SOME CLASSES OF GRAPHS THAT ARE NEARLY CYCLE-FREE

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Abstract

A graph is almost series-parallel if there is some edge that one can add to the graph and then contract out to leave a series-parallel graph, that is, a graph with no $K_4$-minor. In this dissertation, we find the full list of excluded minors for the class of graphs that are almost series-parallel. We also obtain the corresponding result for the class of graphs such that uncontracting an edge and then deleting the uncontracted edge produces a series-parallel graph.

A notable feature of a 3-connected almost series-parallel graph is that it has two vertices whose removal leaves a tree. This motivates consideration of those graphs for which there are two vertices whose removal is cycle-free. We find the full list of excluded minors for the class of graphs that have a set of at most two vertices whose removal is cycle-free.
Chapter 1
Introduction

This chapter contains some basic graph theory and matroid theory terminology that will be used throughout this dissertation. The terminology used for graphs and matroids closely follows [5] and [13].

1.1 Fundamental Graph Definitions

A multigraph $G$ is a pair $(V(G), E(G))$ of sets, where $V(G)$ is a set and $E(G)$ is a multiset whose elements are unordered pairs of elements in $V(G)$. We refer to $V(G)$ and $E(G)$ as the sets of vertices and edges, respectively, of $G$. When it is clear to which graph $G$ we are referring, $V(G)$ and $E(G)$ are abbreviated by $V$ and $E$, respectively. A simple graph is a multigraph in which the edges are distinct pairs of distinct vertices. Throughout this dissertation, when we refer to a graph, we assume it is a multigraph unless otherwise specified.

The number of vertices of a graph $G$ is its order, written $|V(G)|$ and its number of edges is denoted by $|E(G)|$. The graphs we consider are all finite, that is, they all have a finite order and a finite number of edges. For the empty graph, we write $\emptyset$. A graph of order zero or one is called trivial. We will primarily consider non-trivial graphs.

Let $e$ be the edge $\{v, w\}$ where $v$ and $w$ are in $V(G)$. In this dissertation, we denote a single edge between $\{v, w\}$ by $vw$ or $wv$. The edge $e$ is between $v$ and $w$, and we call $v$ and $w$ the endpoints of $e$. An edge is incident with each of its endpoints. An edge whose endpoints are the same vertex is a loop. If $e$ and $f$ are edges, having the same pair of distinct endpoints, then $e$ and $f$ are parallel. For an edge $e$ in a graph $G$, the parallel class of $e$ is the set $e$ together with every element parallel to $e$. Two distinct edges are adjacent if they have an endpoint in common. Two distinct vertices $v$ and $w$ are adjacent, or are neighbors if there is
an edge between \( v \) and \( w \) when this occurs. We also say that there is an edge \( \text{joining} \ v \) and \( w \). The set of neighbors of \( v \) in \( G \) is the \textit{neighborhood} \( N_G(v) \) of \( v \). We abbreviate \( N_G(v) \) to \( N(v) \) when it is understood which graph is meant. If all vertices of \( G \) are pairwise adjacent and \( G \) is simple, then \( G \) is \textit{complete}. A complete graph on \( n \) vertices is denoted \( K_n \). For example, \( K_3 \) is a triangle.

The \textit{degree}, \( d_G(v) \) or \( d(v) \), of a vertex \( v \) is the number of edges meeting \( v \) where a loop is counted twice. The number \( \delta(G) \) is the \textit{minimum degree} of any vertex in \( G \), and \( \Delta(G) \) is the \textit{maximum degree} of any vertex in \( G \). If all the vertices of \( G \) have the same degree \( k \), then \( G \) is \textit{k-regular}, or simply \textit{regular}. A 3-regular graph is called \textit{cubic}.

Let \( G = (V,E) \) and \( G' = (V',E') \) be two graphs. Graphs \( G \) and \( G' \) are \textit{isomorphic}, written \( G \cong G' \), if there are bijections \( \sigma : V(G) \rightarrow V(G') \) and \( \phi : E(G) \rightarrow E(G') \) such that a vertex \( v \) of \( G \) is incident with an edge \( e \) of \( G \) if and only if \( \sigma(v) \) is incident with \( \phi(e) \). We do not normally distinguish between isomorphic graphs. Thus, we often write \( G = G' \), rather than \( G \cong G' \).

A graph \( H \) is a \textit{subgraph} of a graph \( G \), written \( H \subseteq G \), if \( V(H) \subseteq V(G) \) and if each edge of \( H \) is an edge of \( G \). We also say \( G \) is a \textit{supergraph} of \( H \) and \( G \) contains \( H \). If \( V(H) = V(G) \), then we say that \( H \) \textit{spans} \( G \). When it is clear which graph we are referring to, we say that \( H \) is \textit{spanning}. If \( V' \) is a non-empty subset of \( V(G) \), then \( G[V'] \) denotes the subgraph of \( G \) whose vertex set is \( V' \) and whose edge set consists of those edges of \( G \) that have both endpoints in \( V' \). We say that \( G[V'] \) is the subgraph of \( G \) \textit{induced} by \( V' \). Similarly, if \( E' \) is a non-empty subset of \( E(G) \), then \( G[E'] \), the subgraph of \( G \) induced by \( E' \), has \( E' \) as its edge set and the set of endpoints of edges in \( E' \) as its vertex set.

If \( G \) and \( G' \) are graphs, their \textit{union} \( G \cup G' \) is the graph with vertex set \( V(G) \cup V(G') \) and edge set \( E(G) \cup E(G') \). If \( V(G) \) and \( V(G') \) are disjoint, then so are \( E(G) \) and \( E(G') \), and \( G \) and \( G' \) are called \textit{disjoint graphs}. 

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1.2 Deletion, Contraction, and Graph Minors

If $U$ is any set of vertices of $G$, then the graph obtained by deleting all the vertices in $U$ and their incident edges is denoted $G - U$. If $U = \{u\}$ is a single vertex, then we write $G - u$. If $F$ is a set of possible edges of $G$, then $G \setminus F$ is the graph $(V, E \setminus F)$ obtained by deleting the subset $F$, and $G + F$ is the graph $(V, E \cup F)$ obtained by adding the edges $F$ to the graph $G$. For the deletion and addition of a single edge, we write $G \setminus e$ and $G + e$, respectively.

Let $e$ be an edge of a graph $G = (V, E)$ with endpoints $x$ and $y$. We denote the contraction of the edge $e$ by $G/e$. This is the graph obtained from $G$ by contracting the edge $e$ into a new vertex $v_e$, which becomes adjacent to all the former neighbors of $x$ and $y$. The graph $G/e$ is a graph with vertex set $V' = V \setminus \{x, y\} \cup \{v_e\}$ where $v_e$ is the new vertex. Let $f$ be a function which maps every vertex in $V \setminus \{x, y\}$ to itself, and otherwise maps to the new vertex $v_e$. The edge set $E'(G/e) = E \setminus e$ and, for every $z \in V$, the vertex $z' = f(z) \in V'$ is incident to an edge $e' \in E'$ if and only if the corresponding edge $e \in E$ is incident to $z$ in $G$. Let $H \subseteq E(G)$. Then $G/H$ is the contraction of the set $H$ from $G$.

The fundamental substructures of graphs are graph minors which can be obtained by deleting some vertices and edges, and then contracting some further edges. Formally, any sequence of deletions and contractions from $G$ can be written in the form $(G - U) \setminus X/Y$ for some set of vertices $U$ and some pair of disjoint sets of edges $X$ and $Y$. The sets $U$, $X$, and $Y$ may be empty. Graphs of the form $(G - U) \setminus X/Y$ are called minors of $G$. If $U \cup X \cup Y$ is non-empty, then we call $(G - U) \setminus X/Y$ a proper minor of $G$. Note that every subgraph $G$ of a graph is also a minor of $G$, and $G$ is a minor of itself. A graph $G$ has an $N$-minor if $N$ is a minor of $G$ and we say $G$ contains $N$ as a minor or simply $G$ contains $N$.

1.3 Several Important Classes of Graphs

A path is a non-empty graph $P = (V(P), E(P))$ of the form $V(P) = \{x_0, x_1, \ldots, x_k\}$ and $E(P) = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$, where all $x_i$ are distinct. The vertices $x_0$ and $x_k$ are linked by $P$ and are called its ends. The vertices $x_1, \ldots, x_{k-1}$ are the inner vertices of $P$. 

The number of edges of the path is its length, and the path of length \( k \) is denoted \( P_k \). We often refer to the path by the natural sequence of its vertices, writing \( P = x_0x_1 \ldots x_k \) and calling \( P \) a path from \( x_0 \) to \( x_k \) or between \( x_0 \) and \( x_k \).

Two or more paths are internally disjoint if none of them contains an inner vertex of another. Two paths from \( a \) to \( b \), for example, are internally disjoint if and only if \( a \) and \( b \) are their only common vertices.

If \( P = x_0 \ldots x_{k-1} \) is a path, then the graph \( C = P + x_{k-1}x_0 \) is called a cycle, often written \( x_0x_1 \ldots x_{k-1}x_0 \). The length of a cycle is its number of edges (or vertices) and the cycle of length \( k \) is called a \( k \)-cycle and denoted by \( C_k \). A \( C_k^n \) is the graph obtained from a cycle of length \( k \) by replacing every edge of \( C_k \) by \( n \) parallel edges. A wheel, denoted \( W_r \), is a simple graph that is formed by taking an \( r \)-cycle and adding a vertex adjacent to every vertex of the cycle. We call the \( r \)-cycle from which \( W_r \) is formed, the rim of the wheel, and every other edge not in this cycle is a spoke of the wheel.

An acyclic graph, one not containing any cycles, is called a forest. A connected forest is called a tree. The vertices of degree 1 in a tree are its leaves. So every tree with at least two vertices has at least two leaves.

The class of bipartite graphs is a well-known class and has been studied extensively. A graph \( G \) is bipartite if its vertex set has a partition \((A, B)\) into possibly empty sets such that each edge has one endpoint in \( A \) and one endpoint in \( B \); that is, \( G \) has no edge having both endpoints in \( A \) or both endpoints in \( B \). If the graph \( G \) induced on a vertex set contains no edges, then that set is stable. The vertex set of a bipartite graph is the union of two stable sets.

Let \( G \) be a bipartite graph with vertex partition \((A, B)\), where \( A = \{a_1, a_2, \ldots, a_k\} \) and \( B = \{b_1, b_2, \ldots, b_l\} \). The complete bipartite graph \( K_{k,l} \) is the simple bipartite graph where each vertex in \( A \) is adjacent with every vertex in \( B \). Note that any subgraph of \( G \) is also bipartite. Thus, the class of bipartite graphs is closed under edge and vertex deletion. Since
contains no edge of the form \( a_i a_j \), any path in \( G \) must alternate between \( A \)-vertices and \( B \)-vertices, such as \( a_{n_1} b_{n_2} a_{n_3} \ldots b_{n_{m-1}} a_{n_m} \). Clearly any cycle of \( G \) also alternates between \( A \)-vertices and \( B \)-vertices. Thus, a bipartite graph contains no odd cycle. The converse of this also holds, that is, a graph is bipartite if and only if it contains no odd cycles.

Another well-known class of graphs is the class of planar graphs, graphs that can be drawn in the Euclidean plane so that vertices correspond to points of the plane; the edges correspond to arcs connecting vertices; two distinct edges do not intersect except possibly at their endpoints; and no vertex lies in the interior of an edge. Such a drawing in the plane is called a plane graph. Clearly every minor of a plane graph is a plane graph. Thus, the class of planar graphs is closed under taking minors.

### 1.4 Graph Operations

A graph \( G' \) is a subdivision of a graph \( G \) if \( G' \) can be obtained from \( G \) by replacing non-loop edges of \( G \) by paths of non-zero length and replacing loop edges by cycles. The simplification of a graph \( G \), denoted \( \text{si}(G) \), is the graph obtained from \( G \) by deleting all loops and all but one element from each parallel class. We refer to \( \text{si}(G) \) as the underlying simple graph of \( G \).

A plane graph \( G \) has a dual graph \( G^* \), that is, the graph whose vertices are the faces of \( G \) such that, for each edge \( e \in E(G) \), there is an edge \( e' \in E(G^*) \) whose endpoints are the faces that meet \( e \) in \( G \).

A clique-sum of two graphs \( G_1 \) and \( G_2 \) is obtained from the disjoint union of \( G_1 \) and \( G_2 \) by identifying a complete subgraph of \( G_1 \) with a complete subgraph (of the same order) of \( G_2 \), and then deleting the edges of the identified subgraph. If the identified complete subgraph has order \( k \), then the clique-sum is called a \( k \)-sum and is written \( G_1 \oplus_k G_2 \). The 0-sum is the disjoint union of \( G_1 \) and \( G_2 \) and the 1-sum consists of subgraphs \( G_1 \) and \( G_2 \) sharing exactly one vertex.

Let \( G_1 \) and \( G_2 \) be disjoint graphs and let \( p_i \) be a non-loop edge of \( G_i \). Assign a direction to \( p_i \) labeling its tail by \( u_i \) and its head by \( v_i \). The series connection, \( S(G_1, G_2) \), of \( G_1 \) and
with respect to the directed edges $p_1$ and $p_2$ is formed by deleting $p_1$ from $G_1$ and $p_2$ from $G_2$, identifying $u_1$ and $u_2$ as a new vertex $u$, and then adding a new edge $p$ joining $v_1$ and $v_2$. The parallel connection, $P(G_1, G_2)$, of $G_1$ and $G_2$ is formed by deleting $p_1$ from $G_1$ and $p_2$ from $G_2$, identifying the vertices $u_1$ and $u_2$ as the vertex $u$, identifying vertices $v_1$ and $v_2$ as the vertex $v$, and then adding a new edge $p$ joining $u$ and $v$. Thus the parallel connection is obtained by simply identifying $p_1$ and $p_2$ so that their directions agree. Notice that the 2-sum of two graphs $G_1$ and $G_2$, written $G_1 \oplus_2 G_2$ is the deletion of $p$ from the parallel connection of $G_1$ and $G_2$ with respect to the edges $p_1$ of $G_1$ and $p_2$ of $G_2$. We call $p_1$ and $p_2$ basepoints of $G_1$ and $G_2$, respectively, and $p$ the basepoint of the series connection, parallel connection, and 2-sum. For $n \geq 2$, let $G_1, G_2, \ldots, G_n$ be graphs whose edge sets are disjoint except that each has a non-loop directed edge labelled $p$. The parallel connection $P(G_1, G_2, \ldots, G_n)$ of $G_1, G_2, \ldots, G_n$ is obtained by identifying all the edges labelled by $p$ so that their directions agree.

1.5 Graph Connectivity

A non-empty graph $G$ is connected if any two of its vertices are linked by a path in $G$. If $U \subseteq V(G)$ and $G[U]$ is connected, we say that $U$ is connected in $G$.

We call a maximal connected subgraph of a graph $G$ a component of $G$. If $A, B \subseteq V(G)$ and $X \subseteq V(G) \cup E(G)$ are such that every path from $A$ to $B$ in $G$ contains a vertex or an edge from $X$, we say that $X$ separates the sets $A$ and $B$ in $G$. Then $X$ is a separating set or cut set in $G$ if $X$ separates two vertices of $G - X$ in $G$. A cut-vertex is a vertex that separates two other vertices of the same component, and a bridge is an edge separating its ends. We call a set $Y$ of vertices a vertex cut of $G$ if $Y$ separates a component of $G$. If $G \setminus X$ has more components than $G$ for some set $X$ of edges of $G$, then we call $X$ an edge cut of $G$. An edge $e$ for which $\{e\}$ is an edge cut is called a cut-edge. A minimal edge cut is also called a bond of $G$. 
A graph $G$ is $k$-connected for $k \in \mathbb{N}$ if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. In other words, no two vertices of $G$ are separated by fewer than $k$ other vertices. Every (nonempty) graph is 0-connected, and the 1-connected graphs are precisely the non-trivial connected graphs. For a graph $G$, the vertex connectivity, $\kappa(G)$, is defined to be zero when $G$ is disconnected. When $G$ is connected, $\kappa(G)$ is the minimum cardinality of a vertex cut in $G$ unless every two distinct vertices of $G$ are adjacent, in which case, $\kappa(G) = |V(G)| - 1$.

A block of a graph $G$ is a maximal 2-connected subgraph, a parallel class that is not contained in a 2-connected subgraph, a loop, or an isolated vertex. The block-graph of a connected graph $G$ is a tree $T$ whose vertex set is the disjoint union of the blocks of $G$ and those vertices of $G$ that belong to more than one block. The only edges in $T$ are those that join vertices of $G$ to blocks that contain them. We call a block that is a leaf of a connected block-graph an end block.


**Theorem 1.5.1.** Let $G$ be a graph having at least $n + 1$ vertices. Then $G$ is $n$-connected if and only if all pairs of distinct vertices of $G$ are joined by at least $n$ internally disjoint paths.

This theorem implies that a graph $G$ is $k$-connected if and only if, for each pair $v$ and $w$ of distinct vertices, $G$ contains $k$ internally disjoint paths from $v$ to $w$.

### 1.6 Excluded Minors

Kuratowski [10] proved the following characterization of planar graphs in 1930.

**Theorem 1.6.1.** A graph $G$ is planar if and only if it has no subdivision isomorphic to $K_5$ or $K_{3,3}$.

A class of graphs is closed under the minor operation if every minor of a graph in the class is also in the class, and we say that this class is minor-closed. An excluded minor or
forbidden minor of a minor-closed class of graphs is a graph that is not in the class, but all of whose proper minors are in the class.

Instead of dealing with subdivisions, Wagner [22] generalized Theorem 1.6.1 and gave the following excluded-minor characterization of planar graphs in 1937.

**Theorem 1.6.2.** A graph $G$ is planar if and only if it does not have $K_5$ or $K_{3,3}$ as a minor.

In Theorem 1.6.2, $K_5$ and $K_{3,3}$ are excluded minors for the class of planar graphs. Neither of these graphs is in the class of planar graphs, but every proper minor of each graph is planar. Furthermore, $K_5$ and $K_{3,3}$ are the only graphs fitting this description. Much work has been done characterizing various classes of graphs by their excluded or forbidden minors.

The Graph Minors Project [14] of Neil Robertson and Paul Seymour is a set of results published in a series of 23 papers starting in 1983 relating graph minors to topological embeddings. This set of results proved the Graph Structure Theorem and is regarded as some of the most important work ever done in graph theory. In particular, Robertson and Seymour proved that every class of graphs that is closed under taking minors can be characterized by a finite number of excluded minors. The results and tools developed in this series of papers have since been successfully used to attack a large number of problems in graph theory.

### 1.7 Fundamental Matroid Definitions

Much of the motivation of this dissertation arises from matroid theory. Throughout this dissertation, we also use matroid theory to solve several graph-theoretic problems. This section contains an introduction to some basic matroid theory terminology and follows [13] closely.

A matroid $M$ is an ordered pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a collection of independent sets that are subsets of the finite ground set $E$ and satisfy the following three conditions:

(i) $\emptyset \in \mathcal{I}$. 

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(ii) If \( I \in \mathcal{I} \) and \( I' \subseteq I \), then \( I' \) is a member of \( \mathcal{I} \).

(iii) If \( I_1 \) and \( I_2 \) are in \( \mathcal{I} \) and \( |I_1| \leq |I_2| \), then there is an element \( e \) of \( I_2 - I_1 \) such that \( I_1 \cup e \) is a member of \( \mathcal{I} \).

If \( M \) is a matroid on \((E, \mathcal{I})\), then \( M \) is called a matroid on \( E \). A subset of \( E \) that is not in \( \mathcal{I} \) is called dependent. A minimal dependent set in an arbitrary matroid \( M \) is called a circuit of \( M \) and we denote the set of circuits of \( M \) by \( \mathcal{C} \) or \( \mathcal{C}(M) \). The maximal independent sets of \( M \) are called the bases of \( M \) and the sets of bases is denoted by \( \mathcal{B} \) or \( \mathcal{B}(M) \). The bases of \( M \) all have the same cardinality, and this cardinality is equal to the rank of \( M \), written \( r(M) \). The rank of a subset \( X \) of \( E(M) \), written \( r(X) \), is the cardinality of a largest independent set of \( M \) contained in \( X \). Clearly \( X \in \mathcal{I} \) if and only if \( r(X) = |X| \). The definition of matroid given above defines a matroid by its independent sets, but a matroid may also be defined in terms of its sets of bases, its set of circuits, or its rank.

The circuits of a matroid satisfy the following three conditions:

(C1) \( \emptyset \in \mathcal{C}(M) \).

(C2) If \( C_1 \) and \( C_2 \) are in \( \mathcal{C}(M) \) and \( C_1 \subseteq C_2 \), then \( C_1 = C_2 \).

(C3) If \( C_1 \) and \( C_2 \) are distinct members of \( \mathcal{C}(M) \) and \( e \in C_1 \cup C_2 \), then there is a member \( C_3 \) of \( \mathcal{C}(M) \) such that \( C_3 \subseteq (C_1 \cup C_2) - e \).

The closure or span, \( cl_M(X) \) or \( cl(X) \), of a subset \( X \) of \( E(M) \) is the maximal set \( X' \subseteq E(M) \) satisfying \( X \subseteq X' \) and \( r_M(X') = r_M(X) \). If \( X = cl(X) \), then \( X \) is called a flat or closed set of \( M \). A hyperplane of \( M \) is a flat of rank \( r(M) - 1 \). A subset \( X \) of \( E(M) \) is a spanning set if \( cl(X) = E(M) \). Equivalently, \( X \) is a spanning set if and only if \( r(X) = r(M) \). The closure of any basis is the entire matroid, and a set \( X \) is a basis if and only if it is a minimal spanning set. Also, \( X \) is a hyperplane if and only if it is a maximal nonspanning set.
If $M_1$ and $M_2$ are the matroids $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$, then $M_1$ is isomorphic to $M_2$ if there is a bijection $\phi: E_1 \rightarrow E_2$ such that a subset $X$ of $E_1$ is in $\mathcal{I}_1$ if and only if $\phi(X)$ is in $\mathcal{I}_2$.

### 1.8 Several Important Classes of Matroids

An important class of matroids are *representable matroids* or vector matroids, those matroids that can be represented by a finite collection of vectors from a vector space. Let $A$ be an $m \times n$ matrix over a field $F$. The *vector matroid* of $A$, denoted by $M[A]$, is a matroid $(E, \mathcal{I})$, where $E$ is the set of column labels of $M$, and $I$ is the set of subsets $X$ of $E$ for which the multiset of columns labelled by $X$ is a linearly independent set in the vector space $V(m, F)$, the $m$-dimensional vector space over $F$. It is easy to check that the pair $(E, \mathcal{I})$ satisfies (i), (ii), and (iii) and is therefore a matroid. A matroid is said to be *representable over* $GF(q)$, the $q$-element field, if it is isomorphic to the vector matroid of a matrix over $GF(q)$. The vector matroids that are representable over $GF(2)$ and $GF(3)$ are called *binary matroids* and *ternary matroids*, respectively. A matroid is *regular* if it can be represented over the real numbers as the vector matroid of a totally unimodular matrix, one for which all subdeterminants are $\{0, 1, -1\}$.

Another class of matroids is the class of *graphic matroids*, those matroids that can be realized by graphs as follows. Let $G$ be a graph. The *cycle matroid* of a graph $G$, written $M(G)$, arises by taking the ground set $E = E(G)$ and the set of circuits to be the set of edge-sets of the cycles in $G$. Any matroid that is isomorphic to the cycle matroid of a graph is called a graphic matroid. Notice that the independent sets of $M(G)$ are the edge sets of forests in $G$. It is not hard to show that every graphic matroid is regular and every regular matroid is binary. In this dissertation we deal exclusively with graphic and binary matroids.

We say that a matroid $M$ has a specific graph property if there is a graph with that property whose cycle matroid is $M$. For example we say that a matroid $M$ is planar if there is a planar graph with cycle matroid $M(G) \cong M$. 

10
1.9 Matroid Operations and Minors

The dual $M^*$ of a matroid $M$ is the matroid with ground set $E(M)$ whose set of bases is \{\(E(M) - B : B \in \mathcal{B}(M)\)\}. A basis of $M^*$ is a cobasis of $M$, and an independent set in $M^*$ is a coindependent set of $M$. The classes of matroids that are closed under minors and also closed under duality are usually easier to work with than those without one or both properties. In this dissertation, we will introduce and work with classes closed under both properties and consider the dual matroids.

Let $e$ be an element in the ground set $E$ of a matroid $M$. The deletion of a subset $T \subseteq E$, written $M \setminus T$, is the matroid with ground set $E - T$ having \{\(C \in \mathcal{C}(M) : C \cap T = \emptyset\)\} as its set of circuits. For a graph $G$, it is easy to see that $M(G) \setminus T = M(G' \setminus T)$ for any subset $T$ of $E(G)$. The contraction of a subset $T \in E$, written $M/T$, results in a matroid with ground set $E - T$ whose circuits are the minimal non-empty members of \{\(C - T : C \in \mathcal{C}(M)\)\}. The contraction of a subset $T$ of $E$ is also given by $M/T = (M^* \setminus T)^*$. We note that if $G$ is a graph and $T \subseteq E(G)$, then $M(G/T) = M(G)/T$.

The circuits of $M^*$ are the cocircuits of $M$. A three-element circuit is called a triangle and a three-element cocircuit is called a triad.

A matroid $N$ is a minor of a matroid $M$ if $N = M \setminus X/Y$ for some disjoint subsets $X$ and $Y$ of $E(M)$. Some classes of matroids have the property that all of their minors are also in the class, and we say that such classes are closed under minors or minor-closed. It is not hard to check that the class of graphic matroids is minor-closed.

The $r \times r$ identity matrix is denoted $I_r$. Let $A$ be a the matrix $[I_r|D]$ and let $D^T$ be the transpose of $D$. The dual of $M[I_r|D]$ is equal to $M[D^T|I_{|E(M)|-r}]$. It is not difficult to see that if $G$ is a plane graph, then $M^*(G) = M(G^*)$. A matroid that has a graphic dual is called cographic.

A matroid $M$ is a relaxation of a matroid $N$ if, for some circuit-hyperplane $H$ of $N$, the set of bases of $M$ is the set of bases of $N$ together with $H$. 
A loop in a matroid is an element of rank zero. In a graph, a loop edge corresponds to a loop element in the matroid. In a vector matroid, a loop element corresponds to the zero vector. Since a loop in $M$ is in no basis of $M$, it is in every basis of $M^*$, and it is a coloop of $M^*$. A pair of elements are parallel if they form a circuit. In a graph, these elements are in the same parallel class. In a vector matroid, parallel elements correspond to non-zero scalar multiples of the same non-zero vector. For a non-loop element $e$ in a matroid $M$, the parallel class of $e$ is the set $e$ together with every element parallel to $e$.

### 1.10 Matroid Connectivity

A matroid is connected if and only if for every pair of distinct elements in its ground set, there is a circuit containing both elements. If $M$ is not connected, then $M$ is disconnected.

Let $M$ be a matroid with ground set $E$ and rank function $r$. A partition $(X,Y)$ of the ground set $E$ is a $k$-separation if $\min\{|X|,|Y|\} \geq k$ and $r(X) + r(Y) - r(M) \leq k - 1$. If $M$ has a $k$-separation, then $M$ is called $k$-separated or $k$-separable. A matroid is 1-separated if and only if it is disconnected.

The notion of $k$-connectivity in matroids was introduced by Tutte in 1966 [20]. If $M$ is $k$-separated for some $k$, then the Tutte connectivity $\lambda(M)$ of $M$ is $\min\{j : M \text{ is } j \text{-separated}\}$; otherwise we take $\lambda(M)$ to be $\infty$. In general, when we discuss matroid connectivity, we are referring to the Tutte connectivity. If $n$ is an integer exceeding one, we say that $M$ is $k$-connected if $\lambda(M) \geq k$. It is not difficult to check that a matroid is $k$-connected if and only if its dual is $k$-connected, and we have that $\lambda(M) = \lambda(M^*)$. A $k$-connected graph $G$ has no vertices of degree less than $k$ and, more generally, such a graph has no bonds of size less than $k$.

Tutte’s definition of matroid connectivity and the standard definition of graph connectivity are not equivalent and this difference is shown in the following result, which follows from Menger’s Theorem 1.5.1. This shows us that an $n$-connected matroid cannot have small circuits, whereas an $n$-connected graph can.
Corollary 1.10.1. Let $G$ be a graph having no isolated vertices. If $V(G) \geq 3$, then $M(G)$ is 2-connected if and only if $G$ is 2-connected and loopless.

We define $W_r$, the rank-$r$ whirl, to be the matroid formed by starting with the matroid $M(W_r)$, the graphic matroid obtained from the $r$-wheel, and then relaxing the rim of the wheel. Thus, $E(W_r) = E(W_r)$, while the bases of $W_r$ consists of the rim together with all edge sets of spanning trees of $W_r$.

The next two theorems, Tutte’s Wheels and Whirls Theorem [20] and Bixby’s Lemma [1] are basic structural results for 3-connected matroids.

Theorem 1.10.2 (Tutte’s Wheels and Whirls Theorem). The following statements are equivalent for a 3-connected matroid $M$ having at least one element:

(i) For every element $e$ of $M$, neither $M\setminus e$ nor $M/e$ is 3-connected.

(ii) $M$ has rank at least three and is isomorphic to a wheel or a whirl.

The simplification of a matroid $M$, denoted $\text{si}(M)$, is the matroid obtained from $M$ by deleting all loops and all but one element from each parallel class. The cosimplification of a matroid $M$, denoted $\text{co}(M)$, is the matroid $(\text{si}(M^*))^*$. In a 3-connected matroid $M$, an element $e$ is vertically contractible if $\text{si}(M/e)$ is 3-connected.

Theorem 1.10.3. Let $M$ be a 3-connected matroid. For every element $e \in E(M)$, either $\text{co}(M\setminus e)$ or $\text{si}(M/e)$ is 3-connected.

In Tutte’s matroid connectivity, a matroid has connectivity $k$ if and only if its dual has connectivity $k$. However, matroid connectivity does not correspond to graph connectivity. Tutte’s definition of matroid connectivity can be modified to generalize the notion of connectivity in graphs. Below we give the definition of vertical $n$-connectedness, however one loses invariance under duality.
For a positive integer \( k \), we say that a matroid is \textit{vertically} \( k \)-separated if there is a partition \((X, Y)\) of \( E(M) \) such that \( \min\{r(X), r(Y)\} \geq k \) and \( r(X) + r(Y) - r(M) \leq k - 1 \).

It is easy to see that if \( M \) is vertically \( k \)-separated, then \( M \) is \( k \)-separated. The \textit{vertical connectivity} \( \kappa(M) \) of \( M \) is the least positive integer \( j \) such that \( M \) is vertically \( j \)-separated; otherwise we let \( \kappa(M) = r(M) \). In general, a matroid \( M \) is called \textit{vertically} \( n \)-\textit{connected} if \( n \) is an integer for which \( 2 \leq n \leq \kappa(M) \). Vertical \( n \)-connectedness in matroids is a direct generalization of the notion of \( n \)-connectedness in graphs. It is not difficult to show that \( \kappa(M(G)) = \kappa(G) \).

1.11 Graphic Matroid Isomorphism and Roundedness

A graph \( G \) is \textit{2-isomorphic} to the graph \( H \), written \( G \cong_2 H \), if \( H \) can be transformed into a graph isomorphic to \( G \) by a sequence of operations of types (a), (b), and (c), as follows:

(a) \textit{Vertex identification}. If \( v \) and \( v' \) are vertices in distinct components of \( G \), then modify \( G \) by identifying \( v \) and \( v' \) as a new vertex \( v'' \).

(b) \textit{Vertex cleaving}. This is the reverse operation of vertex identification. A graph can only be cleft at a cut-vertex or at a vertex incident with a loop.

(c) \textit{Twisting}. Let \( G \) be the graph obtained from disjoint graphs \( G_1 \) and \( G_2 \) by identifying the vertices \( u_1 \) of \( G_1 \) and \( u_2 \) of \( G_2 \) as the vertex \( u \) of \( G \), and identifying the vertices \( v_1 \) of \( G_1 \) and \( v_2 \) of \( G_2 \) as the vertex \( v \) of \( G \). In a \textit{twisting} \( G' \) of \( G \) about \( \{u, v\} \), we instead identify \( u_1 \) with \( v_2 \) and \( v_1 \) with \( u_2 \). We call \( G_1 \) and \( G_2 \) the \textit{pieces} of the twisting.

Whitney’s 2-Isomorphism Theorem [23], stated below, identifies when two graphs have isomorphic cycle matroids. Shorter proofs have also been given by both Truemper [18] and Wagner [21].
Theorem 1.11.1 (Whitney’s 2-Isomorphism Theorem). Let $G$ and $H$ be graphs having no isolated vertices. Then $M(G)$ and $M(H)$ are isomorphic if and only if $G$ and $H$ are 2-isomorphic.

The following theorem, proved by Edmonds (see Truemper [18]) and Greene [8], implies that a 3-connected graphic matroid uniquely determines a graph.

Theorem 1.11.2. Let $G$ and $H$ be 3-connected loopless graphs without isolated vertices. If $M(G) \cong M(H)$, then $G \cong H$.

A class $\mathcal{N}$ of matroids is $t$-rounded if every member of $\mathcal{N}$ is $(t + 1)$-connected and the following condition holds: If $M$ is a $(t + 1)$-connected matroid having an $\mathcal{N}$-minor and $X$ is a subset of $E(M)$ with at most $t$ elements, then $M$ has an $\mathcal{N}$-minor using $X$.

Seymour [15,17] gave the following characterization of $t$-rounded classes for the case when $t = 1$ or 2.

Theorem 1.11.3. Let $t$ be 1 or 2 and $\mathcal{N}$ be a collection of $(t + 1)$-connected matroids. Then $\mathcal{N}$ is $t$-rounded if and only if the following condition holds: If $M$ is a $(t + 1)$-connected matroid having an $\mathcal{N}$-minor $N$ such that $|E(M) - E(N)| = 1$, and $X$ is a subset of $E(M)$ with at most $t$ elements, then $M$ has an $\mathcal{N}$-minor using $X$. 
Chapter 2
Almost Series-Parallel Graphs

A *series-parallel* graph is formed recursively from a forest by the operations of adjoining a loop, subdividing an edge, or adding an edge in parallel to an existing non-loop edge. Equivalently, series-parallel graphs can be characterized as graphs having no $K_4$-minor [6].

We consider the class $S$ of graphs that are *almost series-parallel*, graphs such that there is some edge that one can add to the graph and then contract out to leave a series-parallel graph. Notice that the operation of adding an edge $e$ joining distinct vertices $u$ and $v$ and then contracting $e$ has the effect of identifying the vertices $u$ and $v$. In their description of the structure of the class of binary matroids, Geelen, Gerards, and Whittle [7] make essential use of an operation they call *perturbation* which consists of adding a set $S$ of elements to a graphic matroid to produce a new binary matroid and then contracting out $S$. It is clear that all series-parallel graphs are almost series-parallel since adding a loop edge to a series-parallel graph and contracting it leaves the original series-parallel graph.

**Lemma 2.0.4.** The class $S$ of almost series-parallel graphs is closed under taking minors.

**Proof.** As $G \in S$, there is an edge $e$ such that $(G+e)/e$ is series-parallel. Consider the graph $G+e$ and suppose $f \in E(G)$. Then clearly $(G+e)/e \backslash f = ((G \backslash f)+e)/e$. As $(G+e)/e$ is series-parallel, so is $(G+e)/e \backslash f$. Hence $G \backslash f \in S$. On the other hand, $(G+e)/e / f = ((G/f)+e)/e$ where we observe that if $e$ and $f$ have the same ends, then $e$ is added as a loop to $G/f$.

In all cases, $((G/f)+e)/e$ is series-parallel, so $G/f$ is series-parallel. We now show that for any vertex $v \in V(G)$, its deletion $G - v$ is almost series-parallel. If $v$ is not incident with $e$, then $((G+e)/e) - v = ((G-v)+e)/e$ is a subgraph of $(G+e)/e$, which is a series-parallel graph, and thus $G - v$ is almost series-parallel. If $v$ is incident with $e$, then $G - v$ is a
subgraph of \(((G + e)/e)\), which is a series-parallel graph, and any series-parallel graph is almost series-parallel. Hence, \( \mathcal{S} \) is minor-closed.

Since \( \mathcal{S} \) is a minor-closed class, Robertson and Seymour’s Graph Minors Theorem implies \( \mathcal{S} \) has a finite number of excluded minors. In this chapter, we find the full list of excluded minors for the graphs that are almost series-parallel.

### 2.1 Main Results

The next result, the main theorem of the chapter, gives the excluded minors for \( \mathcal{S} \). These excluded minors are shown in Figure 2.1.

![Excluded Minors for \( \mathcal{S} \)](FIGURE 2.1: Excluded Minors for \( \mathcal{S} \))
Theorem 2.1.1. The excluded minors for the class of almost series-parallel graphs are the following 11 graphs: $K_4 \oplus_0 K_4$, $K_4 \oplus_1 K_4$, $S(K_4, K_4)$, $K_5$, $K_{2,2,2}$, $R$, $U$, $H_8$, $Q_3$, $S$, and $V$.

Clearly every excluded minor for $S$ is a simple graph with no isolated vertices. To prove the theorem, we divide the argument into cases based on the vertex connectivity of an excluded minor $G$. When the vertex connectivity is not equal to three the excluded minors are fairly easy to determine. Most of the work arises when the vertex connectivity is three. In that case, the argument breaks into two main parts: either $\kappa(G \setminus e) = 3$ for some edge $e$, or $G$ is minimally 3-connected, that is, $\kappa(G \setminus e) = 2$ for every edge $e$.

2.2 Preliminaries

In this section, we introduce some more terminology and results that will be used throughout this dissertation. Much of what we introduce here has to do with the connectivity of a graph $G$ and various decompositions of a graph.

A graph-labelled tree [13] of a 2-connected loopless graph $G$ is a tree $T$ with vertex set \( \{G_1, G_2, \ldots, G_k\} \) for some positive integer $k$ such that

(i) each $G_i$ is a 3-connected simple graph, a cycle, or a set of parallel edges;

(ii) if $G_{j_1}$ and $G_{j_2}$ are joined by an edge $e_i$ of $T$, then $E(G_{j_1}) \cap E(G_{j_2}) = \{e_i\}$ and $\{e_i\}$ is not a bridge of $G_{j_1}$ or $G_{j_2}$; and

(iii) if $G_{j_1}$ and $G_{j_2}$ are non-adjacent, then $E(G_{j_1}) \cap E(G_{j_2})$ is empty.

Let $e$ be an edge of a graph-labelled tree $T$ and suppose that $e$ joins vertices labelled by $N_1$ and $N_2$. Suppose that we contract the edge $e$ of the tree $T$ and relabel by $N_1 \oplus_2 N_2$ the vertex that results by identifying the endpoints of $e$, leaving all other edges and vertex labels unchanged. Then it is not difficult to see that we retain a graph-labelled tree, and it is natural to denote this tree by $T/e$. This process can be repeated, and since the operation
of 2-sum is associative, for every subset \{e_1, e_2, \ldots, e_m\} of \(E(T)\), the graph-labelled tree \(T/e_1, e_2, \ldots, e_m\) is well-defined.

A 2-sum decomposition \([19]\) is a graph-labelled tree \(T\) such that if \(V(T) = \{G_1, G_2, \ldots, G_k\}\) and \(E(T) = \{e_1, e_2, \ldots, e_{k-1}\}\), then

(i) \(E(G) = (E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)) - \{e_1, e_2, \ldots e_{k-1}\}\);

(ii) \(|E(G_i)| \geq 3\) for all \(i\) unless \(|E(G)| < 3\), in which case \(k = 1\) and \(G_1 = G\); and

(iii) \(G\) is the graph that labels the single vertex of \(T/e_1, e_2, \ldots, e_{k-1}\).

The following result of Cunningham and Edmonds \([4]\) and Seymour \([16]\) describes how every 2-connected graph can be written in terms of 2-sums of 3-connected graphs.

**Proposition 2.2.1.** Let \(G\) be a 2-connected graph. Then \(G\) has a 2-sum decomposition \(T\) in which every vertex label is 3-connected, a cycle, or a parallel class.

The next result of Tutte \([19]\) gives a unique 2-sum decomposition called the canonical 2-sum decomposition.

**Proposition 2.2.2.** Let \(G\) be a 2-connected loopless graph. Then \(G\) has a 2-sum decomposition \(T\) in which every vertex label is 3-connected, a cycle, or a bond, and there are no two adjacent vertices that are both labelled by cycles or are both labelled by bonds. Moreover, \(T\) is unique within relabeling of its edges.

The following result of Tutte \([20]\) plays a vital role in this dissertation.

**Lemma 2.2.3** (Tutte’s Triangle Lemma). Let \(M\) be a 3-connected matroid with at least four elements, and suppose that \(\{e, f, g\}\) is a triad of \(M\) such that neither \(M/e\) nor \(M/f\) is 3-connected. Then \(M\) has a triangle that contains \(e\) and exactly one of \(f\) and \(g\).
2.3 Excluded Minors for the Class $S$ with $\kappa(G) \neq 3$

Finding the excluded minors for vertex connectivity not three is straightforward. We consider separately when the vertex connectivity is zero or one, when it is two, and when it is at least four.

**Proposition 2.3.1.** Let $G$ be a simple graph with $\kappa(G) \in \{0, 1\}$. Then $G$ is an excluded minor for the class $S$ if and only if $G \cong K_4 \oplus_0 K_4$ or $K_4 \oplus_1 K_4$.

![Figure 2.2: $K_4 \oplus_0 K_4$ and $K_4 \oplus_1 K_4$](image)

**Proof.** It is easy to check that both $K_4 \oplus_0 K_4$ and $K_4 \oplus_1 K_4$ are excluded minors for $S$. Conversely, let $G$ be an excluded minor for $S$. Clearly $G \cong G_1 \oplus_{\kappa(G)} G_2$ for some graphs $G_1$ and $G_2$ each having at least one edge. If both $G_1$ and $G_2$ have a $K_4$-minor, then it is not difficult to check that $G$ has $K_4 \oplus_{\kappa(G)} K_4$ as a minor. Hence we may assume that $G_1$ has no $K_4$-minor, but $G_2$ does have such a minor. Choose an edge $f$ in $G_1$. Then $G \setminus f \in S$ so there is an edge $e$ that can be added to and contracted from $G \setminus f$ to leave a series-parallel graph. To destroy the $K_4$-minor in $G_2$, the edge $e$ must join two vertices of $G_2$. Then $(G + e)/e$ is series-parallel, so $G \in S$; a contradiction. □

**Proposition 2.3.2.** Let $G$ be a simple graph with $\kappa(G) = 2$. Then $G$ is an excluded minor for the class $S$ if and only if $G \cong S(K_4, K_4)$.

**Proof.** It is straightforward to check that $S(K_4, K_4)$ is an excluded minor for $S$. Now let $G$ be an excluded minor for $S$. As $\kappa(G) = 2$, we can write $G = G_1 \oplus_2 G_2$ where the edge $e$ is the basepoint of the 2-sum, $e$ is not a loop, $a$ and $b$ are the vertex ends of $e$, and $G_1$ and
$G_2$ are 2-connected. Since $G$ is an excluded minor, $((G_1 \oplus_2 G_2) + e)/e$ is not series-parallel so either $G_1/e$ or $G_2/e$ has $K_4$ as a minor. Suppose $G_1/e$ has $K_4$ as a minor. We show next that $K_4$ is also a minor of $G_2$. Suppose not. Then $G_2$ is series-parallel and has a vertex $w$ of degree two in $G$. To see this, observe that the only possible non-trivial parallel class of $G_2$ involves $e$. As $G$ is simple and $G_2$ is series-parallel, $G_2$ is forced to have a degree-2 vertex not incident with $e$. Let $g$ be an edge incident with $w$ and $G' = G/g$. Then $G'$ is in $S$ and so has an edge $f$ so that $(G' + f)/f$ has no $K_4$-minor. The construction of $G'$ means that $(G + f)/f$ has no $K_4$-minor; a contradiction. Thus $G_2$ does indeed have a $K_4$-minor.

We now show, since $G_1/e$ and $G_2$ both have $K_4$ as a minor, that $G$ contains and is isomorphic to $S(K_4, K_4)$. We first show that there is a $K_4$-minor in $G_2$ using the edge $e$. Consider the graphic matroid $M(G_2)$ associated with $G_2$. By Theorem 1.11.3, since $M(G)$ is 2-connected, $M(G)$ has a proper $K_4$-minor using $e$, otherwise $G = K_4$. Therefore, the graph $G$ has a $K_4$-minor using the edge $e$.

Since $G_1/e$ has a $K_4$-minor, $(G_1/e)/X \setminus Y = K_4$ for some subsets $X$ and $Y$ of $E(G_1) - \{e\}$. By uncontracting the edge $e$ from the $K_4$-minor, the graph $G_1/X \setminus Y$ has $e$ as either a subdivided edge of $K_4$ or a pendant edge, that is, an edge adjacent to a vertex of degree one of $K_4$. In the first case, if $e$ is a subdivided edge, then $G$ contains $S(K_4, K_4)$, as desired.

We will show in what follows that $e$ cannot be a pendant edge. Suppose that $e$ is a pendant edge of $G_1/X \setminus Y$. Since every vertex of $G_1$ becomes a vertex of $(G_1/e)/X \setminus Y$, we label the vertices of $G_1/X \setminus Y$ with four labels: 1, 2, 3, and 4, where each label corresponds to a distinct vertex of $(G_1/e)/X \setminus Y = K_4$ to which each vertex of $G_1$ is identified in the contraction of the set $X$. Let the graph $G$ also have the same corresponding labels for the
vertices of $V(G_1)$ in $V(G)$. Since $a$ and $b$ are identified as a single composite vertex in $(G_1/e)/X\setminus Y$, they have the same label, say 1. Since $e$ is a pendant edge of $G_1/X\setminus Y$, one of $a$ or $b$, say $a$, is not adjacent to any vertices labelled by 2, 3, or 4. However, since the original graph is 2-connected, $a$ is adjacent to at least two vertices, both of which are labelled by 1. So there is at least one edge $f$, not the same as $e$, incident with $a$ and also to a vertex labelled by 1. Let $X'$ and $Y'$ be subsets of $E(G_1)$ such that $X' = X \setminus \{e\}$ and $Y' = Y \setminus \{f\}$.

Then the graph $G_1/X'\setminus Y'$ is $K_4 \oplus_1 C_2$ where $C_2$ is a 2-cycle and $E(C_2) = \{e, f'\}$ for some edge $f'$ that runs between $a$ and $b$ in $G_1/X'\setminus Y'$. Now, $G_1/X'\setminus Y'$ has a $K_4$-minor. Also, let $H$ be the graph $G/X'\setminus Y'$. Then $G_2$ remains unaltered in $H$ and $G_2 \setminus e + f'$ has a $K_4$-minor, which gives $K_4 \oplus_1 K_4$ as a proper minor, a contradiction.

Therefore $G$ is an excluded minor for $\mathcal{S}$ if and only if $G \cong S(K_4, K_4)$. \hfill \Box

To find the excluded minors of connectivity at least four, we use the following result of Halin and Jung [9].

**Lemma 2.3.3.** If $G$ is a simple graph with minimum degree 4, then $G$ contains $K_5$ or $K_{2,2,2}$ as a minor.

![Figure 2.4: $K_5$ and $K_{2,2,2}$](image)

It is straightforward to check that both $K_5$ and $K_{2,2,2}$ are excluded minors for the class $\mathcal{S}$. Combining this with the last lemma, we immediately obtain the following corollary from which it follows that $\mathcal{S}$ contains no 4-connected graphs.
Corollary 2.3.4. Let $G$ be a simple graph with $\kappa(G) \geq 4$. Then $G$ is an excluded minor for $S$ if and only if $G \cong K_5$ or $K_{2,2,2}$.

2.4 Members of the Class $S$ with $\kappa(G) = 2$

In this section, we describe the structure of almost series-parallel graphs with vertex connectivity two.

Proposition 2.4.1. Let $G$ be a simple graph that is a member of $S$ and suppose $\kappa(G) = 2$. Then either $G$ is series-parallel, or $G$ can be constructed as follows.

(i) Take the parallel connection with basepoint $p$ of graphs $G_1, G_2, G_3, \ldots, G_n$ where, for each $i \in [1,n]$, the graph $G_i$ is 3-connected and simple, and if $i \geq 2$, then $G_i/p$ is series-parallel.

(ii) Possibly delete the edge $p$.

(iii) At each edge of the resulting graph, attach via 2-sum a 2-connected series-parallel graph or a parallel class.

Moreover, every graph constructed using (i)-(iii) is an almost series-parallel graph with connectivity two.

Proof. As $\kappa(G) = 2$, consider the Cunningham and Edmond’s canonical 2-sum decomposition $T$ of $G$ letting $G_1, G_2, \ldots, G_n$ label the vertices of $T$ where by Proposition 2.2.1 each $G_i$ is a 3-connected graph, a cycle, or a parallel class.

If some $G_i$ is 3-connected, then it has a minor isomorphic to $K_4$ by Tutte’s Wheels Theorem 1.10.2. Moreover, the edge $e$ that one can add to the graph $G$ and contract out to leave a series-parallel graph must be added so that $G_i/e$ has no $K_4$-minor. Suppose two 3-connected members $G_m$ and $G_n$ of the 2-sum decomposition have basepoints $e_m$ and $e_n$. Since $G \in S$ there is one edge that one can add to the graph and then contract out to leave a graph with no $K_4$-minor. Since $(G + e)/e$ has no $K_4$-minor, and $G_m$ has a $K_4$-minor, the
edge $e$ meets two vertices of $G_m$. Similarly $e$ meets two vertices of $G_n$. However, the only edge meeting both two vertices of $G_m$ and two vertices of $G_n$ are the basepoints $e_m$ and $e_n$, respectively. This implies that every 3-connected $G_i$ has the same basepoint, and the edge $e$ that one must add to the graph $G_i$ and contract out to destroy the $K_4$ in each 3-connected $G_i$ is in a 2-cycle with $e$. If there is more than one 3-connected $G_i$, then because $G \in \mathcal{S}$, for every 3-connected $G_j$, the contraction of the basepoint is series-parallel.

Every non-3-connected $G_i$ is a cycle or a parallel class. Notice that a cycle is series-parallel; the 2-sum of two cycles is series-parallel; the 2-sum of a series-parallel graph and a parallel class is series-parallel; and finally, that the 2-sum of two series-parallel graphs is series-parallel. Also, when a series-parallel graph is 2-summed, it creates no new $K_4$-minors and the same edge $e$ that one can add to $G_i$ to destroy each $K_4$ still leaves a graph with no $K_4$-minor. So if $G$ is 2-connected, then it can be formed by the process stated above.

Now we show that every graph constructed by the process stated in (i)-(iii) is almost series-parallel. The graph formed by taking the parallel connection in (i) can be seen to be almost series-parallel by adding in and contracting an edge $f$ parallel to the basepoint $p$. In $G \setminus e$, the addition and contraction of the edge $f$ still leaves a series-parallel graph, so applying (i) and (ii) produces a graph in $\mathcal{S}$. In the contraction $G/f$, attaching a series-parallel graph to a series-parallel graph via 2-sum is again series-parallel, so we are able to attach a series-parallel graph to any edge of $G$ except $e$ via 2-sum. If a series-parallel graph $H$ is attached via 2-sum to the edge $e$, then $G/f$ is the 1-sum of a series-parallel graph, a loop on $e$, and $H/e$, a minor of a series-parallel graph. Thus applying (i)-(iii) produces a graph in $\mathcal{S}$.

\[\square\]

2.5 Members of the Class $\mathcal{S}$ with $\kappa(G) = 3$

In this section, we use the following elementary result. In this result, we begin with three distinct vertices of a graph $G$ labelled $a$, $b$, and $c$, and we consider a minor of $G$. When an
edge $e$ incident with exactly one member $d$ of $\{a, b, c\}$ is contracted, we label the vertex that results by identifying the ends of $e$ by $d$.

**Lemma 2.5.1.** If $G$ is a simple 2-connected graph and $a, b, c \in V(G)$, then $G$ has a minor that is a cycle through $a, b, c$.

**Proof.** Consider distinct edges $e_a$ and $e_b$ incident with $a$ and $b$, respectively. Since $G$ is 2-connected, there is a cycle $C$ containing both $e_a$ and $e_b$. Suppose the vertex $c$ is not in this cycle. By Menger’s Theorem, there are two paths, $P_1$ and $P_2$, from $c$ to $V(C)$ that have only the vertex $c$ in common such that each contains only a single vertex of $V(C)$. If $P_1$ and $P_2$ meet $V(C)$ at both $a$ and $b$, then $G$ clearly contains a cycle through $a, b, c$. Now we may assume that some $P_i$ meets $V(C)$ in a vertex other than $a$ or $b$. If we contract this path, one of the vertices of $C$ is relabeled $c$ and hence, in this minor, $C$ is a cycle through $a, b, c$. \qed

The following lemma is the core of the theorem that we apply when finding the excluded minors with vertex connectivity three.

**Lemma 2.5.2.** Let $G$ be a simple graph with $\kappa(G) = 3$ such that $G$ is a member of $\mathcal{S}$. Then $G$ has two vertices $u$ and $v$ such that $G - \{u, v\}$ is a tree. Moreover, $u$ and $v$ are adjacent to all leaves of the tree.

**Proof.** Suppose first that $|V(G)| = 4$. Since $\kappa(G) = 3$, we must have $G \cong K_4$. In that case, the lemma clearly holds. Thus, we may assume that $V(G) \geq 5$. If $(G + e)/e$ is 3-connected for all possible edges $e$, and $|V((G + e)/e)|$ is at least four, then each such graph $(G + e)/e$ has a $K_4$-minor; so $G \notin \mathcal{S}$. Assume $(G + e)/e$ is not 3-connected for some new edge $e$. We know that $e$ joins two vertices $u$ and $v$ of a 3-vertex cut $\{u, v, w\}$ of $G$. Let $t$ label the vertex of $(G + e)/e$ that results by identifying $u$ and $v$. Now, $(G + e)/e$ is 2-connected but not 3-connected and we consider the block-graph $T$ of $((G + e)/e) - \{t\}$, that is, of $G - \{u, v\}$. Since $G - \{u, v\}$ has $w$ as a cut vertex, this block graph has at least two leaves.
No block of $G - \{u, v\}$ is 3-connected or $G - \{u, v\}$ would contain $K_4$ as a minor. We now show that each of $u$ and $v$ is adjacent to some vertex in each end block of $G - \{u, v\}$. If an end block is simply an edge, then one end of that edge has degree one in $G - \{u, v\}$, so that end is adjacent to both $u$ and $v$ in $G$ since all vertices must have degree at least three in $G$. Suppose an end block $B$ is 2-connected and is not adjacent to some vertex $y$ of $\{u, v\}$. Let $x$ be the vertex in $\{u, v\} - y$. Let $z$ be the vertex of $T$ connected to $B$. Then $\{x, z\}$ is a vertex cut of $G$, which cannot be since $G$ is 3-connected. We conclude that each 2-connected end block of $G - \{u, v\}$ is adjacent to both $u$ and $v$.

Let $B_i$ be a block of the graph of $G - \{u, v\}$. Suppose $B_i$ is 2-connected. Then $B_i$ has at least three vertices. Suppose $a$ and $b$ are distinct cutvertices of $G - \{u, v\}$ belonging to $B_i$. There are paths $P_a$ and $P_b$ that begin at $a$ and $b$, that end in $\{u, v\}$, that meet $V(B_i)$ in $\{a\}$ and $\{b\}$, that meet $\{u, v\}$ in a single vertex, and that meet each other in a subset of $\{u, v\}$. Since $\{a, b\}$ is not a vertex cut of $G$, there is a vertex $c$ of $V(B_i) - \{a, b\}$ such that there is an internally disjoint path from $c$ to $\{u, v\}$ which is internally disjoint from $P_a$ and $P_b$. Call this path $P_c$. Note that the path may have length one. Now, $B_i$ has a minor which is a cycle through $a$, $b$, and $c$. Thus, we have a $K_4$-minor using this cycle and the paths $P_a$, $P_b$, and $P_c$ in $(G + e)/e$. So $G \notin S$; a contradiction.

Suppose $B_i$ contains a single cut-vertex, say $a$, of $G - \{u, v\}$. Then $G$ has a path $P_a$ that begins at $a$, has no other vertices in common with $V(B_i)$, and ends at $u$ or $v$. As $G$ is 3-connected, there are distinct vertices $b$ and $c$ of $V(B_i) - \{a\}$, each of which is adjacent to $u$ or $v$. Let these edges be the paths $P_b$ and $P_c$, respectively. Then again we can use the cycle through $a$, $b$, and $c$ in a minor of $B_i$ and the paths $P_a$, $P_b$, and $P_c$ to get a $K_4$-minor in $(G + e)/e$, so $G \notin S$; a contradiction.

We conclude from the last two paragraphs that each block $B_i$ of $G - \{u, v\}$ consists of a single edge and hence $G - \{u, v\}$ is, in fact, a tree.
Theorem 2.5.3. Let $G$ be a simple graph with $\kappa(G) = 3$. Then $G$ has two vertices $u$ and $v$ such that $G - \{u, v\}$ is a tree if and only if $G \in \mathcal{S}$.

Proof. By the last lemma, if $G \in \mathcal{S}$, it has two such vertices $u$ and $v$. Conversely, assume $G$ has two such vertices $u$ and $v$ and consider the graph $(G + f)/f$ where $f$ is an edge joining $u$ and $v$. Suppose this graph has a $K_4$-minor. Removing the vertex that results from identifying $u$ and $v$ gives a graph isomorphic to $G - \{u, v\}$, which has no cycles. Since $K_4$ has no vertex whose removal has no cycles, we deduce that $(G + f)/f$ has no $K_4$-minor. Hence $G \in \mathcal{S}$. \hfill \square

2.6 Excluded Minors for the Class $\mathcal{S}$ with $\kappa(G) = 3$

To find the excluded minors $G$ for the class $\mathcal{S}$ with $\kappa(G) = 3$ such that there is some edge $e \in E(G)$ where $G \setminus e$ is 3-connected, we use the Theorem 2.5.3. The argument breaks into many cases but each case is straightforward.

Theorem 2.6.1. Let $G$ be a simple graph with $\kappa(G) = 3$ that is an excluded minor for $\mathcal{S}$ such that $G \setminus e$ is 3-connected for some edge $e \in E(G)$. Then $G$ is one of the graphs shown in Figure 2.5.

![Figure 2.5: R and U](#)

Proof. Since $G$ is an excluded minor, $G \setminus e$ is a member of $\mathcal{S}$. By Lemma 2.5.2, $G \setminus e$ has two vertices $u$ and $v$ such that $(G \setminus e) - \{u, v\}$ is a tree $T$. We distinguish the cases when $T$ is a path, when $T$ has exactly three degree-one vertices, and when $T$ has at least four
degree-one vertices. In each case, we find that $G$ is a member of $S$, or $G$ contains and is therefore isomorphic to one of $R, U, K_5, K_{2,2,2}$, or $S(K_4, K_4)$. But $G$ cannot be any of these last three graphs as $\kappa(G) = 3$.

Observe that $e$ joins two vertices of $T$. Otherwise $G - \{u, v\} = T$ and so $G \in S$; a contradiction. Also $T + e$ has a cycle $C$ as a subgraph; otherwise $G - \{u, v\}$ is cycle-free, a contradiction.

**Lemma 2.6.2.** If $T$ is a path, then $G \cong K_{2,2,2}$, $R$, or $U$.

**Proof.** Since $T$ is a path, exactly two vertices of $T + e$ have degree three. Each such degree-3 vertex meets a unique maximal path in $T + e$ that contains no edge of $C$. We call this path a tail of $C$.

**2.6.3.** If $T$ has two tails, then $G \cong R$.

Let the vertices of the path $T$ be in order $l_j, l_{j-1}, \ldots, l_2, l_1, r, s_1, s_2, \ldots, s_n, t, m_1, m_2, \ldots, m_k$ where $e = rt$ and the vertices of the $C$ are $r, s_1, s_2, \ldots, s_n, t$. Since $G \setminus e$ is 3-connected, both $u$ and $v$ are adjacent to $l_j$ and $m_k$, and we may assume that $u$ is adjacent to some $s_i$ where $1 \leq i \leq n$. Partition the vertices of $T$ into three sets, the set $L = \{l_{j-1}, l_{j-2}, \ldots, l_2, l_1, r\}$, the set $M = \{m_1, m_2, \ldots, m_{k-1}, t\}$, and the set $S = \{s_1, \ldots, s_n\}$.

If $v$ is adjacent to both a vertex in the set $L$ and a vertex in the set $M$, then $G$ contains and is isomorphic to the excluded minor $R$. Now, suppose $v$ is adjacent to no vertex in $L \cup M$. Since $G \setminus e$ is 3-connected, $u$ is adjacent to $r$ and $t$. Notice that $v$ is adjacent to no $s_i$ for $i = 1, \ldots, n$; otherwise $G$ contains $R$ as a proper minor. However, if $v$ is adjacent to no $s_i$ for $i = 1, \ldots, n$, then $G - \{u, r\}$ has no cycles, a contradiction by Theorem 2.5.3. We may now assume that $v$ is adjacent to exactly one of $L$ and $M$, say $L$. This implies that $u$ is adjacent to every vertex in $M$. If there is an edge from $v$ to a vertex of $S$, then $G$ contains the excluded minor $K_5$ as can be seen by contracting $vm_k$ and contracting $r, l_1, l_2, \ldots, l_k$ to a single vertex. Hence, we may assume $vs_i$ is not an edge of $G$ for any $i = 1, \ldots, n$. If there
are two or more edges from \( v \) to \( L \), then \( G \) contains the excluded minor \( S(K_4, K_4) \). So \( v \) is adjacent only to \( x_m, z_n \), to exactly one vertex of \( L \), and possibly \( u \). If \( v \) is adjacent to only \( r \), then \( G - \{r, u\} \) is a tree, a contradiction. So \( v \) is not adjacent to \( r \). Also, \( u \) is adjacent to \( r \), and \( v \) is adjacent to some vertex in \( \{l_1, l_2, \ldots, l_{j-1}\} \), which gives a proper \( S(K_4, K_4) \)-minor, a contradiction. Hence, 2.6.3 holds.

We now distinguish three main cases based on the length of the unique cycle \( C \) in \( T + e \) containing \( e \): either \( C \) is a 3-cycle, \( C \) is a 4-cycle, or \( C \) is a cycle of length at least 5.

Suppose first that \( C \) is a 3-cycle. If \( C \) has no tails, then \( G \) has exactly five vertices and, since \( G \) is an excluded minor for \( S \), it follows that \( G \) is isomorphic to \( K_5 \). Suppose \( C \) has exactly one tail. Let the vertices of the path \( T \) be, in order, \( r, s, t, l_1, l_2, \ldots, l_k \), where \( e = rt \). Since \( G \not\in S \), Theorem 2.5.3 implies that the graph \( G - \{r, l_1\} \) contains a cycle. Also, since \( G \) is 3-connected, \( u \) and \( v \) are adjacent to both \( r \) and \( l_k \). Assume the tail has length one. As \( G - \{u, t\} \) contains a cycle as a minor, \( vs \) is an edge of \( G \). By symmetry, \( us \) is an edge of \( G \). Since \( G \setminus e \) is 3-connected, \( t \) has degree at least three in \( G \setminus e \), so we may assume \( ut \) is an edge of \( G \). Then \( G/ vl_1 \cong K_5 \); a contradiction. Assume the tail has length at least two. Let the vertices of \( C \) be \( r, s \) and \( t \) where \( e = rt \) and the vertices of \( l_1, l_2, \ldots, l_k \) are the vertices of the tail. Without loss of generality, we may assume that \( u \) is adjacent to \( s \). Since \( G \setminus e \) is 3-connected, \( u \) or \( v \) is adjacent to \( t \). Also, the vertices \( v \) and \( s \) are not adjacent, otherwise \( G \) has a \( K_5 \)-minor. If none of \( l_1, l_2, \ldots, l_{k-1} \) are adjacent to \( v \), then \( G - \{u, s\} \) has no cycles; a contradiction. We deduce that some \( l_i \) with \( 1 \leq i \leq k - 1 \) is adjacent to \( v \). Hence, \( G \) has as a minor one of the graphs shown in Figure 2.6 after some relabeling the \( l_i \).

The graph in (a) has \( U \) as a minor. The graph in (b) has \( S(K_4, K_4) \) as a minor, as can be seen by deleting the edge joining \( v \) and \( l_1 \).

Now, suppose \( C \) is a 4-cycle. If \( C \) has no tails, notice that \( u \) and \( v \) are not adjacent, otherwise \( G/uv \) contains a \( K_5 \)-minor. Then \( G \) is isomorphic to a subgraph of \( K_{2,2,2} \). Hence, \( G \) is isomorphic to \( K_{2,2,2} \). If \( C \) is a 4-cycle with one tail, then let the vertices of the path \( T \)
be in order \(l_1, l_2, l_3, l_4, \ldots, l_k\), where \(e = l_1l_4\). Since \(G \setminus e\) is 3-connected, \(u\) and \(v\) are adjacent to both \(l_1\) and \(l_k\) and we may assume \(u\) is adjacent to \(l_4\). Also, since \(G\) is 3-connected, \(l_2\) and \(l_3\) have degree at least three in \(G\) so each is adjacent to either \(u\) or \(v\). This implies that \(G\) contains one of the graphs shown in Figure 2.8 as a spanning subgraph, where a bold edge represents a path.

FIGURE 2.7: \(e\) lies in a 4-cycle with one tail. A bold edge represents a path.
In case (a), the vertex $u$ is adjacent to some other vertex; otherwise $G - \{v, l_4\}$ is a tree and $G \in \mathcal{S}$, a contradiction. Now $u$ is adjacent to $l_2, l_3$ or an internal vertex of the path from $l_4$ to $l_5$. If $u$ is adjacent to $l_2$ or $l_3$, then $G$ contains a proper $K_5$-minor. If $u$ is adjacent to a vertex on the path from $l_4$ to $l_5$, then $G$ contains an $S(K_4, K_4)$-minor. We deduce that (a) does not arise.

In cases (b) and (c), we see by contracting the edges from $l_2$ to $l_3$ and from $v$ to $l_5$ that the graph $G$ contains a $K_5$-minor; a contradiction.

Consider case (d). Since the deletion of $u$ and $l_1$ from this graph leaves a tree, $v$ is adjacent to a vertex on the path from $l_2$ to $l_4$, or an internal vertex of the path from $l_4$ to $l_k$. If $v$ is adjacent to $l_2$ or $l_3$, then $G$ has a $K_5$-minor as in (b) or (c). If $v$ is adjacent to a vertex other than $l_5$ on the path from $l_4$ to $l_k$, then $G$ contains the excluded minor $U$ as in Figure 2.6 (a) when $C$ is a 3-cycle with one tail. Therefore, the only vertex $v$ can only be adjacent to $u$, $l_1$, $l_k$, and $l_4$ and $G - \{u, l_4\}$ is a tree. Hence $G \in \mathcal{S}$; a contradiction.

Next suppose $C$ is a cycle of length at least 5. Label the vertices of $C$ in order by $l_1, l_2, \ldots, l_k$, where $e = l_1l_j$. There is at most one tail. If there is a tail, label the vertices of the tail in order by $t_1, t_2, \ldots, t_n$, where $l_k$ is adjacent to $t_1$. By our labeling, the path $T$ has the vertices in order $l_1, l_2, \ldots, l_k, t_1, t_2, \ldots, t_n$ if there is a tail, and $l_1, l_2, \ldots, l_k$ if there is no tail. Since $G \setminus e$ is 3-connected, $u$ and $v$ are adjacent to the ends of the path $T$. We first eliminate the case when every vertex in $l_2, \ldots, l_{k-1}$ is adjacent to only $u$. If this occurs, then because $G - \{u, l_k\}$ has a cycle, there is a tail and $v$ is adjacent to a vertex $t_m$ of the tail for some $m = 1, \ldots, n$. Now, $G$ contains as a proper minor $U$, which can be seen by contracting the cycle to have length three and contracting the tail to have length two. Hence, every vertex in $l_2, \ldots, l_{k-1}$ is not adjacent only to $u$, and by symmetry is not adjacent only to $v$. Combining this with the fact that every vertex of $\{l_2, \ldots, l_k\}$ has degree at least three since $G$ is 3-connected, we have that by symmetry at least two vertices of $\{l_1, \ldots, l_k\}$ are adjacent to $u$, and at least one is adjacent to $v$. Contract the path from $l_k$ to $t_n$, so the resulting
The graph has $u$ and $v$ adjacent to $l_k$. Hence, up to symmetry, we have one of the following graphs shown in Figure 2.8 as a subgraph, where a bold edge represents a path.

![Graph Diagram](image)

**FIGURE 2.8**: $e$ lies in a cycle of length at least 5. A bold edge represents a path.

The graph in (a) has $U$ as a proper minor, which can be seen by deleting the edge $ul_k$, and contracting each bold edge down to a path of length one. The graph in (b) is isomorphic to the excluded minor $R$.

\[ \square \]

**Lemma 2.6.4.** Suppose $T$ has exactly three degree-one vertices. Then $G$ is not an excluded minor for $S$.

**Proof.** Since $T$ has exactly three degree-one vertices, $T$ has exactly one vertex of degree three, call it $r$. Let $l_1, l_2,$ and $l_3$ be the degree-one vertices of $T$. By adding the edge $e$ back into the graph $T \cong (G\setminus e) - \{u, v\}$, we see that $G - \{u, v\}$ contains one of the graphs shown in Figure 2.9 as a spanning subgraph.

Let $s$ and $t$ be the vertices as shown in Figure 2.9. Since $G\setminus e$ is 3-connected, $u$ and $v$ are adjacent to $l_1, l_2,$ and $l_3$.

Suppose (a) occurs. One of $u$ and $v$, say $u$, is adjacent to $s$. If there is an edge from $v$ incident with an internal vertex on the path from $r$ to $l_1$, then, by contracting the edge from $v$ to $l_2$, we see that $G$ has an $S(K_4, K_4)$-minor. If there is an edge from $v$ to an internal vertex of the path from $r$ to $l_3$, then by contracting the path from $l_1$ to $r$ and the edge from 
FIGURE 2.9: $T$ has exactly three degree-one vertices. A bold edge represents a path.

If $v$ is adjacent to $r$, $t$, or $l_3$, then $G$ has an $S(K_4, K_4)$-minor as in (a). Therefore, if $v$ is adjacent to only the vertices $l_1, l_2, l_3$, and possibly the vertex $u$, hence, $G - \{u, r\}$ is a tree and, by Theorem 2.5.3, $G \in S$; a contradiction. Hence, (a) does not occur.

Next, suppose that (b) occurs. Without loss of generality, $u$ is adjacent to $s$. If there is an edge from $v$ to an internal vertex of the path from $t$ to $l_3$, then, $G$ has an $S(K_4, K_4)$-minor. This can be seen by deleting all edges of the path in $T$ from $r$ to $l_2$ and contracting the edges from $l_1$ to $u$ and from $l_2$ to $v$. If there is an edge from $v$ to an internal vertex on the path from $r$ to $l_1$, from $v$ to an internal vertex on the path from $r$ to $l_2$, or from $v$ to an internal vertex on the path from $r$ to $l_t$, then $G$ has an $S(K_4, K_4)$-minor or $K_5$-minor as in (a). So $v$ is adjacent to only the vertices $l_1, l_2, l_3$, and possibly the vertex $u$. Hence, $G - \{u, r\}$ is a tree and, by Theorem 2.5.3, $G \in S$; a contradiction. Thus, (b) does not occur.
Next assume that (c) occurs. Without loss of generality, $u$ is adjacent to $s$. Since $G\setminus e$ is 3-connected, at least one of $u$ or $v$ is adjacent to $t$. If there is an edge from $v$ to $t$, then $G$ has an $S(K_4, K_4)$-minor, which can be seen by deleting the edge from $l_3$ to $v$, deleting the path from $r$ to $t$, and also contracting the path from $l_2$ to $r$ down to a vertex. Hence, $v$ is not adjacent to $t$, so $u$ is adjacent to $t$. Now, $G$ contains as a proper minor the excluded minor $U$, which can be seen by deleting the path from $r$ to $t$, deleting the edge from $l_1$ to $u$, deleting the edge from $l_3$ to $v$, and contracting the edge from $l_2$ to $v$. We deduce that (c) does not arise.

By the same argument that excludes (c), we see that (d) does not occur.

Now assume that (e) occurs. Since $G - \{l_1, l_3\}$ has a cycle, we have without loss of generality that there is an edge from $u$ to another vertex $u'$ of $T$. By contracting the edge between $l_2$ and $v$ and also contracting the path from $u'$ to $r$ to a single vertex, the graph $G$ contains a $K_5$-minor, a contradiction. Hence, (e) does not occur.

Suppose (f) occurs. Since $G - \{l_1, l_3\}$ has a cycle, there is an edge from $u$ to another vertex of $T$. However, this edge is not incident with $r$ and one of: an internal vertex on the path from $r$ to $t$, an internal vertex on the path from $r$ to $l_3$, or an internal vertex on the path from $t$ to $l_2$; otherwise $G$ contains a $K_5$-minor as in case (e). Therefore, there is an edge between $u$ and $t$ or $u$ and an internal vertex on the path from $l_1$ to $t$. The resulting graph has a $K_5$-minor, which can be seen by contracting the edge between $l_1$ and $v$, and contracting the path from $r$ to $l_2$ down to a vertex. Hence, $f$ does not occur.

If (g) occurs, then because both $u$ and $v$ are connected to $l_1, l_2,$ and $l_3$, the graph $G$ contains a $K_5$-minor; a contradiction. In every case $G$ either either in the class $\mathcal{S}$ or contains as a proper minor an excluded minor for $\mathcal{S}$.

Hence, we may assume $T$ has at least four degree-one vertices.
Lemma 2.6.5. Suppose $T$ has at least four degree-one vertices. Then $G$ is not an excluded minor for $S$.

Proof. Let $l_1, l_2, l_3,$ and $l_4$ be degree-one vertices of $T$. If there is a vertex of degree at least four in $T$, call it $r$. Since $G \setminus e$ is 3-connected, both $u$ and $v$ are adjacent all leaves of the tree $T$.

2.6.6. The edge $e$ is incident with a vertex of degree at least three of $T$.

Suppose that $e$ is incident only with vertices of degree one or two in $T$. Then $G - \{u, v\}$ has as a subgraph one of the graphs shown in Figure 2.10.

![Diagrams](image)

FIGURE 2.10: $e$ is incident only with degree-2 vertices of $T$. A bold edge represents a path.

In each of the cases shown, $G$ has a $K_5$-minor, which can be seen by contracting the edge from $l_3$ to $u$ and the edge from $l_3$ to $v$; a contradiction. Hence 2.6.6 holds.

2.6.7. At most one edge of the path $P$ between two vertices of degree greater than two is contained in the cycle $C$ in $G - \{u, v\}$
Let $r_1$ and $r_2$ be vertices of degree at least three in $T$. Suppose two edges of a path between $r_1$ and $r_2$ are contained in the cycle $C$ and $s$ is a vertex, distinct from $r_1$ and $r_2$ on the path contained in the cycle in $G - \{u, v\}$. Then $G - \{u, v\}$ has one of the graphs shown in Figure 2.11 as a subgraph.

![Graphs](image)

**FIGURE 2.11:** $C$ contains at least two edges of the path between $r_1$ and $r_2$. A bold edge represents a path.

In each case, either $s$ has degree greater than two in $T$, in which case there is a path from $s$ to some leaf $l_5$. Since $G\setminus e$ is 3-connected, either $u$ is adjacent to $s$ or $u$ is adjacent to $l_5$. By deleting the edges $ul_2$ and $ul_1$, contracting the path between $r_1$ and $l_2$, and contracting the path between $r_2$ and $l_1$, we see that the excluded minor $R$ is a proper minor. Hence, 2.6.7 holds.

**2.6.8.** There is no cycle in $T + e$ containing two vertices having degree at least three in $T$.

Let $r_1$ and $r_2$ be vertices of degree at least three in $T$. Suppose there is a cycle in $G - \{u, v\}$ containing $r_1$ and $r_2$. Then $G - \{u, v\}$ has one of the following graphs shown in Figure 2.12 as a subgraph.
FIGURE 2.12: \( C \) contains two vertices of degree at least three in \( T \). A bold edge represents a path.

By contracting the edge between \( u \) and \( l_2 \), contracting the edge between \( v \) and \( l_3 \), and contracting the path between \( r_1 \) and \( l_4 \), the graph \( G \) has a \( K_5 \)-minor; a contradiction. Hence 2.6.8 holds.

By eliminating all subgraphs of \( G - \{u, v\} \) in 2.6.6, 2.6.7, and 2.6.8, we see that the graph \( G - \{u, v\} \) has one of the graphs shown in Figure 2.13 as a subgraph.

FIGURE 2.13: \( C \) contains two vertices of degree at least three in \( T \). A bold edge represents a path.
Suppose \( T + e \) is one of (d), (e), and (f). One of \( u \) and \( v \), say \( u \) is adjacent to \( s \). By deleting the edge from \( u \) to \( l_4 \), contracting the path from \( l_4 \) to \( r_2 \), and contracting the edge from \( v \) to \( l_3 \), we see that \( G \) has an \( S(K_4, K_4) \)-minor; a contradiction.

Now, suppose that \( T + e \) is (c). Then \( G \) contains the excluded minor \( S(K_4, K_4) \), which can be seen by deleting the edge from \( u \) to \( l_3 \), deleting the edge from \( v \) to \( l_2 \), deleting the path from \( r \) to \( l_1 \), contracting the edge from \( l_1 \) to \( u \), and contracting the path from \( l_3 \) to \( t \). Hence (c) does not occur.

We are left with the cases when \( T + e \) is one of (a) or (b). In case (b), notice that there are no branches of the tree \( T \) adjacent to an internal vertex of the path from \( l_2 \) to \( r \); otherwise \( G \) has a \( K_5 \)-minor, which can be seen by contracting the path from \( l_1 \) to \( r \) and contracting the edge between \( l_3 \) and \( u \). In both (a) and (b), the vertex \( v \) is adjacent to only leaves of the tree, and possibly \( u \) and \( r \). Now, by the same argument as in Lemma 2.6.4 (a) and (b), this does not arise.

Therefore, \( T \) does not have four or more degree-one vertices.

The case analysis included in the previous Lemmas 2.6.2, 2.6.4, and 2.6.5 finishes the proof of Theorem 2.6.1.

2.7 Minimally 3-Connected Excluded Minors for the Class \( \mathcal{S} \)

In an excluded minor \( G \), if there is an edge \( e \) such that \( G \setminus e \) is 3-connected, then \( G \) is one of the excluded minors in Theorem 2.6.1. Now we assume that \( G \) is minimally 3-connected, that is, there is no edge \( e \) such that \( G \setminus e \) is 3-connected.

Before we prove the main theorem of this section, we find the structure of a minimally 3-connected excluded minor \( G \). The following lemmas characterize the structure of minimally 3-connected excluded minors and play a vital role in finding such excluded minors for the class \( \mathcal{S} \). We begin by showing in Lemma 2.7.1 that for any edge \( e \), in \( G \setminus e \) no two minors of
any two distinct components of the 2-sum decomposition are 3-connected. We use Lemma 2.7.2 to prove Lemma 2.7.3, where we show that for any edge $e$, the deletion $G \setminus e$ is not series-parallel. Then, in Lemma 2.7.4 we show that every edge of $G$ is incident with a vertex of degree three.

**Lemma 2.7.1.** Let $G$ be a simple minimally 3-connected excluded minor for $S$ such that $G \setminus e \cong G_1 \oplus_2 G_2$ for some edge $e$ adjacent to two vertices $a$ and $b$ and graphs $G_1$ and $G_2$. Let $H_1$ be minor of $G_1$ and $H_2$ be a minor of $G_2$. If $H_1$ is 3-connected, then $H_2$ is not 3-connected.

**Proof.** Since $G \setminus e \in S$, there is an edge $f$ joining two vertices $x$ and $y$ such that $((G \setminus e) + f)/f$ has no $K_4$-minor. Suppose both $G_1$ and $G_2$ have 3-connected minors, $H_1$ and $H_2$, respectively.

By Proposition 2.4.1, since $G \setminus e \in S$, this implies that $G$ can be formed by taking the parallel connection of $H_1$ and $H_2$, with basepoint $f$ that may or may not be deleted. Then at each edge of the resulting graph, attach via 2-sum, a 2-connected series-parallel graph, or a parallel class. If $G_1$ and $G_2$ are exactly $H_1$ and $H_2$, then because each of $H_1$ and $H_2$ is 3-connected, there are two vertices whose removal is cycle-free, and those vertices are the vertices $a$ and $b$, so $H_1 - \{a, b\}$ and $H_2 - \{a, b\}$ are cycle-free. Hence, $G - \{a, b\}$ is cycle-free and $G \in S$; a contradiction. As $G$ is simple, this implies that at least one 2-connected series-parallel graph is 2-summed onto some edge. Now, $G - \{a, b\}$ has a cycle, $C$, as a minor, which lies entirely in $G_1$ or $G_2$. Also, $C$ does not lie entirely in $H_1$; otherwise $H_1 - \{a, b\}$ has a cycle, a contradiction. So, $C$ is in a 2-connected series-parallel graph that is 2-summed onto some edge adjacent to neither $a$ nor $b$ of $H_1 \oplus_2 H_2$. Every 2-connected series-parallel graph has in a vertex of degree two, which is not destroyed by 2-summing the graph onto $H_1 \oplus_2 H_2$, and since $\kappa(G) = 3$, this vertex is incident with $f$. Combining the cycle $C$ in the 2-connected series-parallel graph in $G - \{a, b\}$ with the $K_4$-minor in both $H_1$ and $H_2$, the
graph $G$ has $S(K_4, K_4)$ as a proper minor, which can be seen by contracting the edge $e$ and also contracting an edge adjacent to $b$ in $G_2$; a contradiction. Hence, both $H_1$ and $H_2$ are not 3-connected.

\[ \square \]

**Lemma 2.7.2.** Let $G$ be a simple 3-connected graph. If $G$ is an excluded minor for the class $S$, then $G$ does not contain an exact copy of the fan with three spokes shown in Figure 2.14 as a minor where no edge adjacent to $v_a, v_b$ or $v_c$ was deleted or contracted to form such a minor.

**Proof.** Suppose $G$ contains the 3-spoke fan shown in Figure 2.14. Let $v_a, v_b, v_c, v_d, w$, and $u$, be the vertices shown and $a$ be the edge joining $w$ to $v_a$.

![3-spoke fan](image)

**FIGURE 2.14:** 3-spoke fan.

Since $G/a \in S$ is 3-connected, by Proposition 2.5.2 there are two vertices whose deletion is a tree. Suppose the composite formed after contracting $a$, call it $v'$, is not one of the vertices of $G/a$ whose removal is a tree. Then the deletion of the two vertices $x$ and $y$ that leave a tree in $G/a$ also leave a tree in $G$ since contracting $a$ creates no new cycles in $G - \{x,y\}$. By Theorem 2.5.3, $G$ is a member of $S$; a contradiction. Hence, the composite vertex $v'$ is one of the vertices of $G/a$ whose removal is cycle-free.

Now, $v_b, v_c, \text{ and } v_d$ are in a triangle, so the other vertex of $G/a$ whose removal is cycle-free is one of $v_b, v_c, \text{ or } v_d$. If $G/a - \{v', v_b\}$ is a tree, Menger’s theorem implies that there is a path from $u$ to $v_d$ in $G$ avoiding $v_a$ and $v_c$. Therefore $G/a - \{v', v_b\}$ contains a cycle; a
contradiction. If $G/a - \{v', v_d\}$ is a tree, then $G - \{v_d, w\}$ contains no cycles; a contradiction. Similarly, if $G/a - \{v', v_c\}$ is a tree, then $G - \{v_d, w\}$ contains no cycles; a contradiction.

\[\square\]

**Lemma 2.7.3.** If $G$ is an excluded minor, then $G\setminus e$ is not series parallel.

**Proof.** Suppose $G\setminus e$ is series parallel. Using Cunningham and Edmonds’s canonical 2-sum decomposition [4], we see that $G\setminus e$ can be decomposed into a path in which the vertices are labelled alternately by cycles and parallel classes. Each parallel class has three elements and each cycle has three or four elements. The path must have triangles on both ends since $G\setminus e$ is series parallel. Hence, our canonical 2-sum decomposition is: triangle, 3-element parallel class, triangle or square, 3-element parallel class on both ends of the path ending in one of the following two graphs and the next vertex of the 2-sum decomposition will be attached via 2-sum onto the edge $p$ shown in the Figure 2.15.

\begin{figure}[h]
\centering
\begin{tabular}{c c}
\includegraphics[width=0.3\textwidth]{fig215a} & \includegraphics[width=0.3\textwidth]{fig215b} \\
(a) two triangles & (b) a triangle and a 4-cycle \\
\end{tabular}
\caption{\label{fig:215} $G\setminus e$ Decomposition: Path Ends.}
\end{figure}

Suppose first that $G$ ends in (a), two triangles. Now, $p$ is not adjacent to a cycle of length greater than four; otherwise $G$ would contain a 3-spoke fan as in Lemma 2.7.2. Hence, $p$ is adjacent to another triangle, attached via 2-sum with basepoint $p$; otherwise $G$ contains a 3-spoke fan as in Lemma 2.7.2, a contradiction.

Now, $G$ must end in either three triangles as in Figure 2.16 (a) or a triangle and a 4-cycle as in Figure 2.15 (b). If $G$ ends in two copies Figure 2.16 (a), one copy of Figure 2.16 (a) and one copy of Figure 2.15 (b), or two copies of Figure 2.15 (b), then $G$ contains an
(a) 3 triangles

FIGURE 2.16: $G \setminus e$ Decomposition: Path Ends.

$S(K_4, K_4)$-minor; a contradiction. It is easy to see that smaller cases are contained in $S$. Therefore, $G \setminus e$ is not series parallel.

**Lemma 2.7.4.** If $e$ is an edge of a minimally 3-connected excluded minor, then $e$ is incident with a vertex of degree three.

**Proof.** Suppose $G$ is a minimally 3-connected excluded minor. Combining Proposition 2.4.1, Lemma 2.7.1, and Lemma 2.7.3 we know that the deletion of any edge $x$ is the 2-sum of a 3-connected graph $G_1$ and a series-parallel graph $G_2$ where exactly one other series-parallel graph may be 2-summed onto any edge of $G_1$. Also, since $G_2$ is a series-parallel graph, it has a vertex of degree two that is not destroyed by 2-summing $G_1$, which is why only one other series-parallel graph besides $G_2$ can be 2-summed onto $G_1$. Since $G$ is 3-connected, this vertex is incident with $e$ and becomes a vertex of degree three in $G$. Recall that we picked the deleted edge $x$ arbitrarily so every edge of $G$ is incident with a vertex of degree three.

In the following theorem, we find the minimally 3-connected excluded minors for $S$. This completes the list of excluded minors for $S$.

**Theorem 2.7.5.** Let $G$ be a simple minimally 3-connected graph. Then $G$ is an excluded minor for $S$ if and only if $G$ is one of the graphs shown in Figure 2.17.

**Proof.** We use the matroid of a graph $G$ to show that a minimally 3-connected excluded minor $G$ is close to being 3-connected and can be formed from a 3-connected graph as
stated at the end of this paragraph. By a result of Cunningham [3] and Seymour [16],
there is a vertically contractible edge in the dual $M^*(G)$, call it $e$. Now, the simplification,
$\text{si}(M^*(G)/e) = \text{si}(M^*(G/e))$ is 3-connected for some edge $e \in M^*(G)$. In the dual, the
cosimplification $\text{co}(M(G\setminus e)) = \text{co}(M(G)\setminus e)$ is 3-connected for some edge $e \in M(G)$. Now,
$\text{co}(M(G)\setminus e)$, has no coloops since $G\setminus e$ is 2-connected. Also, $G\setminus e$ has at most two series
classes since $G$ is 3-connected. This means that $G$ can be formed from a 3-connected graph
$H$ by either subdividing an edge and adding $e$ from the newly created vertex to another
vertex of $H$; or subdividing two edges and adding $e$ joining the two newly created vertices.

We distinguish two main cases determined by whether or not $e$ lies in a triangle. Let $u$
and $v$ be the vertices incident with $e$. By Lemma 2.7.4, the edge $e$ is incident with a vertex
of degree three, say $v$.

First, suppose $e$ is in a triangle with edges $\{e, g, h\}$, and let $w$ be the third vertex of the
triangle where $g$ is incident with vertices $v$ and $w$, and $h$ is incident with vertices $u$ and $w$.
Suppose $e$ is in two triangles. However, $\text{co}(M(G\setminus e))$ is simple and 3-connected, and if $e$ is
in two triangles, then is a parallel pair of edges in $G\setminus e$; a contradiction. Hence, $e$ is in at
most one triangle. By Lemma 2.2.3 (Tutte’s Triangle Lemma) [20], since $G\setminus e$ and $G\setminus g$ are

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.17.png}
\caption{H$_8$, Q$_3$, S, and V}
\end{figure}
not 3-connected and \{e, g, h\} is a triangle, we have that h is in a triad with either e or g. Hence, either v and w have degree three, or v and u have degree three.

Since u and v have degree three, the vertices u and v are incident with two other edges, call them f and k, respectively. Let x be the other vertex incident with f and y the other vertex incident with k. Since G is an excluded minor, \(G/f \in \mathcal{S}\), so by Theorem 2.5.3 there are two vertices of \(G/f\), call them \(v_f\) and \(v'_f\), whose removal leaves a tree. Without loss of generality \(v_f\) is the composite vertex formed in the contraction; otherwise \(G - \{v_f, v'_f\}\) is a tree, a contradiction. We divide our argument into cases based on the degrees of \(u, v,\) and \(w\).

**2.7.6.** If \(d(u) = d(v)\) and \(d(w) \geq 4\), then \(G \cong \Gamma\).

Suppose first that \(d(u) = d(v) = 3\) and \(d(w)\) is at least four. Notice that this is equivalent to the case when \(d(u) = d(w) = 3\) and \(d(v)\) is at least four. Since \(d(w)\) is at least four, \(w\) is adjacent to two other vertices, call them \(w_1\), and \(w_2\). Now, we have the following graph shown in Figure 2.18 is a subgraph of \(G\).

![Figure 2.18](image)

**FIGURE 2.18:** e is added in a triangle.

Consider the vertex \(v'_f\). Notice that \(v'_f \neq u\); otherwise \(G - \{u, x\}\) is a tree, a contradiction. Similarly, \(v'_f \neq w\); otherwise \(G - \{w, x\}\) is a tree, a contradiction. If \(v'_f = w_1\), then \(w_1\) is adjacent to all leaves of the tree. Also, \(w_1\) must have degree exactly three since \(w_1\) is
adjacent to $w$ and $d(w) \geq 4$ and there are no two adjacent vertices of degree at least four in a minimally 3-connected excluded minor by Lemma 2.7.4. This means that $(G/f) - \{v_f, v'_f\}$ is a path, and hence, $G - \{v_f, w\}$ has no cycles; a contradiction. Similarly, $v'_f \neq w_2$.

Suppose that $v'_f = y$. Now, both $d(x)$ and $d(y)$ are at least four; otherwise $G - \{x, w\}$ or $G - \{y, w\}$ has no cycles, a contradiction. Notice that $x$ and $y$ are not adjacent to $w$; otherwise there would be two adjacent vertices of degree at least four, a contradiction by Lemma 2.7.4. Either $(G/f) - \{v_f, y\}$ is a path or it has at least three degree-one vertices. However, $(G/f) - \{v_f, y\}$ is not a path since $(G/f) - \{v_f, y\} = G - \{v, x, y\}$, and $G - \{v, x, y\}$ has a vertex of degree three, but a path has no such vertex. Hence, $(G/f) - \{v_f, y\}$ has at least three degree-one vertices. Since no two adjacent vertices have degree exceeding three, but $\kappa(G) = 3$, every nonleaf vertex other than $w$ and $u$ of the tree $(G/f) - \{v_f, y\}$ is adjacent to exactly one of $x$ and $y$. Also, as every leaf vertex is adjacent to both $x$ and $y$, by examining the possible graphs, we find that the graph $G$ is isomorphic to the excluded minor $V$.

Now, we may assume that $v'_f \notin \{u, w, w_1, w_2, y\}$. If $v'_f$ is the only other vertex besides $u, v, x, w, w_1, w_2$, and $y$, then $v'_f$ can be adjacent to only $w_1, w_2, x$, and $y$. Hence, $G - \{v, v'_f\}$ is a tree; a contradiction. This implies that there is some other vertex $v''_f \notin \{u, v, w, x, y, w_1, w_2, v'_f\}$. We consider separately when $w_1, w_2$, and $y$ are leaves of the tree, and when they are not. If $w_1$ is not a leaf of the tree, then $G$ contains $S(K_4, K_4)$ as a proper minor; a contradiction. If $y$ is not a leaf of the tree, then, again $G$ contains a proper $S(K_4, K_4)$-minor; a contradiction. Notice that there are no edges from $v'_f$ to $v_f = x$ because no two adjacent vertices of $G$ have degree exceeding three. Hence, $G - \{v_f, w\}$ is a tree; a contradiction. Therefore, 2.7.6 holds.

2.7.7. If $d(u) = d(v) = d(w) = 3$, then $G \cong S$. 

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Now, suppose that \(d(u) = d(v) = d(w) = 3\). There are two paths from \(w\) to \(w'\) and \(u\) to \(y'\), where \(w'\) and \(y'\) are vertices adjacent to leaves of the tree. We have the graph shown in Figure 2.19 is a subgraph of \(G\), where a dotted line represents a path.

![Graph Figure 2.19](image)

FIGURE 2.19: \(e\) is added in a triangle. A dotted line represents a path.

If \(v'_f = u\), then \((G/f) - \{v_f, v'_f\}\) is a path since \(u\) is adjacent to the ends of the path, has degree three, and is adjacent to \(v\). This implies that \(w\) and \(y'\) are the ends of the path. Hence, \(G - \{x, w\}\) is a path; a contradiction. By symmetry, \(v'_f \neq w\). Suppose that \(v'_f = y'\) or by symmetry \(w'\). Either \(G/f - \{v_f, v'_f\}\) is a path or has at least three degree one vertices.

If \(G/f - \{v_f, v'_f\}\) is a path, then either \(G\) contains \(S(K_4, K_4)\) as a minor or \(G - \{v, w'\}\) is a path; both of which are contradictions. If \(G/f - \{v_f, v'_f\}\) has at least three degree-one vertices, then \(G\) contains and is isomorphic to \(S\).

We may now assume \(v'_f \neq \{u, w, x, y'\}\). If \((G/f) - \{v_f, v'_f\}\) is not a path and there is an edge from \(v'_f\) to the path from \(u\) to \(y'\), then \(G\) has a \(K_5\)-minor, so there is no such edge. If there is an edge from \(v'_f\) to the path from \(w\) to \(w'\), then \(G\) contains \(S(K_4, K_4)\) as a proper minor; a contradiction.

Now, assume \((G/f) - \{v_f, v'_f\}\) is a path. If \(v_f\) and \(v'_f\) are not adjacent, then without loss of generality the path from \(w\) to \(w'\) has a third vertex. Hence \(G\) has an \(S(K_4, K_4)\)-minor; a contradiction. If \(v_f\) and \(v'_f\) are adjacent, then \(G - \{x, w\}\) is a tree since there are no
other edges from $v'_f$ besides the edges from $v'_f$ to two leaves of the tree, otherwise there are two adjacent vertices in $G$ of degree at least four. By Lemma 2.7.4 this is a contradiction. Therefore 2.7.8 holds.

We may now assume that $e$ is not in a triangle.

2.7.8. If $e$ is not in a triangle, then $G \cong H_8$, $Q_3$, or $S$.

Notice that $u$ and $v$ are incident with at least two other edges. Let $f$ and $g$ be the two other edges incident with $v$; let $u_f$ the other vertex incident with $f$; and let $u_g$ be the other vertex incident with $g$. Now, $G$ has the following graph shown in Figure 2.20 as a subgraph.

![Diagram](image)

FIGURE 2.20: $e$ is not in a triangle.

There is no triangle containing the edges $g$ and $e$, so $G/g$ is 3-connected and Theorem 2.5.3 has two vertices $v_g$ and $v'_g$ whose deletion is a tree. We may assume $v_g$ is the composite vertex that is obtained by contracting $g$. Otherwise the same two vertices in $G/g$ whose removal from $G/g$ leave a tree, also leave a tree when removed from $G$; a contradiction.

Suppose $G/g - \{v_g, v'_g\}$ has at least three degree-one vertices. If $v_g$ is adjacent to a non-leaf vertex of the tree and is also adjacent to a vertex of degree two of the tree, then the resulting graph obtained by adding all such adjacencies contains a $H_8$-minor. We may now assume that $v_g$ is only adjacent to vertices of degree at least three. Now, if the tree has at least two vertices of degree at least three, then $G$ contains the graph $H_8$ as a minor. If the tree contains exactly one vertex of degree at least three, then the removal of that vertex and the vertex $v'_g$ is a tree; a contradiction.
Now, we may assume that $G/g - \{v_g, v'_g\}$ is a path. Likewise, $G/f - \{v_f, v'_f\}$ is a path. Now $v'_g$ is neither $u_f$ nor $u$, otherwise the deletion of $u_g$ and $v'_g$ is a tree. Let $x_1$ and $x_2$ be the leaves of the path $G/g - \{v_g, v'_g\}$. Notice that $f$ is adjacent to either $x_1$ or $x_2$; otherwise $x_1$ and $x_2$ have degree three in $G/f$ and $G/f - \{v_f, v'_f\}$ is not a path, so without loss of generality suppose $f$ is adjacent to $x_1$. Also, $d(u_f) = d(v) = d(u_g) = 3$ since $\text{co}(M(G\backslash e))$ is 3-connected. So both $g$ and $f$ are not in a triangle in $G$ which implies that both ends of $f$, namely $v$ and $u_f$, have degree three. By symmetry $u_g$ has degree three. We consider separately when $u_g$ is adjacent to $x_2$ of the path $G/g - \{v_g, v'_g\}$, and when it is not.

Suppose $u_g$ is adjacent to $x_2$. If $u_g$ is adjacent to $u'_g$, then $G - \{v'_g, v\}$ is a path; a contradiction. If $u_g$ is not adjacent to $v'_g$, then it is adjacent to another vertex $w$ of the path $G/g - \{v_g, v'_g\}$. If $w$ is closer to $x_2$ than $u$ on the path $G/g - \{v_g, v'_g\}$, then suppose there is another vertex on the path $G/g - \{v_g, v'_g\}$, call it $a$. If $a$ is on the path from $w$ to $u$, then $G$ contains and is isomorphic to $H_8$. If $a$ is on the path from either $x_1$ to $w$ or $u$ to $x_2$, then $G$ contains and is isomorphic to $S$. If there are no other vertices on the path, then $v'_g$ is adjacent to $u$ or $w$ since $G$ is 3-connected. If $v'_g$ is only adjacent to one of $u$ or $w$, then the vertex to which it is adjacent, and $x_1$ or $x_2$, is a two vertex cut of $G$; a contradiction. Hence, $v'_g$ is adjacent to both $u$ and $w$ and has a $K_5$-minor. If $u$ is closer on the path $G/g - \{v_g, v'_g\}$ to $x_2$, then there is another vertex on the path between $x_2$ and $u$ since $e$ is not in a triangle in $G$. The previously mentioned vertex on the path between $x_2$ and $u$ is adjacent to $v'_g$ which gives a proper $S(K_4, K_4)$-minor; a contradiction. Notice that $u \neq w$, otherwise $e$ would be in a triangle.

We may now assume that $u_g$ is not adjacent to $x_2$. If $u_g$ is also adjacent to $v'_g$, then $G - \{v, v'_g\}$ is a tree, a contradiction. If $u_g$ is not adjacent to $v'_g$, then it is adjacent to two intermediate vertices of the path $G/g - \{v_g, v'_g\}$, call them $c$ and $d$. If there is another vertex on the path from $c$ to $d$, then $v_g$ is adjacent to that vertex and $G$ contains and is isomorphic to $Q_3$. If there is another vertex on $G/g - \{v_g, v'_g\}$ minus the path from $c$ to
$d$, then $G$ contains a proper $S(K_4, K_4)$-minor; a contradiction. So we may assume that the only vertices on the path $G/g - \{v_g, v'_g\}$ are $u_f, c, d$ and $u_e$. If $v_g$ is adjacent to exactly one of the intermediate vertices $c$ or $d$, then the removal of that vertex and $u_f$ or $u_e$ is a tree; a contradiction. If $v_g$ is adjacent to both vertices, then $G$ has an $S$-minor, and is isomorphic to $S$.

We restate the main theorem of this chapter, Theorem 2.1.1, for convenience giving the excluded minors for $S$ which follow from Proposition 2.3.1, Proposition 2.3.2, Proposition 2.3.4, Theorem 2.6.1, and Theorem 2.7.5.

**Main Theorem.** The excluded minors for $S$ are the following 11 graphs: $K_4 \oplus_0 K_4$, $K_4 \oplus_1 K_4$, $S(K_4, K_4)$, $K_5$, $K_{2,2,2}$, $R$, $U$, $H_8$, $Q_3$, $S$, and $V$.

**2.8 The Dual Operation**

We extend the excluded minors for $S$ to a related class of graphs, $S^\ast$. The class $S$ consists of those graphs $G$ such that there is a graph $H$ for which $H \setminus e = G$ and $H/e$ is series-parallel for some edge $e \in E(H)$. Consider the class $S^\ast$ of graphs $G$ such that there is a graph $H$ for which $H/e = G$ and $H \setminus e$ is series-parallel for some edge $e \in E(H)$. Then $G \in S^\ast$ if $G$ has a vertex $v$ that can be replaced by two vertices $v_1$ and $v_2$ so that each edge of $G$ incident with $v$ is incident with exactly one of $v_1$ and $v_2$ and the graph obtained by this operation is series-parallel. In general, we refer to this operation as splitting the vertex $v$.

It is easy to check that the class $S^\ast$ is closed under taking minors. By Robertson and Seymour’s Graph Minors Theorem, it has a finite number of excluded minors.

In the proof of the excluded minors for $S^\ast$ it is useful to consider the dual of the graph and we use the following proposition given in [13].

**Proposition 2.8.1.** Let $G$ be a graph having no isolated vertices. If $G$ is series-parallel, then its dual $G^\ast$ is series-parallel.
Lemma 2.8.2. A graph $G$ is series-parallel if and only if its associated graphic matroid $M(G)$ is series-parallel.

Proof. If $G$ is series-parallel, then clearly $M(G)$ is series-parallel by taking $M(G)$ to be the cycle matroid of the graph $G$. Let $M(G)$ be a series-parallel matroid. By Whitney’s 2-Isomorphism Theorem 1.11.1, the graph $G \cong_2 H$ for some series-parallel graph $H$. However, $H$ can be transformed into a graph isomorphic to $G$ by the operations of vertex identification, vertex cleaving, and twisting, and none of these operations create a $K_4$-minor. Hence, $G$ is series-parallel.

Proposition 2.8.3. If $M \setminus f$ is series-parallel and $M / f$ is planar, then $M$ is planar.

Proof. Suppose $M$ is nonplanar. Then $M \setminus X/Y \cong M(K_5)$ or $M(K_{3,3})$ for some subsets $X$ and $Y$ of $E(M)$. Suppose first, that $f \notin X \cup Y$. We have that $M \setminus f$ contains $(M \setminus X/Y) \setminus f$. The previous matroid is $M(K_5) \setminus f$ or $M(K_{3,3}) \setminus f$. However, the deletion of any edge from both $M(K_5)$ and $M(K_{3,3})$ contains $M(K_4)$. Hence, $M \setminus f$ is series-parallel; a contradiction.

Next, suppose that $f \in X$. The matroid $M \setminus f$ contains $M \setminus X/Y$, which is isomorphic to either $M(K_5)$ or $M(K_{3,3})$, and both of those matroids contain $M(K_4)$ as a minor. Hence, $M \setminus f$ is series-parallel; a contradiction. Finally, suppose $f \in Y$. Then $M / f$ contains $M \setminus X/Y$ which is isomorphic to either $M(K_5)$ or $M(K_{3,3})$. By Wagner’s Theorem 1.6.2, the matroid $M / f$ is nonplanar; a contradiction. Therefore, $M$ is a planar, and 2.8.3 holds.

Lemma 2.8.4. Let $G$ be a connected planar graph. If $G$ is a member of $S^*$, then $G^*$ is a member of $S$.

Proof. Since $G \in S^*$, there is graph $H$ and an edge $e \in E(H)$ for which $H/e = G$ and $H \setminus e$ is series-parallel. Since the dual of a planar graph is planar, $G^*$ is planar. Also, by Proposition 2.8.3, the graph $H$ is planar. Consider the graphic matroid associated with $G^*$. Now, $M(G^*) = M(G)^* = M(H/e)^* = M(H)^* \setminus e = M(H^*) \setminus e$. Since the dual of the
series-parallel matroid is series-parallel, the matroid $M(H\setminus e)^* = M(H)^*/e = M(H^*)/e$ is series-parallel. Therefore, there is a graph $H^*$, for which $H^*\setminus e \cong G^*$ and $H^*/e$ is series-parallel. By Whitney’s 2-Isomorphism Theorem, this 2-isomorphism is an isomorphism for some graph $K^*$ and some edge $e \in K^*$ for which $K^*\setminus e = G^*$ and $K^*/e$ is series-parallel, thus $G^* \in S$.

Lemma 2.8.5. Let $G$ be a connected planar graph. If $G$ is an excluded minor for $S^*$, then $G^*$ is an excluded minor for $S$.

Proof. First, we show that $G^* \notin S$. Suppose $G^* \in S$. Then there is a graph $H$ and an edge $e \in E(H)$ for which $H\setminus e = G^*$ and $H/e$ is series-parallel. By taking the dual of Proposition 2.8.3, the graph $H$ is planar. Since $G$ and $H$ are planar graphs, consider the matroids associated with them, and $M(G) = M(G^*)^* = M(H\setminus e)^* = M(H)^*/e = M(H^*)/e$. Also, since series-parallel graphs are closed under duality, $M(H/e)^*$ is series-parallel, but $M(H/e)^* = M(H)^*\setminus e = M(H^*)\setminus e$. Now, we have a graph $H^*/e \cong G$, for which $H^*\setminus e$ is series-parallel. By Whitney’s 2-Isomorphism Theorem, there is some graph $J$ and edge $e \in E(J)$ satisfying $J^*/e = G$ for which $J^*\setminus e = G^*$ is series-parallel, a contradiction.

Next, we show that every proper minor of $G^*$ is a member of $S$. If $F$ is a proper minor of $G^*$, then $F^*$, is a proper minor of $G$. Since $G$ is an excluded minor for $S^*$, the graph $F^* \in S^*$. By Lemma 2.8.4, the dual, $F$ is contained in $S$.

Therefore, since $G^*$ is not in $S$, but every proper minor is in $S$, we have shown that $G^*$ is an excluded minor for $S$.

The following theorem gives the excluded minors for $S^*$ and follows from Theorem 2.1.1 and Lemma 2.8.5.

Theorem 2.8.6. The excluded minors for $S^*$, the class of graphs that have a vertex splitting that is series-parallel, consist of the nine graphs shown in Figure 2.21.
Proof. Let $G$ be an excluded minor for $S^*$. If $G$ is nonplanar, then by Kuratowski’s Theorem [10], $G$ has a $K_{3,3}$ or $K_5$-minor, both of which are excluded minors for $S^*$. If $G$ is planar and disconnected, then $G$ has a $K_4 \oplus 0 K_4$-minor, which is also an excluded minor for $S^*$. We may now assume that $G$ is planar and connected. Since $G$ is an excluded minor for $S^*$, by Lemma 2.8.5, $G^*$ is an excluded minor for $S$. Thus, by taking the duals of the planar excluded minors for $S$, we find the planar excluded minors for $S^*$, shown in Figure 2.21. It is easy to check that each of these graphs is an excluded minor for $S^*$.

\[\Box\]
Chapter 3
Vertex Removal

Recall from Chapter 2 that a feature of a 3-connected almost series-parallel graph is that there are two vertices of the graph whose deletion is a tree. This idea of removing vertices from a graph and destroying all of its cycles gives rise to several new classes of graphs. Let $\mathcal{V}_n$ be the class of graphs such that the deletion of at most $n$ vertices from $G$ produces a graph with no cycles. In this chapter, we find the full list of excluded minors for $\mathcal{V}_1$ and $\mathcal{V}_2$, the classes of graphs such that there are, respectively, at most one vertex and at most two vertices whose removal from the graph gives a cycle-free graph.

3.1 Preliminaries

The following notation closely follows [2]. Denote by $\Omega_k$ the family of graphs containing $k$ vertex-disjoint cycles. The family of graphs not belonging to $\Omega_k$ is denoted by $\overline{\Omega_k}$. Note that $\overline{\Omega_1}$ is just the family of forests.

We repeatedly use the following theorems when finding the excluded minors for $\mathcal{V}_1$ and $\mathcal{V}_2$. The first theorem was proved by Lovász [11] in 1965 and gives a characterization of those graphs that have no two vertex-disjoint cycles, and have minimum degree at least three. The six possibilities that arise in the theorem are illustrated in Figure 3.1.

Theorem 3.1.1. Let $G$ be a graph without two vertex-disjoint cycles. Suppose that $\delta(G) \geq 3$ and there is no vertex meeting all the cycles. Then one of the following six assertions holds.

(i) $G$ has three vertices and multiple edges joining every pair of vertices.

(ii) $G$ is a $K_4$ in which one of the triangles may have multiple edges.

(iii) $G = K_5$. 

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(iv) $G$ is a $K_5^-$, the graph obtained from $K_5$ by deleting an edge, such that some of the edges not adjacent to the missing edge may have multiple edges.

(v) $G$ is a wheel whose spokes may have multiple edges.

(vi) $G$ is obtained from $K_{3,p}$ by adding edges or multiple edges joining vertices in the first class.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.1.png}
\caption{A dotted line shows that there may be multiple edges joining the end vertices.}
\end{figure}

Bollobás [2] generalized the previous result of Lovász in the following theorem and corollary. In the theorem, we consider the empty graph to be a forest.

**Theorem 3.1.2.** A graph $G$ does not contain two vertex-disjoint cycles if and only if either it contains a vertex $x_0$ such that $G - x_0$ is a forest, or it can be obtained from a subdivision $G_0$ of a graph listed in Theorem 3.1.1 by adding a forest and at most one edge joining each tree of the forest to $G_0$.

The following is an immediate consequence of the last result.

**Corollary 3.1.3.** A 2-connected graph $G$ has no two vertex-disjoint cycles if and only if $G$ can be obtained from a subdivision $G_0$ of a graph listed in Theorem 3.1.1.
In the contraction of an edge from an excluded minor for \( \mathcal{V}_n \) there are \( n \) vertices whose removal leaves a graph that is cycle-free. We show that one of those vertices must be the composite vertex formed in the contraction.

**Lemma 3.1.4.** Let \( G \) be an excluded minor for \( \mathcal{V}_n \). For every non-loop edge \( e \in E(G) \), the contraction \( G/e \) has \( n \) vertices whose removal is cycle-free, one of which is the composite vertex that results from identifying the ends of \( e \).

**Proof.** Since \( G \) is an excluded minor for \( \mathcal{V}_n \), the contraction \( G/e \) has \( n \) vertices, \( u_1, \ldots, u_n \), whose removal is cycle-free. Suppose the composite vertex formed in the contraction is not one of the vertices. Then \( G/e - \{u_1, \ldots, u_n\} \) is cycle-free and uncontracting the edge \( e \) creates no new cycles in \( G \). Hence \( G - \{u_1, \ldots, u_n\} \) is cycle-free, a contradiction as \( G \notin \mathcal{V}_n \). \( \square \)

**Lemma 3.1.5.** Let \( G \) be an excluded minor for \( \mathcal{V}_n \). If \( v_1 \) and \( v_2 \) are adjacent vertices of \( G \), then \( G - \{v_1, v_2\} \) has \( n - 1 \) vertices whose removal is cycle-free.

**Proof.** Let \( e \) be an edge joining \( v_1 \) and \( v_2 \). By Lemma 3.1.4 \( G/e \) has \( n \) vertices, \( u_1, \ldots, u_n \), whose removal is cycle-free, one of which is the composite vertex, \( v \), that results by identifying \( v_1 \) and \( v_2 \). Hence, \( (G/e) - v = G - \{v_1, v_2\} \) has \( n - 1 \) vertices whose removal is cycle-free. \( \square \)

The following lemma is used repeatedly to find the excluded minors for \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \).

**Lemma 3.1.6.** For a positive integer \( n \), let \( G \) be an excluded minor for \( \mathcal{V}_n \). Then

(i) each component of \( G \) contains a cycle;

(ii) \( G \) has no vertices of degree one;

(iii) if \( v \) is a degree-two vertex of \( G \), then \( G \) has a loop incident with \( v \) and this loop is a component of \( G \);

(iv) \( G \) has no cut edge.
Proof. Each component of $G$ contains a cycle; otherwise the deletion of an edge in a component without a cycle would be a member of the class $\mathcal{V}_n$. However, adding the edge back creates no cycles; a contradiction.

Suppose $e$ is an edge incident with a degree-one vertex in $G$. Then $e$ is in no cycles. Since $G$ is an excluded minor, $G \setminus e$ has $n$ vertices, $u_1, \ldots, u_n$, whose removal is cycle-free. Thus, $G - \{u_1, \ldots, u_n\}$ is cycle-free; a contradiction. Therefore $G$ has no degree-one vertices.

Suppose $v$ is a degree-two vertex of $G$. Clearly either $v$ is incident to two distinct edges or $G$ has a loop incident with $v$ and this loop is a component of $G$. First, suppose that $f$ joins vertices $v$ and $w$ where $v$ has degree two in $G$. Since $G$ is an excluded minor, $G/f$ has $n$ vertices, $x_1, \ldots, x_n$ whose removal is cycle-free. By Lemma 3.1.4, we may assume that $x_1$ is the composite vertex that results from identifying $v$ and $w$. But now $(G/f) - \{x_1, \ldots, x_n\} = G - \{v, w, x_2, \ldots, x_n\}$. However, since $v$ had degree two in $G$, it has degree one in $G - \{w, x_2, \ldots, x_n\}$ and is in no cycles; a contradiction. Therefore, $G$ has a loop incident with $v$ and this loop is a component of $G$.

Suppose $G$ has a cut edge, call it $g$. Then, $G \setminus g$ has $n$ vertices whose removal is cycle-free. However, since $g$ is a cut edge, it has no cycles and the removal of those same $n$ vertices from $G$ is cycle-free; a contradiction. Hence, $G$ has no cut edge. \qed

3.2 Excluded Minors for $\mathcal{V}_1$

The next theorem follows easily from Theorem 3.1.1 and gives the excluded minors for $\mathcal{V}_1$.

**Theorem 3.2.1.** A graph $G$ is an excluded minor for the class $\mathcal{V}_1$ if and only if $G$ is isomorphic to one of the three graphs shown in Figure 3.2.

**Proof.** If $G$ is disconnected, then, by Lemma 3.1.6 (i), each component of $G$ contains a cycle. Thus $G$ has a minor isomorphic to the disjoint union of two loops, $2L$. Since this graph is easily seen to be an excluded minor for $\mathcal{V}_1$, it is the only disconnected excluded minor.

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FIGURE 3.2: Excluded Minors for $\mathcal{V}_1$

Suppose $G$ is connected. Then $G$ does not have $2L$ as a minor, so $G$ has no two vertex-disjoint cycles. By Lemma 3.1.6 (ii) and (iii), $G$ has no vertices of degree one and two. Using Theorem 3.1.1, it is now a straightforward exercise to check that $G$ is either $K_4$ or $C^2_3$, a doubled triangle. $\square$

3.3 Excluded Minors for $\mathcal{V}_2$ with $\kappa(G) \in \{0, 1\}$ and $\kappa(G) \geq 4$

Finding the excluded minors $G$ for the class $\mathcal{V}_2$ with $\kappa(G) \in \{0, 1\}$ and $\kappa(G) \geq 4$ is not complicated. We consider separately when the connectivity is zero, one, and at least four.

**Theorem 3.3.1.** Let $G$ be an excluded minor for $\mathcal{V}_2$ with $\kappa(G) = 0$. If $G$ is disconnected, then $G$ is isomorphic to one of the three graphs $3L$, $K_4 \oplus 0 L$, or $C^2_3 \oplus 0 L$ as shown in Figure 3.3.

**Proof.** By Lemma 3.1.6 each component of $G$ contains a cycle. Hence, if the number of components of $G$ is at least three, then $G$ is isomorphic to three disjoint loops, $3L$. Suppose the number of components of $G$ is exactly two. No component $C_i$ of $G$ has two vertex-
disjoint cycles; otherwise $G$ has $3L$ as a proper minor. In each $C_i$, either there is one vertex meeting all of the cycles, or $C_i$ contains a $C^2_3$-minor or a $K_4$-minor by Theorem 3.1.2. It follows that $G$ must be isomorphic to the 0-sum of a loop and either $C^2_3$ or $K_4$. It is easy to check that each of these graphs is an excluded minor for the class $V_2$. □

**Theorem 3.3.2.** If a graph $G$ is an excluded minor for $V_2$, then $\kappa(G) \neq 1$.

**Proof.** Suppose $\kappa(G) = 1$. Then there is some vertex $v$ whose deletion disconnects $G$ and $G = G_1 \oplus_1 G_2$ for some connected subgraphs $G_1$ and $G_2$.

We show in what follows that exactly one of $G_1$ and $G_2$ has a vertex meeting all cycles. Moreover, this vertex is $v$. If $G_1$ and $G_2$ both have single vertices meeting all cycles, then $G \in V_2$; a contradiction. Hence, either $G_1$ or $G_2$ has no single vertex meeting all cycles.

Suppose neither $G_1$ nor $G_2$ has a single vertex meeting all cycles. Then both $G_1 - v$ and $G_2 - v$ contain cycle minors. If either $G_1$ or $G_2$ contains two vertex-disjoint cycles, then $G$ contains $3L$ as a proper minor; a contradiction. Hence, either $G_1$ or $G_2$ has no two vertex-disjoint cycles. Moreover, by Lemma 3.1.6, we have that $\delta(G) \geq 3$. Now, by Theorem 3.1.2, either $G_1$ or $G_2$, say $G_1$, contains a minor shown in Figure 3.1. Combining this minor with the cycle in $G_2 - v$, the graph $G$ has as a proper minor either $C^2_3 \oplus_0 L$ or $K_4 \oplus_0 L$; a contradiction. Hence, exactly one of $G_1$ and $G_2$ has a vertex meeting all cycles.

We may now assume that $G_2$ has a vertex meeting all of its cycles and that $G_1$ has no single vertex meeting all of its cycles. Suppose $G_2$ has at least two edges, call one $e$. Then $G/e$ has two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these vertices is the composite vertex formed in the contraction. The other is a vertex $w$ of $G_1$, and $G_1 - w$ is cycle-free. However, since $G_2 - v$ is also cycle-free, $G - \{u, w\} = (G_1 - u) \oplus_1 (G_2 - w)$ is cycle-free; a contradiction. Hence, $G_2$ has exactly one edge. By Lemma 3.1.6, $G_2$ has no cut edges, so $G_2$ is a loop.
Now $G_1 - v$ has no two vertex disjoint cycles; otherwise $G$ contains $3L$ as a proper minor, a contradiction. Also, $G_1 - v$ has no single vertex meeting all cycles, otherwise $G \in \mathcal{V}_2$. Therefore $G_1 - v$ has no two vertex disjoint cycles and no single vertex meeting all cycles and, by Theorem 3.1.2, $G_1 - v$ contains a minor given in Figure 3.1. This, combined with $G_2$, the loop, gives either $C_3^2 \oplus_0 L$ or $K_4 \oplus_0 L$ as a proper minor of $G$; a contradiction. Hence if $G$ is an excluded minor for $\mathcal{V}_2$, then $\kappa(G) \neq 1$.

To find the excluded minors of connectivity at least four, we use the result of Halin and Jung, Lemma 2.3.3. It is straightforward to check that both $K_5$ and $K_{2,2,2}$ are excluded minors for the class $\mathcal{V}_2$. Combining this with Lemma 2.3.3, we immediately obtain the following corollary from which it follows that $\mathcal{V}_2$ contains no 4-connected graphs.

**Corollary 3.3.3.** Let $G$ be a simple graph with $\kappa(G) \geq 4$. Then $G$ is an excluded minor for $\mathcal{V}_2$ if and only if $G \cong K_5$.

In order to prove the main theorem of this section, we will use the following two lemmas. The next results help characterize the structure of the connected excluded minors for $\mathcal{V}_2$.

**Lemma 3.3.4.** If $G$ is a connected excluded minor for $\mathcal{V}_2$, then $G$ has no loops.

*Proof.* Suppose $G$ has a loop $f$ meeting a vertex $w$. Notice that $G - w$ has no two vertex-disjoint cycles; otherwise $G$ would contain $3L$ as a proper minor, a contradiction. Also, there is no single vertex of $G - w$ meeting all cycles of $G - w$; otherwise $G \in \mathcal{V}_2$. By Theorem 3.1.2, $G - w$ can be obtained from a subdivision $G_0$ of a graph listed in Theorem 3.1.1 by adding a forest and at most one edge joining each tree of the forest to $G_0$. Therefore, $G$ contains either $K_4 \oplus_0 L$ or $C_3^2 \oplus_0 L$ as a proper minor; a contradiction. Hence, $G$ has no loops. □
Lemma 3.3.5. If $G$ is a connected excluded minor for $V_2$, then $G$ has no three parallel edges with the same end vertices.

Proof. If $G$ had three parallel edges $\{f, g, h\}$ with the same end vertices, then $G \setminus f$ has two vertices whose removal is cycle-free, one of which is incident with $f$. This is a contradiction since the removal of those same two vertices from $G$ is cycle-free.

3.4 Excluded Minors for $V_2$ with $\kappa(G) = 2$

The following theorem gives the excluded minors with $\kappa(G) = 2$ and relies heavily on Theorem 3.1.2. We use this characterization when we decompose the graph along a 2-separation. The argument itself breaks into two main parts, when the basepoint of the 2-separation is an edge of the graph, and when it is not. We then find the structure of the components of the 2-sum.

Theorem 3.4.1. If $G$ is an excluded minor for $V_2$ with $\kappa(G) = 2$, then $G$ is one of the graphs shown in Figure 3.4.

Proof. As $\kappa(G) = 2$, we can write $G = \bigoplus_{i=1}^{m} G_i$ where there is a single edge $e$ that is the basepoint of this 2-sum, $a$ and $b$ are the vertex ends of $e$, and each $G_i$ and each $G_i/e$ is 2-connected. Since $G$ is an excluded minor, $G - \{u, v\}$ has a cycle for every two vertices $u, v \in V(G)$. Now $G - \{a, b\}$ has a cycle $C$ with vertex set $w_1, w_2, \ldots, w_k$ in, without loss of generality, $G_1$. If $G_2$ has two vertex-disjoint cycles, then $G$ contains $3L$ as a minor, so $G_2$ contains no two vertex-disjoint cycles.

3.4.2. $m = 2$.

Suppose that $m \geq 3$. Since each $G_i$ is 2-connected, there are cycles in $G_2$ and $G_3$ through $a$ and $b$. Let $P_i$ be a path from $a$ to $b$ in $G_i \setminus e$ for $i \in \{2, 3\}$. Let $P_2 = au_1u_2u_3 \ldots u_kb$ in $G_2$ and $P_3 = av_1v_2 \ldots v_kb$ in $G_3$. Then $G/au_1$ has two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these is the composite vertex formed when contracting $au_1$ and
the other is a vertex $w_l$ of $C$ in $G_1$. However, $G - \{a, w_l\}$ has a cycle $C_2$ which lies in $G_2$ since $G - \{u_1, a, w_l\}$ has no cycles. Similarly, $G - \{b, w_m\}$ has a cycle $C_3$ which lies in $G_3$ for some vertex $w_m$ of $C$. Therefore $G$ has three vertex-disjoint cycles as a proper minor; a contradiction. Thus, 3.4.2 holds.

The remainder of the proof of the theorem decomposes into two main cases: when $G$ has an edge joining $a$ and $b$, and when $G$ has no such edge. The next lemmas treats these cases in order.

**Lemma 3.4.3.** If $G$ has an edge $h$ joining $a$ and $b$, then $G$ is isomorphic to one of $S(K_4, C_2^3)$, $S(C_3^2, C_3^2)$, or $Z$.

**Proof.** The excluded minors containing an edge joining $a$ and $b$ are found by decomposing the graph along the 2-sum as given above, and then finding the structure of both $G_1$ and $G_2$. 

FIGURE 3.4: Excluded Minors for $V_2$ with $\kappa(G) = 2$
In 3.4.5, it is shown that $G_2$ has exactly three vertices, and it follows that $G_2$ is isomorphic to $C_3^2$.

The graph $G_2$ has a vertex $v$ meeting all cycles. Otherwise, Corollary 3.1.3 implies that since $G_2$ does not contain two vertex-disjoint cycles. Since $G_1 - \{a, b\}$ contains a cycle $C$, it follows that $G$ contains either a $C_3^2 \oplus L$, or a $K_4 \oplus L$ as a proper minor. This is a contradiction.

**3.4.4. Neither $a$ nor $b$ meets every cycle of $G_2$.**

To see this, suppose $a$ meets all cycles of $G_2$. Since $G$ is an excluded minor, $G/h$ has two vertices whose removal is cycle-free, one of which is the composite vertex, and the other a vertex $w_p$ of the cycle $C$. However, $G - \{w_p, a\}$ is cycle-free; a contradiction. Therefore, 3.4.4 holds.

**3.4.5. $G_2 \cong C_3^2$.**

We now show that every edge of $G_2$ is incident with $a$ or $b$. Suppose $G_2$ has an edge $f$ not incident with $a$ or $b$. By Lemma 3.1.4, $G/f$ has two vertices whose removal is cycle-free, one of which is the composite vertex $v_f$. The other vertex is one of the vertices $w_q$ in the cycle $C$ in $G_2$. Let $v'_f$ and $v''_f$ be the ends of $f$. Since $G - \{w_q, v'_f, v''_f\}$ is cycle-free, $G_1 - w_q$ is cycle-free. We know that $G_2$ has a vertex $v \notin \{a, b\}$ such that $G_2 - v$ is cycle-free. However, $G - \{w_q, v\}$ is cycle-free as this graph is obtained from the two forests $G_1 - w_q$ and $G_2 - v$ by identifying the edge $h$ in each. This contradiction implies that every edge of $G_2$ meets $a$ or $b$.

It follows from the fact that $G_2/e$ is 2-connected that $G_2$ has exactly three distinct vertices, $a, b,$ and $v$. Since $\kappa(G) = 2$, the vertex $v \in G_2$ is adjacent to both $a$ and $b$. Moreover, by Lemma 3.1.6 (iii), $d(v) \neq 2$, so $v$ is in a 2-cycle. By symmetry, suppose $a$ and $v$ are in a 2-cycle. If $b$ and $v$ are not in a 2-cycle, then $a$ meets every cycle of $G_2$, contradicting 3.4.4.
Therefore, \( bv \) is also in a 2-cycle and the exact structure of \( G_2 \) is \( C_3^2 \), a doubled triangle. Hence, we have \( G_2 \cong C_3^2 \), that is, 3.4.5 holds.

We now know the structure of \( G_2 \). In what follows, we find the structure of \( G_1 \) to obtain the remaining excluded minors for \( V_2 \) when there is an edge joining \( a \) and \( b \).

3.4.6. \( V(G_1) = \{a,b\} \cup V(C) \). Moreover, \( G_1 - \{a,b\} = C \), and every vertex of \( C \) is adjacent to \( a \) or \( b \).

Suppose there is a vertex \( s \) of \( G_1 \) that is not in \( \{a,b\} \cup C \). Also suppose an edge \( f \) joins \( s \) to some vertex \( t \) where \( t \) is not in a cycle of \( C \). Since \( G \) is an excluded minor, after contracting the edge \( f \), there are two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these vertices is the composite vertex, \( s' \). However, in \( (G/f) - s' \), there are two vertex-disjoint cycles: the cycle \( C \), and a cycle of \( G_2/e \). This is a contradiction. We now know that all neighbors of \( s \) are in \( V(C) \).

The vertex \( s \) is adjacent to neither \( a \) nor \( b \), since by the above argument every edge of \( G_1 \) is incident with a vertex in \( V(C) \). Now, notice that \( s \) is adjacent to at most two vertices of \( V(C) \); otherwise \( G_1 - \{a,b\} \) contains a \( K_4 \)-minor and so \( G \) has a \( K_4 \oplus_0 L \)-minor, a contradiction. Since \( \kappa(G) = 2 \), we see that \( s \) is adjacent to exactly two vertices of \( C \), call them \( w_i' \) and \( w_i'' \). If the edge joining \( s \) and \( w_i' \) is in a 2-cycle and the edge joining \( s \) and \( w_i'' \) is in a 2-cycle, then \( G \) contains a \( C_3^2 \oplus_0 L \)-minor; a contradiction. Hence, we may assume that there is no 2-cycle with vertex set \( \{s,w_i''\} \). However, by Lemma 3.1.6, we have that (iii), \( d(s) \geq 3 \) so there is a 2-cycle with vertex set \( \{s,w_i'\} \). By contracting an edge joining \( s \) and \( w_i' \) and then removing the newly created composite vertex \( s' \) and \( v \), we get a cycle-free graph. However, \( G - \{w_i',v\} \) also has no cycles since \( s \) has degree one in \( G - \{w_i',v\} \); a contradiction.

If there was an edge \( f \) in \( s\iota(G_1) - \{a,b\} \) not contained in \( C \), then this edge \( f \) runs between two vertices of \( C \) since \( V(G_1) = \{a,b\} \cup V(C) \). However, now \( C \) has a smaller cycle, say \( C' \).
containing the edge \( f \) and a vertex not contained in \( C' \). Since \( C \) was picked arbitrarily, this is a contradiction 3.4.6. Hence, \( \text{si}(G_1) - \{a, b\} \) is a cycle \( C \). Now, if \( \text{si}(G_1) - \{a, b\} \) is not simple, it has a 2-cycle. Let \( C'' \) be the 2-cycle and there is a vertex \( v \) of \( C'' \) not contained in that 2-cycle and since \( C \) was picked arbitrarily, by 3.4.6, this is a contradiction. By Lemma 3.1.6 \( \delta(G) \geq 3 \), so if \( C \) is not a 2-cycle, then every vertex of \( C \) is adjacent to \( a \) or \( b \). If \( C \) is a 2-cycle, again every vertex of \( C \) is adjacent to \( a \) or \( b \) since \( \kappa(G) = 2 \). Thus, 3.4.6 holds.

First, we suppose \( C \) has three distinct vertices. Since every vertex of \( C \) is adjacent to either \( a \) or \( b \), we may assume two vertices of \( C \) are adjacent to \( a \) and a third vertex of \( C \) is adjacent to \( b \). Then \( G \) is \( S(K_4, C^2_3) \).

We are left with the case when \( C \) is a 2-cycle. Let \( V(C) = \{w_1, w_2\} \). If \( C \) is the only pair of parallel edges of \( G_1 \), then \( w_1 \) and \( w_2 \) are adjacent to both \( a \) and \( b \). Hence, \( G_2 \) is \( Z \). We may now suppose there is 2-cycle between \( w_1 \) and \( a \). Suppose first that \( w_2 \) is adjacent to \( b \). There is no 2-cycle from \( w_2 \) to \( b \); otherwise \( G \) contains a \( C^2_5 \)-minor, a 5-cycle where every edge is replaced with a pair of parallel edges, and \( C^2_5 \) is an excluded minor for \( V_2 \). If \( w_2 \) is adjacent to \( a \), then \( G \) is the series connection of copies of \( C^2_3 \), an excluded minor for \( V_2 \). So, \( w_2 \) is adjacent to only \( w_1 \) and \( b \). Hence, \( G - \{w_1, v\} \) is cycle-free; a contradiction. We may now suppose that \( w_2 \) is not adjacent to \( b \). Since \( \kappa(G) = 2 \), \( w_2 \) is adjacent to \( a \) and \( w_1 \) is adjacent to \( b \). Therefore, \( G \) is isomorphic to \( D \). This concludes the argument when \( G \) has an edge joining \( a \) and \( b \).

\[ \square \]

**Lemma 3.4.7.** If \( G \) has no edge joining \( a \) and \( b \), then \( G \) is isomorphic to one of \( S(K_4, K_4) \), \( C^2_5 \), \( S(K_4, C^2_3) \), \( Y \), or \( N \).

**Proof.** We begin by showing that

**3.4.8.** \( G_2 \setminus e \) has a vertex meeting all cycles. Moreover, this vertex is neither \( a \) nor \( b \).
Suppose \( G_2 \setminus e \) has no vertex meeting all cycles. Then, by Corollary 3.1.2, since \( G_2 \setminus e \) has no two vertex-disjoint cycles, \( G_2 \setminus e \) can be obtained from a subdivision \( G_0 \) of a graph listed in Theorem 3.1.1 by adding a forest and at most one edge joining each tree of the forest to \( G_0 \). Therefore \( G_2 \setminus e \) has either a \( C^3 \) or \( K_4 \)-minor. By deleting all edges other than the cycle \( C \) in \( G_1 \) and contracting the cycle down to a loop, we have that \( G \) contains either a \( C^3 \oplus_0 L \)-minor or a \( K_4 \oplus_0 L \)-minor; a contradiction. Therefore 3.4.8 holds.

We find the excluded minors when there is no edge joining \( a \) and \( b \) in two main cases: when \( G_2 \) has a vertex meeting all cycles and when \( G_2 \) has no such vertex.

First, we suppose that \( G_2 \) has a vertex, \( v \), meeting all cycles and find the structure of \( G_2 \) in 3.4.9. We proceed by finding the structure of \( G_1 \), and show in 3.4.10 that every vertex of \( G_1 \) is either \( a \), \( b \), or a vertex of the cycle \( C \) of \( G_1 - \{a, b\} \), and that every vertex of \( C \) is adjacent to \( a \) or \( b \). Using this structure, we extract the excluded minors for \( V_2 \) when \( C \) has size greater than four in 3.4.11, when \( C \) has size three in 3.4.12, and finally when \( C \) has size two in 3.4.13.

3.4.9. If \( G_2 \) has a vertex meeting all cycles, then \( G_2 \cong C^2_3 \), a triangle with two doubled edges and one single edge, the basepoint \( e \).

We begin by showing that every edge of \( G_2 \) is incident with \( a \) or \( b \). Suppose some edge \( e \) of \( G_2 \) is incident with neither \( a \) nor \( b \). Then \( G/e \) has two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these vertices is the composite vertex formed in the contraction. The other vertex is a vertex \( w_i \in V(C) \) for some \( i = 1, \ldots, k \). Now, \( G_1 - w_i \) is cycle-free and \( G_2 - v \) is cycle-free, which implies that \( G - \{w_i, v\} \) is cycle-free; a contradiction.

Now, we show that every vertex of \( G_2 \) is either \( a \), \( b \), or is adjacent to both \( a \) and \( b \). Let \( u \) be a vertex of \( G_2 \), distinct from both \( a \) and \( b \). By the above argument, \( u \) is adjacent to, without loss of generality, \( a \). Suppose \( u \) is not adjacent to \( b \). Since \( G_2 \) is 2-connected, \( u \) is
adjacent to a vertex \( u' \) of \( G_2 \). However, the edge joining \( u \) and \( u' \) is adjacent to neither \( a \) nor \( b \); a contradiction. Hence every vertex of \( G_2 \) is either \( a \), \( b \), or adjacent to both \( a \) and \( b \).

Suppose \( u \) and \( u' \) are two arbitrary vertices of \( G_2 \), both of which are neither \( a \) nor \( b \). Notice that \( u \) and \( u' \) are not adjacent; otherwise the edge joining \( u \) and \( u' \) is incident with neither \( a \) nor \( b \), a contradiction. Also, since \( G_2/e \) is 2-connected, there is at most one vertex distinct from \( a \) and \( b \) in \( G_2 \). Hence, \( G_2 \) has exactly three vertices, \( a \), \( b \), and a third vertex, call it \( u \).

By Lemma 3.1.6, \( u \) has degree at least three, so \( u \) is in a 2-cycle, \( \{f, g\} \), incident to, without loss of generality, \( a \). Suppose the edge from \( u \) to \( b \) is not in a 2-cycle. Then \( u \) has degree three in \( G \). Since \( G \) is an excluded minor for \( \mathcal{V}_2 \), the graph \( G/f \) has two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these vertices is the composite vertex formed in the contraction. The other vertex is a vertex \( w_i \in V(C) \) for some \( i = 1, \ldots k \). So \( G - \{a, u, w_i\} \) is cycle-free. However, as \( u \) is only adjacent to \( a \) and \( b \) and there is only a single edge joining \( u \) and \( b \), we have that \( G - \{w_i, a\} \) is cycle-free; a contradiction. Therefore, the edge from \( u \) to \( b \) is in a 2-cycle. Therefore \( G \) has no loops or parallel classes of size greater than 2. Hence, 3.4.9 gives the structure of \( G_2 \).

3.4.10. If \( G_2 \) has a vertex meeting all cycles, then \( V(G_1) = \{a, b\} \cup V(C) \). Moreover, \( G_1 - \{a, b\} = C \), and every vertex of \( C \) is adjacent to \( a \) or \( b \).

The previous statement is an immediate consequence of the argument given in 3.4.6, in this case where there is an edge \( e \) joining \( a \) and \( b \). We note \( G_2 \) is now \( C_3^{2-} \), the deletion of the edge \( e \), while in the previous argument \( G_2 \) was \( C_3^2 \). However, in this proof, the edge \( e \) is never used. Thus, 3.4.10 holds.

3.4.11. If \( G_2 \) has a vertex meeting all cycles and \( |C| \geq 4 \), then \( G \) is isomorphic to \( N \).

No three vertices of \( C \) are adjacent to \( a \); otherwise \( G \) has \( K_4 \oplus 0L \) as a minor. By symmetry, no three vertices of \( C \) are adjacent to \( b \). Therefore \( C \) has length exactly four. Let \( w_1, w_2, w_3, \)
and $w_4$ be the vertices of $C$, labelled cyclically. Up to symmetry, either $w_1$ and $w_2$ are
adjacent to $a$, and $w_3$ and $w_4$ are adjacent to $b$; or $w_1$ and $w_3$ are adjacent to $a$, and $w_2$ and
$w_4$ are adjacent to $b$. But, if $w_1$ and $w_2$ are adjacent to $a$ and $w_3$ and $w_4$ are adjacent to $b$,
then $G$ contains a $C_5^2$, an excluded minor for $V_2$, as a proper minor; a contradiction. Hence,
$w_1$ and $w_3$ are adjacent to $a$ and $w_2$ and $w_4$ are adjacent to $b$, and $G$ is isomorphic to $N$.
So 3.4.11 holds.

3.4.12. If $G_2$ has a vertex meeting all cycles and $|C| = 3$, then $G \in V_2$.

Let $w_1, w_2$ and $w_3$ be the vertices of $C$. Up to symmetry, we will assume $w_1$ and $w_2$ are
adjacent to $a$, and $w_3$ is adjacent to $b$. Notice that $w_3$ is not adjacent to $a$; otherwise $G$
contains $K_4 \oplus_0 L$ as a proper minor, a contradiction. Notice also that $w_3$ is not in a 2-cycle
with $b$; otherwise $G$ contains $C_5^2$ as a proper minor, a contradiction. Therefore $w_3$ is adjacent
to only the vertices $w_1$, $w_2$, and $b$.

If both $w_1$ and $w_2$ are adjacent to $b$, then $G$ contains $K_4 \oplus_0 L$ as a proper minor; a
contradiction. Hence, at most one of $w_1$ and $w_2$ is adjacent to both $a$ and $b$. If there are no
2-cycles in $G_1$ incident with $a$ or $b$, then $G - \{w_1, v\}$ or $G - \{w_2, v\}$ is cycle-free.

Therefore we may assume that there is a 2-cycle with vertex set $\{a, w_1\}$. If $w_2$ is adjacent
to both $a$ and $b$, then $G$ contains $C_5^2$ as a proper minor; a contradiction. This can be seen
by deleting the edge joining $w_2$ and $a$, and contracting the edge joining $w_2$ and $w_3$. If both
$w_1$ and $a$ are in a 2-cycle and $w_2$ and $a$ are in 2-cycles, then $G$ contains $C_5^2 \oplus_0 L$ as a proper
minor; a contradiction. This can be seen by deleting the edge $w_3b$ and contracting the edge
$w_3w_2$. Therefore $G_1$ is exactly a triangle $C = w_1w_2w_3$ with a 2-cycle joining $w_1$ and $a$, a
single edge $w_2a$, and a single edge $w_3b$. However, in this case $G - \{w_1, v\}$ is cycle-free so
$G \in V_2$; a contradiction. Therefore, 3.4.12 holds.

3.4.13. If $G_2$ has a vertex meeting all cycles and $|C| = 2$, then $G$ is isomorphic to $C_5^2$. 

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Let \( w_1 \) and \( w_2 \) be the vertices of the 2-cycle \( C \). Since \( G - \{w_1, v\} \) has a cycle, without loss of generality, \( w_2 \) and \( a \) are in a 2-cycle. Similarly, either \( w_1 \) and \( a \), or \( w_1 \) and \( b \) are in a 2-cycle. If \( w_1 \) and \( a \) are in a 2-cycle, then \( G \) contains \( C_3^2 \oplus \alpha L \) as a proper minor; a contradiction. This can be seen by deleting the 2-cycle from \( a \) to \( v \). Therefore, \( w_1 \) and \( b \) are in a 2-cycle, and \( G \) is isomorphic to the excluded minor \( C_3^2 \). Hence, 3.4.13 holds.

We are now left with the case when \( G \) has no vertex meeting all cycles and can be obtained from a subdivision \( G_0 \) of a graph listed in Theorem 3.1.1 (a)-(f). However, \( G \) is not \( C_3^2 \); otherwise \( G \) has an edge joining \( a \) and \( b \). Also, \( G \) is not one of the graphs listed in (c), (d), or (f); otherwise \( G_0 \) contains a \( K_4 \)-minor, and so \( G \) contains \( K_4 \oplus \alpha L \) as a proper minor, a contradiction. Therefore, \( G \) is one of the graphs in (b) or (e).

### 3.4.14. If \( G \) has no vertex meeting all cycles, then \( V(G_1) = \{a, b\} \cup V(C) \). Moreover, \( si(G_1 - \{a, b\}) = C \), and every vertex of \( C \) is adjacent to \( a \) or \( b \).

The previous statement is an immediate consequence of the argument given in 3.4.6, when there is an edge \( e \) joining \( a \) and \( b \). We note that \( G \) now contains a \( K_4 \)-minor and hence a \( C_3^{3^{-}} \)-minor, whereas in 3.4.6, we have that \( G \) is \( C_3^2 \). However, the proof of 3.4.6, the edge \( e \) is never used. Thus, 3.4.14 holds.

If \( G \) is one of the graphs in (e), then the underlying simple graph is a wheel. If this wheel has at least four spokes, then the basepoint \( e \) of the 2-sum is a rim edge and not a spoke; otherwise \( G_2 \setminus e \) contains a \( K_4 \)-minor and \( G \) contains \( K_4 \oplus \alpha L \) as a proper minor, a contradiction. If \( G \) is (e) a wheel of size at least four and the basepoint is a rim edge, then by contracting an edge of the wheel not adjacent to the center vertex, \( v_c \), or the vertices \( a \) or \( b \), we see that \( G \) has two vertices whose removal is cycle-free. One of these vertices is the composite vertex, and the other is a vertex \( w_i \) of the cycle \( C \). However, the removal of \( v_c \) and \( w_i \) is still cycle-free in \( G \) as \( G_2 \setminus e - \{v_c\} \) is cycle-free and \( G \in \mathcal{V}_2 \); a contradiction.
Hence, $\text{si}(G_2)$ is $K_4$. We now consider the structure of $G_1$. Suppose the cycle $C$ in $G_1$ has at least four vertices. Since each vertex of $C$ is adjacent to either $a$ or $b$, it is easy to check that $G$ contains either a $K_4 \oplus 0$ as a minor, a $C_5^2$-minor, or an $N$-minor. Therefore, we may assume $C$ is either a 2-cycle or a 3-cycle. If $C$ is a 3-cycle, then, by symmetry, we may assume two vertices of $C$ are adjacent to $a$ and one is adjacent to $b$, which implies that $G$ is $S(K_4, K_4)$.

If $C$ is a 2-cycle, then let $w_1, w_2 \in V(C)$ and, by symmetry, we may assume there is an edge from $w_1$ to $a$ and from $w_2$ to $b$. Let $v \neq a$ or $b$, where $v$ is a vertex of $G_2$. Notice that there cannot be 2-cycles with vertex sets $\{w_1, a\}$ and $\{w_2, b\}$; otherwise $G$ contains $C_5^2$ as a proper minor, a contradiction. If $G$ has both edges from $w_1$ to $b$ and $w_2$ to $a$, then $G$ is the excluded minor $Y$. Also, if $G$ has $w_1$ and $a$ in a 2-cycle and an edge from $w_2$ to $a$, then $G$ is $S(K_4, C_3^2)$. Now, we are left with $G_1$ having edges from $w_1$ to $a$ and from $w_2$ to $b$. There may also be either a 2-cycle between $w_1$ and $a$ and no other additional edges in $G_1$; or there may be an edge from $w_2$ to $a$ and no other additional edges in $G_1$.

Now we consider $G_2$. Suppose $G_2$ is one of the graphs in (e) and the basepoint $e$ of the 2-sum is a rim edge and not a spoke. Then $G' \setminus e - v$ is cycle-free, where $v$ is the center vertex and $v \neq a$ or $b$. Recall that $G_1$ is a 2-cycle with vertices $w_1$ and $w_2$ with two additional adjacent vertices $a$ and $b$ where $w_1$ is adjacent to $a$ and $w_2$ is adjacent to $b$. Also, there may be either a 2-cycle between $w_1$ and $a$ and no other additional edges in $G_1$; or there may be an edge from $w_2$ to $a$ and no other additional edges in $G_1$. In the first case, $G - \{w_1, v\}$ is cycle-free; and in the second case, $G - \{w_2, v\}$ is cycle-free; a contradiction. Hence, the basepoint $e$ is a spoke edge, and not a rim edge, which means that either $a$ or $b$ is the center vertex. The graph $G_2$ has at most three 2-cycles, all adjacent to the center vertex. Since there is no edge joining $a$ and $b$, the basepoint $e$ not in a 2-cycle. If there are no 2-cycles in $G_2$ or exactly one 2-cycle in $G_2$, then it is easy to see that $G_2$ is isomorphic to one of the graphs in (e), with basepoint $e$ being a rim edge. Hence, $G_2$ has exactly two 2-cycles, both
of which are adjacent to $a$ or $b$. If two 2-cycles are adjacent to $a$ or two 2-cycles are adjacent to $b$, then $G$ has $C_3^2 \oplus_0 L$ as a proper minor; a contradiction. This can be seen by deleting the edge joining $a$ and $w_1$ and the edge joining $a$ and $w_2$. 

This case analysis completes the determination of the excluded minors $G$ of $V_2$ with $\kappa(G) = 2$. 

3.5 Excluded Minors for $V_2$ with $\kappa(G) = 3$

In this section, we find the excluded minors $G$ for $V_2$ with $\kappa(G) = 3$. The simple 3-connected excluded minors for $V_2$ can be determined using the 3-connected excluded minors for $S$. To obtain the non-simple excluded minors, we show that a non-simple excluded minor consists of a 2-cycle and another cycle, vertex disjoint from the first, with all other edges joining a vertex of the 2-cycle to vertex of the other cycle.

Theorem 3.5.1. Let $G$ be a simple graph with $\kappa(G) = 3$. Then $G$ is an excluded minor for the class $V_2$ if and only if $G$ is isomorphic to one of the graphs shown in Figure 3.5.

To prove this theorem, we will use the following two lemmas.

Lemma 3.5.2. Let $G$ be a simple excluded minor for $V_2$ with $\kappa(G) = 3$. Then $G$ is isomorphic to $R$, $H_8$, $Q_3$, or $S$.

Proof. By Lemma 2.5.2, the graph $G \notin S$, but every proper minor is a member of $S$. Hence, $G$ is a 3-connected excluded minor for $S$. Notice that $U$ has $K_4 \oplus_0 L$ as a minor, while $V$ has $3L$ as a minor. However, it is straightforward to check that the remaining excluded minors for $S$ with $\kappa(G) = 3$, shown in Figure 3.5 (a)-(d), are also excluded minors for $V_2$. 

Lemma 3.5.3. Let $G$ be a non-simple excluded minor for $V_2$ with $\kappa(G) = 3$. Then $G$ is isomorphic to $X$, $K_4^2$ or $W$. 

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Proof. By Lemma 3.3.4, $G$ has no loops. Thus, $G$ has a pair $\{f, g\}$ of parallel edges with end vertices $u$ and $v$. By Lemma 3.3.5, no two vertices of $G$ are joined by three or more parallel edges. As $G$ is an excluded minor for $\mathcal{V}_2$, the graph $G - \{u, v\}$ has a cycle.

3.5.4. Let $C$ be a cycle in $G - \{u, v\}$. Every edge of $G$ is incident with $u, v$, or a vertex of $V(C)$.

Suppose there is an edge $e$ that is not incident with $u, v$, or a vertex of $C$. Now, in $G/e$ there are two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these vertices is the composite vertex $w$ resulting from contracting $e$. But $(G\setminus e) - w$ has vertex disjoint cycles on $\{u, v\}$ and $V(C)$, so no single vertex deletion from $(G\setminus e) - w$ is cycle-free, a contradiction. Thus, 3.5.4 holds.
Let \( V(G) = \{u, v\} \cup V(C) \cup X \), where \( X = \{x_1, x_2, \ldots, x_n\} \) and \( V(C) = \{w_1, \ldots, w_k\} \) for some \( n \geq 1 \) and \( k \geq 2 \). Since \( \text{si}(G) \) is 3-connected, each \( x_i \) for \( 1 \leq i \leq n \) is adjacent to at least three distinct vertices. No two distinct \( x_i \) and \( x_j \) are adjacent; otherwise the edge joining \( x_i \) to \( x_j \) is not incident with \( u, v \), or a vertex of \( V(C) \), contradicting 3.5.4. Also, no \( x_i \) is adjacent to three or more distinct vertices of \( V(C) \); otherwise \( G \) contains \( K_4 \oplus \odot_0 L \) as a proper minor, a contradiction. Hence, each \( x_i \) can only be adjacent to \( u, v \), and two distinct vertices of \( V(C) \). Since \( G \) is 3-connected, each \( x_i \) has either three or four neighbors.

**3.5.5.** \( x_i \) is not in a 2-cycle containing \( u \) or \( v \).

Suppose \( x_i \) is in a 2-cycle containing, without loss of generality \( u \). Since \( \text{si}(G) \) is 3-connected, \( u \) is adjacent to a vertex \( y \) that is different from \( v \) and \( x_i \). Let \( h \) be an edge joining \( u \) and \( y \). Upon deleting \( h \), there are two vertices, \( v_1 \) and \( v_2 \) say, whose removal leaves a cycle-free graph. Clearly \( \{v_1, v_2\} \cap \{u, y\} = \emptyset \); otherwise there are two vertices of \( G \) whose removal is cycle-free, a contradiction. Since \( u \) and \( v \) are in a 2-cycle and \( u \) and \( x_i \) are in a 2-cycle, but \( u \notin \{v_1, v_2\} \), we must have that \( \{v_1, v_2\} = \{v, x_i\} \). But \( (G \setminus h) \setminus \{v_1, v_2\} \) has \( C \) as a cycle; a contradiction. Hence, 3.5.5 holds.

**3.5.6.** If \( x_i \) is adjacent to \( u \) and \( v \), then it is also adjacent to exactly two vertices of \( V(C) \).

Suppose that some \( x_i \) is adjacent to \( u, v \), and exactly one vertex of \( V(C) \), say \( w_1 \). After contracting an edge joining \( x_i \) to \( w_1 \), there are two vertices whose removal is cycle-free. By Lemma 3.1.4, one of them is the composite vertex formed in the contraction. The other is either \( u \) or \( v \), say \( u \), since \( u \) and \( v \) are in a 2-cycle. This also implies that \( G - \{u, x_i, w_1\} \) is cycle-free. However, since \( G \) is an excluded minor, \( G - \{u, w_1\} \) has a cycle and this cycle contains \( x_i \), but \( x_i \) is only adjacent to \( v \) in \( G - \{u, w_1\} \). Therefore \( x_i \) is in a 2-cycle with \( v \), but this contradicts 3.5.5. Therefore, \( x_i \) is adjacent to some other vertex, and by 3.5.4 it must be a vertex of \( V(C) \). Since \( x_i \) is adjacent to at most four distinct vertices, it is adjacent to exactly four vertices, and 3.5.6 holds.
Henceforth we may assume that, for any cycle $C$ in $G - \{u, v\}$, any $x_i \in X$ is adjacent to exactly two vertices of $V(C)$ and one or two vertices of $\{u, v\}$. Choose the vertices $u$ and $v$ in a 2-cycle and the cycle $C$ of $G - \{u, v\}$ so that $|V(C)|$ is a minimum. Then $C$ has no chords.

Since each $x_i$ is adjacent to exactly two vertices of $V(C)$, there is a cycle of $G - \{u, v\}$ through $x_i$ that has at most $2 + \left\lfloor \frac{|V(C)|}{2} \right\rfloor$ edges. Since $|V(C)|$ is a minimum, it follows that $V(C) \leq 4$. Hence, if $X \neq \emptyset$, then $|V(C)| = 2, 3$ or $4$.

3.5.7. If $|V(C)| = 2$, then $G \cong K^2_4$.

Let $V(C) = \{w_1, w_2\}$. Suppose $X \neq \emptyset$. Since every $x_i$ is adjacent to exactly two vertices of $V(C)$, the vertex $x_i$ is adjacent to both $w_1$ and $w_2$. By switching the roles of the 2-cycle containing $\{u, v\}$ and the 2-cycle containing $\{w_1, w_2\}$, we see that similarly $x_i$ is adjacent to both $u$ and $v$. Hence, every $x_i$ is adjacent to $w_1, w_2, u,$ and $v$. If there are two or more distinct $x_i$, then by deleting an edge joining $u$ and $v$, we see that the resulting graph has $Y$ as a proper minor; a contradiction. Therefore, there is at most one $x_i$. Suppose that there is exactly one $x_i$, implying $|V(G)| = 5$. By 3.5.5, the vertex $x_i$ is not in a 2-cycle with $u$ or $v$. Similarly, by switching the roles of the 2-cycle containing $\{u, v\}$ and the 2-cycle containing $\{w_1, w_2\}$, we see that $x_i$ is not in a 2-cycle with $w_1$ or $w_2$. Therefore, $x_i$ is not in a 2-cycle. Since $G - \{u, w_1\}$ has a cycle, there is an edge joining $v$ and $w_2$. By symmetry there is an edge joining $v$ and $w_1$, an edge joining $u$ and $w_1$, and an edge joining $u$ and $w_2$. However, this graph has a proper $K_5$ minor, which can be seen by deleting an edge joining $u$ to $v$ and an edge joining $w_1$ to $w_2$; a contradiction. Now, $X$ is empty and $|V(G)| = 4$. Therefore, $G \cong K^2_4$. Thus 3.5.7 holds.

Henceforth, we may assume that if $X \neq \emptyset$, then $|V(C)| \in \{3, 4\}$. We may also assume that $G$ has no two vertex-disjoint 2-cycles; otherwise the choice of $C$ is violated.

3.5.8. If $|V(C)| = 4$ and $X \neq \emptyset$, then $G \cong B$. 

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Let the vertices of $C$, in cycle order, be $w_1, w_2, w_3,$ and $w_4$. Since $|V(C)|$ is a minimum, there is no $x_i$ joining two adjacent vertices of $C$; otherwise there is a smaller cycle of size three through $x_i$ in $G - \{u, v\}$. Therefore $x_i$ is adjacent to two nonadjacent vertices of $C$.

Assume $X \neq \emptyset$. Let $x_i$ be a vertex adjacent to $w_1$ and $w_3$. Suppose some $x_j$ different from $x_i$ is adjacent to $w_2$. Since $x_j$ is also adjacent to two nonadjacent vertices by the argument in the previous paragraph, it is adjacent to $w_4$. However, since we picked $C$ to be an arbitrary cycle of minimum size, by taking $C$ to be the cycle through $\{w_1, w_2, x_j, w_4\}$, there is an edge from $x_i$ to $w_3$ that is not adjacent to $C$ or $\{u, v\}$; a contradiction. Hence, every $x_i \in X$ is adjacent to the same two nonadjacent vertices of $C$. Say every $x_i$ is adjacent to $w_1$ and $w_3$.

Since $\kappa(\text{si}(G)) = 3$, each vertex in $\{w_2, w_4\} \cup X$ is adjacent to $u$ or $v$. Thus, without loss of generality, there are at least two vertices of $\{w_2, w_4\} \cup X$ adjacent to $v$.

We show in what follows that $\{w_2, w_4\} \cup X$ contains a vertex adjacent to $u$ and a vertex adjacent to $v$. Suppose no vertex in $\{w_2, w_4\} \cup X$ is adjacent to $u$. Then $u$ is adjacent to only $v, w_1,$ and $w_3$. Moreover, $v$ is adjacent to all vertices in $\{u, w_2, w_4\} \cup X$ as well as possibly $w_1$ or $w_3$. Now $G - \{v, w_1\}$ has a cycle. This cycle, call it $D$, must be a 2-cycle. If $D$ does not meet $u$, then the choice of $C$ is contradicted. Thus, $D$ meets $u$ and so has vertex set $\{u, w_3\}$. By symmetry, $G$ has a cycle with vertex set $\{u, w_1\}$. But then deleting the composite vertex after contracting an edge joining $v$ and $w_2$ leaves a graph without a single vertex whose removal is cycle-free; a contradiction.

We may not suppose that there are two vertices, $z_1$ and $z_2$, in $\{w_2, w_4\} \cup X$ that are adjacent to $v$ and a different vertex, say $z_3$ in $\{w_2, w_4\} \cup X$ that is adjacent to $u$. Let $h$ be an edge joining $u$ and $z_3$. As $G$ is an excluded minor, upon contracting $h$, there are two vertices whose removal is cycle-free. By Lemma 3.1.5, the graph $G - \{u, x_3\}$ has a single vertex whose removal is cycle-free. If $|X| \geq 2$, then $G - \{u, z_3\}$ has a $K_4$-minor a contradiction. Thus, $|X| = 1$. 

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Suppose $\{w_2, w_4\} \cup X$ contains distinct vertices $z_1$ and $z_2$, that are both adjacent to $u$ and $v$. Take $z_3 \in (\{w_2, w_4\} \cup X) - \{z_1, z_2\}$. Since $w_1$ and $z_3$ are adjacent, $G - \{w_1, z_3\}$ has a single vertex whose removal is cycle-free. But there is no such vertex. Hence, at most one vertex in $\{w_2, w_4\} \cup X$ is adjacent to both $u$ and $v$.

Next assume that $v$ is adjacent to $z_3$. By the previous argument, $u$ is adjacent to neither $z_1$ nor $z_2$. Thus, without loss of generality, $u$ is adjacent to $w_3$. By Lemma 3.1.5, $G - \{w_1, z_2\}$ has a single vertex whose removal is cycle-free. But $G - \{w_1, z_1\}$ has no such vertex. We deduce that $v$ is not adjacent to $z_3$.

If $u$ is adjacent to $z_1$ or $z_2$, then $G$ has $B$ as a subgraph, so $G \cong B$. Thus, we may assume that $u$ is adjacent to neither $z_1$ nor $z_2$. Without loss of generality, $u$ is adjacent to $w_3$. If $d(v) > 4$, consider the graph obtained by removing an edge $s$ incident with $v$, leaving edges joining $v$ to $z_1$ and $z_2$ and a 2-cycle on $\{u, v\}$. Then $G \setminus s$ has two vertices, $v_1$ and $v_2$ say, whose removal is cycle-free, neither of which is $v$. Then $u \in \{v_1, v_2\}$. But $(G \setminus s) - u$ has no single vertex whose removal is cycle-free. Hence $d(v) = 4$. Now, $G - \{v, w_3\}$ has a cycle, so either $\{u, z_3\}$ is in a 2-cycle, or $u$ is adjacent to $w_1$. Assume that $\{u, z_3\}$ is in a 2-cycle. Upon deleting an edge $t$ joining $u$ and $w_3$ gives a graph having an $N$-minor. We deduce that $\{u, z_3\}$ is not in a 2-cycle, so $u$ is adjacent to $w_1$. By Lemma 3.1.5, the graph $G - \{v, z_2\}$ has a single vertex whose removal is cycle-free, however, there is no such vertex; a contradiction.

3.5.9. If $|V(C)| = 3$, then $X = \emptyset$.

Suppose first that $|X| \geq 2$. If every $x_i$ is adjacent to the same two vertices of $V(C)$, call them $w_1$ and $w_2$, then delete the edge joining $w_1$ and $w_2$. In $G \setminus w_1 w_2$, there are two vertices whose removal leaves a cycle-free graph, neither of which is $w_1$ or $w_2$; otherwise $G \in \mathcal{V}_2$, a contradiction. Since $\{u, v\}$ is in a 2-cycle, one of the vertices is either $u$ or $v$, say $u$. However, $(G \setminus w_1 w_2) - u$ has no vertex, in $V(G) - \{w_1, w_2\}$ whose removal is cycle-free; a contradiction. So, if $x_i$ is adjacent to $w_1$ and $w_2$, then, without loss of generality, $x_j$ is adjacent to $w_1$ and
Since $\text{si}(G)$ is 3-connected, $G - \{x_i, x_j\}$ is connected. Consider a path of minimum length in $G - \{x_i, x_j\}$, from $\{u, v\}$ to $V(C)$. Suppose this path is incident to $v$ and one of $w_2$ or $w_3$, say $w_3$. Then, upon contracting an edge joining $x_i$ and $w_2$, there are two vertices whose removal leaves a cycle-free graph. One of these is the composite vertex formed in the contraction. However, the removal of this vertex has two vertex-disjoint cycles, and there is no other vertex whose removal is cycle-free; a contradiction. Hence, we may assume this path is incident to $u$ and $w_1$. Upon deleting an edge $t$ of this path incident to $w_1$, there are two vertices whose removal is cycle-free. Again, $w_1$ is not one of these vertices; otherwise $G \in V_2$, a contradiction. However, the removal of no two vertices of $V(C) - w_1$ from $G \setminus t$ leaves a cycle-free graph; a contradiction. Therefore there is at most one $x_i$. We may now assume there is exactly one $x_i$, so the graph $G$ has exactly six vertices. Suppose $x_i$ is adjacent to $u$, $v$, $w_1$, and $w_2$, and, without loss of generality, $w_3$ is adjacent to $v$. Now, $w_3$ has only three neighbors as there is no edge joining $w_3$ to $u$; otherwise $G$ has a proper $Y$-minor. Suppose $d(v) \geq 5$. Then there is an edge $q$ incident with $v$ after whose removal from $G$, there is still the 2-cycle through $\{u, v\}$, an edge from $v$ to $x_i$, and an edge from $v$ to $w_3$. However, $G \setminus q$ has two vertices whose removal is cycle-free, one of which is $u$ since $\{u, v\}$ is in a 2-cycle. However, $(G \setminus q) - u$ has a $K_4$-minor, and there is no single vertex whose removal is cycle-free; a contradiction. Therefore, $d(v) = 4$. The only other possible edges are incident with $u$. As $G - \{v, x_i\}$ has a path from $\{w_1, w_2\}$ to $u$, we may assume that $u$ is adjacent to $w_1$. If $u$ is also adjacent to $w_2$, then $G$ has $K_5$ as a proper minor, which can be seen by contracting the edge $vw_3$ and deleting one of the edges joining $u$ and $v$. Therefore $u$ is only adjacent to $v, w_1$, and $x_i$, and by 3.5.5, the only other possible edge of $G$ creates a 2-cycle containing $\{u, w_1\}$. It follows that $G - \{v, w_1\}$ is cycle-free, a contradiction. Therefore, we may assume $x_i$ is adjacent to only $u, w_1$, and $w_2$. Suppose $u$ is also adjacent to $w_1$. If $u$ is adjacent to $w_3$, then $(G \setminus uw_1) - v$ has a vertex whose removal is cycle-free. However, $(G \setminus uw_1) - v$ has a $K_4$-minor, so there is no such
vertex; a contradiction. Therefore, $v$ is adjacent to $w_3$, but $u$ is not. If $u$ is adjacent to $w_2$, then upon contracting an edge joining $w_3$ and $v$, there are two vertices whose removal leaves a cycle-free graph. By Lemma 3.1.4, one of those vertices is the composite vertex formed in the contraction. However, after removing the composite vertex, there is still a $K_4$-minor so there is no other vertex whose removal is cycle-free; a contradiction. Therefore, if $u$ is adjacent to $w_1$, then $u$ is adjacent to only $v$, $x_i$, and $w_1$. Now, $G - \{v, w_1\}$ is cycle-free as the only 2-cycles of $G$ are incident to $u$ or $v$ and incident with no $x_i$. Hence $G \in \mathcal{V}_2$; a contradiction. Thus, $u$ is not adjacent to $w_1$. By symmetry, $u$ is not adjacent to $w_2$. However, $u$ is adjacent to three distinct vertices, so $u$ is adjacent to $w_3$. Hence, $u$ is adjacent to only $v$, $x_i$, and $w_3$.

Since $v$ is adjacent to at least three distinct vertices, it is adjacent to one of $w_1$ and $w_2$, say $w_2$. Also, $v$ is adjacent to either $w_1$ or $w_3$. If $v$ is adjacent to both $w_1$ and $w_3$, after contracting the edge $ux_i$, there are two vertices whose removal is cycle-free, one of which is the composite vertex $r$ by Lemma 3.1.4. So $(G/ux_i) - r$ has a $K_4$-minor and there is no other vertex whose removal is cycle-free; a contradiction. Hence, $v$ is adjacent to exactly one of $w_1$ and $w_3$. If $v$ is adjacent to $w_1$, then, since $G - \{u, w_2\}$ has a cycle, $\{v, w_1\}$ is the vertex set of a 2-cycle. By symmetry, $\{v, w_2\}$ is the vertex set of a 2-cycle. However, now $G$ has $W$ as a proper minor, which can be seen by contracting the edge $ux_i$; a contradiction. Hence, $v$ is adjacent to only $u$, $w_1$, and $w_3$. Since $G - \{u, w_1\}$ has a cycle, $\{v, w_3\}$ is the vertex set of a 2-cycle as we have eliminated all other possibilities. However, this graph has $S(K_4, C_3^2)$ as a proper minor, which can be seen by deleting an edge joining $v$ to $w_1$; a contradiction. Therefore, 3.5.9 holds.

Combining 3.5.7-3.5.9, we may assume $|V(C)| \geq 3$ and $X = \emptyset$. Recall that, since $|V(C)|$ is minimal, $G$ has no two vertex-disjoint 2-cycles. Also, $C$ has no chords so every vertex of $C$ is adjacent to $u$ or $v$. Recall that $f$ and $g$ are the edges joining $u$ and $v$ and the vertices of $C$, in cyclic order are $w_1, w_2, \ldots, w_k$. 77
Since \( G \) is an excluded-minor for \( \mathcal{V}_2 \) and \( \kappa(\text{si}(G)) = 3 \), the graph \( G \setminus f \) has two vertices whose removal is a tree, call it \( T_f \). Since neither \( u \) nor \( v \) is one of these vertices, they are \( w_a \) and \( w_b \) for some integers \( a \) and \( b \). Observe the following.

3.5.10. Every leaf of \( T_f \) is adjacent to both \( w_a \) and \( w_b \).

Since \( G \setminus f \) is 3-connected and every vertex \( w_i \in V(C) \) is adjacent to exactly two other vertices of \( V(C) \) and possibly \( u \) and \( v \), every \( w_i \) is adjacent to exactly 3 or 4 vertices for every \( i \in [k] \). Hence, the tree \( T_f = (G \setminus f) - \{w_a, w_b\} \) has at most four leaves by 3.5.10.

Suppose \( T_f \) has four leaves. Then every leaf of the tree is adjacent to both \( w_a \) and \( w_b \). Therefore, the leaves of the tree are \( u, v, w_c, \) and \( w_d \) for some integers \( c \) and \( d \). However, \( u \) and \( v \) are both leaves of the tree, and are also adjacent in \( G \setminus f \); a contradiction as no two leaves are adjacent. Thus, \( T_f \) does not have four leaves. Therefore, either the tree \( T_f \) is a path, or it has exactly three leaves.

3.5.11. If \( T_f \) is a path, then \( G \) is isomorphic to \( K_4^2 \), \( X \), or \( W \).

If the path has length one, then clearly \( G \) is isomorphic to \( K_4^2 \) so we will assume the path has length at least two.

Since \( u \) and \( v \) are adjacent in \( T_f \), they are not both leaves of \( T_f \). Suppose one of \( u \) or \( v \), say \( u \), is a leaf of the path \( T_f \). Then, the other leaf is \( w_j \) for some integer \( j \). As \( w_a \) and \( w_b \) are adjacent to the ends of the path, \( w_j \) is adjacent to both \( w_a \) and \( w_b \). Thus the neighbor of \( w_j \) on the path \( T_f \) must be \( v \). Hence, \( T_f \) has two edges, so \( |V(G)| = 5 \) and \( w_a \) and \( w_b \) are adjacent. Since \( G - \{v, w_b\} \) has a cycle and also every 2-cycle of \( G \) meets \( \{u, v\} \), the vertices \( \{u, w_a\} \) are in a 2-cycle. By symmetry, \( \{u, w_b\} \) is in a 2-cycle. Since \( G \) has no two vertex-disjoint 2-cycles, it follows that \( G \) is a subgraph of \( W \). As \( W \) is an excluded minor, we conclude that \( G = W \).

Now, we may assume that neither \( u \) nor \( v \) is an end of the path. This implies that the edge \( g \) lies in the interior of the path \( T_f \), so the path must have length at least three. In
fact, the path must have length exactly three because, if the path has length at least four, it has an end $w_l$ that is not adjacent to either $u$ or $v$; a contradiction. Thus $|V(G)| = 6$. Let $w_c$ and $w_d$ be the ends of the path for some integers $c$ and $d$.

By 3.5.10, $w_a$ and $w_b$ are adjacent to the ends of the path, and both $w_a$ and $w_b$ are adjacent to $u$ or $v$. If both $w_a$ and $w_b$ are adjacent to $u$ and $v$, then $G$ is isomorphic to the excluded minor $X$. Therefore $\text{si}(G) + f$ is as shown in Figure 3.6, where the dotted edge may or may not be present. We may assume that there is another parallel class other than $\{f, g\}$, otherwise $G \in \mathcal{V}_2$.

![Figure 3.6: Illustration of si$(G) + f$ when $T_f$ is a path. A dotted edge may or may not be present in the graph.](image)

Since $G$ is an excluded minor for $\mathcal{V}_2$, the graph obtained by deleting $\{v, w_c\}$ from $G$ has a cycle, so either $\{w_d, w_b\}$, $\{w_c, u\}$, or $\{w_c, w_b\}$ is in a 2-cycle. As $G$ has no two vertex-disjoint 2-cycles, $\{w_c, u\}$ is in a 2-cycle. If the dotted edge is absent, then, by symmetry, $\{v, w_d\}$ is in a 2-cycle and we have two vertex-disjoint 2-cycles; a contradiction. Thus, the dotted edge $e$ is present. After deleting $\{v, w_c\}$, the resulting graph has $\{w_a, u\}$ in a 2-cycle. Now, $G/w_cw_b$ has $W$ as a minor, a contradiction. Thus, 3.5.11 holds.

3.5.12. $T_f$ does not have exactly three leaves.

Suppose the tree $T_f$ has exactly three leaves. Since $\text{si}(G\backslash f)$ is 3-connected, each vertex of $\text{si}(G\backslash f)$ has degree at least three. By 3.5.10, $w_a$ and $w_b$ are adjacent to every leaf of the tree.
Thus, at most two of these leaves are in $V(C)$. Therefore $u$ or $v$, say $u$, is a leaf of the tree. Hence $u$ is adjacent to only $v$, $w_a$, and $w_b$. Since $\kappa(\text{si}(G)) = 3$, the vertex $v$ is adjacent to every $w_i$ where $i \notin \{a, b\}$. If there were two adjacent $w_i$ and $w_j$ for $i, j \notin \{a, b\}$, then, since $v$ is adjacent to both $w_i$ and $w_j$, there is a cycle through $v$, $w_i$, and $w_j$ in $(G \setminus f) - \{w_a, w_b\}$, a contradiction. Hence, $|V(C)| = 4$ and $|V(G)| = 6$. There are two $w_i$ that are different from $w_a$ and $w_b$, call them $w_c$ and $w_d$ for some integers $c$ and $d$. Both $w_c$ and $w_d$ are adjacent to both $w_a$ and $w_b$ as they are nonadjacent vertices.

Since $u$ is a leaf of $T_f$, its neighbor in $T_f$ is $v$, and $v$ is also adjacent to both $w_c$ and $w_d$ in $T_f$. Then $\text{si}(G) + f$ is the graph shown in Figure 3.7 where the dotted edge may or may not be present.

![Figure 3.7: Illustration of $\text{si}(G) + f$ when $T_f$ has three leaves. A dotted edge may or may not be present in the graph.](image)

Since $G - \{v, w_b\}$ has a cycle, $\{u, w_a\}$ is in a 2-cycle. By symmetry, $\{u, w_b\}$ is in a 2-cycle. If there is an edge joining $v$ to $w_b$, then $G$ has $X$ as a proper minor, which can be seen by contracting $w_aw_c$; a contradiction. Therefore there is no edge joining $v$ to $w_b$. By symmetry there is no edge joining $v$ to $w_a$. Hence, the graph $\text{si}(G) + f$ is the graph shown in Figure 3.7 with no dotted edges. Now, since $G - \{u, w_c\}$ has a cycle, $\{v, w_d\}$ is in a 2-cycle, so $G$ has two vertex-disjoint 2-cycles; a contradiction. Hence, 3.5.12 holds.
This concludes the proof of Lemma 3.5.3, finishing the classification of the excluded minors for $V_2$ with $\kappa(G) = 3$ for Theorem 3.5.1.

By combining Theorems 3.3.1, 3.3.2, 3.4.1, and 3.5.1, we get the main result of this chapter, the excluded minor characterization for $V_2$.

**Theorem 3.5.13.** The excluded minors for $V_2$, the class of graphs $G$ such that $G - \{u_1, u_2\}$ has no cycles for some $u_1, u_2 \in V(G)$, are the twenty-one graphs shown in Figure 3.8.
References


Vita

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