, $A_q^+ = \exp \mathfrak{a}_q^+$, and $W_{K \cap H} = N_{K \cap H}(\mathfrak{a}_q)/Z_{K \cap H}(\mathfrak{a}_q) = W(\mathfrak{a}_q, \mathfrak{g}_+)$ the Weyl group. We then have the following important generalization of the $KAK$-decomposition for $G$:

Theorem 3.2.3 ($KA_qH$-decomposition). Every element $g \in G$ has a unique decomposition as $g = kah$ with $k \in K$, $a \in A_q$, and $h \in H$. Here $a$ is unique up to conjugation with an element from $W_{K \cap H}$. The mapping

$$(kZ_{K \cap H}(\mathfrak{a}_q), a) \rightarrow kah \in G/H$$

maps $K/Z_{K \cap H}(\mathfrak{a}_q) \times A_q^+$ onto $G/H$, and it maps $K/Z_{K \cap H}(\mathfrak{a}_q) \times A_q^+$ diffeomorphically onto an open and dense subset of $G/H$.

For $\alpha$ in $\mathfrak{a}_q^+$, let $g_\alpha$

$$(3.2) \quad g_\alpha = \{ Y \in \mathfrak{g} \mid [H,Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{a}_q \},$$

and let $\Sigma(\mathfrak{a}_q, \mathfrak{g})$ denote the system of nonzero roots $\alpha$ for which $g_\alpha \neq \{0\}$. Note that $\tau \vartheta(g_\alpha) = g_{-\alpha}$, so that $g_\alpha$ decomposes as $g_\alpha = g_\alpha^+ \oplus g_{-\alpha}$. Let $m_{\alpha}$ denote the dimension of $g_\alpha$ (also called the multiplicity of $\alpha$), and put $m_\alpha^+ = \dim g_\alpha^+$ so that $m_\alpha = m_\alpha^+ + m_{-\alpha}$. Here $m_\alpha^+$ is the multiplicity of $\alpha$ as a member of $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+)$. Furthermore, let

$$(3.3) \quad f(Y) = \prod_{\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})} \sinh^{m_\alpha^+} \alpha(Y) \cosh^{m_{-\alpha}} \alpha(Y) \text{ for } Y \in \mathfrak{a}_q.$$

Then we have the following fundamental result of Flensted-Jensen ([FJ80]):

$$dx$$
Theorem 3.2.4. The invariant measure $dx$ on the semisimple symmetric space $X = G/H$ is given by

$$\int_X f(x) \, dx = \int_K \int_{a_q^+} f(k \exp Y \cdot o) J(Y) \, dY \, dk,$$

where $dY$ is a fixed Lebesgue measure on $a_q^+$, where $dk$ is a Haar measure on $K$ and $J(Y)$ is given by (3.3).

Definition 3.2.5. A Cartan subspace of $G/H$ is a maximal abelian subset $\mathfrak{a}$ of $q$ consisting of semisimple elements. The rank of $G/H$ is the real dimension of any Cartan subspace of $G/H$.

Main example 1: The group-case Let $\mathcal{G}$ be a Lie group, let $G = \mathcal{G} \times \mathcal{G}$, and define an involution $\tau : G \to G$ by $\tau (x, y) = (y, x)$. The fixpoint group $H = G^\tau$ is then the diagonal in $G$, and the symmetric space $G/H$ is isomorphic to $\mathcal{G}$ through the mapping $(x, y) \mapsto x y^{-1}$, with $\mathcal{G}$ viewed as a homogeneous space for the left-times-right action of $\mathcal{G} \times \mathcal{G}$.

Main example 2: The real hyperboloids Let $p$ and $q$ be positive integers, and define the Lorentzian form of signature $(p, q)$ on $\mathbb{R}^{p+q}$ by

$$\beta_{p,q}(x, y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i.$$

Let $G = SO_e(p, q)$ denote the connected group of real-valued $(p+q) \times (p+q)$ matrices preserving the form $\beta_{p,q}$, and let $H = SO_e(p-1, q)$ denote the stabilizer subgroup of $G$ in the point $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{p+q}$. Here we view $H$ as a subgroup of $G$ via the natural identification

$$SO_e(p-1, q) \ni L \mapsto \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \in SO_e(p, q).$$

Let $K = SO(p) \times SO(q)$ be the maximal compact subgroup of elements in $G$ being fixed by the Cartan involution $\theta$ on $G$ defined by $\theta(g) = (g^1)^{-1}$.

Let $J = \text{diag}(1, -1, \ldots, -1)$, and define an involution $\tau$ on $G$ by $\tau (g) = J g J$. Then $\tau$ commutes with $\theta$ and $G/H$ becomes a semisimple symmetric space with respect to $\tau$. The map $g \mapsto g \cdot (1, 0, \ldots, 0)$ (usual matrix multiplication from the left) induces an embedding of $G/H$ as the hypersurface in $\mathbb{R}^{p+q}$ given by

$$X = \{ x \in \mathbb{R}^{p+q} \mid \beta_{p,q}(x, x) = 1 \}.$$

In fact, it is easily verified that $G = SO_e(p, q)$ acts transitively on the set $X$, and the stabilizer of $G$ in $e_1$ is $H$ by construction. Hence $G/H$ is diffeomorphic to $X$.

With $SO_e(p, q)/SO_e(p-1, q)$ being realized as $X \subset \mathbb{R}^{p+q}$, we can give a more concrete form of Theorem 3.2.4; using the polar coordinates $(v, r) \in S^{p-1} \times \mathbb{R}^+$ on the first $p$ entries and $(w, s) \in S^{q-1} \times \mathbb{R}^+$ on the last $q$ entries, respectively, the Lebesgue measure $dx = dx_1 \cdots dx_{p+q}$ on $\mathbb{R}^{p+q}$ an invariant measure on $X$ by $dv \, dw \, \cosh^{p-1} t \sinh^{q-1} t \, dt$.

We will also need to have detailed knowledge of the joint decomposition (3.1) for the symmetric spaces $SO_e(p, q)/SO_e(p-1, q)$; the result is listed in Figure 3.1 below. In particular, the rank of $SO_e(p, q)/SO_e(p-1, q)$ is one.
With \( g = sl(p, q) \), and \( \tau \) and \( \theta \) as defined above, the joint eigenspace decomposition of \( g \) in (3.1) is described as follows:

\[
\begin{align*}
\mathfrak{e} \cap \mathfrak{q} &= \left\{ \begin{pmatrix} 0_{1,1} & \ast & \cdots & \ast & 0_{1,q} \\
\ast & 0_{p-1,p-1} & 0_{p-1,q} \\
\vdots & 0_{p-1,p-1} & 0_{p-1,q} \\
\ast & 0_{q-1,p} & 0_{q-1,q} \\
0_{1,1} & \ast & 0_{q-1,p} & 0_{q-1,q} \\
\end{pmatrix} \right\} \\
\mathfrak{e} \cap \mathfrak{h} &= \left\{ \begin{pmatrix} 0_{1,1} & 0_{1,p-1} & 0_{1,q} \\
0_{p-1,1} & \ast & 0_{p-1,q} \\
0_{1,1} & 0_{1,p-1} & \ast \\
0_{1,1} & \ast & 0_{q-1,p} & 0_{q-1,q} \\
\end{pmatrix} \right\} \\
\mathfrak{p} \cap \mathfrak{h} &= \left\{ \begin{pmatrix} 0_{1,1} & 0_{1,p-1} & 0_{1,q} \\
0_{p-1,1} & 0_{p-1,p-1} & \ast \\
0_{1,1} & \ast & 0_{q-1,p} & 0_{q-1,q} \\
\end{pmatrix} \right\} \\
\mathfrak{p} \cap \mathfrak{q} &= \left\{ \begin{pmatrix} 0_{1,1} & 0_{1,p-1} & \ast \\
0_{p-1,1} & 0_{p-1,p-1} & 0_{p-1,q} \\
0_{1,1} & \ast & 0_{q-1,p} & 0_{q-1,q} \\
\end{pmatrix} \right\}
\end{align*}
\]

Figure 3.1: Decomposition of \( so(p, q) \)

### 3.3 Causal Structures and Orientations

Let \( M \) be a smooth manifold, let \( T_m(M) \) denote the tangent space of \( M \) at \( m \in M \), and let \( T(M) \) denote the tangent bundle.

**Definition 3.3.1.** A smooth causal structure on \( M \) is a map

\[
\mathcal{M} \ni m \mapsto C(m) \subset T_m(M)
\]

that assigns a nontrivial closed convex cone \( C(m) \) in \( T_m(M) \) to each point \( m \) in \( M \), and is smooth in the following sense: There exists an atlas \( \{U_i, \varphi_i\} \) and a cone \( C \) in \( \mathbb{R}^n \) with open sets \( U_i \subset M \) and smooth maps

\[
\varphi_i : U_i \times \mathbb{R}^n \to T(M)
\]
satisfying \( \varphi_i(m, v) \in T_m(M) \) and \( C(m) = \varphi_i(m, C) \).

The causal structure is generating, proper, respectively regular, if \( C(m) \) is generating, proper, respectively regular, for all \( m \in M \).

A map \( f : M \to M \) is causal if \( d_m f(C(m)) \subset C(f(m)) \) for all \( m \in M \), where \( d_m \) is the differential at the point \( m \in M \).

If a Lie group \( G \) acts smoothly on \( M \), written \( (g, m) \mapsto g \cdot m \), we denote the diffeomorphism \( m \mapsto g \cdot m \) by \( \ell_g \).

**Definition 3.3.2.** Let \( M \) be a smooth manifold with a causal structure, and let \( G \) be a Lie group acting on \( M \). The causal structure is \( G \)-invariant if all the maps \( \ell_g, g \in G \), are causal.

Usually the causal structure is not given or constructed in terms of the actual assignment \( m \mapsto C(m) \) of cones but rather in terms of the Lie algebra structure, as follows. Let \( o = \{H\} \in G/H \).

**Theorem 3.3.3.** Let \( M = G/H \) be a homogeneous space. Then

\[
C \mapsto (aH \mapsto d_o \ell_a(C))
\]
defines a bijection between \( \text{Cone}_H(T_o(M)) \) and the set of \( G \)-invariant regular causal structures on \( M \).

21
The result is part of the folk-lore surrounding the theory of causal spaces, but a proof can be found in, say, [HÖ97] and [Kan91].

Let \( \mathcal{M} = G/H \) and \( C \in \text{Cone}_G(T_0(\mathcal{M})) \). An absolutely continuous curve\(^1\) \( \gamma : [a, b] \to \mathcal{M} \) is \( C \)-causal if \( \gamma'(t) \in C(\gamma(t)) \) whenever the derivative exists.

**Definition 3.3.4.** We write \( m \prec_s n \) if there exists a \( C \)-causal curve \( \gamma \) from \( m \) to \( n \) in \( \mathcal{M} \).

The relation \( \prec_s \) is reflexive and transitive, and is called a *causal orientation* on \( \mathcal{M} \).

For vector spaces we have a special way to obtain a causal orientation: Let \( V \) be a finite dimensional vector space and let \( C \) be a closed convex cone in \( V \). We define a causal \( \text{Aut}(C) \)-invariant orientation \( \prec \) on \( V \) by

\[
  u \prec v \iff v - u \in C.
\]

Then \( \prec \) is antisymmetric precisely when \( C \) is proper.

**Observation:** The light cone \( C \) in \( \mathbb{R}^{n+1} \) defines an \( SO_e(1, n) \)-invariant ordering in \( \mathbb{R}^{n+1} \). We refer to \( \mathbb{R}^{n+1} \) equipped with this ordering as the \((n + 1)\)-dimensional Minkowski space.

**Definition 3.3.5.** Let \( \mathcal{M} \) be a smooth manifold.

1. A causal orientation \( \leq \) on \( \mathcal{M} \) is topological if its graph

\[
\mathcal{M}_\leq = \{(m, n) \in \mathcal{M} \times \mathcal{M} \mid m \leq n\}
\]

is closed in \( \mathcal{M} \times \mathcal{M} \).

2. A space \( (\mathcal{M}, \leq) \) with a topological causal orientation is called a causal space. If \( \leq \) is a partial order (that is, antisymmetric in addition to being reflexive and transitive), \( (\mathcal{M}, \leq) \) is called an ordered space.

3. Let \( (\mathcal{M}, \leq) \) and \( (\mathcal{N}, \leq) \) be two causal spaces and let \( f : \mathcal{M} \to \mathcal{N} \) be a continuous map. Then \( f \) is order preserving (or monotone) if

\[
m_1 \leq m_2 \implies f(m_1) \leq f(m_2).
\]

4. Let \( G \) be a group acting on \( \mathcal{M} \). A causal orientation \( \leq \) on \( \mathcal{M} \) is \( G \)-invariant if

\[
m \leq n \implies \forall a \in G: a \cdot m \leq a \cdot n.
\]

5. A triple \( (\mathcal{M}, \leq, G) \) is a causal \( G \)-manifold (or a causal \( G \)-space) if \( \leq \) is a topological \( G \)-invariant causal orientation.

---

\(^1\)A continuous map \( \gamma : [a, b] \to \mathcal{M} \) is *absolutely continuous* if, for every chart \( \phi : U \to \mathbb{R}^n \), the map

\[
\eta = \phi \circ \gamma : \gamma^{-1}(U) \to \mathbb{R}^n
\]

has absolutely continuous coordinate functions that have locally bounded derivatives.
We will now return to the hyperbolic spaces and study the meaning of these concepts. Let \( p \) and \( q \) be positive integers, \( n = p + q \), and write elements of \( \mathbb{R}^n \) as \( v = (x, y) \) with \( x \in \mathbb{R}^p \) and \( y \in \mathbb{R}^q \). For \( r > 0 \) we define

\[
Q_{-r}^{p,q} = Q_{-r} := \{ v \in \mathbb{R}^{n+1}_+ | \beta_{p,q+1}(v, v) = -r^2 \}
\]

and

\[
Q_{+r}^{p,q} = Q_{+r} := \{ v \in \mathbb{R}^{n+1}_+ | \beta_{p,q+1}(v, v) = r^2 \}.
\]

**Lemma 3.3.6.** For \( p, q \in \mathbb{N} \), the group \( SOe(p+1,q) \) acts transitively on \( Q_{+r} \). The isotropy subgroup of \( SOe(p+1,q) \) at \( (r,0,\ldots,0) \) is isomorphic to \( SOe(p,q) \). Thus

\[
Q_{+r} \cong SOe(p+1,q)/SOe(p,q).
\]

**Proof.** Assume, without loss of generality, that \( r = 1 \). According to Witt’s Theorem, there exist \( A \in SOe(p+1) \) and \( D \in SOe(q) \) such that \( Ax = \|x\|e_1 \) and \( Dy = \|y\|e_{p+q} \). Since the matrix \( \left( \begin{array}{c} A \ 0 \\ 0 \ D \end{array} \right) \) belongs to \( SOe(p+1,q) \) we may assume that \( x = \lambda e_1 \) and \( y = \mu e_{p+q} \) for suitable positive numbers \( \lambda \) and \( \mu \) satisfying \( \lambda^2 - \mu^2 = 1 \).

It follows that \( a(e_1) = v \) with

\[
a = \begin{pmatrix}
\lambda & 0 & \mu & 0 \\
0 & I_p & 0 & 0 \\
\mu & 0 & \lambda & 0 \\
0 & 0 & 0 & I_{q-1}
\end{pmatrix} \in SOe(p+1,q).
\]

Hence the action of \( SOe(p+1,q) \) is transitive.

The statement regarding the stabilizer follows from similar calculations. \( \blacksquare \)

Consider the linear isomorphism

\[
L_n : \mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto (x_n, \ldots, x_1) \in \mathbb{R}^n.
\]

In particular, \( L_{n+1}(r,0,\ldots,0) = (0,\ldots,0,r) \), and the group conjugation map

\[
\text{Ad}(L_{n+1}) : a \mapsto L_{n+1}aL_{n+1}^{-1}
\]

establishes the group isomorphism \( O(q+1,p) \cong O(p,q+1) \). Furthermore, \( \text{Ad}(L_{n+1}) \) maps the stabilizer of \( (r,0,\ldots,0) \) onto the \( O(p,q) \)-stabilizer of \( (0,\ldots,0,r) \). It follows that

\[
Q_{+r} = O(p,q+1)/O(p,q) \cong SOe(p,q+1)/SOe(p,q).
\]

Hence the hyperboloids \( Q_{\pm r} \) are homogeneous spaces.

We will now show that \( Q_{\pm r} \) are, in fact, symmetric spaces. To this end it suffices to work with \( O(p+1,q)/O(p,q) \cong Q_{+r} \), since a similar result for \( Q_{-r} \) will then follow by conjugation with \( L_{n+1} \).

Let \( \tau \) be conjugation on \( O(p+1,q) \) with \( I_{1,n} \). Then

\[
(O(p+1,q)^\tau)_e = SOe(p,q) \subset O(p+1,q)_{re_1} \subset O(p+1,q)^\tau.
\]
We also have that \( h = so(p, q) = g^T \) (since \( \tau \) on \( g = so(p + 1, q) \) is also given by conjugation with \( I_{1,n} \)). Define

\[
q(v) = \begin{pmatrix} 0 & -I(vI_p,q) \\ v & 0 \end{pmatrix}
\]

for \( v \in \mathbb{R}^n \) to obtain a linear isomorphism \( \mathbb{R}^n \ni v \rightarrow q(v) \in g \) with the additional properties that \( q(a v) = aq(v)a^{-1} \) for \( a \in SO_e(p, q) \) and \( \beta_{p,q}(v, w) = -\frac{1}{2} \text{Tr}(q(v)q(w)) \).

For \( n \geq 2, q = n - 1 \) and \( p = 1 \), we define the cone \( C \) in \( \mathbb{R}^n \) by

\[
C = \{ v \in \mathbb{R}^n | \beta_{1,q}(v, v) \geq 0, x \geq 0 \}
\]

and let

\[
\Omega = C^\circ = \{ v \in \mathbb{R}^n | \beta_{1,q}(v, v) > 0, x > 0 \}.
\]

Here \( C \) is called the forward light cone in \( \mathbb{R}^n \).

**Remark.** One can show that only for this choice of \( p \) and \( q \) will \( C \) be a global causal structure on the hyperbolic space.

**Lemma 3.3.7.** The cone \( C \) is self-dual.

**Proof.** An element \( v = (x, y) \) belongs to \( C \) if and only if \( x \geq \|y\| \). If \( v \) belongs to the intersection \( C \cap (-C) \), then \( 0 \leq x \leq 0 \), and thus \( x = 0 \). But then \( y = 0 \), and \( C \cap (-C) = \{0\} \). Hence \( C \) is a proper cone.

For \( v, v' \in C \) we have

\[
(v', v) = x'x + (y', y) \geq \|y'\|\|y\| + (y', y) \geq 0,
\]

implying that \( C \) is contained in \( C^\circ \).

Conversely, let \( v = (x, y) \) belong to \( C^\circ \). Then \( x \geq 0 \). We may now assume, without loss of generality, that \( y \) is nonzero. Choose \( w \in \mathbb{R}^n \) such that \( pr_1(w) = \|y\| \) and \( pr_2(w) = -y \).

Then \( w \) belongs to \( C \) and satisfies the inequality

\[
0 \leq (w, v) = x\|x\| - \|y\|^2 = (x - \|y\|)\|y\|.
\]

Hence \( x \geq \|y\| \), and we conclude that \( y \) belongs to \( C \), so that \( C^\circ \subset C \). Therefore \( C^\circ = C \). \[\square\]

**Remark.** It is shown in the same manner that \( \Omega \) is self-dual. Notice that \( C \) is invariant under the usual action of \( SO_e(1, q) \) (via matrix multiplication from the left) and under the action of \( \mathbb{R}^+ = \{\lambda I_n | \lambda > 0 \} \).

**Proposition 3.3.8.** The group \( \mathbb{R}^+SO_e(1, q) \) acts transitively on \( \Omega \) whenever \( q \geq 2 \). In particular, \( \Omega \) is homogeneous.

**Proof.** Assume \( q \geq 2 \) and let

\[
a_t = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \in SO_e(1, q).
\]

Then \( a_t \cdot \begin{pmatrix} h \end{pmatrix} = \lambda^t(\cosh t, \sinh t, 0, \ldots, 0) \) belongs to \( \Omega \) for all \( t \in \mathbb{R} \). The claim now follows from the fact that \( SO(o) \) acts transitively on the sphere \( S^{q-1} \) and \( \begin{pmatrix} h \\ 0 \end{pmatrix} \) belongs to \( SO_e(1, q) \) for all \( A \) in \( SO(o) \). \[\square\]
The natural embedding $SO(q) \hookrightarrow SO_e(1,q)$ exhibits $SO(q)$ as a maximal compact subgroup of $SO_e(1,q)$ that fixes the unit vector $e_1 \in \Omega$. One easily verifies that the stabilizer of $SO_e(1,q)$ in $e_1$ is $SO(q)$. Since an $SO_e(1,q)$-invariant regular cone in $\mathbb{R}^n$ always contains an $SO(q)$-invariant multiple of $e_1$, it follows from the homogeneity of $\Omega$ that (for $q > 1$) the only $SO_e(1,q)$-invariant closed regular and convex cones are $C$ and $-C$.

### 3.4 The Causal Compactification of a Compactly Causal Symmetric Space

Consider a semisimple symmetric space $G/H$, write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{t} \oplus \mathfrak{p}$, and identify the tangent space $T_e(G/H)$ with $\mathfrak{q}$ as before. Let $\text{Cone}_H(\mathfrak{q})$ denote the set of $H$-invariant closed and regular convex cones in $\mathfrak{q}$.

**Definition 3.4.1.** Let $(G,H,\tau)$ be a causal symmetric space with $G$ semisimple. Then $(G,H,\tau)$ is compactly causal if there exists a cone $C$ in $\text{Cone}_H(\mathfrak{q})$ such that $C^\circ \cap \mathfrak{t} \neq \emptyset$, and non-compactly causal if there exists a cone $C$ in $\text{Cone}_H(\mathfrak{q})$ such that $C^\circ \cap \mathfrak{p} \neq \emptyset$. If $(G,H,\tau)$ is both compactly causal and non-compactly causal, $(G,H,\tau)$ is said to be of Cayley type.

The complete classification can be found in [HÓ97], Chapter 3 and 4.

Let $(\mathfrak{g}_1,\mathfrak{t}_1,\theta)$ be an irreducible orthogonal symmetric Lie algebra of noncompact Hermitian type, and let $\mathfrak{g}_{1\mathbb{C}} = \mathfrak{g}_1 \otimes \mathbb{C}$ denote the complexification of $\mathfrak{g}_1$. Let $G_{1\mathbb{C}}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}_{1\mathbb{C}}$, let $(G_1,K_1,\theta)$ be associated to $(\mathfrak{g}_1,\mathfrak{t}_1,\theta)$, and assume that $G_1$ is contained in $G_{1\mathbb{C}}$. We will always denote the Cartan involution $\theta$ for $G_1$ or $\mathfrak{g}_1$ and its extension to the respective complexification by the same letter. Let $\mathfrak{g}_1 = \mathfrak{t}_1 \oplus \mathfrak{p}_1$ be the Cartan decomposition of $\mathfrak{g}_1$ with respect to $\theta$.

Let $\mathfrak{t} \subset \mathfrak{t}_1$ be a Cartan subalgebra and let $\Delta(\mathfrak{g}_{1\mathbb{C}},\mathfrak{t}_1)$ denote the corresponding root system. Choose a positive system $\Delta^+(\mathfrak{g}_{1\mathbb{C}},\mathfrak{t}_1)$ such that the noncompact roots, $\Delta^+(\mathfrak{p}_{1\mathbb{C}},\mathfrak{t}_1)$, dominate the compact roots, $\Delta^+(\mathfrak{t}_{1\mathbb{C}},\mathfrak{t}_1)$. Let $\{Y_1, \ldots, Y_r\} \subset \Delta^+(\mathfrak{p}_{1\mathbb{C}},\mathfrak{t}_1)$ be a maximal system of strongly orthogonal roots, and choose $E_{i,j} \in \mathfrak{g}_{1\mathbb{C},Y_j}$ such that $X_j := E_j + E_{-j}$ belongs to $\mathfrak{t}_1$, $Y_j := i(E_j - E_{-j})$ belongs to $\mathfrak{p}_1$, and such that $[E_j,E_{-j}] = H_j$. Here $H_j \in \mathfrak{t}$ is defined by

$$\langle H_j,H \rangle = 2 \frac{Y_j(H)}{\langle Y_j,Y_j \rangle}$$

for all $H \in \mathfrak{t}$.

Recall the (full) Cayley transform $\mathbf{c}$. An $\mathfrak{sl}(2,\mathbb{R})$-calculation, combined with the strong orthogonality of the $Y_j$'s, shows that $\text{Ad}(\mathbf{c})iH_j = X_j$ and $\text{Ad}(\mathbf{c})it^- = a$, where $t^- := \sum \mathbb{R}iH_j \subset \mathfrak{t}$.

Now consider the triangular decomposition of $\mathfrak{g}_{1\mathbb{C}}$ given by

$$\mathfrak{g}_{1\mathbb{C}} = \mathfrak{p}_{1\mathbb{C}}^+ \oplus \mathfrak{t}_{1\mathbb{C}} \oplus \mathfrak{p}_{1\mathbb{C}}^-$$

where

$$\mathfrak{p}_{1\mathbb{C}}^+ := \sum_{\gamma \in \Delta^+(\mathfrak{p}_{1\mathbb{C}},\mathfrak{t}_1)} \mathfrak{g}_{1\mathbb{C},Y} \text{ and } \mathfrak{p}_{1\mathbb{C}}^- := \sum_{\gamma \in \Delta^+(\mathfrak{p}_{1\mathbb{C}},\mathfrak{t}_1)} \mathfrak{g}_{1\mathbb{C},-Y}.$$
Let $P^+\subset K_{1C}$ be the corresponding analytic subgroups of $G_{1C}$, let $x_0 = eK_{1C}P_1^+ \subset G_{1C}/K_{1C}P_1^-$: $M^*$ be the coset of the identity in the Borel realization of the Hermitian symmetric space $G_1/K_1$. The Silov boundary $\tilde{S}_1$ of $G_1/K_1$ is the closed boundary $G_1(\mathfrak{e}x_0)$. An $\mathfrak{s}\mathfrak{l}(2,\mathbb{R})$-calculation reveals that
\[
\mathfrak{e}_j = \exp(-E_j) \exp(\log \sqrt{2}E_j) \exp(-E_j).
\]
In the Harish-Chandra realization of $G_1/K_1$ in $p_1^+$, the Silov boundary is therefore given as th\$\tilde{S}_1$ of $G_1/K_1$ is the closed boundary $G_1(\mathfrak{e}x_0)$. An $\mathfrak{s}\mathfrak{l}(2,\mathbb{R})$-calculation reveals that
\[
\mathfrak{e}_j = \exp(-E_j) \exp(\log \sqrt{2}E_j) \exp(-E_j).
\]

**Theorem 3.4.2.** The stabilizer of the boundary point $eK_{1C}P_1^-$ of $G_1/K_1$ is $P' := P_{1C} \cap G_1$. If $G_1$ is simple, $P'$ is a maximal parabolic subgroup of $G_1$.

**Corollary 3.4.3.** The Silov boundary of $G_1/K_1$ can be described as $\tilde{S}_1 \cong G_1/P'$.

Also note that the Langlands decomposition for $P'$ is given by $P' = M' \times A' \times Q_1^-$, where $A' := \exp \mathbb{R}X^0$ and $M'A' = Z_{G_1}(X^0)$.

Since $(g_1, t_1, \theta)$ is of tube type, the inner automorphism
\[
\eta := \text{Ad}(\frac{\pi iX^0}{2})
\]
is an involution, and $H_1 := G_1^\eta$ coincides with $Z_{G_1}(X^0)$. Furthermore $\theta \eta = \eta \theta$. The triple $(G_1, H_1, \eta)$ thus defines an irreducible symmetric space of Cayley type.

Decompose $X^0 := \sum_{j=1}^r X_j$ into $(\pm)$-eigenvectors $Y_\pm$ of $ad X^0$ and define $H_1$-invariant cones by
\[
C_\pm := \text{conv}(\text{Ad}(H_1)_{\mathbb{R}^+}Y_\pm)
\]
(the closed convex hull of the set $\text{Ad}(H_1)_{\mathbb{R}^+}Y_\pm$). Then $C_k := C_+ - C_- \subset q_1^+ \oplus q_1^- =: q_1$ is a regular $H_1$-invariant cone. Using the identification of $q_1$ with the tangent space $T_{eH_1}(G_1/H_1)$ of $G_1/H_1$ at $eH_1$, we thus use $C_k$ to define the compactly causal structure on $G_1/H_1$.

**Theorem 3.4.4.** The canonical inclusion $\iota : G/H \hookrightarrow G_1/G_1^\eta$ is causal and the map $\Phi := \Phi_1 \circ \iota : G/H \hookrightarrow \tilde{S}_1$ is a $G$-equivariant causal compactification of $G/H$.

**Proof.** See Theorem 2.3 in [BÓ01].

Here $\Phi_1 : G_1/G_1^\eta \hookrightarrow \tilde{S}_1 = G_1/P'$ is the canonical projection.

**Example:** It is explained in [Bet97], Chapter 6, and in [Bet03], Example 5.2 how to choose $g_1$, $\tau$ and $\sigma$ in order to ensure that $(g, h)$ corresponds to $(\mathfrak{s}\mathfrak{o}(2, \mathfrak{n}), \mathfrak{s}\mathfrak{o}(1, \mathfrak{n}))$. We let $G_1 = SO(2, n+1)$ and $\theta(g) = t g^{-1}$. Then $H_1 = G_1^\eta \cong SO(1, 1) \times O(1, n)$, and $(G_1, H_1, \eta)$ is of Cayley type.
Let $\sigma$ denote the inner automorphism
\[
\sigma = \text{Ad} \left( \begin{pmatrix} I_2 & 0 \\ -I_n & 1 \end{pmatrix} \right).
\]
Then $G_1^\sigma \simeq SO(2, n)$, and $SO_e(2, n)/SO_e(1, n)$ has a causal compactification in $SO(2, n+1)/P'$.

Writing $\mathcal{S}_1 = G_1(c^{-1}x_0)$, we define the Cayley transformed boundary by $\mathcal{S}_1^c = cG_1(c^{-1}x_0) = (\text{Ad}(c)G_1)x_0$. Left multiplication $L_c$ on $M^*$ defines a diffeomorphism of $\mathcal{S}_1^c$ onto $\mathcal{S}_1$ if $G_1/K_1$ is of tube type. Indeed, $G_1/K_1$ being of tube type is equivalent to the property that $\text{Ad}(c^2)t_1 = t_1$, which implies that $\text{Ad}(c^2)g_1 = g_1$, and thus $L_c\mathcal{S}_1^c = (\text{Ad}(c^2)G_1)(cx_0) = G_1(cx_0) = \mathcal{S}_1$.

### 3.5 Causal Structure on the Silov Boundary

Assume $(g_1, t_1, \theta)$ is of tube type, so that $\text{ad}X^0$ has eigenvalues $0$ and $\pm 2$. The inner automorphism $\eta := \text{Ad}(\exp \frac{\pi i}{2} X^0)$ is involutive and $H_1 := G_1^\eta = Z_{G_1}(X^0)$. Furthermore $\theta \tau = \tau \theta$, and $(G_1, H_1, \eta)$ is an irreducible symmetric space of Cayley type. Let $Y^0 := \sum_{j=1}^r Y_j$ be the decomposition of $Y^0$ into eigenvectors $Y_+$ and $Y_-$ corresponding to the $+2$, respectively, $-2$-eigenvectors of $\text{ad}X^0$, and define $C_\pm = \text{conv}\text{Ad}(H_1)e^{\mathbb{R}^+} Y_\pm$ (the closed convex hull of $\text{Ad}(H_1)e^{\mathbb{R}^+} Y_\pm$). The cones $C_\pm$ are $H_1$-invariant, so $C_k := C_+ - C_-$ is a regular $H_1$-invariant cone in $q_1 := q_1^+ \oplus q_1^-$. We then use the canonical identification of $q_1$ with the tangent space $T_e H_1(G_1/H_1)$ to define a compactly causal structure of $G_1/H_1$ using $C_k$.

Let $\mathcal{S}_1 = G_1(c^{-1}x_0)$ denote the Silov boundary of $G_1/K_1$ and let $\mathcal{S}_1^c = cG_1(c^{-1}x_0) = (\text{Ad}(c)G_1)(x_0)$ denote the Cayley transform of the Silov boundary. Observe that by definition, $\mathcal{S}_1^c$ is the Silov boundary of the unbounded realization of $G_1/K_1$. Using the differential of $\xi^c : p_1^+ \to M^*$, $X \mapsto (\exp X) \cdot x_0$, we identify the tangent space $T_{x_0}M^*$ with $p_1^+$. The tangent space $T_{x_0}\mathcal{S}_1^c$ thereby corresponds to $\mathcal{S}_1^c : = p_1^+ \cap \text{Ad}(c)g_1$. The cone defining the causal structure of $\mathcal{S}_1^c$ is then given by the closure of the orbit of $K_1^c := K_1c \cap \text{Ad}(c)G_1$ through $E := -i\xi^{-1}(cx_0) = i\sum_j E_j$. Equivalently, we may notice that the cone $C_+$ defined above is regular and $H_1$-invariant in $q_1^+$, and thus invariant under the linear isotropy representation of $P'$ in $T_{p'}(G_1/P')$. The cone $C_+$ therefore defines a $G_1$-equivariant causal structure on $G_1/P'$ (unique up to a change of sign). This is the method adopted in [Kan91]

**Observation.** One can also construct the causal structure on $\partial S_{\Omega_1}$ directly: Let $T_{\partial S_{\Omega_1}}$ denote the unbounded realization of $G_1/K_1$, and identify the tangent space of $\partial S_{\Omega_1} \simeq \mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$ in every point. The closed cone $\overline{\partial S_{\Omega_1}} \subset T_{0}(\partial S_{\Omega_1})$ thus defines the invariant causal structure on the Silov boundary through translations.

### 3.6 Existence of Lowest Weight Representations and Spherical Vectors

**Definition 3.6.1.** If $\delta(K)$ has a nonzero fixed vector, then $\delta$ is called a spherical representation, and the vector being fixed by $\delta(K)$ a spherical vector.
Theorem 3.6.2. Let $\delta$ be an irreducible representation of $G$ on a finite-dimensional complex vector space $V$.

1. $\delta(K)$ has a nonzero fixed vector if and only if $\delta(M)$ leaves the highest-weight vector of $\delta$ fixed.

2. Let $\lambda$ be a linear form on $\mathfrak{h}_\mathbb{R}$. Then $\lambda$ is the highest weight of an irreducible finite-dimensional spherical representation of $G$ if and only if

\[(3.4)\quad \lambda(i(\mathfrak{h} \cap \mathfrak{k})) = 0 \quad \text{and} \quad \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \quad \text{for all} \quad \alpha \in \Sigma^+.
\]

The proof can be found in [Hel02], p.535-537.

Recall the set $\Delta^+ (p_1 \mathbb{C}, t_1 \mathbb{C})$ of positive non-compact roots. We let

$$\rho_{1,n} = \frac{1}{2} \sum_{\alpha \in \Delta^+ (p_1 \mathbb{C}, t_1 \mathbb{C})} m_\alpha \alpha = \frac{1}{2} \left( 1 + \frac{d(r-1)}{2} \right) (y_1 + \cdots + y_r)$$

where $d$ is the dimension of the root space of $\frac{1}{2}(y_i \pm y_j)$, $i \neq j$. Furthermore, let $\rho^+ = \frac{1}{2} \left( 1 + \frac{d(r-1)}{2} \right) (y_1^c + \cdots + y_r^c)$, where $\alpha^c = \alpha \circ \text{Ad}(c^{-1})$. We denote by $(\pi_m, V_m)$ the irreducible finite-dimensional representation of $G_{1\mathbb{C}}$ with lowest weight $-m \rho^+$ if it exists.

Theorem 3.6.3. 1. There exists an irreducible finite dimensional representation $(\pi_2, V_2)$ of $G_{1\mathbb{C}}$ with lowest weight $-2 \rho^+$;

2. Assume $g \neq \text{sp}(2n, \mathbb{R})$ and $g \neq \text{so}(2,2k+1)$ for $k, n \geq 1$. Then there exists a finite dimensional irreducible representation of $G_{\mathbb{C}}$ with lowest weight $-\rho^+$.

In conjunction to the previous result we have the following important theorem:

Theorem 3.6.4. Let the notation be as above. Then the following statements hold:

1. The weight space $V_{m,-m\rho^+}$ is left point-wise fixed by the identity component of $M'_{\mathbb{C}}$, where $M'_{\mathbb{C}}$ is defined by the Langlands decomposition $P' = M' \times \exp(\mathbb{R}X^0) \times Q_1$. For $m = 4$ it is fixed by $M'_{\mathbb{C}}$. If $g_1 \neq \text{sp}(2n, \mathbb{R})$ and $g_1 \neq \text{so}(2,2k+1)$ for $n, k \geq 1$, the same conclusion holds for the case $m = 2$;

2. There is a number $m \in \{1, 2, 4, 8\}$ such that $\pi_m$ is $G_{\mathbb{C}}$-spherical. In this case we can choose a scalar product $\langle \cdot, \cdot \rangle$ and vectors $v_m \in V_{m,-m\rho^+}$ and $\xi_m \in V_{m^C}$ such that $\langle \cdot, \cdot \rangle$ is invariant under the action of the analytic subgroup of $G_1$ with Lie algebra $\mathfrak{t}_1 + i\mathfrak{p}_1$ and such that $\langle v_m | \xi_m \rangle = 1$.

The first result is [ÓØ99], Theorem 2.6, and the second result is part of [BÓ01], Proposition 4.1. Using the classification of Cayley type spaces, we can list the values for $r$ and $d$ (and thus determine the various values for $m)$:

28
Example (cont.): Let \( G_1 = SO_e(2, n+1) \) with Cartan involution \( \theta(g) = {}^t g^{-1}, X^0 = 2(E_{n+3} + E_{n+3,1}) \), and
\[
\tau = \text{Ad} \begin{pmatrix} I_{n+2} & 0 \\ 0 & -1 \end{pmatrix}.
\]
Thereby \( b = \mathbb{R} X^0 = a^d \), and the for the Riemannian dual algebra \( g_1^d \) we therefore get, restricting the roots of \( \Delta^+(g_1, a_p) = \{ \gamma_1^c, \gamma_2^c, \frac{1}{2}(\gamma_1^c \pm \gamma_2^c) \} \) to \( b \), that \( \Delta^+(g_1^d, b) = \Delta^+(g_1, b) = \{ \frac{1}{2}(\gamma_1^c + \gamma_2^c) \} \).

Applying Helgason's theorem to \( (g_1^d, b) \) then reveals that \( \pi_m \) is \( G \)-spherical if and only if
\[
m \left( 1 + \frac{d(r-1)}{2} \right) \in \mathbb{Z}^+.
\]
But for \( g_1 = \text{so}(2, n+1) \) is follows from Figure 3.2 above that \( r = 2 \) and \( d = n - 1 \). Therefore \( \pi_2 \) is always \( G \)-spherical, whereas \( \pi_1 \) is \( G \)-spherical if and only if \( n \) is odd.

3.7 The Function \( \varpi_m \)

Now select \( m \in \mathbb{N} \) such that \( (\pi_m, V_m) \) exists as a representation of \( G_1 \) and is \( G \)-spherical. Fix a lowest weight vector \( v_m \in V_{m,-\rho^+} \) and a \( G_C \)-spherical vector \( \xi_m \in V_m^{G_C} \) such that \((v_m | \xi_m) = 1\). Then define a function \( \varpi_m : \Gamma_1 \to \mathbb{C} \) by \( \varpi_m(X) = (\pi_m(\exp X)v_m | \xi_m) \).

**Lemma 3.7.1.** The matrix coefficient map \( \varpi_m \) is always a polynomial (in \( \mathbb{N}_1 \)-coordinates) on \( \mathbb{R}^{n+1} \).

**Sketch of proof.** We have the commuting diagram

\[
\begin{array}{c}
\varpi_1 \\
\downarrow \exp \\
\mathbb{R}^{n+1} \\
\end{array} \rightarrow \begin{array}{c}
\mathbb{N}_1 \\
\end{array}
\]

and we must show that the map \( \varpi_1 \ni X \mapsto \varpi_m(\exp X) \in \mathbb{C} \) is polynomial. But \( \varpi_m(X) = (e^{d\pi_m(X)}v_m | \xi_m) \) is a matrix coefficient of \( e^{d\pi_m} \) and we are therefore saying that \( e^{d\pi_m(X)} \) is
a polynomial mapping in \( X \). Since \( N_1 \) is nilpotent, and “\( \text{ad}(\text{nilpotent}) \)” implies “nilpotent” according to a theorem of Lie, there exists an integer \( N \) such that \( d\pi_m(X)^N = 0 \) on \( V_m \). Thus

\[
e^{d\pi_m(X)} = \sum_{k=0}^{N} \frac{d\pi_m(X)^k}{k!}
\]

is indeed a polynomial mapping in \( X \).

Before we state the main result let us recall the abstract action of \( G \) on \( \overline{N}_1 \): For \( g \in G_1 \) and \( \pi \in \overline{N}_1 \), the action \( g \cdot \pi \) is defined through the Bruhat decomposition \( G_1 \sim \overline{N}_1 M_1 A_1 N_1 \) as

\[
g\pi = (g \cdot \pi) m_1(g\pi) a_1(g\pi) n_1(g\pi)
\]

with \( m_1(g\pi) \in M_1, a_1(g\pi) \in A_1 \), and \( n_1(g\pi) \in N_1 \). Since \( M_1 A_1 \) normalizes \( N_1 \) is follows easily that \( g \cdot \pi = (g\pi)(m_1(g\pi)a_1(g\pi))^{-1}n_1 \) with \( n_1 = n_1(g\pi)^{-1} \in N_1 \).

In the following we let \( Y \) denote the projection map from \( G/H \) into \( \partial_s T\Omega \) (where we now think of \( \overline{N}_1 \) as a model for the Silov boundary \( \partial_s T\Omega \) of \( G_1/K_1 \), so that \( \partial_s T\Omega \approx \overline{N}_1 \) is open dense in \( G_1/P' = S_1 \).

Let \( a_1(g,X) = a_1(g \exp X)^{2\rho_1,n} \) where \( (\cdot)^{2\rho_1,n} \) is the character \( \ell \mapsto \ell^{2\rho_1,n} = \det \text{Ad}(\ell)|_{n} \) of \( L_1 \), and let \( a_1^{m/2}(g,X) = a_1(g \exp X)^{m\rho_1,n} \) the square-root of \( \ell \mapsto \ell^{2m\rho_1,n} \). Note\(^3 \) that \( a_1(g,X) \) is the determinant of the jacobian of the action of \( g \) on \( \overline{N}_1 \).

**Theorem 3.7.2.** Assume \( m \in \mathbb{Z}^+ \) is such that \( (\pi_m,V_m) \) exists and is \( G_\mathbb{C} \)-spherical. Then \( \varpi_m \) is holomorphic on \( \overline{N}_1 \mathbb{C} \) and has the following properties:

1. \( \Upsilon(G/H) = \{ X \in \overline{N}_1 \mid \varpi_m(X) \neq 0 \} \);

2. For \( g \in G_\mathbb{C} \) and \( X \in \overline{N}_1 \) such that \( g \cdot X \) is defined, \( \varpi_m \) satisfy the transformation rule

\[
\varpi_m(g \cdot X) = a_1^{m/2}(g,X)\varpi_m(X).
\]

**Proof.** The space \( G/H \) is embedded as an open dense \( G \)-orbit \( \Omega \) in \( G_1/P' \). Then \( \Omega_N := \overline{N}_1 \cap \Omega \) is also open dense, so there is only one open \( G \)-orbit in \( G_1/P' \). Let \( M(g) := (\pi_m(g) v_m|_{\mathcal{E}_m}) \). A trivial consequence is that \( \Omega \) is contained in \( \overline{N}_1 \mid M(\overline{N}_1) \neq 0 \), thus contained in the set \( \{ g \in G_1 \mid M(g) \neq 0 \} \), where the set on the right is \( G \)-invariant. It follows that the set \( G_1 \setminus M^{-1}(\{0\}) \) is left-\( P' \)-invariant, right-\( G \)-invariant and open dense. Therefore \( \Omega \subset G \setminus M^{-1}(\{0\}) \).

We now claim that, in fact, \( \Omega = G \setminus M^{-1}(\{0\}) \). To this end, suppose \( p \) belongs to \( G_1 \setminus \Omega \). We will show that \( M(p) = 0 \). Since \( \Omega \) is dense, there exists a sequence \( (x_k) \) in \( \Omega \) with the property that \( x_k \to p \). Each element \( x_k \) decomposes according to the \( GM_1 A_1 N_1 \)-decomposition as \( x_k = g_k m_k a_k n_k \). Here \( M(x_k) = a_k^{-2m\rho} \), and we now assert that \( (a_n) \) tends out of every compact subset.

Since \( A_1 \) is one-dimensional this claim is saying that \( a_k^{2m\rho} \) tends out of every compact subset of \( (0,\infty) \). But \( M(x_k) \to M(p) \subset \mathbb{C} \) (in \( \mathbb{R} \)), so \( a_k^{2\rho_1} \) converges in \( \mathbb{C} \) (in \( \mathbb{R} \)) to zero. On the other hand it converges in \( \mathbb{C} \) (in \( \mathbb{R} \)) to \( M(p) \), so we conclude that \( M(p) = 0 \).

\(^3\)One usually does this calculation while construction the non-compact realization of a principal series representation on \( \mathcal{N} \), as in [Ola87] or [vdB96]. I would like to thank professor Ólafsson for pointing this fact out to me.
The proof of the above can essentially be found\textsuperscript{4} in [Óla87] (but some ideas can be traced back to [Wal73], if not even earlier) and goes as follows: Assume $a_k$ does not tend out of every compact subset. Then all $a_k$’s belong to a fixed compact subset $C$ of $A_1$. Passing to a subsequence, if necessary, we may assume that $a_k \to 0$.

Here $(gm_k) \in G$ is fixed under the involution,
\[
\left(\frac{g m_k a_k n_k a_k^{-1}}{\tilde{g}_k} \right) \to \left(\frac{p a^{-1}}{\tilde{p}}\right).
\]

Now apply $\sigma$ to get
\[
\sigma(\tilde{n}_k)^{-1} \tilde{n}_k \to \sigma(\tilde{p})^{-1} \tilde{p}
\]
(the $\tilde{g}_k$ cancel out), where the sequence on the left is a converging sequence in $\mathbb{N} N$.

The product map $\mathbb{N} \times \mathbb{N} \to G$ (injective multiplication) is a proper map, that is, the pre-image of a compact set is compact. Hence
\[
\sigma(\tilde{p})^{-1} \tilde{p} = \bar{n} \tilde{n}
\]
for suitable elements $\bar{n}$ and $\tilde{n}$. Therefore
\[
\sigma(\bar{n})^{-1} \sigma(\tilde{n})^{-1} = \sigma(\bar{n} \tilde{n})^{-1} = \bar{n} \tilde{n}
\]
implies by injectivity that $\bar{n} = \sigma(\tilde{n})^{-1}$ and $\sigma(\tilde{p} \tilde{n}^{-1}) = (\tilde{p} \tilde{n}^{-1})$.

But then it follows that $\tilde{p} \tilde{n}^{-1}$ belongs to $G_1^\sigma$, implying that $\tilde{p}$ belongs to $G_1^\sigma \tilde{n}$. We conclude that $p \in G_1^\sigma \tilde{n} a \subset G_1^\sigma P_1$.

This contradicts the fact that $G_1^\sigma P_1$ is a union of open $G \times P_1$-orbits and therefore contained in $\Omega$.

The second statement follows more or less directly from some standard facts about lowest weights. To be precise\textsuperscript{5}, $N_1$ pushes the weights up, $A_1$ acts by scalars, and $M_1$ fixes the highest weight vectors, according to Helgason’s theorem. Therefore
\[
\varpi_m(g \cdot X) = (\pi_m(g \cdot \exp X) v_m | \xi_m)
= (\pi_m(\exp X) \pi_m(a_1(g \exp X)^{-1}) \pi_m(m_1(g \exp X)^{-1}) \pi_m(n_1^\prime) v_m | \xi_m)
= (\pi_m(\exp X) \pi_m(a_1(g \exp X)^{-1}) v_m | \xi_m)
= a_1(g \exp X)^{m_1,n} (\pi_m(\exp X) v_m | \xi_m)
= a_1(g \exp X)^{m_1,n} \varpi_m(X).
\]

As a corollary we therefore get the following result (which we choose to label as a theorem):

\textsuperscript{4}In our case $P'$ is a maximal parabolic subgroup that is, at the same time, $\sigma \tau$-minimal. The results in the quoted reference will then apply to the current proof.

\textsuperscript{5}See the proof of Theorem V.4.1 in [Hel02] for details.
Main Theorem 1. Let $G/H$ denote an irreducible compactly causal symmetric space, and let $\Upsilon$, $T_{\Omega_1}$, and $N_1$ be as above. The map $\Upsilon$ is then a causal $G$-equivariant embedding of $G/H$ onto an open dense subspace of $\partial_s T_{\Omega_1} \simeq N_1$.

Indeed, the fact that $\text{Im} \ Upsilon$ is open and dense follows from the first statement in Theorem 3.7.2 and Lemma 3.7.1. Furthermore, that $\Upsilon$ is $G$-equivariant follows from the construction. It thus suffices to show the causality of $\Upsilon$. That follows, in turn, from the observation made on page 27.

3.8 Identification of $L^2$-Spaces

We now realize the regular representation of $G$ on $L^2(G/H)$ as a sub-representation of a natural representation of $G_1$ acting on $L^2(\partial_s T_{\Omega_1})$. More precisely, we have the following main result.

Main Theorem 2. 1. The $G$-invariant measure on $\Upsilon(G/H) \subset \partial_s T_{\Omega_1}$ is, up to normalization, given by

$$\int_{\partial_s T_{\Omega_1}} f(x) |\varpi_m(x)|^{-2/m} \, d\mu(x),$$

where $d\mu(x)$ is the Euclidean measure on $\partial_s T_{\Omega_1} \simeq N_1$.

2. A unitary representation of $G_1$ on $L^2(\partial_s T_{\Omega_1}, d\mu)$ is given by

$$(\lambda_0(g) f)(x) = |J_1(g^{-1}, x)|^{1/2} f(g^{-1}.x).$$

3. Identify $G/H$ with its image $\Upsilon(G/H)$, and let $\lambda$ denote the left-regular representation of $G$ on $L^2(G/H)$. Then

$$f \mapsto f |\varpi_m|^{1/m}$$

is a $G$-equivariant isometry of $(L^2(\partial_s T_{\Omega_1}), \lambda_0|_G)$ onto $(L^2(G/H), \lambda)$.

Proof. Temporarily let $\nu(x) = |\varpi_m(x)|^{-2/m} d\mu(x)$. For $g \in G$ such that $g.x$ exists, it follows from the second statement in Theorem 3.7.2 and that

$$\nu(g.x) = |\varpi_m(g.x)|^{-2/m} d\mu(g.x) = |a_1(g, x)^{m/2} \varpi_m(x)|^{-2/m} |a_1(g, x)| d\mu(x) = |\varpi_m(x)|^{-2/m} d\mu(x) = \nu(x),$$

so $\nu$ is $G$-invariant. It is easily seen that $\lambda_0$ defines a unitary representation, and the third statement now follows from a simple calculation.

Note that, according to the third statement, $f \mapsto f |\varpi_m|^{1/m}$ realizes $(L^2(G/H), \lambda)$ as a sub-representation of $(L^2(\partial_s T_{\Omega_1}), \lambda_0)$.

For specific choices of $G$, one can use $m = 1$ in the statements above. That is important because it allows us to remove the numerical values in the definition of $\lambda_0$ and thus obtain a holomorphic square-root. We have seen previously that, for $G_1 = SO_e(2, n + 1)$, this is equivalent to the requirement that $n$ be an odd number.
Chapter 4

The Finer Details in the Geometric Realization of $SO_e(2, n)/SO_e(1, n)$

4.1 Introduction

In this chapter we calculate the structure theory of $SO_e(2, n)/SO_e(1, n)$ in order to construct a model of the Silov boundary and a concrete realization of the $G_1$-action on the boundary in this model. More specifically, we mention recall the general definition here.

Definition 4.1.1. Let $g \in G_1$ and $\pi \in N_1$ be such that $g \pi \in N_1 P_1$. Then $g$ is said to act on $\pi$, and the action thus obtained is uniquely defined uniquely by the requirement

$$(g \cdot \pi) P_1 = g \pi P_1,$$

that is, $g \cdot \pi$ is the $N_1$-component of $g \pi$ according to the Bruhat decomposition $G_1 \supset N_1 P_1$.

Using the representation $\partial_s T_{\Omega_1} \cong G_1 / P_1 \supset N_1$, we thus obtain through Definition 4.1.1 a “canonical” action of $G_1$ on the boundary $\partial_s T_{\Omega_1}$. We have already seen that $\partial_s T_{\Omega_1} \cong \mathbb{R}^{n+1}$, so there is a “canonical” action of $G_1$ on $\mathbb{R}^{n+1}$. The action thus obtained is the one for which we will construct a particularly useful model in the present chapter.

4.2 Details on the Structure Theory

In addition to the general remarks on the structure theory in Section 3.2, we add some details on the subgroups used in the general construction described in . More specifically, we supplement the details summarized in Figure 3.1 with calculations of $M_1$, $A_1$, $N_1$, and $N_1$ together with the associated Lie algebras. In the next section, Section 4.3, we use these details to describe various group-actions explicitly.

As mentioned several times already, we identify $G$ and $H$ with subgroups of $SO_e(2, n + 1)$.
using the natural embedding

\[ G = SO(2, n) \hookrightarrow \begin{pmatrix} SO(2, n) \\ \vdots \\ 1 \end{pmatrix}, \quad H = SO(1, n) \hookrightarrow \begin{pmatrix} 1 \\ \vdots \\ SO(1, n) \end{pmatrix} \]

We will also have to consider a maximal parabolic subgroup \( P_1 = M_1 A_1 N_1 \) of \( G_1 \) (with its Langlands decomposition) where, explicitly,

\[
M_1 = \left\{ m_1^\ell(\varepsilon) = \begin{pmatrix} \varepsilon \\ \ell \\ \varepsilon \end{pmatrix} \mid \varepsilon \in \{\pm 1\}, \ell \in SO(1, n) \right\},
\]

\[
A_1 = \left\{ a_1^t = \begin{pmatrix} \cosh t & \sinh t \\ -\sinh t & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\},
\]

and

\[
N_1 = \left\{ n_1(x) = \begin{pmatrix} \frac{1}{\sqrt{2}} \beta_{1,n}(x, x) & x & \frac{1}{\sqrt{2}} \beta_{1,n}(x, x) \\ x & I_{n+1} & x \\ \frac{1}{\sqrt{2}} \beta_{1,n}(x, x) & x & 1 + \frac{1}{\sqrt{2}} \beta_{1,n}(x, x) \end{pmatrix} \mid x \in \mathbb{R}^{n+1} \right\}.
\]

Here \( \beta_{1,n} \) denotes the Lorentz form of signature \((1, n)\) on \( \mathbb{R}^{n+1} \), and \( \tilde{x} = (x_1, -x_2, \ldots, -x_{n+1}) \).

The last subgroup of \( G_1 \) we currently need to consider, is \( \overline{N}_1 := \theta(N_1) \), where \( \theta \) is the usual Cartan involution. Explicitly, we have the representation

\[
\overline{N}_1 = \left\{ \overline{n}_1(y) = \begin{pmatrix} \frac{1}{\sqrt{2}} \beta_{1,n}(y, y) & \tilde{y} & -\frac{1}{\sqrt{2}} \beta_{1,n}(y, y) \\ -\tilde{y} & I_{n+1} & -\tilde{y} \\ \frac{1}{\sqrt{2}} \beta_{1,n}(y, y) & -\tilde{y} & 1 + \frac{1}{\sqrt{2}} \beta_{1,n}(y, y) \end{pmatrix} \mid y \in \mathbb{R}^{n+1} \right\}.
\]

### 4.3 A priori Group-Actions

Using the detailed knowledge of the structure theory from Section 4.2, we calculate the various relevant group-actions and their orbits. In Section 4.5 we construct an action on the Silov boundary that is supposed, among other things, to reflect the a priori orbit picture determined by the structure theory.

**The action of \( M_1 A_1 \) on \( N_1 \):** It is a well-known fact that, in general, the Levi part \( L_1 = M_1 A_1 \) normalizes the nilradical \( N_1 \) and therefore acts on \( N_1 \) by \( \ell_1.n_1 = \ell_1 n_1 \ell_1^{-1}, \ell_1 \in L_1, n_1 \in N_1 \).

If \( N_1 \) is abelian (as is currently the case), then \( P_1 \) acts on \( N_1 \). Although we can also show this directly in the calculations that follow shortly, we will in fact use that \( M_1 A_1 \) normalizes \( N_1 \) to make the computations more manageable.
· **The action of** \((M_1)_e\). We will first study the action of the connected component, \((M_1)_e\), of \(M_1\). Given \(\ell \in SO_e(1, n)\) and \(x \in \mathbb{R}^{n+1}\), consider the elements \(m^1(\ell)\) and \(n_1(x)\) defined earlier. Then

\[
m^1(\ell).n_1(x) = \begin{pmatrix}
1 - \frac{1}{2} \beta_{1,n}(x,x) & x\ell^{-1} - \frac{1}{2} \beta_{1,n}(x,x) \\
-\ell \tilde{x}^t & \ell \tilde{x}^t
\end{pmatrix}
\begin{pmatrix}
\ell \tilde{x}^t \\
\ell \tilde{x}^t
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} \beta_{1,n}(x,x) \\
\frac{1}{2} \beta_{1,n}(x,x)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 - \frac{1}{2} \beta_{1,n}(x,x) & x\ell^{-1} - \frac{1}{2} \beta_{1,n}(x,x) \\
-\ell \tilde{x}^t & \ell \tilde{x}^t
\end{pmatrix}
\begin{pmatrix}
\ell \tilde{x}^t \\
\ell \tilde{x}^t
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} \beta_{1,n}(x,x) \\
\frac{1}{2} \beta_{1,n}(x,x)
\end{pmatrix}
\]

It therefore turns out that the action of \((M_1)_e\) on \(N_1\) is completely described by the usual action of \(SO(1, n)\) on \(\mathbb{R}^{n+1}\) (since \(\beta_{1,n}(\tilde{x}, \tilde{x}) = \beta_{1,n}(x, x)\) and \(\beta_{1,n}\) is \(SO(1, n)\)-invariant). In (\(\ast\)), we have used the following easy observation: A matrix \(\ell \in SO(1, n)\) satisfies the relation

\[
\ell^t \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix} = \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix} \ell^t \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix}
\]

so \(\ell^{-1} = \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix} \ell^t \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix}\) and

\[
x\ell^{-1} = x \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix} \ell^t \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix} = \tilde{x}^t \ell^t \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix}
\]

\[
= (\ell \tilde{x})^t \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix} = (\ell \tilde{x})^t.
\]

We also use the fact that \((\ell \tilde{x})^t = \ell x\), since

\[
\ell \tilde{x} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix} x = \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} x
\]

and

\[
(\ell \tilde{x})^t = \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} x \begin{pmatrix} 1 & -I_n \\ -I_n & 1 \end{pmatrix}^t
\]

\[
= \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} x^t \begin{pmatrix} A^t & C^t \\ -B^t & -D^t \end{pmatrix} = \tilde{x}^t \begin{pmatrix} A^t & C^t \\ -B^t & -D^t \end{pmatrix} = \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} \tilde{x}
\]

\[
= (\ell x)^t.
\]
The action of $M_1$ on $N_1$  Next we study the action of the elements $m_{-1}^1(\ell) \in M_1$. In this case
\[
m_{-1}^1(\ell).n_1(x) = \begin{pmatrix}
1 - \frac{1}{2} \beta_{1,n}(x,x) & -x\ell^{-1} & \frac{1}{2} \beta_{1,n}(x,x) \\
\ell \tilde{x}^t & I_{n+1} & -\ell \tilde{x}^t \\
-\frac{1}{2} \beta_{1,n}(x,x) & -x\ell^{-1} & 1 + \frac{1}{2} \beta_{1,n}(x,x)
\end{pmatrix} = n_1(-\ell x).
\]

The action of $A_1$ on $N_1$ For $t \in \mathbb{R}$ and $x \in \mathbb{R}^{n+1}$ we consider the elements $a_1^t$ and $n_1(x)$ defined elsewhere. Then
\[
a_1^t.n_1(x) = \begin{pmatrix}
1 - \frac{1}{2} e^{2t} \beta_{1,n}(x,x) & e^t x & \frac{1}{2} e^{2t} \beta_{1,n}(x,x) \\
-e^{t} \tilde{x}^t & I_{n+1} & e^{t} \tilde{x}^t \\
-\frac{1}{2} e^{2t} \beta_{1,n}(x,x) & e^t x & 1 + \frac{1}{2} e^{2t} \beta_{1,n}(x,x)
\end{pmatrix} = n_1(e^t x).
\]

We conclude that the action of $A_1$ on $N_1$ is completely described by the usual action of $\mathbb{R}^+$ on $\mathbb{R}^{n+1}$.

The action of $N_1$ Since $N_1$ is abelian, there is nothing to study, in terms of a conjugate action (it is trivial). We also write $n_1(x)n_1(y) = n_1(x + y)$.

The $M_1A_1$-orbits on $\overline{N}_1$ First notice that $N_1$ does not act trivially on $\overline{N}_1$ (in fact, $N_1$ does not act on $\overline{N}_1$ at all). In order to study the orbits of $M_1A_1$ on $\overline{N}_1$ we proceed in the same way as before. Since the calculations are the same, we we simply state the various results:
\[
m_{1}^1(\ell).\pi(y) = \begin{pmatrix}
1 - \frac{1}{2} \beta_{1,n}(y,y) & \tilde{\ell} y & -\frac{1}{2} \beta_{1,n}(y,y) \\
-\tilde{\ell} y & I_{n+1} & -\tilde{\ell} y \\
\frac{1}{2} \beta_{1,n}(y,y) & -\tilde{\ell} y & 1 + \frac{1}{2} \beta_{1,n}(y,y)
\end{pmatrix} = \pi_1(\ell y)
\]
\[ m_1^{-1}(\ell).\pi_1(y) = \begin{pmatrix}
1 - \frac{1}{2} \beta_{1,n}(y,y) & -\tilde{\gamma} \ell^{-1} & -\frac{1}{2} \beta_{1,n}(y,y) \\
\ell y & \ell y & \ell y \\
\frac{1}{2} \beta_{1,n}(y,y) & \tilde{\gamma} \ell^{-1} & 1 + \frac{1}{2} \beta_{1,n}(y,y)
\end{pmatrix} = \pi_1(-\ell y) \]

so the \( M_1 \)-orbits on \( \mathbb{N}_1 \) are described by the \( SO(1,n) \)-orbits on \( \mathbb{R}^{n+1} \) - in fact, in precisely the same way as was the case with the \( M_1 \)-orbits on \( \mathbb{N}_1 \).

Furthermore, \( a_1.\pi_1(y) = \pi_1(e^{-t}y) \) for \( t \in \mathbb{R} \) and \( y \in \mathbb{R}^{n+1} \), so the \( A_1 \)-orbits on \( \mathbb{N}_1 \) are also completely determined by the orbits on \( \mathbb{R}^{n+1} \). We summarize these facts in the following figure:

<table>
<thead>
<tr>
<th>Subgroup acting on ( \mathbb{M}_1 )</th>
<th>as ( \mathbb{A}_1 ) acting on ( \mathbb{N}_1 )</th>
<th>( \mathbb{M}_1 ) acting on ( \mathbb{N}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{M}_1 )</td>
<td>( \mathbb{N}_1 )</td>
<td>( m_1(\ell).n_1(x) = n_1(\ell x) )</td>
</tr>
<tr>
<td>( \mathbb{A}_1 )</td>
<td>( \mathbb{N}_1 )</td>
<td>( a_1.n_1(x) = n_1(e^t x) )</td>
</tr>
<tr>
<td>( \mathbb{N}_1 )</td>
<td>( \mathbb{N}_1 )</td>
<td>( m_1(\ell).\pi_1(y) = \pi_1(\ell y) )</td>
</tr>
<tr>
<td>( \mathbb{A}_1 )</td>
<td>( \mathbb{N}_1 )</td>
<td>( a_1.\pi_1(y) = \pi_1(e^{-t} y) )</td>
</tr>
<tr>
<td>( \mathbb{N}_1 )</td>
<td>( \mathbb{N}_1 )</td>
<td>( \mathbb{A}_1 ) does not act globally</td>
</tr>
</tbody>
</table>

and thus briefly turn our attention to the action of \( \mathbb{R}^+SO(1,n) \) on \( \mathbb{R}^{n+1} \).

**Lemma 4.3.1.** The group \( H' = \mathbb{R}^+SO(1,n) \) acts on \( \mathbb{R}^{n+1} \) by matrix multiplication, and the sets

\[
\begin{align*}
\mathcal{O}_1 &= \{ x \in \mathbb{R}^{n+1} : \beta_{1,n}(x,x) > 0, x_1 > 0 \}, \\
\mathcal{O}_2 &= \{ x \in \mathbb{R}^{n+1} : \beta_{1,n}(x,x) > 0, x_1 < 0 \}, \text{ and} \\
\mathcal{O}_3 &= \{ x \in \mathbb{R}^{n+1} : \beta_{1,n}(x,x) < 0 \}
\end{align*}
\]

are open orbits on \( \mathbb{R}^{n+1} \) under this action of \( H' \). In fact, if \( e_i \) is the vector \( (\delta_{ij})_{j=1}^{n+1} \) in \( \mathbb{R}^{n+1} \) then \( \mathcal{O}_1 = H' \cdot e_1, \mathcal{O}_2 = H' \cdot (-e_1) \) and \( H' \cdot e_{n+1} = \mathcal{O}_3 \).

Furthermore, \( \mathbb{R}^{n+1} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \) up to a set of measure zero.

**Proof.** The first two identities are verified in exactly the same way, we will show that \( H' \cdot e_1 = \mathcal{O}_1 \) and \( H' \cdot e_{n+1} = \mathcal{O}_3 \).
To this end, we see that

$$H' \cdot e_1 = \left\{ \lambda_{n+1} \begin{pmatrix} a & b_1 & \cdots & b_n \\ c_1 & d_{11} & \cdots & d_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & d_{n1} & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} : \lambda > 0, \begin{pmatrix} * & * \end{pmatrix} \in SO(1, n) \right\}$$

$$= \left\{ \lambda \begin{pmatrix} a \\ c_1 \\ \vdots \\ c_n \end{pmatrix} : \lambda > 0, \ldots \right\}$$

It follows that

$$0 < 1 = A^t A - C^t C = a^2 - (c_1, \ldots, c_n)(c_1, \ldots, c_n)^t = a^2 - \sum_{k=1}^n c_k^2,$$

and therefore

$$\beta \left( \lambda \begin{pmatrix} a \\ c_1 \\ \vdots \\ c_n \end{pmatrix}, \lambda \begin{pmatrix} a \\ c_1 \\ \vdots \\ c_n \end{pmatrix} \right) = \lambda^2 \left( a^2 - \sum_{k=1}^n c_k^2 \right) > 0,$$

from which we infer that $H' \cdot e_1 \subset \mathcal{O}_1$.

On the other hand, if $x = (x_1, x_2, \ldots, x_{n+1})$ belongs to $\mathcal{O}_1$, then $x_1 > 0$, in particular, and we may therefore write $x = x_1(1, \frac{x_2}{x_1}, \ldots, \frac{x_{n+1}}{x_1})$. Now simply select a matrix in $SO_e(1, n)$ with the first column vector being given by $(1, \frac{x_2}{x_1}, \ldots, \frac{x_{n+1}}{x_1})$ to write $x$ as an element in $H' \cdot e_1$. Thus $\mathcal{O}_1 \subset H' \cdot e_1$ and the first part of the claim follows.

Next we prove that $H' \cdot e_{n+1} = \mathcal{O}_3$. To this end, first notice that if $x \in \mathcal{O}_3$ and $\lambda > 0$, then $\beta(\lambda x, \lambda x) = \lambda^2 \beta(x, x) < 0$, so $\lambda x \notin \mathcal{O}_3$. The group $\mathbb{R}^+$ acts by dilations, so the action of the $\mathbb{R}^+$-part in $H'$ clearly leaves $\mathcal{O}_3$ invariant, and it therefore suffices to prove that $SO_e(1, n)$ acts transitively on $\mathcal{O}_3$.

Multiplying with the number $\lambda = (-\beta(x, x))^{-\frac{1}{2}}$, if necessary, we may assume that $\beta(x, x) = -1$. Let $x' = (x_2, \ldots, x_{n+1})^t$. Then $\|x'\|^2 - x_1^2 = 1$, so there is a real number $t$ with the property that $\|x'\| = \cosh t$ and $x_1 = \sinh t$.

In particular, the elements $(\cosh t)e_{n+1}$ and $x'$ both belong to the set

$$\{ y \in \mathbb{R}^n : \|y\| = \cosh t \} \subset \mathbb{R}^{n+1}.$$ 

The group $SO_e(n)$ acts by rotations on this set, hence transitively, and we therefore choose a matrix $k \in SO_e(n) \subset SO_e(1, n)$ so that $k(\cosh t)e_{n+1} = x'$. But then

$$k \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} e_{n+1} = k(\sinh t, 0, \ldots, 0, \cosh t)^t = x,$$

so $SO_e(1, n)$ acts transitively on $\mathcal{O}_3$. Hence $H' \cdot e_{n+1} = \mathcal{O}_3$. \hfill \blacksquare

Recall from Chapter 3 that $\mathcal{O}_1$ (‘the forward light cone’) and $\mathcal{O}_2$ (‘the backward light cone’) are both homogeneous self-dual convex cones in $\mathbb{R}^{n+1}$.
4.4 Explicit Model for the Silov Boundary

Define the subset $S \subset \mathbb{R}^{n+3}$ by

$$S = \{v \in \mathbb{R}^{n+1} \mid \beta_{2,n+1}(v, v) = 0, v_1 + v_{n+3} = 1\}.$$

Furthermore, define a map

$$\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+3}$$

by

$$\varphi(x) = \left(\begin{array}{c}
\frac{1}{2}(1 - \beta_{1,n}(x, x)) \\
x \\
\frac{1}{2}(1 + \beta_{1,n}(x, x))
\end{array}\right).$$

Then $(\varphi(x))_1 + (\varphi(x))_{n+3} = 1$ by construction, and

$$\beta_{2,n+1}(\varphi(x), \varphi(x)) = \left[\frac{1}{2}(1 - \beta_{1,n}(x, x))\right]^2 + \beta_{1,n}(x, x) - \left[\frac{1}{2}(1 + \beta_{1,n}(x, x))\right]^2 = 0.$$

from which it follows that $\text{Im } \varphi \subset S$.

**Lemma 4.4.1.** $\text{Im } \varphi = S$.

**Proof.** Let $v$ belong to $S$. We have to construct a vector $x \in \mathbb{R}^{n+1}$ so that

$$v = \left(\begin{array}{c}
\frac{1}{2}(1 - \beta_{1,n}(x, x)) \\
x \\
\frac{1}{2}(1 + \beta_{1,n}(x, x))
\end{array}\right)$$

so the only possible choice for $x$ is $x = v'$. The claim is therefore that $v_1 = \frac{1}{2}(1 - \beta_{1,n}(x, x)) = \frac{1}{2}(1 - \beta_{1,n}(v', v'))$ and $v_{n+3} = \frac{1}{2}(1 + \beta_{1,n}(x, x)) = \frac{1}{2}(1 + \beta_{1,n}(v', v'))$. Both identities are easily verified as follows: By the choice of $v$ and $x$, $0 = \beta_{2,n+1}(v, v) = v_1^2 + \beta_{1,n}(x, x) - v_{n+3}^2$ and therefore $\beta_{1,n}(x, x) = v_{n+3}^2 - v_1^2 = (v_1 + v_{n+3})(v_{n+3} - v_1) = v_{n+3} - v_1$. Hence $\frac{1}{2}(1 - \beta_{1,n}(x, x)) = \frac{1}{2}(v_1 + v_{n+3} - (v_{n+3} - v_1)) = v_1$ and $\frac{1}{2}(1 + \beta_{1,n}(x, x)) = \frac{1}{2}(v_1 + v_{n+3} + v_{n+3} - v_1) = v_{n+3}$. ■

The proof reveals that $\varphi$ is also injective, so $\varphi$ is a bijection of $\mathbb{R}^{n+1}$ onto $S$.

**Lemma 4.4.2.** The inverse of $\varphi$, whenever defined, is $\mathbb{R}^{n+3} \ni v \mapsto \frac{1}{v_1 + v_{n+3}} v'$, where $v' = (v_2, \ldots, v_{n+2})$.

**Proof.** Temporarily denote the map $v \mapsto \frac{v}{v_1 + v_{n+3}}$ by the letter $\psi$. On one hand,

$$\psi(\varphi(x)) = \psi\left(\begin{array}{c}
\frac{1}{2}(1 - \beta_{1,n}(x, x)) \\
x \\
\frac{1}{2}(1 + \beta_{1,n}(x, x))
\end{array}\right) = \frac{1}{2}(1 - \beta_{1,n}(x, x)) + \frac{1}{2}(1 + \beta_{1,n}(x, x)) = x.$$

39
and on the other hand

\[ \varphi(\psi(v)) = \varphi \left( \frac{1}{v_1 + v_{n+3}} \begin{pmatrix} v_2 \\ \vdots \\ v_n \\ v_{n+2} \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2} (1 - \beta_{1,n} \left( \frac{1}{v_1 + v_{n+3}} v', \frac{1}{v_1 + v_{n+3}} v' \right) \\ \frac{1}{2} (1 + \beta_{1,n} \left( \frac{v_1 + v_{n+3}}{v_1 + v_{n+3}} v', \frac{1}{v_1 + v_{n+3}} v' \right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} (1 - \beta_{1,n}(v', v')) \\ v' \end{pmatrix} \]

since \( v \) belongs to \( S \). What remains is to show that \( v_1 = \frac{1}{2} (1 - \beta_{1,n}(v', v')) \) and \( v_{n+3} = \frac{1}{2} (1 + \beta_{1,n}(v', v')) \). But since \( v \) belongs to \( S \), it follows that

\[
0 = \beta_{2,n+1}(v, v) = v_1^2 + \beta_{1,n}(v', v') - v_{n+3}^2
\]

\[
= (1 - v_{n+3})^2 + \beta_{1,n}(v', v') - v_{n+3}^2
\]

\[
= 1 - 2v_{n+3} + \beta_{1,n}(v, v').
\]

Similarly, one shows that \( v_1 = \frac{1}{2} (1 - \beta_{1,n}(v', v')) \).

From now on \( S \) is equipped with the subspace topology from \( \mathbb{R}^{n+3} \).

**Lemma 4.4.3.** Let \( u = \frac{1}{2}(e_1 + e_{n+3}) \). Then \( \{\overline{\pi}_1(y)u \mid y \in \mathbb{R}^{n+1} \} = S \).

**Proof.** First we show the inclusion “\( \subseteq \)”. We have that

\[
\overline{\pi}_1(y)u = \left( 1 - \beta_{1,n}(y, y), 2y^t, 1 + \beta_{1,n}(y, y) \right)^t = (v_1, \ldots, v_{n+3})^t
\]

with \( v_1 + v_{n+3} = \frac{1}{2} (1 - \beta_{1,n}(y, y)) + \frac{1}{2} (1 + \beta_{1,n}(y, y)) = 1 \). Furthermore

\[
\beta_{2,n+1}(\overline{\pi}_1(y)u, \overline{\pi}_1(y)u) = \left( \frac{1}{2} (1 - \beta_{1,n}(y, y)) \right)^2 + \beta_{1,n}(y, y)^2 - \left[ \frac{1}{2} (1 + \beta_{1,n}(y, y)) \right]^2
\]

\[
= 0.
\]

and it follows that \( \overline{\pi}_1(y)u \subset S \).

To show the other inclusion, “\( \supseteq \)”, take a vector \( v \in \mathbb{R}^{n+3} \) with \( \beta_{2,n+1}(v, v) = 0 \) and \( v_1 + v_{n+3} = 1 \), and write \( v \) as \( v = (v_1, 2y, v_{n+3})^t \) with \( y \in \mathbb{R}^{n+1} \). Then

\[
0 = \beta_{2,n+1}(v, v)
\]

\[
= v_1^2 + 4\beta_{1,n}(y, y) - v_{n+3}^2
\]

\[
= (v_1 - v_{n+3}) + 4\beta_{1,n}(y, y),
\]

from the assumption on \( v \), and therefore \( \beta_{1,n}(y, y) = \frac{1}{4} (v_{n+3} - v_1) \). So in order to prove the lemma we need to construct \( y \in \mathbb{R}^{n+1} \) with such property. But that is easy; let \( y = (y_1, \ldots, y_{n+1}) \) be given by \( y_1 = \frac{1}{2} \sqrt{|v_1 - v_{n+3}|} \) and \( y_j = 0 \) for \( j = 2, \ldots, n+1 \).
Observe that \( \overline{\pi}_1(y) \mapsto \overline{\pi}_1(y)u \) is simply the map \( \overline{\pi}_1 \ni X \mapsto (\exp X)u \in S \). We therefore have the following result:

**Corollary 4.4.4.** The map 
\[
\overline{N}_1 \ni \overline{\pi}_1(y) \mapsto \overline{\pi}_1(y)u \in S
\]
is an analytic diffeomorphism.

The map \( \varphi \) is thus a smooth map onto \( S \).

### 4.5 A Group-Action on the Silov Boundary

Define a smooth map \( \Phi : G'_1 \to \mathbb{R}^{n+3} \) by
\[
\Phi(g) = \frac{gu}{(gu)_1 + (gu)_{n+3}},
\]
where \( G'_1 \) is the subset of elements \( g \in G_1 \) such that \((gu)_1 + (gu)_{n+3} \neq 0\). Then clearly \( (\Phi(g))_1 + (\Phi(g))_{n+3} = 1 \), and
\[
\beta_{2,n+1}(\Phi(g),\Phi(g)) = \left( \frac{1}{(\Phi(g))_1 + (\Phi(g))_{n+3}} \right)^2 \beta_{2,n+1}(gu,gu)
\]
\[
= \left( \frac{1}{(\Phi(g))_1 + (\Phi(g))_{n+3}} \right)^2 \beta_{2,n+1}(u,u)
\]
where we have used that \( \beta_{2,n+1} \) is \( \text{SO}(2,n+1) \)-invariant. Therefore \( \text{Im} \Phi \subset S \).

Once again, let \( P_1 = M_1A_1N_1 \) be the maximal parabolic subgroup introduced in the beginning, and write \( \overline{N}_1 = \Theta(N_1) \) (as in previous sections).

**Lemma 4.5.1.** \( \Phi(g) = \Phi(e) \) if and only if \( g \in M_1A_1N_1 \).

**Proof.** Simply use the explicit description of the subgroups \( M_1, A_1, N_1 \) to show the “if”-part in the statement:

- \( g = m_1^1(\ell) \): Here \( gu = u \), so \( \Phi(g) = \Phi(e) \).
- \( g = m_1^{-1}(\ell) \): Then \( gu = -u \), so \( \Phi(g) = \Phi(-e) \) where
  \[
  \Phi(-e) = \frac{-u}{(-u)_1 + (-u)_{n+1}} = u = \Phi(e).
  \]
- \( g = a_1^1 \): Here \( gu = e^t u \) with
  \[
  \Phi(gu) = \frac{1}{\frac{1}{2}e^t + \frac{1}{2}e^t}e^t u = u = \Phi(u).
  \]
- \( g = n_1(x) \): Then \( gu = u \).
To complete the proof we use the Iwasawa decomposition $G_1 = K_1 M_1 A_1 N_1$ and will show that if $\Phi(k_1) = \Phi(e)$ for some $k_1 \in K_1 = \text{SO}(2) \times \text{SO}(n+1)$, then $k_1$ contained in $P_1$. So consider an element $k_1 \in K_1$ of the form

$$k_1 = \begin{pmatrix} A \\ B \end{pmatrix}, A \in \text{SO}(2), B \in \text{SO}(n+1).$$

In this case

$$k_1 u = \frac{1}{2} \begin{pmatrix} a_{11} \\ a_{21} \\ b_{1,n+1} \\ \vdots \\ b_{n+1,n+1} \end{pmatrix},$$

which is equal to $u$ if and only if $a_{11} = 1$, $a_{21} = 0$, and $b_{1,n+1} = \cdots = b_{n,n+1} = 0$, $b_{n+1,n+1} = 1$. Since $A$ and $B$ are orthogonal, it is easy to see that $A = I_2$ and $B = I_{n+1}$.

**Observation.** $N_1 M_1 A_1 N_1 \subset G'_1$.  

It is well-known that $N_1 M_1 A_1 N_1$ is an open, dense subset of $G_1$, so $G'_1$ is also an open dense subset of $G_1$. Therefore $\Phi$ induces a map, also denoted by $\Phi$, from $G'_1/P_1$ into $\mathbb{S}$, allowing us to think of $\mathbb{S}$ as a model of the Silov boundary $\tilde{S}_1 \approx G_1/P_1$ (up to measure zero, very important). We will now use this connection to introduce the action of $G_1$ on the model $\mathbb{S}$ of $\tilde{S}_1$.

Let $g$ belong to $G_1$ and select a vector $v$ in $\mathbb{S}$. The action of $g$ on $v$, written $g \cdot v$, is then

$$g \cdot v \overset{\text{def}}{=} \frac{gv}{(gv)_1 + (gv)_{n+3}} \text{ if } (gv)_1 + (gv)_{n+3} \neq 0.$$

**Lemma 4.5.2.** The operation $\cdot$ defines an action on $\mathbb{S}$ for all $g$ in an open dense subset of $G_1$.

**Proof.** It is easily seen that $g \cdot v$ belongs to $\mathbb{S}$ if $(gv)_1 + (gv)_{n+3} \neq 0$; indeed,

$$(g \cdot v)_1 + (g \cdot v)_{n+3} = \left( \frac{gv}{(gv)_1 + (gv)_{n+3}} \right)_1 + \left( \frac{gv}{(gv)_1 + (gv)_{n+3}} \right)_{n+3}$$

$$= \frac{1}{(gv)_1 + (gv)_{n+3}} ((gv)_1 + (gv)_{n+3}) = 1,$$

and

$$\beta_{2,n+1}(g \cdot v, g \cdot v) = \frac{1}{((gv)_1 + (gv)_{n+3})^2} \beta_{2,n+1}(gv, gv)$$

$$= \frac{1}{((gv)_1 + (gv)_{n+3})^2} \beta_{2,n+1}(v, v) = 0.$$
Let us proceed to show that \( e \cdot v = v \) and \( g_1 \cdot (g_2 \cdot v) = (g_1 g_2) \cdot v \) for all \( g_1, g_2 \in G_1 \) and \( v \in S \). First notice that \( e \cdot v = \frac{v}{(v_1 + (v)_{n+3})} = v \) since \( v \in S \). Furthermore,

\[
g_1 \cdot (g_2 \cdot v) = g_1 \cdot \left( \frac{g_2 v}{(g_2 v)_1 + (g_2 v)_{n+3}} \right) = \frac{g_1 \left( \frac{g_2 v}{(g_2 v)_1 + (g_2 v)_{n+3}} \right)}{(g_1) \cdot \left( \frac{g_2 v}{(g_2 v)_1 + (g_2 v)_{n+3}} \right)} = \frac{g_1 g_2 v}{(g_1 g_2 v)_1 + (g_1 g_2 v)_{n+3}} = (g_1 g_2) \cdot v,
\]

so, is a \( G_1 \)-action on \( S \).

**Remark.** It does not follow that \( (G_1/P_1, \cdot) \) and \( (S, \cdot) \) are isomorphic as \( G_1 \)-spaces through the map \( \Phi \); the \( G_1 \)-action is not always defined (but for the \( L^2 \)-analysis carried out later on it will be sufficient).

**Definition 4.5.3.** Let \( \cdot \) denote the operation of an element \( g \in G_1 \) on \( \mathbb{R}^{n+1} \) defined by

\[
G_1 \times \mathbb{R}^{n+1} \ni (g, x) \mapsto g \cdot x \overset{\text{def}}{=} \varphi^{-1}(g \cdot \varphi(x)).
\]

**Lemma 4.5.4.** The operation \( \cdot \) is an almost everywhere defined \( G_1 \)-action on \( \mathbb{R}^{n+1} \) (in the same sense as \( \cdot \) above).

**Proof.** It is clear from the construction that \( g \cdot x \) is in \( \mathbb{R}^{n+1} \). Furthermore, \( e \cdot x = \varphi^{-1}(e \cdot \varphi(x)) = \varphi^{-1}(\varphi(x)) = x \) for all \( x \in \mathbb{R}^{n+1} \), and, for \( g_1, g_2 \in G_1 \) and \( x \in \mathbb{R}^{n+1} \), we have that

\[
g_1 \cdot (g_2 \cdot x) = g_1 \cdot \varphi^{-1}(g_2 \cdot \varphi(x)) = \varphi^{-1}(g_1 \cdot \varphi(\varphi^{-1}(g_2 \cdot \varphi(x)))) = \varphi^{-1}((g_1 g_2) \cdot \varphi(x)) = (g_1 g_2) \cdot x
\]

proving the lemma.

We thus have a commutative diagram describing the two \( G_1 \)-actions: that is, \( g \cdot \varphi(x) = \varphi(g \cdot x) \) for all \( g \in G_1 \) and \( x \in \mathbb{R}^{n+1} \).

**Theorem 4.5.5.** Let \( g \) belong to \( G_1 \) and let \( \varpi_1(y) \) an element in \( \varpi \). Then \( g \) acts on \( \varpi_1 \) in the sense of Definition 4.1.1 if and only if \( g \) acts on \( y \) in the sense of Definition 4.5.3 and Lemma 4.5.4, in which case

\[
g \cdot \varpi_1(y) = \varpi_1(g \cdot \varphi(y)) = \varpi_1(\varphi(g \cdot y)).
\]

43
The action \( \cdot \) from Definition 4.5.3 is therefore a concrete model for the abstract action described in Definition 4.1.1.

We will now continue to study this action in more detail, and first seek to relate the action on \( \mathbb{R}^{n+1} \) to the orbits.

**Theorem 4.5.6 (Consistency Theorem).** The Levi factor \( L_1 = M_1 A_1 \) of \( P_1 \) acts on \( \mathbb{R}^{n+1} \), and the open orbits are precisely the two sets \( \mathcal{O}_1 \cup \mathcal{O}_2 \) and \( \mathcal{O}_3 \).

**Proof.** Consider an element \( x \in \mathcal{O}_1 \cup \mathcal{O}_2 \) written as \( x = \lambda me_1 \) with \( \lambda \neq 0 \) and \( m \in SO(1, n) \).

Then

\[
\varphi(x) = \begin{pmatrix}
\frac{1}{2}(1 - \beta_{1,n}(\lambda me_1, \lambda me_1)) \\
\frac{1}{2}(1 + \beta_{1,n}(\lambda me_1, \lambda me_1)) \\
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}(1 - \lambda^2 \beta_{1,n}(me_1, me_1)) \\
\frac{1}{2}(1 + \lambda^2 \beta_{1,n}(me_1, me_1)) \\
\end{pmatrix},
\]

whereby

\[
m_1^\ell \cdot \varphi(x) = \begin{pmatrix}
\frac{1}{2}(1 - \lambda^2) \\
\frac{1}{2}(1 + \lambda^2) \\
\end{pmatrix} \quad \text{and} \quad a_t \cdot \varphi(x) = \begin{pmatrix}
\frac{1}{2}(1 - \lambda^2 e^{-2t}) \\
\frac{1}{2}(1 + \lambda^2 e^{-2t}) \\
\end{pmatrix}.
\]

It follows that

\[
m_1^\ell(x) = \varphi^{-1}(m_1^\ell \cdot \varphi(x)) = \varphi^{-1}\left( \begin{pmatrix}
\frac{1}{2}(1 - \lambda^2) \\
\frac{1}{2}(1 + \lambda^2) \\
\end{pmatrix} \right) = \varepsilon \lambda \ell me_1
\]

and

\[
a_t x = \varphi^{-1}(a_t \cdot \varphi(x)) = \varphi^{-1}\left( \begin{pmatrix}
\frac{1}{2}(1 - \lambda^2 e^{-2t}) \\
\frac{1}{2}(1 + \lambda^2 e^{-2t}) \\
\end{pmatrix} \right) = \lambda e^{-t} me_1.
\]
It now follows easily that \( m_{1}^{\ell}(\ell).x \) and \( a_{t}.x \) both remain in \( O_{1} \cup O_{2} \).

It follows in precisely the same way that \( O_{3} \) is invariant under the action of \( M_{1} \) and \( A_{1} \); the only change is that we now write the vector \( x \) as \( x = \lambda me_{n+1} \), with \( \lambda \neq 0 \) and \( m \in SO(1, n) \), so that \( \varphi(x) = (\frac{1}{2} (1 + \lambda^{2}), \lambda me_{n+1}, \frac{1}{2} (1 - \lambda^{2}))^{t} \).

\[ \text{Theorem 4.5.7.} \]

Let the notation be as above. Then

\[
m_{1}^{\ell}(\ell).x = \varepsilon \ell x \quad \text{where} \quad \varepsilon = \pm 1, \ell \in SO(1, n),
\]

\[
a_{t}.x = e^{-t}x \quad \text{where} \quad t \in \mathbb{R}
\]

\[
\overline{m}_{1}(y).x = x - y \quad \text{where} \quad y \in \mathbb{R}^{n+1}.
\]

\[ \text{Proof.} \]

Only the last identity still lacks a proof. Here,

\[
\overline{m}_{1}(y) \varphi(x) = \begin{pmatrix}
1 - \frac{1}{2} \beta_{1,n}(y,y) & \tilde{y} & -\frac{1}{2} \beta_{1,n}(y,y) \\
-\tilde{y} & I_{n+1} & -y \\
\frac{1}{2} \beta_{1,n}(y,y) & -\tilde{y} & 1 + \frac{1}{2} \beta_{1,n}(y,y)
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} (1 - \beta_{1,n}(x,x)) \\
x \\
\frac{1}{2} (1 + \beta_{1,n}(x,x))
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{2} (1 - \beta_{1,n}(y,y))(1 - \beta_{1,n}(x,x)) + \langle \tilde{y}, x \rangle - \frac{1}{2} \beta_{1,n}(y,y)(1 + \beta_{1,n}(x,x)) \\
-\frac{1}{2} (1 - \beta_{1,n}(x,x))y^{t} + x^{t} - \frac{1}{2} (1 + \beta_{1,n}(x,x))y^{t} \\
\frac{1}{2} \beta_{1,n}(y,y)(1 - \beta_{1,n}(x,x)) - \langle \tilde{y}, x \rangle + \frac{1}{2} (1 + \beta_{1,n}(y,y))(1 + \beta_{1,n}(x,x))
\end{pmatrix}
\]

where it turns out that \( (\overline{m}_{1}(y) \varphi(x))_{1} + (\overline{m}_{1}(y) \varphi(x))_{n+3} = 1 \). Therefore, again by easy calculations, it follows that

\[
\overline{m}_{1}(y).x = \varphi^{-1}(\overline{m}_{1}(y) \cdot \varphi(x)) \quad \text{here} \quad \varphi^{-1}(\varphi(x - y)) = x - y.
\]

\[ \text{Proof.} \]

Being a subgroup of \( G_{1} \), it makes sense to study the action of the group \( G \) on \( S \) and \( \mathbb{R}^{n+1} \) too. As we will discover shortly, \( G \) does not act in a nice way (unfortunately).

First note that not all of \( G \) acts on \( \mathbb{R}^{n+1} \). For example, with \( g = \text{diag}(-1, 1, \ldots, 1, -1) \in SO_{e}(2, n) \leftrightarrow SO_{e}(n+1) \) and \( x \in \mathbb{R}^{n+1} \) we see that \( g \varphi(x) = (-\frac{1}{2} (1 - \beta_{1,n}(x,x)), x_{1}, \ldots, x_{n}, -x_{n+1}, \frac{1}{2} (1 + \beta_{1,n}(x,x)))^{t} \), and then \( g.x = \beta_{1,n}(x,x)^{-1}(x_{1}, \ldots, x_{n}, -x_{n+1})^{t} \). If \( \beta_{1,n}(x,x) = 0 \), the action of this particular group element \( g \) on \( x \) is therefore not defined.

First consider an element \( m_{1}^{\ell}(\ell) \in M \), written as \( m_{1}^{\ell}(\ell) = \begin{pmatrix} \ell & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \) with \( \ell \in SO(1, n - 1) \).

Then

\[
m_{1}^{\ell}(\ell) \varphi(x) = \begin{pmatrix}
\frac{1}{2} (1 - \beta_{1,n}(x,x)) \\
\ell x \\
\varepsilon x_{n+1} \\
\frac{1}{2} (1 + \beta_{1,n}(x,x))
\end{pmatrix}
\]
where $x' := (x_1, \ldots, x_n)$, and therefore

$$m^\xi(\ell) \cdot \varphi(x) = \frac{1}{\frac{1}{2}(1 - \beta_{1,n}(x,x)) + \frac{1}{2}(1 + \beta_{1,n}(x,x))} \left( \frac{\xi}{2} (1 - \beta_{1,n}(x,x)) \right)$$

$$= \left( \frac{1}{2} (1 - \beta_{1,n}(x,x)) \right) \frac{\ell x'}{\varepsilon x_{n+1}}$$

$$= \left( \frac{1}{2} (1 - \beta_{1,n}(x,x)) \right) \frac{\ell x'}{x_{n+1}},$$

from which it then follows that

$$m^\xi(\ell).x = \begin{pmatrix} \ell x' \\ x_{n+1} \end{pmatrix}$$

$$\varepsilon = 1 \equiv \begin{pmatrix} \ell x' \\ x_{n+1} \end{pmatrix}.$$

We describe the actions of $A$ and $N$ in a similar manner, and the result is that

$$a_\ell \cdot \varphi(x) = \frac{1}{\frac{1}{2}(1 - \beta_{1,n}(x,x)) + \frac{1}{2}(1 + \beta_{1,n}(x,x))} \left( \frac{1}{2} (1 - \beta_{1,n}(x,x)) \right)$$

$$= \left( \frac{1}{2} (1 - \beta_{1,n}(x,x)) \right) \frac{x'}{x_{n+1} \cdot \sinh t}$$

and

$$a_\ell \cdot x = \left( \frac{1}{2} (1 - \beta_{1,n}(x,x)) \right) \frac{x'}{x_{n+1} \cdot \cosh t}.$$

Finally (and this is the most complicated part)

$$n(y) \cdot \varphi(x) = \frac{1}{\frac{1}{2}(1 - \beta_{1,n}(x,x)) + \frac{1}{2}(1 + \beta_{1,n}(x,x))} \left( \frac{1}{2} (1 - \beta_{1,n}(x,x)) \right)$$

$$= \left( \frac{1}{2} (1 - \beta_{1,n}(x,x)) \right) \frac{\tilde{y} + x'}{x_{n+1} \cdot \tilde{y}}$$

$$= \left( \frac{1}{2} (1 - \beta_{1,n}(x,x)) \right) \frac{\tilde{y} + x'}{x_{n+1} \cdot \tilde{y}}$$

and

$$n(y) \cdot x = \left( \frac{1}{2} (1 - \beta_{1,n}(x,x)) \right) \frac{\tilde{y} + x'}{x_{n+1} \cdot \tilde{y}}$$

Although it might be difficult to see this, there is no vector in $\mathbb{R}^{n+1}$ that is $MN$-invariant without being $A$-invariant at the same time.

### 4.6 Action of the Non-Trivial Weyl Group Element

It is well-known that the Weyl group for $G_1 = SO_e(2, n + 1)$ has two elements, namely the identity and the non-trivial element represented, say, by $w = \begin{pmatrix} I_{n+1} \\ -1 \end{pmatrix}$. Using the results
from the previous section, the action of $w$ on $S$ as well as on $\mathbb{R}^{n+1}$ is easily determined: Let $v = \varphi(x)$ belong to $S$. Then

$$wv = \left( \begin{array}{cc} 1 & I_{n+1} \\ -1 & \end{array} \right) \left( \begin{array}{c} \frac{1}{2} (1 - \beta_{1,n}(x,x)) \\ \frac{1}{2} (1 + \beta_{1,n}(x,x)) \end{array} \right) = \left( \begin{array}{c} \frac{1}{2} (1 + \beta_{1,n}(x,x)) \\ \frac{1}{2} (1 - \beta_{1,n}(x,x)) \end{array} \right) x$$

with $(w \varphi(x))_1 + (w \varphi(x))_{n+3} = \frac{1}{2} (1 + \beta_{1,n}(x,x)) - \frac{1}{2} (1 - \beta_{1,n}(x,x)) = \beta_{1,n}(x,x)$, and therefore

$$w \cdot \varphi(x) = \frac{1}{\beta_{1,n}(x,x)} \left( \begin{array}{c} \frac{1}{2} (1 + \beta_{1,n}(x,x)) \\ \frac{1}{2} (1 - \beta_{1,n}(x,x)) \end{array} \right) x.$$ 

Similarly,

$$w^{-1} \cdot \varphi(x) = -\frac{1}{\beta_{1,n}(x,x)} \left( \begin{array}{c} -\frac{1}{2} (1 + \beta_{1,n}(x,x)) \\ \frac{1}{2} (1 - \beta_{1,n}(x,x)) \end{array} \right) x.$$ 

Here we are allowing one of the “forbidden” group elements to act, but then it an only act on a subset of $\mathbb{R}^{n+1}$. Such behavior is not that exotic though; it is already seen for $SL(2, \mathbb{R})$. The Weyl group, as a subgroup of $G_1$, also acts on $\mathbb{R}^{n+1}$:

$$w \cdot x = \varphi^{-1}(w \cdot \varphi(x))$$

$$= \frac{1}{\beta_{1,n}(x,x)} \left( \frac{1}{2} (1 + \beta_{1,n}(x,x)) x \right)$$

and

$$w^{-1} \cdot x = \varphi^{-1}(w^{-1} \cdot \varphi(x))$$

$$= -\frac{1}{\beta_{1,n}(x,x)} \left( \frac{1}{2} (1 - \beta_{1,n}(x,x)) x \right)$$

We have thus proved the following

**Proposition 4.6.1.** Let $w = \left( \begin{array}{cc} 1 & I_{n+1} \\ -1 & \end{array} \right)$ be a representative for the non-trivial element in the Weyl group for $SO_v(2, n + 1)$. The action of $w$ and $w^{-1}$ on $S$ and on $\mathbb{R}^{n+1}$, respectively, is given by

$$w \cdot \varphi(x) = \frac{1}{\beta_{1,n}(x,x)} \left( \begin{array}{c} \frac{1}{2} (1 + \beta_{1,n}(x,x)) x \end{array} \right), \quad w^{-1} \cdot x = -\frac{1}{\beta_{1,n}(x,x)} x$$
and
\[ w^{-1} \cdot \varphi(x) = -\frac{1}{\beta_{1,n}(x,x)} \begin{pmatrix} -\frac{1}{2}(1 + \beta_{1,n}(x,x)) \\ \sigma'(x) \\ \frac{1}{2}(1 - \beta_{1,n}(x,x)) \end{pmatrix}, \quad w^{-1} \cdot x = -\frac{x}{\beta_{1,n}(x,x)}. \]

In the coordinates \((v_1, \ldots, v_{n+3})\) on \(S\), we would get the formulas
\[ w \cdot v = \frac{1}{v_{n+3} - v_1} \begin{pmatrix} v_{n+3} \\ v_2 \\ \vdots \\ v_{n+2} \\ -v_1 \end{pmatrix} \quad \text{and} \quad w^{-1} \cdot v = \frac{1}{v_1 - v_{n+3}} \begin{pmatrix} -v_{n+3} \\ v_2 \\ \vdots \\ v_{n+2} \\ v_1 \end{pmatrix}. \]

**Lemma 4.6.2.** The Jacobian of the map \(\psi : x \mapsto w \cdot x, x \in \mathbb{R}^{n+1}\), is given by
\[ \det D\psi = \frac{(-1)^{n+1}}{\beta_{1,n}(x,x)^{n+1}} \]

and the Jacobian of the map \(\phi : x \mapsto w^{-1} \cdot x\) is given by
\[ \det D\phi = \frac{(-1)^n}{\beta_{1,n}(x,x)^{n+1}}. \]

**Proof.** Induction after \(n\). ■

Recall from Theorem 4.5.7 that \(M_1\) and \(A_1\) act linearly by multiplication on \(\mathbb{R}^{n+1}\) and that \(N_1\) acts by translation on \(\mathbb{R}^{n+1}\). It is therefore easy to compute the jacobian of the action of each of these; the result is the following

**Lemma 4.6.3.** Let \(m \in M_1\) have the form \(m = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\) with \(L \in SO_e(1, n)\), let \(a_t \in A_1\) have the form \(a_t = \exp(tY)\) (as usual), and write elements in \(N_1\) as \(n(y)\). Then
\[ \det \text{jac}(x \mapsto m \cdot x) = \det L = 1, \]
\[ \det \text{jac}(x \mapsto a_t \cdot x) = e^{-t} \]
\[ \det \text{jac}(x \mapsto n(y) \cdot x) = 1. \]

Alternatively, we may recall the general integration formula for \(N\), that
\[ \int_{N_1} f(g \cdot n) a_{N_2}(g \bar{n})^{-2 \rho_1} d\bar{n} = \int_{N_1} f(\bar{n}) d\bar{n}. \]

The Jacobian determinant is then \((g, n) \mapsto a_{N}(g \bar{n})^{-2 \rho}\). Since \(g n(y) u = e^{t(g \cdot y)} n(g \cdot y) u\) where \(g n(y) = n(g \cdot y) m = a'_{t(g \cdot y)} n\), and \((a')^{2 \rho_1} = e^{\dim n \cdot t} = e^{(n+1)t}\), we therefore get an 'abstract' formula for the Jacobian. Since we need to take a square root, we once again the importance of the choice \(n\) being an odd number.

48
4.7 The Realization of the Hyperboloids in the Silov Boundary

We have already discussed the abstract definition of the mapping $\Upsilon$ previously, but we can now make the description more explicit.

Write $\omega = (\omega_1, \ldots, \omega_{n+1}) \in \mathbb{R}^{n+1}$ and $H \ni h = \begin{pmatrix} 1 & L \\ L & 1 \end{pmatrix}$, $L \in SO_e(1, n)$. Then $h.\omega = L\omega$. We want to make sure that $H \subset G\omega$, so in particular we must impose the requirement that $h.\omega = \omega$. Since the only $SO_e(1, n)$-fixed vector in $\mathbb{R}^{n+1}$ is the zero vector, we conclude that $\omega = 0$.

Next we need to make sure that $G\omega = H$, so we want to calculate, for arbitrary $n \geq 1$, the stabilizer of $SO_e(2, n)$ in $0$; the claim is that we get the group $SO_e(1, n)$. Writing an element $g$ in the form

$$
g = \begin{pmatrix} a_{11} & a_{12} & b_{11} & \cdots & b_{1n} & 0 \\
a_{21} & a_{22} & b_{21} & \cdots & b_{2n} & 0 \\
c_{11} & c_{12} & d_{11} & \cdots & d_{1n} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_{1n} & c_{2n} & d_{n1} & \cdots & d_{nn} & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
$$

it is easily seen that $g.0 = 0$ precisely when $a_{21} = c_{11} = \cdots = c_{n1} = 0$. Since column vectors in $g$ are mutually orthogonal it then follows that $a_{12} = 0$, and in fact $b_{11} = \cdots = b_{2n} = 0$ too.

Writing $g$ as $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with

$$
A = \begin{pmatrix} a_{11} & 0 \\
0 & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \cdots & 0 & 0 \\
b_{21} & \cdots & b_{2n} & 0 \end{pmatrix},
$$

$$
C = \begin{pmatrix} 0 & c_{12} \\
\vdots & \vdots \\
0 & c_{n2} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} d_{11} & \cdots & d_{1n} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
d_{n1} & \cdots & d_{nn} & 0 \\
0 & \cdots & 0 & 1 \end{pmatrix}
$$

it follows from the requirement that $AA^t - BB^t = I_2$ that

$$
\begin{pmatrix} a_{11}^2 & 0 \\
0 & a_{22}^2 - \sum_{j=1}^{n} b_{2j}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix},
$$

from the requirement that $AC^t - BD^t = 0$ it follows that

$$
\begin{pmatrix} 0 & \cdots & 0 \\
a_{22}c_{12} - \sum_{j=1}^{n} b_{2j}d_{1j} & \cdots & a_{22}c_{n2} - \sum_{j=1}^{n} b_{2j}d_{nj} & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \end{pmatrix},
$$

and finally, from the requirement that $CC^t - DD^t = -I$ it follows that

$$
\begin{pmatrix} c_{12}^2 - \sum_j d_{1j}^2 & \cdots & c_{12}c_{n2} - \sum_j d_{1j}d_{nj} & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
c_{12}c_{n2} - \sum_j d_{nj}d_{1j} & \cdots & c_{n2}^2 - \sum_j d_{nj}^2 & 0 \\
0 & \cdots & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -I_n & 0 \\
0 & -1 \end{pmatrix}.
$$

49
Now define matrices $A'$, $B'$, $C'$, and $D'$ by

$$A' = \begin{pmatrix} a_{22} \end{pmatrix}, \quad B' = \begin{pmatrix} b_{21} & \cdots & b_{2n} \end{pmatrix},$$

$$C' = \begin{pmatrix} c_{12} \\ \vdots \\ c_{1n} \end{pmatrix} \quad \text{and} \quad D' = \begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{pmatrix}.$$  

Looking very closely at (4.1), (4.2), and (4.3), it is now seen that

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in O(1,n),$$  

as was to be expected.

We now use the additional constraint that $\det g = 1$ to conclude that if $g_{11} = 1$, then

$$\det \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = 1,$$

implying that $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in SO(1,n)$. (Consult [Hel02], p. 202-203, for a discussion of the connected components of $O(2,n)$).

The explicit formula for the map $Y$ in this special model for the Silov boundary is therefore simply

$$Y : G/H : gH \mapsto g \cdot 0 \in \mathbb{R}^{n+1}.$$  

### 4.8 The Explicit Formula for the Function $\varpi$  

Although we have an abstract description of $\varpi$ from the previous chapter, it will be important for calculations later on to have an explicit formula related to the coordinates on the Silov boundary in question. In terms of the terminology from the last sections of Chapter 3, we thus mention the following crucial (yet basic) observation:

**Lemma 4.8.1.** The vector $v_m = e_1 + e_{n+3}$ is a lowest weight vector for $\pi_2$, and $\xi_m = e_{n+3}$ is $G_C$-spherical.

We thus get a nice formula for $\varpi$ as follows: Identifying $\mathbb{R}^{n+1}$ and $N_1$, we have

$$\varpi(x) = \langle \pi_m(\overline{\pi_1}(x))v_m | \xi_m \rangle$$

$$= \left\langle \begin{pmatrix} 1 - \frac{1}{2} \beta_{1,n}(x,x) & -\frac{1}{2} \beta_{1,n}(x,x) \\ -x^t & I_{n+1} \\ \frac{1}{2} \beta_{1,n}(x,x) & -x^t \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \vdots \\ 0 & 1 \end{pmatrix} \right\rangle$$

$$= 1 + \beta_{1,n}(x,x).$$

We can also give a more hands-on calculation with $\varpi_m$ in the following way: If

$$g = a_t = \begin{pmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & I_{n} & 0 & 0 \\ \sinh t & 0 & \cosh t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then
then it is easily computed that \( a_t \cdot 0 = \frac{1}{1 + \cosh t}(0, \ldots, 0, \sinh t)^t \). Therefore, in this case,

\[
\beta_{1,n}(g \cdot 0, g \cdot 0) = -\frac{\sinh^2 t}{(1 + \cosh t)^2}
\]

and

\[
\varpi(a_t \cdot 0) = 1 - \frac{\sinh^2 t}{(1 + \cosh t)^2} = \frac{2(1 + \cosh t)}{(1 + \cosh t)^2},
\]

which is nonzero for \( t \) any real number.

Next consider an element \( g \) from \( M \). In this case,

\[
g = m(\ell) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

with \( \ell \in \text{SO}_e(1, n-1) \),

and \( m \cdot 0 = 0 \) so that \( \varpi(n \cdot 0) = 1 \neq 0 \).

Now consider an element \( g \) in \( N \), written as

\[
g = n(x') = \begin{pmatrix} 1 - \frac{1}{2}\beta_{1,n-1}(x',x') & x' & \frac{1}{2}\beta_{1,n-1}(x',x') & 0 \\ -\tilde{x}' & I_n & \tilde{x}' & 0 \\ -\frac{1}{2}\beta_{1,n-1}(x',x') & x' & 1 + \frac{1}{2}\beta_{1,n-1}(x',x') & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

with \( x' \in \mathbb{R}^n \). In this case

\[
n(x') \varphi(0) = n(x') u = \begin{pmatrix} \frac{1}{2}(1 - \frac{1}{2}\beta_{1,n-1}(x',x')) \\ -\frac{1}{2}\tilde{x}' \\ -\frac{1}{4}\beta_{1,n-1}(x',x') \\ \frac{1}{2} \end{pmatrix}
\]

so that

\[
m \cdot 0 = \frac{1}{1 - \frac{1}{4}\beta_{1,n-1}(x',x')} \begin{pmatrix} -\frac{1}{2}\tilde{x}' \\ -\frac{1}{4}\beta_{1,n-1}(x',x') \end{pmatrix}.
\]

(Note that the action of \( m \) is not defined if \( \beta_{1,n-1}(x',x') = 4 \).)

Now

\[
\beta_{1,n}(m \cdot 0, m \cdot 0) = \frac{\frac{1}{4}x_1^2 - \frac{1}{4}x_2^2 - \cdots - \frac{1}{4}x_n^2 - \frac{1}{16}\beta_{1,n-1}(x',x')^2}{(1 - \frac{1}{4}\beta_{1,n-1}(x',x'))^2} = \frac{\beta_{1,n-1}(x',x')}{4 - \beta_{1,n-1}(x',x')}.
\]

This expression will attain the value zero whenever \( \beta_{1,n-1}(x',x') = 0 \), but will never attain the value -1; if it did, it would follow that \( \beta_{1,n-1}(x',x') = \beta_{1,n-1}(x',x') - 4 \), which is clearly a contradiction.
Summarizing, we have seen that if \( g \in P = MAN \subset SO_e(2, n + 1) \), then \( \varpi(g, 0) \neq 0 \).
Finally, let \( g \in N \) be of the form
\[
g = \pi(y') = \begin{pmatrix}
1 - \frac{1}{2} \beta_{1,n-1}(y', y') & -\frac{1}{2} \beta_{1,n-1}(y', y') & 0 \\
-\frac{1}{2} \beta_{1,n-1}(y', y') & I_n & -y' \\
0 & 0 & 1
\end{pmatrix}
\]
with \( y' \in \mathbb{R}^n \). In the same way as before, we see that
\[
\pi(y').0 = \frac{1}{1 - \frac{1}{2} \beta_{1,n-1}(y', y')} \begin{pmatrix}
-\frac{1}{2} y' \\
\frac{1}{2} \beta_{1,n-1}(y', y') \\
0
\end{pmatrix}
\]
with
\[
\beta_{1,n}(\pi(y').0, \pi(y').0) = \frac{\beta_{1,n-1}(y', y')}{4 - \beta_{1,n-1}(y', y')}.
\]
which is the same as \( \beta_{1,n}(n(y').0, n(y').0) \) that we dealt with just before.
Now we are able to say that if \( g \in NMAN \) is such that \( g.0 \) is defined, then \( \varpi(g, 0) \neq 0 \). Hence
\[
G/H \cong G.0/G^0 = \{ x \in \mathbb{R}^{n+1} \mid \varpi(x) \neq 0 \}.
\]
For a vector \( x \in \mathbb{R}^{n+1} \) we let \( x' = (x_1, \ldots, x_n) \). Consider the set
\[
\mathcal{U} = \{ x \in \mathbb{R}^{n+1} \mid \beta_{1,n-1}(x', x') = 4 \}.
\]
Then
\[
\mathcal{U} \cap \mathcal{O}_1 = \{ x \in \mathbb{R}^{n+1} \mid \beta_{1,n}(x, x) > 0, x_1 > 0, \beta_{1,n-1}(x', x') = 4 \}
\]
\[
= \{ x \in \mathbb{R}^{n+1} \mid \beta_{1,n-1}(x', x') = 4, x_{n+1} \in (-2, 2) \}
\]
\[
= \{ x' \in \mathbb{R}^n \mid \beta_{1,n-1}(x', x') = 4 \} \times (-2, 2)
\]
which has measure zero as a subset of \( \mathcal{O}_1 \). Similarly, \( \mathcal{U} \cap \mathcal{O}_2 \) is seen to have measure zero as a subset of \( \mathcal{O}_2 \). Finally,
\[
\mathcal{U} \cap \mathcal{O}_3 = \{ x \in \mathbb{R}^{n+1} \mid \beta_{1,n}(x, x) < 0, \beta_{1,n-1}(x', x') = 4 \}
\]
\[
= \{ x \in \mathbb{R}^{n+1} \mid \beta_{1,n-1}(x', x') = 4, x_{n+1} > 2 \text{ or } x_{n+1} < -2 \},
\]
also a set of measure zero in \( \mathbb{R}^{n+1} \).

### 4.9 A Remark on \( S \)
Recall the set \( S \) defined as
\[
S = \{ v \in \mathbb{R}^{n+3} \mid \beta_{2,n+1}(v, v) = 0, v_1 + v_{n+3} = 1 \}.
\]
Also consider the set $S_1$ defined as

$$S_1 = \{ v \in \mathbb{R}^{n+3} | \beta_{2,n+1} = 0, v \neq 0 \}.$$ 

By studying the $A_1$-action on $S_1$, it is easily seen that

$$S_1 \simeq G_1 / M_1 N_1$$

and that $S$ is a “projective model” of $S_1$. By induction in stages, it follows that

$$L^2(G_1 / M_1 N_1) \simeq \int_{\hat{A}_1} \pi_y dy$$

that is, $L^2(S_1)$ is the most continuous part of $L^2(G_1 / H_1)$ with $G_1 / H_1 = SO_e(2,n+1) / SO_2(1,n+1)$. The latter statement is not so important, but what it does show is that $L^2(S)$ (which is isomorphic to $L^2(G/H)$ as a $G$-representation) is embedded in the principal series representations for the symmetric space $SO_e(2,n+1) / SO_e(1,n+1)$. In fact, $L^2(S)$ turns out to “be” a single principal series representation, namely the one with trivial parameter. So in some sense we are computing the branching law $G_1 \Downarrow G$ for a principal series representation of $G_1 / H_1$, and by adopting ideas from [ØZ95] one can estimate the $K$-types in this realization.
Chapter 5

The Holomorphic Discrete Series and Embedding of Generalized Hardy Spaces

5.1 Introduction

In this chapter we introduce the notion of a generalized Hardy space associated to a compactly causal symmetric space $G/H$, and we construct the holomorphic discrete series of representations associated to $G/H$. The standard references include [ÓØ88a], [ÓØ91], and [HÓØ91], but the reader might also consult [Joh00] for a more detailed and essentially self-contained exposition.

5.2 Generalized Hardy Spaces

Let $C$ be a fixed regular, $G$-invariant convex cone in $g$, let $\kappa : G_C \rightarrow G_C/H_C$ denote the canonical mapping and set $\Xi(C) = \kappa(G\exp iC \subseteq G_C)$ denotes the corresponding complex Ol’shanskiı semigroup. Then $\Gamma(C) = G\exp(-C)$ and $\Gamma(C)^{-1} = \{\exp(-ix)g^{-1} \mid g \in G, X \in C\}$.

Now choose an element in $\Gamma(-C)$ of the form $\gamma = g_0\exp(-iX_0)$ with $g_0$ in $G$ and $X_0$ in $C$. Since $g_0\exp(-iX_0) = \exp(-iX_0)g_0$ it follows that $\Gamma(-C)$ is a subset of $\Gamma(C)^{-1}$. The other inclusion follows in the same way, so $\Gamma(-C) = \Gamma(C)^{-1}$. But then $\Gamma(-C^\circ)\Xi(C)$ is a subset of $\Xi(C^\circ)$ and $\Gamma(C)$ thus acts by left-translation on functions defined on $\Xi(C^\circ)$: Given a function $f : \Xi(C^\circ) \rightarrow \mathbb{C}$ and an element $\gamma$ from the semigroup $\Gamma(C^\circ)$ we define a map $\gamma.f : X \rightarrow \mathbb{C}$ by $(\gamma.f)(x) = f(\gamma^{-1}x)$. We will denote this action by $T$, i.e., $(T(\gamma)f)(z) = f(\gamma^{-1}z)$, where $\gamma$ is an element of $\Gamma(C)$ and where $z$ belongs to $\Xi(C^\circ)$.

Definition 5.2.1. Assume that $C$ is a regular $G$-invariant convex cone in $g_0$. We denote by $H(C)$ the space of complex-valued holomorphic functions $f$ defined on $\Xi(C^\circ)$ satisfying the inequality

$$\|f\|_H := \sup_{\gamma \in \Gamma(C^\circ)} \|\gamma.f\|_{L^2(G/H)} < \infty.$$
Theorem 5.2.2.  
1. The space $H(C)$ introduced above is a Hilbert space with respect to the (inner product induced by the) norm $\|\cdot\|_H$.

2. There is an isometry $I : H(C) \to L^2(G/H)$ such that $I f = \lim_{n \to \infty} y_n.f$ for any sequence $(y_n)_{n \in \mathbb{N}}$ in $\Gamma(C^\circ)$ converging to 1. Here the limit is taken in $L^2(G/H)$.

3. $I$ is an intertwining operator for the $G$-actions, i.e., $IT(g) = \lambda(g)I$ for every $g$ in $G$ (where $\lambda$ denotes the left-regular representation).

4. $T$ is a holomorphic representation $T$ of $\Gamma(C)$ on $H(C)$.

Proposition 5.2.3. The Hardy space $H(C)$ is a reproducing kernel Hilbert space.

Corollary 5.2.4. With notation as above, every evaluation map $ev_x$ on $H(C)$ is continuous.

5.3 The Holomorphic Discrete Series

It is well known that the holomorphic discrete series of $SU(1, 1)$ can be realized in Hardy spaces on the unit disc (see for example [Kna86] or [Sug90]). There is a related construction, due to Rossi and Vergne ([VR73] see also Nomura’s paper [Nom89]), that realizes the holomorphic discrete series of a Hermitian Lie group on Hardy-type spaces on the unbounded realization of $G/K$. In all the cases, be it in the bounded or the unbounded realization, the Fourier transform on the relevant Silov boundary is important, and several Paley-Wiener type results exist. As already pointed out, the case where $G/K$ is of tube type is particularly nice, since the Silov boundary of the tube domain is an abelian group. Working with the Fourier transform on the boundary then becomes Euclidean harmonic analysis.

In the present chapter we first recall the construction of the holomorphic discrete series associated to an affine symmetric space $G/H$ (due to Matsumoto ([Mat81]) and, independently, Ólafsson and Ørsted ([ÓØ88a] and [ÓØ91]). We will follow the construction given in the latter two references. We then proceed to generalize the results of Rossi, Vergne, and Nomura to give an explicit realization of the holomorphic discrete series representations in Hardy-type spaces on the tube domain $T_\Omega$. While implicitly alluded to in the introduction of [Óla91], we still include the precise statements and proofs.

Let $\delta$ be a unitary representation of $K$ in the finite dimensional Hilbert space $V$. We may then extend $\delta$ to a holomorphic representation of $K_C$ in $V$, still denoted $\delta$. Let $\tilde{\delta}$ be the contragredient representation of $K_C$ in $V^*$ and assume that it contains a nonzero $K_C \cap H_C$-invariant vector $v^\circ$, i.e., $\tilde{\delta}(k)(v^\circ) = v^\circ$ for every $k$ in $K_C$ (this is equivalent to the requirement that $\delta$ contains a $K_C \cap H_C$-invariant nonzero vector, because we may identify $V$ with $V^*$). We define $\Phi_\delta = \Phi : G/H \to V^*$ by the formula $\Phi(x) := \tilde{\delta}(k_H(x^{-1})^{-1})v^\circ$, where $k_H$ is the projection onto the $H_C$-factor in the decomposition $G \subseteq H_CK_CP^-$. Then $\Phi$ is well-defined because $v^\circ$ was assumed to be invariant under $H_C \cap K_C$ and furthermore obviously holomorphic because $\delta$ and $h_H$ are both holomorphic. For every $x$ in $G/H$ and $k$ in $K$ we have the relation $\Phi(kx) = \tilde{\delta}(k)\Phi(x)$ because

$$\Phi(kx) = \tilde{\delta}(k_H((kx)^{-1})^{-1})v^\circ = \tilde{\delta}(k_H(x^{-1}k^{-1})^{-1})v^\circ = \delta(k)\delta(k_H(x^{-1})^{-1})v^\circ = \tilde{\delta}(k)\Phi(k).$$
For $v$ in $\mathcal{V}$ we let $\Phi_v := \langle v, \Phi(\cdot) \rangle$, i.e., $\Phi_v(x) = \langle v, \tilde{\delta}(k_H(x^{-1})^{-1})v^\circ \rangle = \langle \delta(k_H(x^{-1}))v, v^\circ \rangle$. Then $l_k \Phi_v = \Phi_{\delta(k)v}$, i.e., $\Phi_v$ is of type $\delta$.

Let $n^\pm := \sum_{\alpha \in \Delta^+} g_{\pm \alpha}$ and $n^\pm := n^\pm \cap \mathfrak{t}$. Then $n^\pm = n_c^\pm \oplus p^\pm$ and by using the Poincaré-Birkhoff-Witt Theorem we get

\begin{align*}
(5.1) \quad U(\mathfrak{g}) &= U(\mathfrak{t}) \oplus (n^- U(\mathfrak{g}) + U(\mathfrak{g}) n^+) \\
(5.2) \quad &= U(\mathfrak{t}) \oplus (p^- U(\mathfrak{g}) + U(\mathfrak{g}) p^+) , \\
(5.3) \quad U(\mathfrak{t}) &= U(\mathfrak{t}) \oplus (n_\mathfrak{c}^- U(\mathfrak{t}) + U(\mathfrak{t}) n_\mathfrak{c}^+) .
\end{align*}

Let $q^\delta$, $q_t$ and $q^t$ be the corresponding projection onto the first factor. If we need to emphasize the dependence on the choice of positive root systems we will use the more cumbersome notation $q^\delta(\Delta^+(\mathfrak{g}, v))$, for example. Then $q^\delta = q^t \circ q_t$. Let $\delta$ in $\mathfrak{t}^*$. For $\lambda$ in $\mathfrak{t}^*$ we define $l_\lambda : S(\mathfrak{t}) \rightarrow S(\mathfrak{t})$ by $l_\lambda(h) = h - \lambda(h)$. For $\mathfrak{l} = \mathfrak{g}$ or $\mathfrak{l} = \mathfrak{t}$ we define $\mu^i$ by $\mu^i := l_{\rho_i} \circ q^i$. Then $\mu^i$ is an isomorphism from $Z(\mathfrak{l})$ onto $S(\mathfrak{l})^{W(\mathfrak{l}, \mathfrak{t})}$, $q_t(Z(\mathfrak{g}))$ is a subset of $Z(\mathfrak{t})$ (seen by direct calculation), and $\mu^\delta = l_{\rho_\mathfrak{c}} \circ q^t \circ q_t$. Because we will only need the second assertion we will not prove the other statements.

**Lemma 5.3.1.** Using the above notation, $q_t(Z(\mathfrak{g}))$ is a subset of $Z(\mathfrak{t})$, the center of $U(\mathfrak{t})$.

For $\lambda$ in $\mathfrak{t}$ define $\chi_\lambda : Z(\mathfrak{l}) \rightarrow \mathbb{C}$ by $\chi_\lambda^i(z) := \mu^i(z)(\lambda)$. Then every character of $Z(\mathfrak{l})$ is of the form $\chi_\lambda$, and two characters $\chi_\lambda$ and $\chi_\mu$ are equal if and only if there is an element $w$ in the Weyl-group $W(\mathfrak{l}, \mathfrak{t})$ such that $\lambda = w \cdot \mu$. This $\chi_\lambda$ is independent on the system of positive roots used in the construction.

**Theorem 5.3.2.** Let the notation be as above. Let $\mu$ be the highest weight of $(\delta, \mathcal{V})$ and assume that $(\mu + \rho, \alpha) < 0$ for every noncompact positive root $\alpha$. Then the following statements hold:

1. $\Phi$ belongs to $(\mathcal{A}(G/H, \delta, \chi_{\lambda+\rho_\mathfrak{g}}) \cap L^2(G/H)) \otimes \mathcal{V}^* \otimes \mathfrak{p}^+ \Phi = 0$.

2. Let $b = \exp(\sum s_i y_i)$ belong to $B$. Then

$$\Phi_b(b) = \bigg\langle \delta \bigg( \prod_{i=1}^s \exp \left( \frac{1}{2} \log(\cosh(2s_i))t_i \right) \bigg), v, v^\circ \bigg\rangle .$$

If $v$ is of weight $\nu$, then

$$\Phi_v(b) = \langle v, v^\circ \rangle \prod_{i=1}^s \cosh(2s_i)^{\frac{1}{2}v(t_i)} .$$

3. $\Phi_v$ belongs to $L^2(G/H)$.

4. Let $\mathcal{H}_\delta(\delta) := \{ \Phi_v \mid v \in \mathcal{V} \}$ and let

$$\mathcal{H}_\delta := \text{span}\{ U(\mathfrak{g}) \{ \Phi_v \mid v \in \mathcal{V} \} \}$$

be the completion of $U(\mathfrak{g}) \mathcal{H}_\delta(\delta)$ in $L^2(G/H)$. Then $\mathcal{H}_\delta$ is irreducible and unitary with infinitesimal character $\chi_{\lambda+\rho_\mathfrak{g}}$, the map $v \rightarrow \Phi_v$ from $\mathcal{V}$ into $\mathcal{H}_\delta(\delta)$ is a $K$-isomorphism and $\mathcal{H}_\delta(\delta)$ is the lowest $K$-type of $\mathcal{H}_\delta$.  

56
(5) \( p^+ \mathcal{H}_\delta(\delta) = 0 \).

(6) If \( \nu \) is a highest weight of a \( K \)-type occurring in \( \mathcal{H}_\delta \), then there exists positive integers \( n_\alpha \) and noncompact positive roots \( \alpha \) such that \( \nu = \mu - \sum n_\alpha \alpha \).

**Definition 5.3.3.** The representations \( \varepsilon_\delta \) of \( G \) on \( \mathcal{H}_\delta \) are called the holomorphic discrete series of \( G/H \) with lowest \( K \)-type \( \delta \). Explicitly, they are defined by \( \varepsilon_\delta(g) := \lambda(g) \bigg|_{\mathcal{H}(\delta)} \).

### 5.4 An Embedding of the Domain \( \Xi \)

We have already given an important description of \( \Upsilon(G/H) \) in the Silov boundary, and in this section we extend the map \( \Upsilon \) to provide an embedding of the complex domain \( \Xi \), too. First we collect a few results from [BÓ01], and then we switch to the unbounded realization.

**Lemma 5.4.1.** Denote by \( \text{pr}_z \) respectively \( \text{pr}_s \) the projection onto \( z \) respectively \( g_s \) corresponding to the decomposition \( g = z \oplus g_s \). Then the following holds:

1. \( \text{pr}_z(C) = z \);  
2. \( C_s := \text{pr}_s(C) \) is a regular \( H_s \)-invariant cone in \( q_s \). In particular \( C_{\min,s} \subset \text{pr}_s(C) \subset C_{\max,s} \).

**Proof.** Lemma 3.1. ■

The cone \( C_k \) is minimal in \( q_{1\eta} \) and is generated by \( \text{Ad}(G_{1\eta}^\eta)Z^0 \). A minimal extension of \( C_k \) to a \( G_1 \)-invariant cone in \( g_1 \) is \( W_k \), the minimal cone in \( g_1 \) generated by \( \text{Ad}(G_1)Z^0 \). As \( Z^0 \) belongs to \( q \cap t \) is follows that \( W_k \) is \( \sigma \)-invariant. For a subset \( D \) of \( W_k \) we define

\[ \Gamma_1(D) = G_1 \exp(iD) \subset G_{1C}. \]

It is known that the Ol’shanskiĭ \( \Gamma_1(W_k) \) is a closed semigroup in \( G_{1C} \) and that

\[ \Gamma_1(W_k) = \Gamma_1(W_k^\sigma) \approx G_1 \times iW_k^\sigma, \]

where the diffeomorphism is given by \( (g, iX) \mapsto g \exp(iX) \).

**Theorem 5.4.2.** For \( W := W_k \cap g \) the following holds:

1. \( W = W_k^\sigma = \text{pr}_g(W_k) \), where \( \text{pr}_g : g_1 \to g \) denotes the orthogonal projection.
2. \( W \) is a regular \( G \)-invariant cone in \( g \) such that \( W \cap q = \text{pr}_q(W) = C \), where \( \text{pr}_q : g \to q \) denotes the orthogonal projection, i.e., \( W \) is a \( G \)-invariant extension of \( C \).
3. \( W^\sigma = \text{Ad}(G)(t \cap W^\circ) \).
4. Let \( \Gamma(W) := G \exp(iW) \). Then

\[ \Gamma(W) = \Gamma_1(W_k)^\sigma = \Gamma_1(W_k) \cap G_{1C}^\sigma. \]

Thus \( \Gamma(W) \) is a closed semigroup in \( G_{1C} := G_{1C}^\sigma \).
5. $\Gamma(W)^\circ = G \exp(iW^\circ) =: \Gamma(W^\circ)$, and

$$G \times iW^\circ \to \Gamma(W^\circ) \ , \ (g, iX) \mapsto g \exp(iX)$$

is a diffeomorphism.

**Proof.** Theorem 3.3.  ■

We now switch to the unbounded realizations, and mention the following important result

**Theorem 5.4.3.** Let the notation be as above. Then $\Gamma(W)$ is a sub-semigroup of $\{ \gamma \in G_{1C} \mid \gamma^{-1}D_u \subset D_u \}$, where $D_u$ denotes the unbounded realization of $G_{1}/K_1$.

**Proof.** First we notice that $\Gamma(1/W_k)$ is a sub-semigroup of $\{ \gamma \mid \gamma^{-1}D_u \subset D_u \}$ [formally, the proof of this follows along the lines of a similar statement about the semigroup acting on $D_p$, the bounded realization. See Theorem 1.2 in the paper [HÖØ91] by Hilgert, Ólafsson, and Ørsted]. On the other hand we have just mentioned that $\Gamma(W)$ is a closed semigroup in $G_{1C}$, from which it follows that $\Gamma(W)$ is a closed sub-semigroup in $\Gamma(1/W_k)$. Hence $\Gamma(W)$ is a sub-semigroup of $\{ \gamma \in G_{1C} \mid \gamma^{-1}D_u \subset D_u \}$.  ■

Let $\Xi(C) := \Gamma(-W)eH_C \subset G_{1C}/H_C$. Then $\Xi(C) \cong G \times_H (-iC)$ and $\Xi(C^\circ) = \Xi(C)^\circ$ where $\Xi(C^\circ) := \Gamma(-W^\circ)eH_C$. Thus

**Corollary 5.4.4.** The complex domain $\Xi$ is contained in the generalized upper half-plane $D_u$.

**Proof.** Follows from the Theorem just before.  ■

**Theorem 5.4.5.** Let the notation be as above. Then

$$Y(\Xi(C^\circ)) = \{ z \in T_{\Omega_1} \mid \varpi_m(z) \neq 0 \}.$$

**Theorem 5.4.6.** Identify the complex domain $\Xi(C^0)$ with its image $Y(\Xi(C^0))$. If $g_1 \neq \mathfrak{sp}(2n, \mathbb{R})$ and $g_1 \neq \mathfrak{so}(2, 2k + 1)$, and if $\pi_1$ is $G$-spherical, then

$$I : H_2(T_{\Omega_1}) \to H(C), \ f \mapsto f\varpi_1|_{\Xi(C^0)}$$

is a $G$-equivariant isometry.

**Proof.** The group $G$ acts in $H(C)$ by translations on the left, so for a function $f$ in $H_2(T_{\Omega_1})$ we have that

$$(\lambda(f)I f)(x) = f(g^{-1}.x)\varpi_1(g^{-1}.x)$$

$$= J_1(g^{-1}, x)^{1/2} f(g^{-1}.x)\varpi_1(x)$$

$$= I(\lambda(g)f)(x).$$

It follows that $I$ is an isometry.  ■

For the general case (that is, when $\pi_1$ might not exist) we let $\widehat{H(C)}$ denote the covering space of $H(C)$ with $\Gamma(W)$-action, where

$$\widehat{H(C)} = \{ f \in \widehat{\Xi(C^0)} \to \mathbb{C} \mid f \text{ is holomorphic and } \|f\|_{\widehat{H(C)}} < \infty \}.$$

Here $\|f\|_{\widehat{H(C)}} := \sup_{y \in \Gamma_0} \|y \cdot f\|_{L^2(G/\Gamma_0)}$, and the analogy to Theorem 5.4.6 we get the following
Theorem 5.4.7. If \( \pi_m \) is \( G \)-spherical then

\[
\tilde{I} : H_2(T_{\Omega_1}) \rightarrow \tilde{H}_{odd}(C), \quad f \mapsto m \sqrt{\varpi_m} \left| \tilde{\Xi}(C^0) \right|
\]

is a \( \tilde{G} \)-equivariant isometry.

We refer to [BÓ01], p.301ff for additional details.

5.5 Embedding of the Generalized Hardy Spaces

In the following section, let \( f \) belong to the generalized Hardy space \( H(C) \). Then

\[
\|f\|^2 = \sup_{\gamma \in \Gamma} \int_{G/H} |\gamma \cdot f(x)|^2 \, d\mu(x)
\]

\[
= \sup_{\gamma \in \Gamma} \int_{G/H} |f(\gamma^{-1}x)|^2 \, d\mu(x)
\]

\[
= \sup_{\gamma \in \Gamma} \int_{\partial_s T_{\Omega_1}} |f(\gamma^{-1}z)|^2 |w(z)|^{-2} \, d\mu_{\partial_s T_{\Omega_1}}(z)
\]

\[
= \sup_{\gamma \in \Omega_1} \int_{\partial_s T_{\Omega_1}} \left| \frac{f(x + iy)}{w(x)} \right|^2 \, d\mu_{\partial_s T_{\Omega_1}}(x).
\]

To estimate the last quantity, we will need a fairly strong uniform bound on the quantity \( \left| \frac{w(x)}{w(x + iy)} \right| \), with \( x \in \mathbb{R}^{n+1} \) and \( y \in \Omega_1 \).

We may assume, without loss of generality, that \( y = \lambda e_1 \) for some \( \lambda \in \mathbb{R} \), and therefore

\[
\frac{w(x)}{w(x + iy)} = \frac{1 + \beta_{1,n}(x, x)}{1 + \beta_{1,n}(x, x) - \lambda^2 + 2i\lambda x_1}.
\]

Now fix \( x \in \mathbb{R}^{n+1} \), and write \( k = 1 + \beta_{1,n}(x, x) \). Then (5.4) becomes

\[
\left| \frac{w(x)}{w(x + iy)} \right| = \left| \frac{(k - \lambda^2 - 2i\lambda x_1)k}{(k - \lambda^2)^2 + 4\lambda^2 x_1^2} \right|
\]

\[
= \left| \frac{k(k - \lambda^2)}{(k - \lambda^2)^2 + 4\lambda^2 x_1^2} \right| + 2 \left| \frac{\lambda k x_1}{(k - \lambda^2)^2 + 4\lambda^2 x_1^2} \right|
\]

which is clearly bounded in \( \lambda \). We have thus proved that, for each fixed \( x \in \mathbb{R}^{n+1} \), there exists a constant \( C_x \) such that

\[
\frac{1}{|w(x)|} \geq C_x \frac{1}{|w(x + iy)|}
\]

uniformly in \( y \in \Omega_1 \).

The next step is to improve the estimate to become uniform in the \( x \)-variable too. To be precise, we assert that

\[
\sup_{x + iy \in T_{\Omega_1}} \left| \frac{w(x)}{w(x + iy)} \right| < \infty.
\]
To this end we simply have to notice that the quantities
\[
\left| \frac{(1 + \beta_{1,n}(x,x))(1 + \beta_{1,n}(x,x) - \lambda^2)}{(1 + \beta_{1,n}(x,x) - \lambda^2)^2 + 4\lambda^2 - x_1^2} \right|
\]
and
\[
\left| \frac{(1 + \beta_{1,n}(x,x))\lambda x_1}{(1 + \beta_{1,n}(x,x) - \lambda^2)^2 + 4\lambda^2 x_1^2} \right|
\]
are both uniformly bounded in \( x \in \mathbb{R}^{n+1} \).

It now follows that
\[
\|f\|^2 = \sup_{\gamma \in \Omega_1} \int_{\partial_s T_0_1} \left| \frac{f(x + iy)}{\varpi(x)} \right|^2 d\mu_{\partial_s T_0_1}(x)
\]
\[
= \sup_{\gamma \in \Omega_1} \int_{\partial_s T_0_1} \left| \frac{f(x + iy)}{\varpi(x + iy)} \right|^2 \left| \frac{\varpi(x)}{\varpi(x + iy)} \right|^2 d\mu_{\partial_s T_0_1}(x)
\]
\[
\geq \text{const} \cdot \sup_{\gamma \in \Omega_1} \int_{\partial_s T_0_1} \left| \frac{f(x + iy)}{\varpi(x + iy)} \right|^2 d\mu_{\partial_s T_0_1}(x),
\]
proving that \( z \mapsto f(z)/\varpi(z) \) belongs to the Hardy space \( H_2(T \Omega_1) \) on the tube domain \( T \Omega_1 \). We therefore have a map from \( H(C) \) into \( H_2(T \Omega_1) \):
\[
I : H(C) \to H_2(T \Omega_1) \quad , \quad f \mapsto f/\varpi.
\]

Furthermore,
\[
\|f\|^2 = \lim_{y \to 0} \int_{\partial_s T_0_1} \left| \frac{f(x + iy)}{\varpi(x + iy)} \right|^2 d\mu_{\partial_s T_0_1}(x)
\]
\[
= \lim_{y \to 0} \int_{\partial_s T_0_1} \left| \frac{\varpi(x + iy)}{\varpi(x)} \right|^2 \left| \frac{f(x + iy)}{\varpi(x + iy)} \right|^2 d\mu_{\partial_s T_0_1}(x)
\]
since \( \lim_{y \to 0} \left| \frac{\varpi(x + iy)}{\varpi(x)} \right|^2 = 1 \), implying that the map \( I : H(C) \to H_2(T \Omega_1) \) is an isometric embedding.

We have thus proved the following main result:

**Main Theorem 3.** Assume \( G/H = SO_\varepsilon(2,n)/SO_\varepsilon(1,n) \) with \( n \) odd. Then the classical Hardy space \( H_2(T \Omega_1) \) and the generalized Hardy space \( H(C) \) are isometrically isomorphic.

It is proved in [HÓØ91] that the \( G \)-equivariant boundary value map
\[
\mathbb{I} : H(C) \to L^2(G/H)
\]
given by \( \mathbb{I}f = \lim \gamma_j \cdot f \) for a sequence \( \{\gamma_j\} \) in \( \Gamma^\circ \) converging to 1, relates the Hardy space \( H(C) \) with the representations \((\varepsilon_\delta, \mathcal{H}_\delta)\) in the holomorphic discrete series by
\[
\mathbb{I}(H(C)) = \bigoplus_\delta \mathcal{H}_\delta.
\]

It then follows from Main Theorem 3 that \((\varepsilon_\delta, \mathcal{H}_\delta)\) is being realized in the classical Hardy space \( H_2(T \Omega_1) \).
Chapter 6

The Plancherel Decomposition of $SO_e(2, n)/SO_e(1, n)$

6.1 Introduction

We have described that embedding of $SO_e(2, n)/SO_e(1, n)$ into $\mathbb{R}^{n+1}$ and also described the embedding of the regular representation in this model. In this chapter, we use the orbit structure on $\mathbb{R}^{n+1}$, as described in Section 4.3, to decompose the regular representation of $G$ accordingly.

The general philosophy is easily described: Let $S$ denote the Silov boundary of a tube domain $T_\Omega$, let $L$ denote the Levi-part of the parabolic subgroup $P$ (as described in Chapter 2), let $\mathcal{O}$ denote an open orbit on $S$. Finally, let $dx$ denote the Euclidean measure on $S$, and let $\langle \cdot, \cdot \rangle$ denote the $\text{Aut}(\Omega)$-invariant form on $S$ describing the cone $\Omega$. We then have a natural Fourier transform on $S$ defined by

$$\hat{f}(y) = \int_S f(x)e^{-i\langle x, y \rangle} \, dx,$$

and since $S \simeq N$ is abelian, one uses abelian harmonic analysis\(^1\) to show that

$$L^2_\mathcal{O} := \{ f \in L^2(S, dx) \mid \text{supp}\hat{f} \subset \overline{\mathcal{O}} \}$$

is an irreducible representation of the opposite parabolic subgroup $\overline{P}$. Since we have a special version of the Silov boundary in mind (namely $\mathbb{R}^{n+1}$), we will write out the details in this type of argument in Sections 6.2 and 6.3 below.

As indicated in Chapter 4, it is often useful to think of the Levi-part $L_1$ of the parabolic subgroup $P'$ as the group $\mathbb{R}^+SO_e(1, n)$. The construction of $L^2_\mathcal{O}$ is then suggested to us by a look at [?], and it also becomes clear why we include a Fourier transform in the definition. We explore this point of view in the next section, followed in Section 6.3 with the important connection to the main results from Chapter 4 and Chapter 5.

\(^1\)The essential ingredient in a proof is to notice that $N$ acts by translation on $S$, and therefore by multiplication by characters on the Fourier-side. Since the characters separate points, a standard application of the Stone-Weierstrass Theorem and Schur’s Lemma will show irreducibility.
6.2 The Decomposition Corresponding to the Orbits $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$

As already noted, the group $H' = \mathbb{R}^+ \text{SO}(1, n)$ acts on $\mathbb{R}^{n+1}$ by matrix multiplication. It is known (see for example [FÓ03]) that a left Haar measure on $G'$ is given by

$$ d\mu_{G'}(h, x) = \frac{1}{(2\pi)^{n+1}|\det h|} dh dx, $$

where $dh$ is a left Haar measure on $H'$ and $dx$ is the usual Lebesgue measure on $\mathbb{R}^{n+1}$.

Now let

$$ \hat{f}(y) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} f(x) e^{-i(x, y)} dx $$

be the Fourier transform of $f \in L^1(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1})$, and let

$$ \tilde{f}(y) = \int_{\mathbb{R}^{n+1}} f(x) e^{-i\beta(x, y)} dx. $$

Furthermore, let

$$ [\rho(h, x)f](y) = |\det h|^{-\frac{1}{2}} f(h^{-1}(y - x)) $$

be the quasi-regular representation of $G' = \mathbb{R}^+ \text{SO}(1, n) \ltimes \mathbb{R}^{n+1}$ on $L^2(\mathbb{R}^{n+1})$, and put

$$ \hat{\rho}(h, x)\hat{f} := \rho(h, x)g \text{ and } \tilde{\rho}(h, x)\tilde{f} := \rho(h, x)f. $$

Then we have the following result.

**Lemma 6.2.1.** Let $f \in L^1(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1})$. Then

$$ (6.1) \quad \hat{\rho}(h, x)\hat{f}(y) = |\det h|^{\frac{1}{2}} e^{-i(x, y)} \hat{f}(h^t y) $$

and

$$ (6.2) \quad \tilde{\rho}(h, x)\tilde{f}(y) = |\det h|^{\frac{1}{2}} e^{-i\beta(x, y)} \tilde{f}(h^{-1} y). $$

**Proof.** The two identities are proved in exactly the same manner, so let us just prove the latter identity (the first one is proved in [FÓ03]). First notice that the form $\beta$ is separately linear in each variable. It follows that

$$ \tilde{\rho}(h, x)\tilde{f}(y) = \int_{\mathbb{R}^{n+1}} \rho(h, x)f(u)e^{-i\beta(u, y)} du $$

$$ = \int_{\mathbb{R}^{n+1}} |\det h|^{-\frac{1}{2}} f(h^{-1}(u - x)) e^{-i\beta(u, y)} du $$

$$ = e^{-i\beta(x, y)} \int_{\mathbb{R}^{n+1}} |\det h|^{-\frac{1}{2}} f(h^{-1}u) e^{-i\beta(u, y)} du $$

$$ = e^{-i\beta(x, y)} \int_{\mathbb{R}^{n+1}} |\det h|^{-\frac{1}{2}} f(u) e^{-i\beta(hu, y)} du $$

$$ = e^{-i\beta(x, y)} \int_{\mathbb{R}^{n+1}} |\det h|^{-\frac{1}{2}} \det h |f(u)| e^{-i\beta(u, h^t y)} du $$

$$ = |\det h|^{\frac{1}{2}} e^{-i\beta(x, y)} \tilde{f}(h^{-1} y). $$
We are now able to state the decomposition of $L^2(\mathbb{R}^{n+1})$ as a representation of $G'$.

**Proposition 6.2.2 (See [FÓ03]).** The quasi-regular representation of $G' = \mathbb{R}^+ \text{SO}(1, n) \ltimes \mathbb{R}^{n+1}$ on $L^2(\mathbb{R}^{n+1})$ decomposes into irreducible parts according to

$$L^2(\mathbb{R}^{n+1}) \cong_{G'} L^2_{\bar{\mathcal{O}}_1}(\mathbb{R}^{n+1}) \oplus L^2_{\bar{\mathcal{O}}_2}(\mathbb{R}^{n+1}) \oplus L^2_{\bar{\mathcal{O}}_3}(\mathbb{R}^{n+1}).$$

**Proof.** Let us simply show that the subspace $L^2_{\bar{\mathcal{O}}_1}(\mathbb{R}^{n+1})$ is invariant under $\rho(h,x)$; that the remaining two parts are also invariant follow from similar arguments.

Let $h \in \mathbb{R}^+ \text{SO}(1, n)$, $x \in \mathbb{R}^{n+1}$, $f \in L^2_{\bar{\mathcal{O}}_1}(\mathbb{R}^{n+1})$ and $y \in \mathbb{R}^{n+1}$. Then $\rho(h,x)f \in L^2(\mathbb{R}^{n+1})$ and

$$\rho(h,x)f(y) = |\det h|^\frac{1}{2} e^{-i(x,y)} \hat{f}(h^t y).$$

By assumption, $\text{supp} \hat{f} \subset \bar{\mathcal{O}}_1$, and since $H'$ acts on $\mathcal{O}_1$ it follows at once that $\text{supp}(\hat{f}(h^t \bullet)) \subset \bar{\mathcal{O}}_1$. Hence $\text{supp}(\hat{\rho}(h,x)\hat{f}) \subset \bar{\mathcal{O}}_1$ and so $\rho(h,x)f \in L^2_{\bar{\mathcal{O}}_1}(\mathbb{R}^{n+1})$.

We now proceed to show that $L^2_{\bar{\mathcal{O}}_1}(\mathbb{R}^{n+1})$ is irreducible\(^2\). It follows from (6.1) that, in particular,

$$\rho(e,x)f(y) = e^{-i(x,y)} \hat{f}(y) \quad \text{and} \quad \rho(h,0)f(y) = |\det h|^\frac{1}{2} \hat{f}(h^t y).$$

Now let $S$ be a closed invariant subset of $L^2_{\bar{\mathcal{O}}_1}$, let $\hat{f}$ belong to $S$, and let $h$ belong to $L^1(\mathbb{R}^{n+1}, dx)$. Then

$$M_h \hat{f}(y) = \hat{h}(y) \hat{f}(y) = \int_{\mathbb{R}^{n+1}} h(x)e^{-i(x,y)} \hat{f}(y) \, dx = \int_{\mathbb{R}^{n+1}} h(x)\rho(e,x)\hat{f}(y) \, dx$$

is a limit in $L^2(\mathbb{R}^{n+1}, dx)$ of linear combinations (Riemann sums) of vectors of the form $(\rho(e,x)f)^\wedge$, and thus belongs to $S$. It follows that $M_h \hat{f}$ is in $S$ for every $h \in L^1(\mathbb{R}^{n+1}, dx)$.

Since $\|\hat{h}\|_\infty \leq \|h\|_1$, it follows that $M_h \hat{f}$ is in $S$ for a given $\hat{f} \in S$ and every bounded continuous function $h$. Consequently, $M_h \hat{f}$ is in $S$ for every bounded measurable function $h$, and it follows that if $P$ is an orthogonal projection commuting with all the operators $\rho(e,x)$, then $P$ also commutes with all multiplication operators $M_h$. Therefore, $P$ has to be of the form $P \equiv M_E$ for some measurable set $E$, and $P \hat{f} = 1_E \hat{f}$ for all $f \in L^2(\mathbb{R}^{n+1}, dx)$.

Let $P$ be the orthogonal projection onto $S$. Then

$$|\det h|^\frac{1}{2} 1_E(h^t y)\hat{f}(h^t y) = |\det h|^\frac{1}{2} P \hat{f}(h^t y) = [\rho(h,0)P \hat{f}]^\wedge(y) = P[\rho(h,0)f]^\wedge(y) = 1_E(y)|\det h|^\frac{1}{2} \hat{f}(h^t y)$$

so $1_E = 1_{\lambda \cdot E}$ for all $x \in H$. It follows that up to measure zero - $E$ is either the set $\{0\}$ or the orbit $\mathcal{O}_1$. This shows that $L^2_{\bar{\mathcal{O}}_1}$ is irreducible. \(\blacksquare\)

\(^2\)The idea of the argument we are about to give is well known classically – as written, say, in [FÓ03] – and the generalization is essentially straightforward.
6.3 The Regular Representation

It has been shown that (almost all of) \( G_1 \) acts on \( \mathbb{R}^{n+1} \), so we can introduce a left-regular representation, \( \lambda \), of \( G_1 \) on \( L^2(\mathbb{R}^{n+1}) \). The most immediate definition of \( \lambda \) would be to let \( (\lambda(g)f)(x) = f(g^{-1} \cdot x) \), where \( \cdot \) denotes the action constructed in the previous chapter. But then \( \lambda \) would not be unitary (the Lebesgue measure on \( \mathbb{R}^{n+1} \) is not invariant under the action of \( G_1 \)).

Write \( J(g, x) \) for the determinant of the jacobian for the map \( x \mapsto g \cdot x \) (the action of \( G_1 \) on \( \mathbb{R}^{n+1} \)). Then

\[
(\lambda(g)F)(x) = J^{-1/2}(g^{-1} \cdot x)F(g^{-1} \cdot x)
\]
defines a unitary representation of \( G_1 \) on \( L^2(\mathbb{R}^{n+1}) \).

The first result is the statement that \( \lambda \), restricted to the parabolic subgroup \( P_1 \) opposite of \( P_1 \) decomposes irreducibly according to the decomposition of \( \rho \) as a representation of \( \mathbb{R} \times SO(1, n) \rtimes \mathbb{R}^{n+1} \). More precisely, we have the following main result.

**Theorem 6.3.1.** The restriction of the (quasi-) regular representation \( \lambda \) of \( G_1 \) on \( L^2(\mathbb{R}^{n+1}) \) to \( P_1 = M_1 A_1 N_1 \) decomposes irreducibly into a direct sum

\[
L^2(\mathbb{R}^{n+1}) = L^2_0(\mathbb{R}^{n+1}) \oplus L^2_1(\mathbb{R}^{n+1}) \oplus L^2_2(\mathbb{R}^{n+1})
\]

**Proof.** The result follows almost at once from Proposition 6.2.2 but let us give the details anyway. Essentially, all we need is a version of Lemma 6.2 adjusted to reflect the fact that we are now using the \( G_1 \)-action on \( \mathbb{R}^{n+1} \) instead of the one used in the previous section. We easily get the following relations between the action of \( P_1 \) and the Fourier transform on \( \mathbb{R}^{n+1} \):

\[
(\hat{\lambda(m_1(\ell)f))(y) = \int_{\mathbb{R}^{n+1}} (\lambda(m_1(\ell)f)(x)e^{i\beta_{1,n}(x,y)}) \, dx
\]

\[
= \int_{\mathbb{R}^{n+1}} J(m_1(\ell)^{-1}, x) \cdot f(m_1(\ell).x) \cdot e^{i\beta_{1,n}(x,y)} \, dx
\]

(Lemma 4.6.3) \[= \int_{\mathbb{R}^{n+1}} f(x) |\det \ell|^e^{i\beta_{1,n}(x,y)} \, dx\]

\[= \int_{\mathbb{R}^{n+1}} f(x) e^{i\beta_{1,n}(x,\ell^t y)} \, dx\]

\[= \hat{f}(\ell^t y)\]
\[
(\lambda(a_t)f)(y) = \int_{\mathbb{R}^{n+1}} (\lambda(a_t)f)(x)e^{i\beta_1(x,y)} \, dx
= \int_{\mathbb{R}^{n+1}} f(a_t^{-1}x)^{1/2} f(a_t^{-1}x)e^{i\beta_1(x,y)} \, dx
\]

(Lemma 4.6.3) \[
e^{t/2} \int_{\mathbb{R}^{n+1}} f(a_{-t}x)e^{i\beta_1(x,y)} \, dx = e^{-t/2} \int_{\mathbb{R}^{n+1}} f(x)e^{i\beta_1(n^{-1}x, y)} \, dx
= e^{-t/2} \hat{f}(e^{-t}y),
\]

and

\[
(\lambda(\overline{n}(y'))f)(y) = \int_{\mathbb{R}^{n+1}} (\lambda(\overline{n}(y'))f)(x)e^{i\beta_1(x,y)} \, dx
= \int_{\mathbb{R}^{n+1}} f(\overline{n}(y'))^{1/2} f(\overline{n}(y'))^{-1} f(x)e^{i\beta_1(x,y)} \, dx
\]

(Lemma 4.6.3) \[
e^{i\beta_1(n^{-1}x, y)} \int_{\mathbb{R}^{n+1}} f(x)e^{i\beta_1(x,y)} \, dx
= e^{-i\beta_1(n^{-1}x, y)} \hat{f}(y).
\]

The remainder of the proof now follows from Proposition 6.2.2 and its proof.

Since \(L^2(N_1)\) and \(L^2(\overline{N}_1)\) are identical as representations of \(G_1\), we infer from the Theorem that \(\lambda|_{\mathcal{P}_1}\) also decomposes in the manner specified in the statement.

Our next main result is that the decomposition of \(\lambda|_{\mathcal{P}_1}\), described in Theorem 6.3.1, even holds as a decomposition of \(\lambda\) as a representation of \(G_1\) itself.

**Theorem 6.3.2.** The regular representation \(\lambda\) of \(G_1\) on \(L^2(\mathbb{R}^{n+1})\) decomposes into irreducible components according to

\[L^2(\mathbb{R}^{n+1}) = L^2_{\partial_1}(\mathbb{R}^{n+1}) \oplus L^2_{\partial_2}(\mathbb{R}^{n+1}) \oplus L^2_{\partial_3}(\mathbb{R}^{n+1}).\]

**Proof.** Let \(\nu_j\) denote the natural representation of \(G_1\) in the classical Hardy space \(H_2(T_{\partial_j})\), \(j = 1, 2\), as defined in a previous chapter. Let \(\beta_j : H_2(T_{\partial_j}) \rightarrow L^2(\mathbb{R}^{n+1})\) denote the boundary value map defined by

\[
\beta_j(F)(x) := \lim_{y \to 0, y \in \partial_j} F(x + iy).
\]

65
We have seen elsewhere that $\text{Im} \beta_j = L^2_\partial_j(\mathbb{R}^{n+1})$. On the other hand, it is easily seen that $\beta_j$ intertwines $\nu_j$ and $\lambda$ in the sense that

$$\beta_j(\nu_j(g)F) = \lambda(g)\beta_j(F)$$

for all $g \in G_1$ and $F \in H_2(T_\partial_j)$. Select a function $f$ in $L^2_{\partial_j}(\mathbb{R}^{n+1})$. In order to verify the statement in the present Theorem, it suffices to show that $\lambda(g)f$ also belongs to $L^2_{\partial_j}(\mathbb{R}^{n+1})$ - because then it follows automatically that also $L^2_{\partial_3}(\mathbb{R}^{n+1})$ is $\lambda(G_1)$-invariant. Write $f$ as $f = \beta_j(F)$ with $F$ in $H_2(T_\partial_j)$. Then

$$\lambda(g)f = \lambda(g)\beta_j(F) = \beta(\nu(g)F).$$

Since $\nu_j$ is a representation on $H_2(T_\partial_j)$, we know that $\nu_j(g)F$ belongs to $H_2(T_\partial_j)$. But then $\lambda(g)f$ belongs to the image of $\beta_j$, which is $L^2_{\partial_j}(\mathbb{R}^{n+1})$. Thus $L^2_{\partial_j}(\mathbb{R}^{n+1})$ is invariant. ■

### 6.4 Identification of the Series of Representations

We are now, finally, able to relate the orbit structure on the Silov boundary $\partial_s T_{\Omega_i}$ with the Plancherel decomposition of $L^2(G/H)$ for $G/H = SO(e(2,n))/SO(e(1,n))$. First of all, we use the fact that $\partial_1$ and $\partial_2$ are homogeneous self-dual cones, combined with the Paley-Wiener Theorem from Chapter 2, to conclude that $L^2_{\partial_1}(\mathbb{R}^{n+1}) = H_2(T_{\partial_1})$ and $L^2_{\partial_2}(\mathbb{R}^{n+1}) = H_2(T_{\partial_2})$ as representations of $G$. Next we use Theorem 3.4 in [HÓØ91] to conclude that $H(C)$ is the direct sum of the full holomorphic discrete series attached to $G/H$, and similarly that $H(-C)$ is the direct sum of the full anti-holomorphic discrete series of $G/H$. So in this way, we have the identifications

$$L^2(G/H) \cong_G L^2(\mathbb{R}^{n+1}) \cong_G \bigoplus(\text{holomorphic discrete series}) \bigoplus L^2_{\partial_3}(\mathbb{R}^{n+1}).$$

In particular, we thereby obtain an isometric embedding of the holomorphic discrete series of $G/H$ into a space of functions on the Silov boundary $\partial_s T_{\Omega_i} \cong \mathbb{R}^{n+1}$ via the Euclidean Fourier transform. This is the generalization of results by Rossi and Vergne that was mentioned as item 2 on page 5 in the Introduction.

We can also identify the last remaining piece, $L^2_{\partial_3}(\mathbb{R}^{n+1})$: Since $G/H$ is rank one in this case, it is known from the general Plancherel theorem that $L^2_{\partial_3}(\mathbb{R}^{n+1})$ has to be $G$-isomorphic to the most continuous part of $L^2(G/H)$, which is the direct integral of the principal series representations associated to $G/H$.

Another would be to notice that it actually follows from the results above that $L^2_{\partial_1}(\mathbb{R}^{n+1}) \oplus L^2_{\partial_2}(\mathbb{R}^{n+1})$ decomposes discretely. It would therefore suffice to show that $L^2_{\partial_j}(\mathbb{R}^{n+1})$ had a purely continuous spectral decomposition. One way to accomplish this would be to study the spectral decomposition of radial part of the Laplace-Beltrami operator $\Delta$ on $G/H$, as in [Ros78] and [Str73], but after $\Delta$ was carried by $Y$ to an invariant differential operator on $\mathbb{R}^{n+1}$.

### 6.5 A Remark on the Plancherel Formula for $\bar{P}_1$

It was mentioned in the introduction that a main source of inspiration in the initial stages of the work was to study the Plancherel formula for the non-unimodular group $\mathbb{R}^+SO_e(1,n) \times \mathbb{R}$.
The underlying idea in our model for the Silov boundary in the present chapter was indeed to obtain a simple description of the restriction to $P_1$ of the regular representation of $G$ on $L^2(G/H)$, realized in the boundary. The way we constructed the group-action on $\mathcal{O}_1$, it dropped out immediately that $\mathbb{R}^+SO_e(1,n)$ corresponded to the Levi-part in $P_1$ and that the $\mathbb{R}^{n+1}$-part in the semi-direct product $\mathbb{R}^+SO_e(1,n) \ltimes \mathbb{R}^{n+1}$ corresponded to $\mathbb{R}^{n+1} \approx \partial_s T_\Omega$. The only thing that prevents us from concluding that the two representations are the same is the obvious problem that the notion of ‘equivalence’ between representations assumes that the representations in question are representations of the same group. That is not the case with $P_1$ resp. $\mathbb{R}^+SO_e(1,n) \ltimes \mathbb{R}^{n+1}$.

Leaving aside the semi-direct product, it seems highly plausible that the quasi-regular representation $\mu$ of $P_1$ and the restriction of $\lambda$ to $P_1$ are, in fact, unitarily equivalent. The Plancherel decomposition for $\mu$ thus ought to translate to a similar statement for $\lambda|_{P_1}$, which in turn would provide a finer description of the spectral properties of the representations $L^2_{\partial_1}(\mathbb{R}^{n+1})$. 

67
Chapter 7

Suggestions for Further Work

7.1 The Plancherel Formula for \( SO_e(2, n)/SO_e(1, n) \)

It would obviously be very nice to be able to pull out the Plancherel formula from the previous chapters. What we can do at this point is to give formulas for the discrete part (since projections onto the Hardy spaces are known to be given by the Cauchy-Szegő kernel), but the continuous part is causing problems. For \( K \)-invariant functions, the projection onto \( L^2_{\mathcal{O}_3}(\mathbb{R}^{n+1}) \) is described by some sort of an orbital integral, though:

\[
F(0) = \int_{\mathcal{O}_3} \hat{F}(\xi) \, d\xi \quad \text{Fourier inversion}
\]

\[
= \int_0^\infty \int_{H/H_{\mathcal{O}_3}} \hat{F}(rh \cdot \xi_3) \, dh \, r^n \, dr
\]

\[
= \int_0^\infty \mathcal{O}_3 \hat{F}(r\xi_3) \, r^n \, dr.
\]

The last integral can be expressed as a generalized Riesz potential on \( \mathbb{R}^{n+1} \), whose inversion formula should appear somewhere in the literature.

**Notation:** Since we will not work directly in the framework provided by the symmetric space setup in \([\text{Orl87}]\), we first have to make certain that we use the correct notation throughout the paper. First of all, the tangent space \( q \) of \( G/H \) is the Lie algebra

\[
q = \left\{ q(x) = \begin{pmatrix} 0 & x_1 & \cdots & x_{n+1} \\ x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n+1} & 0 & \cdots & 0 \end{pmatrix} \bigg| x \in \mathbb{R}^{n+1} \right\} \approx \mathbb{R}^{n+1}
\]

and in \([\text{Orl87}]\), \( n = p + q \) is defined to be the dimension of \( q \) for the space \( G/H = SO_e(p, q + 1)/SO_e(p, q) \). In our version we therefore use \( q = 1 \), \( p = n \), and \( n \mapsto n + 1 = \dim q \).

Write an element \( h \) in \( H = SO_e(1, n) \) as an element in \( G = SO_e(2, n) \) by

\[
h = \begin{pmatrix} 1 \\ \ell \end{pmatrix}, \quad \ell \in SO_e(1, n)
\]
and consider an element \( q(x) \) in \( q \). The element \( h \) then acts naturally on \( q(x) \) (via the adjoint action of \( G \) on \( q \)) by

\[
h \cdot q(x) = hq(x)h^{-1} = \begin{pmatrix} 1 & 0 \\ \ell & \ell^{-1} \end{pmatrix} \begin{pmatrix} x^t \\ x^{n+1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \ell & \ell^{-1} \end{pmatrix}^{-1} = q(\ell x)
\]

by a simple calculation of \( \ell^{-1} \). The adjoint action of \( H \) on \( q \) is therefore the usual action of \( SO_e(1,n) \) on \( \mathbb{R}^{n+1} \) (via matrix multiplication).

**Generalized Riesz Potentials:** Let \( x_1, \ldots, x_{n+1} \) be the standard coordinates on \( \mathbb{R}^{n+1} \) and let \( \langle \cdot, \cdot \rangle \) denote the quadratic form on \( \mathbb{R}^{n+1} \) with signature \((1,n)\),

\[
\langle x,x \rangle = x_1^2 - x_2^2 - \cdots - x_{n+1}^2.
\]

We let \( R_+ = \{ x \in \mathbb{R}^{n+1} \mid \langle x,x \rangle > 0 \} \) and \( R_- = \{ x \in \mathbb{R}^{n+1} \mid \langle x,x \rangle < 0 \} \). On \( R_+ \) we let \( r_+ = \langle x,x \rangle^{1/2} \), and on \( R_- \) we let \( r_- = -\langle x,x \rangle \) (that is, \( r = r(x) = |\langle x,x \rangle|^{1/2} \)). Furthermore, define the “wave operator” \( \Box \) by

\[
\Box = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_{n+1}^2}.
\]

We introduce polar coordinates on \( R_+ \) through the map \( (H_+ \cdot \mathbb{R}^+) \ni (x,r) \mapsto rx \), where \( H_+ = \{ x \in \mathbb{R}^{n+1} \mid \langle x,x \rangle = 1 \} \). It is easy to verify that \( H_+ = H \cdot e_1 \simeq H/M \), where \( M = Z_H(e_1) \).

Just as for the usual polar coordinates on \( \mathbb{R}^{n+1} \), we have that \( dx = r^n dr dh \).

**Lemma 7.1.1.** The radial part of \( \Box (\text{on } W) \) is given by

\[
\Delta(\Box) = \frac{d^2}{dr^2} + \frac{n}{r} \cdot \frac{d}{dr}.
\]

**Corollary 7.1.2.** We have \( \Box(r^{s-(n+1)}) = (s-2)(s-(n+1))r^{s-(n+1)-2} \).

For \( s \in \mathbb{C} \) we let

\[
H_{n+1}(s) = 2^{s-1} \pi^{n-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s-n+1}{2}\right)
\]

and immediately note that \( (s-2)(s-(n+1))H_{n+1}(s-2) = H_{n+1}(s) \).

**Definition 7.1.3.** Let \( f \in C_0^\infty(\mathbb{R}^{n+1}) \cap S(\mathbb{R}^{n+1}) \). The generalized Riesz potential, \( I^{s}_{+} f \), is then defined by

\[
I^{s}_{+} f = \frac{1}{H_{n+1}(s)} \int_{R_+} f(x) r^{s-(n+1)} \, dx.
\]

Clearly \( I^{s}_{+} f \) is well defined for \( \text{Re } s > n + 1 \), and is analytic in this region.
Proposition 7.1.4. The generalized Riesz potential $I_s^n f$ defined above is well defined and analytic in the region \( \{ s \in \mathbb{C} \mid \Re z > n + 1 \} \), has a meromorphic continuation to \( \mathbb{C} \), and satisfies the functional equation
\[
I_s^n (\Box f) = I_s^{n-2} f.
\]

Corollary 7.1.5. With notation as above,
\[
I_s^{n+2m} (\Box^m f) = I_s^n f \text{ for all } m \in \mathbb{N},
\]
and \( f \mapsto I_s^n f, s \) fixed, is a tempered distribution.

It is easy to see that \( H_{n+1} \) has a pole in \( s = 0 \) (or order one or two), so that \( s \mapsto \frac{1}{H_{n+1}(s)} \) has a zero in \( s = 0 \). Trying to study \( I_0^n f \) therefore leads to an examination of the pole at \( s = 0 \) of the integral
\[
s \mapsto \int_{R_+} f(x) r^{s-(n+1)} \, dx.
\]

Lemma 7.1.6. There exists a constant \( c \) such that
\[
I_0^n f = c f(0) \quad (\text{that is, } I_0^n = c \delta \text{ as an identity between distributions}).
\]

A closer examination reveals that, in fact,
\[
c = 2 \sin(n \pi/2).
\]

We also need a Riesz potential for \( R_- \), so define \( I_s^f \) by
\[
I_s^f = \frac{1}{H_{n+1}(s)} \int_{R_-} f(x) r_-^{s-(n+1)} \, dx,
\]
where \( r_-^2(x) = -\langle x, x \rangle \) for \( x \in R_- \).

Lemma 7.1.7.
\[
I_0^n f = 2 \sin(\frac{n \pi}{2}) f(0) = 2 \sin(\frac{n \pi}{2}) f(0) = 2 f(0).
\]

For \( n \) odd (that is, \( p \) odd in Orloff’s notation) we therefore get \( I_0^n f = (-1)^{n-1} f(0) \).

Inversion Formulas for Generalized Riesz Potentials: Now we calculate: First we notice that
\[
I_s^f = \frac{1}{H_{n+1}(s)} \int_{R_-} f(x) r_-^{s-(n+1)} \, dx
\]
\[
= \frac{1}{H_{n+1}(s)} \int_{O_3} f(x) \langle x, x \rangle^{s-(n+1)} \, dx
\]
\[
= \frac{1}{H_{n+1}(s)} \int_0^\infty \int_{H/M} f(rhe_{n+1}) r^n r_-^{s-(n+1)} \, dh \, dr
\]
\[
= \frac{1}{H_{n+1}(s)} \int_0^\infty r_-^{s-1} \int_{H/M} f(rhe_{n+1}) \, dh \, dr,
\]
so in particular,
\[
\int_0^\infty r^n \int_{H/M} f(rhe_{n+1}) \, dh \, dr = H_{n+1}(n+1) I_{n+1} f = 2^n \pi^{n-1} \Gamma(\frac{n+1}{2}) \Gamma(\frac{3}{2}) I_{n+1} f.
\]

It thus becomes a question about inverting the Riesz integral potential \( I_{n+1} \).
7.2 The $H$-Invariant Distribution Character

Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of the Lie group $G$ on the Hilbert space $\mathcal{H}_\pi$. Then the space of $C^\infty$-vectors of $(\pi, \mathcal{H}_\pi)$ is given by

$$\mathcal{H}_\pi^\infty = \{ v \in \mathcal{H}_\pi \mid g \mapsto \pi(g)v \text{ is } C^\infty \}.$$

It is a dense subspace of $\mathcal{H}_\pi$ on which both the Lie group $G$ and its Lie algebra $\mathfrak{g}$ acts, and it can be given a natural topology for which it becomes a Fréchet space. As a subspace of $\mathcal{H}_\pi$, the space $\mathcal{H}_\pi^\infty$ is dense. The topological anti-linear dual of $\mathcal{H}_\pi^\infty$ is called the space of distribution vectors of $(\pi, \mathcal{H}_\pi)$ and is denoted by $\mathcal{H}_\pi^{-\infty}$. Since $\mathcal{H}_\pi^\infty$ is dense in $\mathcal{H}_\pi$, we get a continuous embedding $\mathcal{H}_\pi \hookrightarrow \mathcal{H}_\pi^{-\infty}$.

The space $\mathcal{H}_\pi^\infty$ is $G$-invariant so the restriction of $\pi$ to $\mathcal{H}_\pi^\infty$ defines a representation that we denote by $\pi^\infty$. By transposition we now get a representation $\pi^{-\infty}$ of $G$ on $\mathcal{H}_\pi^{-\infty}$ by

$$\langle \pi^{-\infty}(g)T, v \rangle = \langle T, \pi^\infty(\tilde{\varphi}_0)(v) \rangle.$$

**Lemma 7.2.1.** For each $\varphi \in \mathcal{D}(G)$ and each $v \in \mathcal{H}_\pi$, the vector $\pi(\varphi)v$ belongs to $\mathcal{H}_\pi^\infty$. Moreover, the space spanned by all such vectors $\pi(\varphi)v$ is dense in $\mathcal{H}_\pi^\infty$; this space is called the Gårding space of the representation $(\pi, \mathcal{H}_\pi)$.

For each $\varphi \in \mathcal{D}(G)$ and each $T \in \mathcal{H}_\pi^{-\infty}$ we may now form the distribution vector $\pi^{-\infty}(\varphi)(T) \in \mathcal{H}_\pi^{-\infty}$ by

$$\langle \pi^{-\infty}(\varphi)(T), v \rangle = \int_G \varphi(g) \langle \pi^{-\infty}(g)T, v \rangle \, dg = \langle T, \pi^\infty(\tilde{\varphi}_0)(v) \rangle$$

for all $v \in \mathcal{H}_\pi^\infty$. Here $\tilde{\varphi}$ is defined by $\tilde{\varphi}(g) = \varphi(g^{-1})$.

**Lemma 7.2.2.** The distribution vector $\pi^{-\infty}(\varphi)(T)$ belongs to $\mathcal{H}_\pi^\infty$ whenever $\varphi \in \mathcal{D}(G)$ and $T \in \mathcal{H}_\pi^{-\infty}$.

Hence, each $T \in \mathcal{H}_\pi^{-\infty}$ defines a (continuous) linear map $A_T : \mathcal{D}(G) \to \mathcal{H}_\pi$ by $A_T(\varphi)(\varphi) = \pi^{-\infty}(\tilde{\varphi})$.

Now consider the space $(\mathcal{H}_\pi^{-\infty})^H$ of $H$-invariant distribution vectors for $\pi$.

**Lemma 7.2.3.** Let $\pi \in \hat{\hat{G}}$. There is a bijective anti-linear map from $(\mathcal{H}_\pi^{-\infty})^H$ onto the space of continuous equivariant linear maps from $\mathcal{H}_\pi^\infty$ to $C^\infty(G/H)$.

**Proof.** For $v' \in \mathcal{H}_\pi^{-\infty}$ and $v \in \mathcal{H}_\pi^\infty$, define $T_{v,v'} \in C^\infty(G)$ by $T_{v,v'}(g) = v'(\pi(g^{-1})v)$. Then $T$ is linear in $v'$ and anti-linear in $v$. It is clear that if $v'$ is fixed under $H$, then the map $v \mapsto T_{v,v'}$ is a continuous equivariant linear map from $\mathcal{H}_\pi^\infty$ to $C^\infty(G/H)$.

Conversely, if such a map $j : \mathcal{H}_\pi^\infty \to C^\infty(G/H)$ is given, then we obtain an element $v'$ in $(\mathcal{H}_\pi^{-\infty})^H$ by defining $v'$ as $v'(v) = j(v)(e)$. ■

**Definition 7.2.4.** A symmetric pair $(G, H)$ for which $(\mathcal{H}_\pi^{-\infty})^H$ is finite-dimensional for all $\pi \in \hat{\hat{G}}$ is called a generalized Gelfand-pair.
Proposition 7.2.5. The space \( \mathcal{H}^{\pi}_{\tau} \) is finite dimensional for all \( \pi \in \hat{\mathcal{G}} \).

Proof. Fix a nonzero \( K \)-finite vector \( v \) in \( \mathcal{H}^{\pi}_{\tau} \). Then it follows from Lemma 7.2.3 and its proof that the map

\[
(\mathcal{H}^{\pi}_{\tau})^H \ni v' \mapsto T_{v,v'} \in C^\infty(G/H)
\]

is injective. Since \( \pi \) is irreducible, it has an infinitesimal character \( \chi \), and therefore \( T_{v,v'} \) is a \( K \)-finite eigenfunction for the center \( Z(g) \) of the universal enveloping algebra \( U(g) \). In fact, the space space of functions on \( G/H \) that are \( K \)-finite of a given type and eigenfunctions for \( Z(g) \) with a given infinitesimal character is finite dimensional. ■

Once again let \( v \) belong to \( (\mathcal{H}^{\pi}_{\tau})^H \). Then \( \pi^{\infty}(\varphi)v \) belongs to \( (\mathcal{H}^{\pi}_{\tau})^\infty \) for all \( \varphi \) in \( C^\infty(G/H) \) and we define the \( H \)-invariant distribution \( \Theta_{\pi} \) by

\[
(7.1) \quad C^\infty_c(G/H) \ni \varphi \mapsto \Theta_{\pi}(\varphi) := \langle \pi^{\infty}(\varphi)v, v^* \rangle \in \mathbb{C}.
\]

Definition 7.2.6. The distribution \( \Theta_{\pi} \) defined in (7.1) is called the spherical distribution character of \( \pi \).

The “distribution character”-problem is then to find an explicit formula – THE CHARACTER FORMULA FOR \( \pi \) – for the distribution \( \Theta_{\pi} \).

Let \( \pi_{\xi,\lambda} \) be a representation from the unitary principal series of \( G/H \). As we have seen, \( \pi_{\xi,\lambda} \) may then be realized as an irreducible sub-representation of \( (\nu_3, L_2^2(\mathbb{R}^{n+1})) \), and we will now examine the character of \( \pi_{\xi,\lambda} \) in this realization.

Identifying \( G/H \) with its image in \( \mathbb{R}^{n+1} \), we first notice that an \( H \)-invariant distribution \( \Theta_{\pi} \) on \( G/H \) thus corresponds to an \( H \)-invariant distribution \( \Lambda_{\pi} \) on \( \mathbb{R}^{n+1} \). As we have already discovered, the group \( H \) acts linearly on \( \mathbb{R}^{n+1} \) as well as functions defined on \( \mathbb{R}^{n+1} \) (via the representation of \( G_1 \) on \( L^2(\mathbb{R}^{n+1}) \)). Hence \( \Lambda_{\pi} \) is a distribution on \( \mathbb{R}^{n+1} \) that is invariant under the usual action of \( SO_c(1,n) \) by matrix multiplication, i.e., \( \Lambda_{\pi} \) is an \( SO_c(1,n) \)-invariant distribution on \( \mathbb{R}^{n+1} \).

Remark added in proof: It turns out that \( SO(1,n) \)-invariant distributions on \( \mathbb{R}^{n+1} \) have been studied in the literature for a long time. See [Ten60], [KV92] and [docg92], for example.

7.3 The Plancherel Decomposition for Higher Rank Spaces

Some ideas along the lines we will outline are already present in [VR76], albeit hidden, and more recently in [Sah92]. We have changed the basic construction of representations associated to the orbits to fit into the general framework of previous chapters, though.

7.3.1 Introduction: Computations with \( G = SL(2, \mathbb{R}) \)

Here we consider the usual subgroups \( K, A, N, \overline{N}, A, \) and \( M \) given by

\[
K = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \middle| \theta \in [0, 2\pi) \right\}, \quad A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \middle| t \in \mathbb{R} \right\},
\]

72
\[ N = \left\{ \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \mid \xi \in \mathbb{R} \right\}, \quad \overline{N} = \left\{ \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \mid \xi \in \mathbb{R} \right\}, \quad M = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \]

We also put \( L = MA \), that is,
\[ L = \left\{ \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\}, \]
and notice that \( L \cap K = \{ \pm I \} = M \). Furthermore notice that
\[ \overline{n} = \left\{ \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} \mid \xi \in \mathbb{R} \right\}. \]

The standard triple for \( \mathfrak{sl}(2, \mathbb{C}) \) is
\[ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \]
so in this case it is obvious that every element in \( \overline{n} \) is \( L \cap K \)-conjugate to an element of the form \( rf \).

Now let
\[ f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } f_{-1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \]
and recall that \( SL(2, \mathbb{R}) \) acts on \( \overline{n} \) through its adjoint action, that is,
\[ \overline{n} \ni X \mapsto \text{Ad}_\ell(X) = \ell X \ell^{-1}, \ell \in L. \]

For \( \ell \in L \) we now get that
\[
\text{Ad}_\ell(f_1) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \\
= \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 & e^{-t} \\ e^t & 0 \end{pmatrix} \\
= \begin{pmatrix} 0 & 0 \\ e^{-2t} & 0 \end{pmatrix},
\]
so clearly the \( L \)-orbit through \( f_1 \in \overline{n} \) is the set
\[ \left\{ \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \mid \lambda > 0 \right\} \cong (0, \infty). \]

Similarly, the \( L \)-orbit through \( f_{-1} \) is the set
\[ \left\{ \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix} \mid \lambda > 0 \right\} \cong (-\infty, 0). \]
7.3.2 General Notation

Let \((g, t)\) be an irreducible Hermitian symmetric pair of tube type, and let \(h^s = a^s \oplus t^s\) be a maximally split Cartan subalgebra of \(g\). Let \(n = \text{dim}\ a^s\). Then it is known that the restricted root system is of type \(C_n\), so we may choose a basis \(\{\varepsilon_1, \ldots, \varepsilon_n\}\) for \((a^s)^*\) such that \(\Sigma(g, a^s) = \{\varepsilon_i \pm \varepsilon_j\} \cup \{\pm 2\varepsilon_j\}\). The root spaces for \(\pm \varepsilon_i \pm \varepsilon_j\) have a common dimension, denoted \(d\), and the root spaces for \(\pm 2\varepsilon_j\) are one-dimensional. Hence, to each root \(2\varepsilon_j\) there is associated in a natural way an \(sl(2)\)-triple\(^1\) \(\{h_j, e_j, f_j\}\) in \(g\). These \(S\)-triples commute, and

\[
\{ h = \sum h_j, e = \sum e_j, f = \sum f_j \}
\]

also defines an \(S\)-triple.

The Cayley transform is defined to be the element \(c = \exp \frac{\pi i}{4} (e + f) \in \text{Ad}(g_{\mathbb{C}})\). Let \(t = ic(a^s)\). Then \(h = t + t^s\) is a compact Cartan subalgebra in \(g\) and \(\{g_i | g_i := c \circ (2\varepsilon_i), i = 1, \ldots, n\}\), is a maximal set of strongly orthogonal roots in \(\Sigma(h_{\mathbb{C}}, g_{\mathbb{C}})\).

The eigenvalues of \(\text{ad}(h)\) on \(g\) are \(-2, 0,\) and \(2\). Let \(n, l,\) and \(n\) denote the corresponding eigenspaces. Then \(n\) and \(n\) are both abelian subalgebras, and \(l + n\) and \(l + n\) are maximal parabolic subalgebras in \(g\).

Let \(G\) and \(K\) be the simply connected groups with Lie algebras \(g\) and \(k\), respectively, and let \(P = LN\) and \(\overline{P} = L\overline{N}\) be the maximal parabolic subgroups of \(G\) corresponding to \(l + n\) and \(l + \overline{n}\), respectively. Then \(G/K\) is a symmetric tube type domain of rank \(n\) and \(G/P\) is its Silov boundary.

7.3.3 The Orbits and the Representations

Now \(L\) (resp. \(K\)) has a unique positive (resp. unitary) character \(\nu\) (resp. \(\mu\)) whose differential is \(\varepsilon_1 + \cdots + \varepsilon_n\) (resp. \(\gamma_1 + \cdots + \gamma_n\)). The restriction of \(\mu\) to \(L \cap K\) extends uniquely to a unitary character of \(L\), also denoted by \(\mu\), that is trivial on the identity component of \(L\).

Theorem XI.5.5 in [BK66] shows that each element in \(n\) is \(L \cap K\)-conjugate to an element of the form \(r_1 f_1 + \cdots + r_n f_n\), where \(r_j\) are real numbers that are uniquely determined up to permutation. Since \(L = (L \cap K)A(L \cap K)\), it follows that the elements

\[
f_{pq} = (f_1 + \cdots + f_p) - (f_{n-q+1} + \cdots + f_n)
\]

form a set of representatives for the \(L\)-orbits on \(n\).

Now let \(S_{pq}\) denote the stabilizer of \(f_{pq}\) in \(L\), and let \(s_{pq}\) denote the (real) Lie algebra of \(S_{pq}\). If \(p + q = n\), that is, if we are considering a semi-definite orbit, then \(s_{pq}\) is a real form of \(s_{n0} = l \cap t\).

\(^1\)We have in mind the following general construction: Given a simple system \(\Pi = \{\alpha_1, \ldots, \alpha_n\}\) we construct for each \(\alpha_i\) the associated triple \(\{h_i, e_i, f_i\}\) by

\[
h_i = \frac{2}{|\alpha_i|^2} H_{\alpha_i},
\]

\(e_i\) a nonzero root vector for \(\alpha_i\), and \(f_i\) a nonzero root vector for \(-\alpha_i\) with \(B(e_i, f_i) = \frac{1}{|\alpha_i|^2}\). See for example p. 140 in the green book by Knapp.
If \( p + q < n \), we proceed in the following manner: Let \( l_{ij} \) denote the root space in \( l \) for \( \varepsilon_i - \varepsilon_j \), and let
\[
\begin{align*}
l_1 &= \text{span}\{h_{i,j} | i,j \in \{p + 1, p + 2, \ldots, n - q\}\}, \\
l_2 &= \text{span}\{h_{i,j} | i,j \notin \{p + 1, p + 2, \ldots, n - q\}\}, \\
\mathfrak{u} &= \text{span}\{l_{ij} | i \in \{p + 1, p + 2, \ldots, n - q\}, j \notin \{p + 1, p + 2, \ldots, n - q\}\}.
\end{align*}
\]
Then \( l_2 \cap \mathfrak{s}_{pq} \) is a real form of \( l_2 \cap \mathfrak{t} \), and \( \mathfrak{s}_{pq} = l_1 + (l_2 \cap \mathfrak{s}_{pq}) + \mathfrak{u} \).

For \( p + q \leq n \), \( \mathfrak{s}_{p0} \cap \mathfrak{s}_{0q} \) is contained in \( \mathfrak{s}_{pq} \), \( l_2 \cap \mathfrak{s}_{p0} \cap \mathfrak{s}_{0q} \) is compact, and \( \mathfrak{s}_{p0} \cap \mathfrak{s}_{0q} = l_1 + (l_2 \cap \mathfrak{s}_{p0} \cap \mathfrak{s}_{0q}) + \mathfrak{u} \).

Let \( O_{pq} \) denote the \( L \)-orbit through \( f_{pq} \), and let \((\pi_{\varepsilon,t}, I(\varepsilon,t))\) denote the induced representation \( \text{Ind}_{\mathfrak{O}}^G(\mu^\varepsilon \otimes \nu^t) \) (normalized \( C^\infty \)-induction). Using the Bruhat decomposition of \( G \), and the fact that \( \exp : n \to N \) is a bijective diffeomorphism, we can realize \( I(\varepsilon,t) \) as a subspace of \( C^\infty(n) \). It is now easy to describe the action of \( P \): \( N \) acts by translation, and since the modular function for \( \overline{P} \) is given by \( \nu^{-2r} \) where \( r = 1 + \frac{d}{2}(n - 1) \), it follows that
\[
\pi_{\varepsilon,t}(l)\eta(x) = \mu^\varepsilon(l)\nu^{t-r}(l)\eta(l^{-1} \cdot x) \quad \text{for } l \in L.
\]

Now fix \( p \) and \( q \), and let \( t = 1 + \frac{d}{2}(n - 1 - p - q) \). Then
\[
\pi_{\varepsilon,t}(l)\eta(x) = \mu^\varepsilon(l)\nu^{-\frac{d}{2}(p + q)}\eta(l^{-1} \cdot x).
\]

Note that \((\pi_{\varepsilon,t}, H)\) is a unitary representation of \( P \).

Let \( H_{pq} \) denote the Hilbert space of functions whose Fourier transform is supported in \( O_{pq} \). The important fact is then that each \( H_{pq} \) can be realized as a sub-representation of this particular induced representation \( \pi_{\varepsilon,t} \), that is, to each orbit \( O_{pq} \) we associate a representation \((\pi_{\varepsilon,t}, H_{pq})\). Obviously one has to be careful about \( H_{pq} \) being invariant, but this follows from a somewhat tedious calculation.

**Lemma 7.3.4.** \((\pi_{\varepsilon,t}, H_{pq})\) is irreducible upon restriction to the identity component \( P_0 \).

**Sketch of proof.** It follows at once from construction that the Levi-part \( L \) is well-behaved. The action of \( N \) consists of multiplication by characters on \( \mathcal{O} \), and since these characters separate points (as \( N \) is abelian; think about the Gelfand-Raikov Theorem), it follows from the Stone-Weierstrass theorem that \( T \) is itself the operator of multiplication by a bounded Borel function. Finally, since \( L_0 \) acts transitively on \( \mathcal{O} \), we see that this function is a constant, proving the lemma.

But then clearly \((\pi_{\varepsilon,t}, H_{pq})\) is itself irreducible, and by collecting all the orbits \( O_{pq} \), we get a direct decomposition of \( \pi_{\varepsilon,t} \) into irreducible representations \((\pi_{\varepsilon,t}, H_{pq})\).

Such an idea would not work for \( G = SO_\varepsilon(2, n) \) in the geometric realization, since the map \( Y \) maps into a Silov boundary that is related to a “bigger” group \( G_1 \). For \( G/H \) of Cayley type, on the other hand, it is easy to see that the Silov boundary \( \partial_\varepsilon T_{\Omega_1} \) is in fact the boundary of a generalized bi-upper half-plane \( \partial_\varepsilon T_{\Omega_1} \times \partial T_{\Omega_1} \). Applying the above idea to each of the two factors \( \partial_\varepsilon T_{\Omega_1} \) would therefore realize the regular representation of \( G \) on \( L^2(G/H) \approx L^2(\partial_\varepsilon T_{\Omega_1} \times \partial_\varepsilon T_{\Omega_1}) \) as a direct sum of tensor products of the form \((\pi_{\varepsilon,t}, H_{pq}) \otimes (\pi_{\varepsilon,t}, H_{p'q'})\). The Plancherel
decomposition of $L^2(G/H)$ would therefore translate into a study of the branching law for the restriction of these tensor products to the “diagonal” $G$ in $G_1 = G \times G$.

Such branching laws have recently attracted a fair amount of attention, and there is an elaborate theory (in large part due to Kobayashi) that covers discrete decomposability in such branching laws. The methods appear to be non-conclusive for continuous parts of the spectrum, however, so it seems unlikely that such a method would account for all of the representations in the discrete series. For example, it is known ([Rep78]) that the tensor product of two mock-discrete series representations of $SL(2, \mathbb{R})$ decomposes as a direct integral of principal series representations of $SL(2, \mathbb{R})$. Continuous decomposability is thus far from being understood, in general.
Bibliography


78


Vita

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