

RATIONAL APPROXIMATION SCHEMES FOR SOLUTIONS
OF ABSTRACT CAUCHY PROBLEMS AND EVOLUTION EQUATIONS

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Abstract

In this dissertation we study time and space discretization methods for approximating solutions of abstract Cauchy problems and evolution equations in a Banach space setting. Two extensions of the Hille-Phillips functional calculus are developed. The first result is the Hille-Phillips functional calculus for generators of bi-continuous semigroups, and the second is a C -regularized version of the Hille-Phillips functional calculus for generators of C -regularized semigroups. These results are used in order to study time discretization schemes for abstract Cauchy problems associated with generators of bi-continuous semigroups as well as C -regularized semigroups. Stability, convergence results, and error estimates for rational approximation schemes for bi-continuous and C -regularized semigroups are presented. We also extend the Trotter-Kato Theorem to the framework of C -regularized semigroups and combine it with the time discretization methods previously mentioned in order to obtain fully discretized schemes, provided by A -stable rational functions. Among the applications, we outline how to use rational approximation schemes to approximate solutions of nonlinear ODE's, and we show the significance of the results for bi-continuous semigroups for obtaining new numerical inversion formulas for the Laplace transform (with sharp error estimates). Furthermore, rational approximation schemes for integrated semigroups are presented with applications to the second order abstract Cauchy problem.

Introduction

“Orthodox Mathematicians, when they cannot find the solution of a problem in a plain algebraical form, are apt to take refuge in a definite integral, and call that the solution.”

Oliver Heaviside (1850-1925)

The abstract Cauchy problem

$$(ACP) \begin{cases} u'(t) &= Au(t) \\ u(0) &= x \in D(A), \end{cases} \quad (1)$$

where A is a linear operator with domain $D(A)$ and range in a Banach space X , has been at the center of a tremendous effort made by mathematicians ever since Einar Hille published his celebrated monograph “Functional Analysis and Semigroups” in 1948 ([52]). This effort is seconded by thousands of papers on the subject and with over a hundred of research monographs, textbooks, and conference proceedings dealing extensively with the semigroup approach to (ACP). Paul Erdős once said “Problems worthy of attack prove their worth by fighting back”, and (ACP) shows its worth by continuing to provide mathematics with a wide array of open problems and uncharted territories. Günter Lumer, one of the key contributors to the basic theory of (ACP) and operator semigroups, once said after giving a talk at LSU that he “would like to be allowed to live long enough to fully understand (ACP).” Chuckling he continued “because then, I would be allowed to live forever.”

Abstract Cauchy problems can be roughly divided into two classes: well-posed ones and ill-posed ones. A problem is called well-posed if it admits a unique classical solution for a large set of initial data x which then depends continuously on the initial data. Clearly, if A is a bounded linear operator on a Banach space X , then $D(A) = X$ and (ACP) has a unique classical (entire) solution

$$u_x(t) = e^{tA}x = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n}A \right)^{-n} x \quad (x \in X) \quad (2)$$

that satisfy $\|u_{x_1}(t) - u_{x_2}(t)\| = \|e^{tA}x_1 - e^{tA}x_2\| \leq e^{t\|A\|}\|x_1 - x_2\|$; i.e., for bounded linear operators, (ACP) is perfectly well-posed. For unbounded operators, the situation becomes considerably more difficult. The “more unbounded” the operator A , the “more uncharted” the present understanding of the qualitative and quantitative mathematical theories concerning (ACP) become. The class of unbounded linear operators for which (ACP) is best understood is the class of generators A of strongly continuous semigroups, considered first by E. Hille [51, 52] and T. Yosida [102]. A densely defined linear operator A on a Banach space X is defined to be the generator of a strongly continuous semigroup T if the exponential formula (2) holds for all $x \in X$; i.e., a densely defined linear operator A generates a strongly continuous semigroup if

$$T(t)x := e^{tA}x := \lim_{n \rightarrow \infty} \left(I - \frac{t}{n}A \right)^{-n} x \quad (3)$$

exists for all $x \in X$ and $u(t) = T(t)x$ solves (ACP) for all $x \in D(A)$. In the literature, the exponential formula (3) is called a rational approximation scheme since it is based on the approximation of the exponential function $t \rightarrow e^{tz}$ by rational polynomials $r^n\left(\frac{t}{n}z\right)$ with $r(z) = \frac{1}{1-z}$.

The purpose of this dissertation is the study of rational approximation methods for solutions of abstract Cauchy problems where the operator A belongs to a wider class of operators than just generators of strongly continuous semigroups.

Our starting point for developing approximation methods for solutions of the abstract Cauchy problem are the groundbreaking works of R. Hersh and T. Kato [47] and P. Brenner and V. Thomée [10] concerning rational approximation methods for strongly continuous semigroups. In [10], P. Brenner and V. Thomée showed that if A generates a strongly continuous semigroup T and r is an \mathcal{A} -stable¹ rational approximation of the exponential function of order $q \geq 1$, then

$$r^n\left(\frac{t}{n}A\right)x - T(t)x = O\left(\frac{t^{q+1}}{n^q}\right) \text{ as } n \rightarrow \infty, \quad x \in D(A^{q+1}). \quad (4)$$

In particular, (4) implies that the backward Euler approximation (2) satisfies that

$$\left(I - \frac{t}{n}A\right)^{-n}x - T(t)x = O\left(\frac{t^2}{n}\right) \text{ as } n \rightarrow \infty. \quad (x \in D(A^2)) \quad (5)$$

The main tool used by Brenner and Thomée, as well as by Hersh and Kato, was the Hille-Phillips functional calculus developed by E. Hille and R.S. Phillips in [52, 53, 83]. This functional calculus provides a powerful tool to study $r^n\left(\frac{t}{n}A\right)$ when r is an \mathcal{A} -stable rational function and A is the generator of a strongly continuous semigroup. R. Hersh and T. Kato remarked in [47]:

“Earlier versions of this work...used the Dunford calculus... By using instead the Hille-Phillips calculus... we now get sharper estimates and simpler proofs”

The Hille-Phillips functional calculus states that if A is the generator of a strongly continuous semigroup T of type $(M, 0)$ on X , and α is a normalized function of bounded variation such that $f(z) = \int_0^\infty e^{zt} d\alpha(t)$ for $\text{Re}(z) \leq 0$, then $\Psi : \mathcal{F} \rightarrow \mathcal{L}(X)$ defined by

$$f(A) := \Psi(f)x := \int_0^\infty T(t)x d\alpha(t) \quad (6)$$

is an algebra homomorphism, where \mathcal{F} is the algebra of analytic functions of the form $f(z) = \int_0^\infty e^{zt} d\alpha(t)$ for $\text{Re}(z) \leq 0$. Moreover, if $f(A) := \Psi(f)$ then $\|f(A)\| \leq M \int_0^\infty d|\alpha|(t) < \infty$. For a proof see [60, 83].

Many theories have been developed in order to generalize strongly continuous semigroups, e.g. [4, 15, 24, 27, 41, 50, 59, 66, 67, 69, 71, 78, 80]. In this dissertation

¹A rational function r is called \mathcal{A} -stable if $|r(z)| \leq 1$ for $\text{Re}(z) \leq 0$. If in addition $r(z) = e^z + o(z)$ as $z \rightarrow 0$, then r is said to be \mathcal{A} -acceptable. Moreover, r is an approximation to the exponential function of order $q \geq 1$ if $r(z) = e^z + O(|z|^{q+1})$ as $z \rightarrow 0$. For example, if $r(z) = \frac{1}{1-z} = 1+z+z^2+\dots$, then $r(z)-e^z = \frac{z^2}{2} + \frac{5z^3}{6} + \dots = O(z^2)$ as $z \rightarrow 0$; i.e. $q = 1$.

we will concentrate on the study of two of them, namely bi-continuous semigroups and C -regularized semigroups. Bi-continuous semigroups include a variety of interesting cases such as Feller semigroups [1, 74] including the Ornstein-Uhlenbeck semigroup [25, 68], the heat semigroup [40], adjoint semigroups [9, 75], implemented semigroups [9], weakly continuous semigroups [13], as well as certain evolutions semigroups [16] and semigroups induced by non-linear flows studied by J.R. Dorroh and J.W. Neuberger [30, 31, 32]. The C -regularized semigroups include the classes of integrated and distributional semigroups, see [28, 69].

Section 1.1 of this dissertation describes the theories of bi-continuous semigroups and C -regularized semigroups. Both approaches “fix” the failure of the continuity property of the map $t \rightarrow T(t)x$ for some $x \in X$. The basic idea of bi-continuous semigroups, developed by F. Kühnemund in [66, 67], is to consider a coarser topology τ on the Banach space X for which the maps $t \rightarrow T(t)x$ are continuous in the τ -topology; and for which the norm of every element in X can be calculated by using the dual space $(X, \tau)'$. In this way, bi-continuous semigroups extend the concept of strongly continuous semigroups since for the case of strongly continuous semigroups, τ can always be considered as the norm topology on X . On the other hand, the idea of C -regularized semigroups is to consider a bounded and injective operator C for which the maps $t \rightarrow T(t)Cx$ are continuous for every $x \in X$. In other words, the semigroup T is “regularized” by an operator C that maps X into the continuity domain of T . Once more, the concept of C -regularized semigroups extends the one of strongly continuous semigroups because C can be taken to be the identity operator on X if T is a strongly continuous semigroup.

In order to be able to construct a functional calculus for generators of bi-continuous semigroups and C -regularized semigroups, Section 1.2 describes the Banach Algebras of normalized functions of bounded variation introduced by R.S. Phillips in [83]. The construction is based on the Riesz Representation Theorem and therefore it is different than the original description developed by Phillips in [83] as well as the construction developed by M. Kovács in [60]. The starting point of this dissertation is developed in Section 1.3, where the Hille-Phillips Functional Calculus is extended for generators of bi-continuous semigroups. The construction takes advantage of the proof given by Kovács in [60, 61] for the strongly continuous case. Furthermore, Section 1.4 develops a regularized version of the Hille-Phillips functional calculus for generators of C -regularized semigroups. The results of Chapter 1 can be summarized as follows.

Theorem (Hille-Phillips Functional Calculus). *Let A be the generator of a bi-continuous semigroup T (resp. a C -regularized semigroup W) of type (M, ω) on X . If $\alpha \in NBV^\omega$ is such that $f(z) = \int_0^\infty e^{zt} d\alpha(t)$ for $Re(z) \leq \omega$, then $\Psi : \mathcal{F}^\omega \rightarrow \mathcal{L}(X)$ (resp. $\Psi : \mathcal{F}^\omega \rightarrow \mathcal{L}_C(X)$) defined by*

$$f(A)x := \Psi(f)x := \int_0^\infty T(t)x d\alpha(t) \quad \left(\text{resp. } f(A)Cx := \int_0^\infty W(t)x d\alpha(t) \right) \quad (7)$$

is an algebra homomorphism. Moreover, $\|f(A)\| \leq M\|\alpha\|_\omega$ (resp. $\|f(A)C\|$).

Chapter 2 contains the main results of this dissertation concerning time and space discretization of bi-continuous semigroups as well as C -regularized semigroups. Section 2.1 introduces the different types of \mathcal{A} -stable rational approximations to the exponential such as the Composite Exponential Approximations developed by A. Iserles in [54], the Padé approximants to the exponential and the Hermite-Padé approximants. Section 2.2 develops crucial estimates for the variation norm of the inverse Laplace-Stieltjes transform of rational functions originally developed in [10] for measures and also in [60, 63] for normalized functions of bounded variation. By using the Riesz Representation Theorem developed in Section 1.1 we show that these two results are actually equivalent. Section 2.3 shows the main results for rational approximation schemes of bi-continuous semigroups provided by \mathcal{A} -stable rational functions. It discusses stability, sharpness and error estimates for the approximation schemes in the norm sense. Moreover, for the stable² schemes we obtain convergence in the τ topology for all initial data and for smooth initial data we obtain convergence in the norm sense. In this way, we fully lift the results of P. Brenner and V. Thomée [10] to generators of bi-continuous semigroups. In particular, we show that (4) holds when A generates a bi-continuous semigroup. Section 2.4 contains the main results concerning approximation schemes for C -regularized semigroups. It covers stability properties and sharp error estimates for regularized time discretization schemes for C -regularized semigroups given by \mathcal{A} -stable rational functions. Thus, the results of Brenner and Thomée are also lifted to the C -regularized case. In particular, we obtain that if W is a C -regularized semigroup generated by A then $r^n \left(\frac{t}{n}A\right)Cx - W(t)x = O\left(\frac{t^{q+1}}{n^q}\right)$ as $n \rightarrow \infty$ for $x \in D(A^{q+1})$, which is a regularized³ version of (4). Section 2.5 develops an extension of the Trotter-Kato theorem for C -regularized semigroups in order to obtain space discretization and then we show fully discretized schemes for approximating C -regularized semigroups by combining this result with the ones obtained in Section 2.4.

Chapter 3 develops some of the applications of the results obtained in Chapter 2 for some cases of bi-continuous and C -regularized semigroups. Section 3.1 studies approximation methods for semigroups induced by nonlinear flows. In particular, we show error estimates for the approximation found by J.R. Dorroh and J.W. Neuberger in [30, 31, 32]. Section 3.2 applies the results developed in Section 2.3 to the (left) translation semigroup in order to obtain inversion formulas of the vector-valued Laplace transform with error estimates for smooth functions on the Banach space of continuous and bounded functions. This section contains the author's contribution to the joint work with F. Neubrander and K. Özer [58].

Theorem (Laplace Transform Inversion). *Let \widehat{f} be the Laplace transform of $f \in C_b(\mathbb{R}_0^+; X)$ and define $f_0 := \lim_{\lambda \rightarrow \infty} \lambda \widehat{f}(\lambda)$. If r is an \mathcal{A} -stable rational approximation to the exponential function of order $q \geq 1$, then there exist $K > 0$, $b_i \in \mathbb{C}$*

²An scheme is called stable if the variation norm of the n^{th} convoluted inverse Laplace-Stieltjes transform of the \mathcal{A} -stable rational function is uniformly bounded.

³Notice that the error estimate is independent of the operator C .

with $\operatorname{Re}(b_i) > 0$ and constants $C_{n,i,j} \in \mathbb{C}$ (independent of f) such that

$$\left\| C_{n,0,0}f_0 + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq nr}} C_{n,i,j} \left(\frac{n}{t}\right)^j \frac{(-1)^{j-1}}{(j-1)!} \widehat{f}^{(j-1)}\left(\frac{n}{t}b_i\right) - f(t) \right\|_X \leq K \frac{t^{q+1}}{n^q} \|f^{(q+1)}\|_\infty. \quad (8)$$

This result is implemented in a small Mathematica algorithm and it has direct applications to the problem of approximate solutions of evolution equations of convolution type [58]; i.e., linear evolution equations that can be treated directly with Laplace transform methods. Section 3.3 shows the particular form of the approximations schemes when considering generators of integrated semigroups, including explicit schemes to approximate the solutions of the second order abstract Cauchy problem. In particular, approximation schemes to approximate solutions of the wave equation are studied. Furthermore, we develop integrated schemes for approximating integrated semigroups with error estimates for sufficiently smooth initial data.

A typical example of an evolution equation of convolution type is given by Volterra integro-differential equations of scalar type

$$u'(t) = g(t) + \int_0^t a(t-s)Au(s)ds, \quad u(0) = 0, \quad (9)$$

which has characteristic equation $\lambda \widehat{u}(\lambda) = \widehat{g}(\lambda) + \widehat{a}(\lambda)A\widehat{u}(\lambda)$, with solution $\widehat{u}(\lambda) = (\lambda I - \widehat{a}(\lambda)A)^{-1}\widehat{g}(\lambda)$. Thus, (8) provides straight forward time-discretization procedures for obtaining u (provided that u is known to be sufficiently smooth). Prior to the availability of the Laplace transform inversion (8), the most common way to obtain rational approximations of the solution of (9) was by rewriting it into the abstract Cauchy problem, and then apply the Hersh-Kato and Brenner-Thomée results or the generalizations given in Chapter 2. Clearly, there are many evolution equations that are not of convolution type and for which, therefore, the Laplace transform inversion are of little use. These equations (like all evolution equations) can be treated by studying the associated abstract Cauchy problem (ACP) on a properly chosen state space X and, in particular, by applying the results of Section 2. Typical examples of evolution equations that are not of convolution type are semigroups induced by nonlinear flows and semigroups induced by non-autonomous problems ($u'(t) = A(t)u(t)$, $u(0) = x$).

Finally, we do not attempt to give a detailed analysis of rational approximation methods for any of these cases. The purpose of this dissertation is to add to the understanding of the basic tools that will be necessary when studying each of these classes of problems in more detail. However, as all problems worthy of attack, they will fight back and it will require many more research in order to fully understand rational approximation schemes for abstract Cauchy problems and evolution equations. We hope that this dissertation contributes to this process in a meaningful manner.

Chapter 1

Extensions of the Hille-Phillips Functional Calculus

The basic idea of a functional calculus is to define operators $f(A)$ for a certain class \mathcal{F} of complex-valued functions f and a suitable class \mathcal{A} of linear operators defined on a Banach space X such that algebraic properties of the class \mathcal{F} are preserved; i.e., $(f + g)(A) = f(A) + g(A)$, $(\lambda f)(A) = \lambda f(A)$, and $(fg)(A) = f(A) \circ g(A)$ for all $f, g \in \mathcal{F}$ and $A \in \mathcal{A}$.

Probably, the most basic functional calculus is the one provided by the algebra \mathcal{F} of polynomials and the class $\mathcal{A} := \mathcal{L}(X)$ of bounded linear operators defined on a Banach space X . If $A \in \mathcal{L}(X)$ and $p(z) := \sum_{i=0}^n \alpha_i z^i$ is a polynomial on \mathbb{C} , then the map $\Psi : p \mapsto p(A) := \sum_{i=0}^n \alpha_i A^i$ defines an algebra homomorphism between \mathcal{F} and $\mathcal{L}(X)$. However, if one considers a function f which is not a polynomial or if A is not assumed to be bounded, then the meaning of $f(A)$ is no longer as straightforward as before. In fact, the previously mentioned example is a particular case of the “The Riesz-Dunford Functional Calculus” which takes advantage of the Cauchy integral representation for analytic functions. It states that for each open set $U \subset \mathbb{C}$, whose boundary γ consists of a finite number of rectifiable Jordan curves (positively oriented), one can define an algebra homomorphism Ψ between the algebra \mathcal{F}_U of complex-valued analytic functions on U and the class $\mathcal{A} := \mathcal{L}_U(X)$ of bounded linear operators A on a Banach space X with spectrum $\sigma(A) \subset U$ by $\Psi : f \mapsto f(A) := \frac{1}{2\pi i} \int_{\gamma} f(\lambda) R(\lambda, A) d\lambda$, where $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A , see [38].

The “Hille-Phillips Functional Calculus” (H-P Functional Calculus) developed in [53, 83] provides an interpretation of $f(A)$ by means of the class \mathcal{A}_ω consisting of operators A that are generators of strongly continuous semigroups T of type (M, ω) (i.e. $\|T(t)\| \leq M e^{\omega t}$, $t \geq 0$) on a Banach space X , and the algebra of complex-valued analytic functions \mathcal{F}^ω with a Laplace-Stieltjes representation $f(z) = \int_0^\infty e^{zt} d\mu(t)$ for $\operatorname{Re}(z) \leq \omega$; where μ belongs to the convolution Banach algebra of weighted bounded regular complex Borel measures on \mathbb{R}_0^+ of the form $\mu(E) = \int_E e^{-\omega s} d\mu_0$, and where μ_0 is a bounded regular complex Borel measure on \mathbb{R}_0^+ . In this way, the H-P Functional Calculus states that for each $A \in \mathcal{A}_\omega$ generating a semigroup T of type (M, ω) and for each $f \in \mathcal{F}^\omega$ the map $\Psi : f \rightarrow f(A)$ with

$$f(A)x = \int_0^\infty T(t)x d\mu(t) \tag{1.1}$$

defines an algebra homomorphism from \mathcal{F}^ω into $\mathcal{L}(X)$. R. S. Phillips showed in [83] that “The Hille-Phillips Functional Calculus” can be expressed in terms of normalized functions of bounded variation instead of measures, see also M. Kovács [60, 61]. A.V. Balakrishnan [5] and E. Nelson [76] extended “The Hille-Phillips

Functional Calculus” by allowing a wider class of “functions” but by maintaining the class of generators of strongly continuous semigroups.

In this chapter we develop two new extensions of “The Hille-Phillips Functional Calculus” by allowing in both cases a wider class of operators \mathcal{A} than generators of strongly continuous semigroups. The first section contains the basic results and definitions concerning (i) generators of bi-continuous semigroups and (ii) generators of C -regularized semigroups. In the second section, we employ the Riesz Representation Theorem to describe the algebra of normalized functions of bounded variation by using an approach different than those used by R.S. Phillips in [83] and M. Kovács in [60, 61]. Finally, the last two sections contains the extensions of “The Hille-Phillips Functional Calculus” to generators of bi-continuous semigroups and to generators of C -regularized semigroups.

1.1 Operator Semigroups

Consider the abstract Cauchy problem

$$(\text{ACP}) \begin{cases} u'(t) &= Au(t) & (t \geq 0) \\ u(0) &= x, \end{cases}$$

where A is a linear operator with domain $D(A)$ and range in a Banach space X . If A generates a strongly continuous semigroup T , then the solution of (ACP) is given by

$$u(t) = T(t)x := e^{tA}x := \lim_{n \rightarrow \infty} \left(I - \frac{t}{n}A \right)^{-n} x,$$

where u is a classical solution in $C([0, \infty), D(A)) \cap C^1([0, \infty), X)$ if $x \in D(A)$ and u is a mild solution in $C([0, \infty), X)$ solving $u(t) = A \int_0^t u(s)ds + x$, ($t \geq 0, x \in X$). For a comprehensive study of strongly continuous semigroups and their generators see [39, 44, 82].

The first part of this section is dedicated to collect the basic properties of strongly continuous semigroups. Let $\mathcal{L}(X)$ denote the Banach space of bounded operators on a Banach space X . A semigroup T is a map from $[0, \infty)$ into $\mathcal{L}(X)$ that satisfies the functional equation

$$(\text{FE}) \begin{cases} T(t+s) &= T(t)T(s) \\ T(0) &= I \end{cases}$$

for all $t, s \geq 0$ (where I denotes the identity on X). The semigroup T is strongly continuous if the maps $t \rightarrow T(t)x$ are continuous on $\mathbb{R}_0^+ := [0, \infty)$ for each $x \in X$, and it is of type (M, ω) if there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \in \mathbb{R}_0^+$. The generator of a strongly continuous semigroup T is defined by

$$Ax := \lim_{h \rightarrow 0^+} \frac{T(h)x - x}{h}$$

with domain $D(A)$ consisting of those $x \in X$ for which the limit exists. One can think of $T(t)x$ as

$$e^{tA}x := \lim_{n \rightarrow \infty} \left(I - \frac{t}{n}A \right)^{-n} x,$$

where A is closed, densely defined, linear operator with a “sufficiently nice” resolvent

$$\lambda \rightarrow R(\lambda, A) := (\lambda I - A)^{-1}.$$

In fact, the celebrated Hille-Yosida Theorem (see [39, p.77]) states that

$$T(t)x = e^{tA}x := \lim_{n \rightarrow \infty} \left(I - \frac{t}{n}A \right)^{-n} x = \lim_{n \rightarrow \infty} \left(\frac{t}{n} \right)^n R\left(\frac{n}{t}, A\right)^n x$$

exists for all $x \in X$ if and only if $D(A)$ is dense in X and there exists $M, \omega > 0$ such that $R(\lambda, A)$ exists and

$$\|(\lambda - \omega)^n R(\lambda, A)^n\| \leq M$$

for all $\lambda > \omega$.

Remark 1.1.1. The algebraic property $T(t+s) = T(t)T(s)$ amplifies many topological properties of the semigroup T . For instance, if T is a strongly measurable semigroup on $(0, \infty)$, then it is strongly continuous on $(0, \infty)$ (not necessarily on \mathbb{R}_0^+). Moreover, T is strongly continuous if and only if it is weakly continuous at $t = 0$, see [82].

The importance of the shift semigroup is given by the applications of the results developed in Chapter 1 and Chapter 2 given in Section 3.3 concerning the numerical inversion of the Laplace transform.

Example 1.1.2. If $X := C_{\text{ub}}(\mathbb{R}_0^+)$ is the Banach space of bounded, uniformly continuous, and complex-valued functions with $\|x\|_\infty := \sup_{s \in \mathbb{R}_0^+} |x(s)|$, then the shift semigroup

$$T(t)x : s \rightarrow x(s+t) \quad (t \in \mathbb{R}_0^+)$$

is a strongly continuous semigroup with generator

$$A = \frac{d}{ds} \text{ and } D(A) = \{x \in C_{\text{ub}}(\mathbb{R}_0^+) : x' \in C_{\text{ub}}(\mathbb{R}_0^+)\}.$$

However, if one chooses X to be $C_b(\mathbb{R}_0^+)$, the Banach space of bounded, continuous, complex-valued functions with norm $\|x\|_\infty$, then the shift semigroup fails to be strongly continuous. For example, by shifting rapidly oscillating functions x with $\|x\|_\infty = 1$ (like $x : s \rightarrow e^{is^2}$) one obtains that $\|T(t)x - T(s)x\| = 2$ for all $t, s \geq 0$ with $t \neq s$. Therefore, for certain $x \in X$, the map $t \rightarrow T(t)x$ is nowhere continuous (and hence, by Remark 1.1.1, the shift semigroup is also not strongly measurable on $(0, \infty)$). Notice that $D(A) := \{x \in C_b^1(\mathbb{R}_0^+) : x' \in C_b(\mathbb{R}_0^+)\}$ is not dense in $C_b(\mathbb{R}_0^+)$, and that the part of A in $\overline{D(A)} = C_{\text{ub}}(\mathbb{R}_0^+)$ generates a strongly continuous semigroup on $\overline{D(A)}$, see also Corollary 13 of [67]. Observe that if $X := L^\infty(\mathbb{R}_0^+)$ then the same counterexample holds.

Example 1.1.2 shows that in order to discuss the shift semigroup on spaces such as $C_b(\mathbb{R}_0^+)$ or $L^\infty(\mathbb{R}_0^+)$ one needs a more general framework than the one provided by strongly continuous semigroups. The remainder of this section compiles the basic results concerning two extensions of strongly continuous semigroups known as bi-continuous semigroups and C -regularized semigroups.

The class of C -regularized semigroups was introduced by G. Da Prato in [24] and further developed by R. deLaubenfels among others. See [27] for references and a comprehensive introduction to C -regularized semigroups. The C -regularized approach provides a flexible and powerful framework to study (ACP) for operators not generating strongly continuous semigroups, including generators of integrated and distributional semigroups.

Bi-continuous semigroups were introduced by F. Kühnemund in [66, 67]. They are special cases of C -regularized semigroups with two remarkable properties. First of all, bi-continuous semigroups allow for a rich qualitative theory since they are still sufficiently regular to be treated by classical Laplace transform methods, see, e.g., [1, 40, 41, 42, 66, 67]. Second, there is a variety of interesting semigroups belonging to this class, e.g., Feller semigroups [1, 74], the heat semigroup [40], adjoint semigroups [9, 75], implemented semigroups [9], weakly continuous semigroups [13] including the Ornstein-Uhlenbeck semigroup [25, 68], as well as certain evolutions semigroups [16] and semigroups induced by non-linear flows studied by J.R. Dorroh and J.W. Neuberger [30, 31, 32].

Bi-continuous semigroups: As observed in Example 1.1.2, the shift semigroup is not strongly continuous on $L^\infty(\mathbb{R}_0^+)$ or $C_b(\mathbb{R}_0^+)$. However, it has still enough regularity to be weakly continuous and thus to be integrated in a classical sense.

Let T be a semigroup on a Banach space X ; i.e., the operators $T(t)$ are in $\mathcal{L}(X)$ and satisfy $T(0) = I$, $T(t + s) = T(t)T(s)$ ($t, s \geq 0$). In some cases (e.g. the shift semigroup on $C_b(\mathbb{R}_0^+)$) the maps $t \rightarrow T(t)x$ are not strongly continuous for all $x \in X$. Basic examples of these semigroups are the shift semigroup, convolution semigroups, adjoint semigroups, and semigroups induced by flows. The basic idea behind bi-continuous semigroups is to consider a suitable coarser topology τ on the space X such that the maps $t \rightarrow T(t)x$ are τ -continuous on \mathbb{R}_0^+ for each $x \in X$; i.e., the semigroup is τ -strongly continuous but not necessarily strongly continuous on X .

Definition 1.1.3. A locally convex topology τ defined via a family \mathcal{P}_τ of seminorms (cf. [93, p.48]) on a Banach space X is said to be coherent (with the norm topology) if

- (i) the space (X, τ) is sequentially complete on $\|\cdot\|$ -bounded sets; i.e., every $\|\cdot\|$ -bounded τ -Cauchy sequence converges in (X, τ) ,
- (ii) the topology τ is Hausdorff and $p(x) \leq \|x\|$ for all $x \in X$ and $p \in \mathcal{P}_\tau$, and

(iii) the topological dual $(X, \tau)'$ is norming for $(X, \|\cdot\|)$; i.e., for all $x \in X$,

$$\|x\| = \sup\{|\langle x, \phi \rangle| : \phi \in (X, \tau)' \text{ and } \|\phi\|_{(X, \|\cdot\|)'} \leq 1\}.$$

The condition (ii) implies that $(X, \tau)' \subseteq (X, \|\cdot\|)'$. Moreover, if $\Phi := \{\phi \in (X, \tau)' : \|\phi\|_{(X, \|\cdot\|)'} \leq 1\}$, then (ii) implies that

$$\sup_{\phi \in \Phi} |\langle x, \phi \rangle| \leq \|x\| \tag{1.2}$$

for all $x \in X$. In this way, the dual space $(X, \tau)'$ is norming for $(X, \|\cdot\|)$ if the supremum reaches the norm in (1.2). A subset Φ_0 of Φ for which $\sup_{\phi \in \Phi_0} |\langle x, \phi \rangle| = \|x\|$ for all $x \in X$ is called a coherent norming set for X .

Example 1.1.4. A coherent topology τ for $X = C_b(\mathbb{R}_0^+)$ is the topology of uniform convergence on compact subsets K of \mathbb{R}_0^+ ; i.e., the topology given by the family of seminorms $p_K \in \mathcal{P}_\tau$, where $p_K(x) := \sup_{s \in K} |x(s)|$, $K \subset \mathbb{R}_0^+$ is compact. Furthermore, a coherent norming set $\Phi_0 \subset \Phi$ is given by the point measures $\phi_a : X \rightarrow \mathbb{C}$ defined by $\phi_a(x) = x(a)$.

From now on, and unless stated otherwise, it will be always assumed that a τ topology is coherent with the norm topology of the Banach space X .

Definition 1.1.5. A semigroup T of type (M, ω) is bi-continuous (with respect to τ) if

- (i) T is τ -strongly continuous; i.e., $t \rightarrow T(t)x$ is τ -continuous on \mathbb{R}_0^+ for all $x \in X$, and
- (ii) T is locally bi-equicontinuous; i.e., for every $t_0 \geq 0$, $\varepsilon > 0$, $p \in \mathcal{P}_\tau$ and for every $\|\cdot\|$ -bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ with $\tau\text{-}\lim_{n \rightarrow \infty} x_n = x$ there exists $N \in \mathbb{N}$ such that $\sup_{0 \leq t \leq t_0} p(T(t)(x_n - x)) < \varepsilon$ for all $n \geq N$.

In particular, if T is a bi-continuous semigroup and $t \geq 0$, then the operator $T(t)$ is continuous from $(X, \|\cdot\|)$ into $(X, \|\cdot\|)$ and from (X, τ) into (X, τ) . Moreover, every strongly continuous semigroup on a Banach space is a bi-continuous semigroup by taking the locally convex topology τ as the topology given by the norm. However, not every bi-continuous semigroup is strongly continuous.

Example 1.1.6. The shift semigroup is not strongly continuous but bi-continuous with respect to τ on $C_b(\mathbb{R}_0^+)$ where τ is the topology of uniform convergence on compact sets. To see this, consider $K \subset \mathbb{R}_0^+$ compact and $x \in C_b(\mathbb{R}_0^+)$. By the uniform continuity of x on compact subsets, it follows that $p_K(T(t_n)x - T(t)x) = \sup_{s \in K} |x(s + t_n) - x(s + t)| \rightarrow 0$ as $t_n \rightarrow t$. Thus, T is τ -strongly continuous on $C_b(\mathbb{R}_0^+)$. In order to show bi-equicontinuity, let $t_0 \in \mathbb{R}_0^+$ and $K \subset \mathbb{R}_0^+$ be compact. Define $W := \bigcup_{t \in [0, t_0]} K + t$ and let $\{x_n\}_{n \in \mathbb{N}}$ be a norm-bounded sequence with $\tau\text{-}\lim_{n \rightarrow \infty} x_n = x$. For $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that $p_W(x_n - x) = \sup_{s \in W} |x_n(s) - x(s)| \leq \varepsilon$ for $n \geq N$. Then $\sup_{t \in [0, t_0]} p_K(T(t)(x_n - x)) = \sup_{t \in [0, t_0]} \sup_{s \in K} |x_n(s) - x(s)| \leq p_W(x_n - x) \leq \varepsilon$. Thus, the shift semigroup is bi-continuous on $(C_b(\mathbb{R}_0^+), \|\cdot\|_\infty, \tau)$.

More examples of bi-continuous semigroups can be found in the work of Bálint Farkas [40] and Franziska Kühnemund [66]. The following list compiles some examples from their work, for more details and examples see [40, 66].

- **Adjoint Semigroups.** Let X be a Banach space and let X' be its topological dual. If T is a strongly continuous semigroup on X , then T' defined by $T'(t) := [T(t)]'$ the adjoint operator on X' is bi-continuous on X' with respect to the weak*-topology on X' .
- **Implemented Semigroups.** Let X be a Banach space and let T, S be strongly continuous semigroups on X . Then the implemented semigroup defined on $\mathcal{X} := \mathcal{L}(X)$ by $\mathcal{U} : t \rightarrow T(t)YS(t)$ for each $t \geq 0$ and $Y \in \mathcal{X}$ is a bi-continuous semigroup on \mathcal{X} with respect to the strong operator topology; i.e., the coherent topology τ is induced by the family of semi-norms $P_{\tau_{\text{S-top}}} := \{p_y : y \in Y\}$ where $p_y(Y) = \|Yy\|_X$.
- **Ornstein-Uhlenbeck Semigroup.** Let \mathcal{H} be a separable Hilbert space and let $X := C_b(\mathcal{H})$ be the Banach space of all the continuous and bounded functions defined on \mathcal{H} with the uniform norm. Let S be a strongly continuous semigroup on \mathcal{H} with generator A and let Q be a bounded self-adjoint positive operator on \mathcal{H} . Consider the covariance operator \mathcal{Q} defined by $\mathcal{Q}(t)x := \int_0^t S(r)QS'(r)xdr$ where S' is the adjoint semigroup of S , $x \in \mathcal{H}$, $t \geq 0$ and each $\mathcal{Q}(t)$ is of trace-class. Then there exist Gaussian measures $\mathcal{N}(S(t)x, Q(t))$ for all $t \geq 0$ and $x \in \mathcal{H}$ and the Ornstein-Uhlenbeck semigroup P defined on X by

$$P(t)f(x) = \int_{\mathcal{H}} f(y)\mathcal{N}(S(t)x, Q(t))dy$$

is a bi-continuous semigroup with respect to the topology given by uniform convergence on compact sets.

- **Semigroups induced by non-linear flows.** A jointly continuous flow on a topological space Ω is a map $\phi : \mathbb{R}_0^+ \times \Omega \rightarrow \Omega$, $\phi_t(v) := \phi(t, v)$, that is jointly continuous and satisfies the semigroup property $\phi_{t+s} = \phi_t\phi_s$, $\phi_0 = Id$. If $X := C_b(\Omega)$, then the semigroup S defined by $S(t) : x \rightarrow x \circ \phi_t$ is called the semigroup induced by the flow ϕ . F. Kühnemund shows in [66, 67] that these semigroups, considered also by J. R. Dorroh and J. W. Neuberger in [30, 31, 32, 77], are bi-continuous on $C_b(\Omega)$ and

$$\tau\text{-}\lim_{n \rightarrow \infty} \left[\frac{n}{t}R\left(\frac{n}{t}, A\right) \right]^n x = S(t)x$$

for all $x \in X$, where the limit of (3.1) is uniform for t in compact sets of \mathbb{R}_0^+ . In this case, the topology τ is defined as the finest locally convex topology on $C_b(\Omega)$ agreeing with the compact-open topology τ_c on $\|\cdot\|_\infty$ -bounded sets. Also Ω has to be a Polish space, see [30, 40] for more details.

C -regularized semigroups: As previously mentioned, a framework more general than bi-continuous semigroups is the one of C -regularized semigroups. In order to grasp the idea behind C -regularized semigroups consider the Banach space $X := C_0(\mathbb{R}, \mathbb{C})$ of all continuous complex-valued functions x such that $\lim_{|s| \rightarrow \infty} x(s) = 0$ with $\|x\|_\infty = \sup_{s \in \mathbb{R}} |x(s)|$. Let $a : \mathbb{R} \rightarrow \mathbb{C}$ be a continuous path given by $a(s) = a_1(s) + ia_2(s)$ and define $Ax = ax$ with $D(A) = \{x \in X : ax \in X\}$. The the semigroup generated by the multiplication operator A is given by

$$T(t)x(s) := e^{ta(s)}x(s) \quad (s \in \mathbb{R}) \quad (1.3)$$

with $D(T(t)) = \{x \in X : e^{ta}x \in X\}$. If $\operatorname{Re}|a(s)| \leq \omega$ for all $s \in \mathbb{R}$, then $T(t)$ is a bounded linear operator with $\|T(t)\| \leq e^{\omega t}$. and $t \rightarrow T(t)x$ is continuous for all $x \in X$ since $\|T(t)x - T(t_0)x\| = \sup_{s \in \mathbb{R}} |e^{ta(s)}x(s) - e^{t_0a(s)}x(s)| \rightarrow 0$ as $t \rightarrow t_0$. If a is unbounded then $T(t)$ is unbounded since $e^{ta(s)}$ is an unbounded function on \mathbb{R} . It can be shown that the (ACP) associated with the multiplication operator on X has solutions for all initial data in a dense set (the set of all functions x such that $\lim_{|s| \rightarrow \infty} e^{ta(s)}x(s) = 0$). For example, if $a(s) = s$ then $T(t)x(s) = e^{ts}x(s)$ is an unbounded linear operator with $D(T(t)) = \{x \in X : s \rightarrow e^{ts}x(s) \in X\}$. However, if $C : X \rightarrow X$ is defined by $Cx(s) = e^{-s^2}x(s)$, then $e^{ts}e^{-s^2}$ is a bounded function for all $t \geq 0$ and the map $W : t \rightarrow e^{t(\cdot)}e^{-(\cdot)^2}x(\cdot)$ is continuous on $[0, \infty)$ for every $x \in C_0(\mathbb{R})$; i.e. W is strongly continuous. In this way, the basic idea behind C -regularized semigroups is to consider a bounded and injective linear operator C which regularizes the original unbounded semigroup $T(t)$ so that $T(t)C \in \mathcal{L}(X)$ for all $t \geq 0$.

The rest of this section collects the basic properties and definitions of C -regularized semigroups. From now on, C will be a bounded and injective operator defined on a Banach space X .

Definition 1.1.7. A strongly continuous map $W : [0, \infty) \rightarrow \mathcal{L}(X)$ is called a C -regularized semigroup if $W(0) = C$ and $W(t)W(s) = CW(t+s)$ for all $t, s \geq 0$. In particular, $W(t)C = CW(t)$ for all $t \geq 0$. A linear operator $A : D(A) \subseteq X \rightarrow X$ is the generator of W if

$$Ax = C^{-1} \left[\lim_{t \rightarrow 0} \frac{W(t)x - Cx}{t} \right],$$

where $D(A)$ denotes the maximal domain of A in X . Moreover, if there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|W(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$, then the semigroup W is said to be of type (M, ω) .

Notice that a C -regularized semigroup W generated by A is of type (M, ω) if and only if $A - \omega$ generates a C -regularized semigroup of type $(M, 0)$.

Proposition 1.1.8 ([27], Thm. 3.4). *Let W be a C -regularized semigroup generated by A , $x \in D(A)$ and $t \geq 0$. Then A is closed, $\operatorname{Im}(C) \subseteq \overline{D(A)}$, $W(t)x \in D(A)$, $W(t)Ax = AW(t)x$, and $W(t)x = Cx + \int_0^t W(s)Ax ds$.*

Corollary 1.1.9. *If $x \in D(A)$ then $u := t \rightarrow W(t)x$ is a classical solution of the abstract Cauchy problem*

$$(ACP_C) \begin{cases} u'(t) &= Au(t) \quad (t \geq 0) \\ u(0) &= Cx. \end{cases}$$

Moreover, if $x \in D(A^k)$ for some $k \in \mathbb{N}$, then $u \in C^k([0, \infty), X)$ and $u^{(k)}(t) = A^k u(t)$.

Examples of regularized semigroups are bi-continuous semigroups (see [66, Thm. 1.28]), integrated semigroups (see [27, 28]) and distributional semigroups (see [69]).

Remark 1.1.10. Although a substantial number of abstract Cauchy problems can be studied when the operator A is the generator of either a strongly continuous, bi-continuous, or C -regularized semigroup, Proposition 1.1.8 shows that operators A which are not closed cannot be directly studied by any of those approaches. However, as a consequence of the results obtained in Chapter 1 and Chapter 2 for these type of semigroups, Section 3.3 develops novel inversion formulas for the vector-valued Laplace transform which can be used for both solving and approximating the solutions of the abstract Cauchy problem when the operator A is not closed. For a discussion of the existence and uniqueness of solutions of the abstract Cauchy problem governed by non-closed (non-closable) operators A , see [7].

1.2 The Banach Algebra of Normalized Functions of Bounded Variation

Let $\alpha : [0, R] \rightarrow \mathbb{C}$ be a function of bounded variation with total variation denoted by $\text{Var}_{[0, R]}(\alpha)$. The space of normalized functions of bounded variation is denoted by $\text{NBV}[0, R]$, and

$$\text{NBV}_{\text{loc}} := \bigcap_{R > 0} \text{NBV}[0, R];$$

i.e., $\alpha(0) = 0$ and $\alpha(t) = \frac{\alpha(t^+) + \alpha(t^-)}{2}$ ($t > 0$). Notice that if $\alpha \in \text{NBV}_{\text{loc}}$, then $V_\alpha(t) := \text{Var}_{[0, t]}(\alpha)$ also belongs to NBV_{loc} . If $\alpha \in \text{NBV}_{\text{loc}}$ then $\text{Var}(\alpha)$ denotes $\text{Var}_{[0, \infty)}(\alpha)$. The Riemann-Stieltjes convolution on NBV_{loc} is defined by

$$(\alpha * \beta)(t) := \int_0^t \alpha(t-s) d\beta(s)$$

and satisfies $V_\gamma(t) \leq V_\alpha(t)V_\beta(t)$, where $\gamma := (\alpha * \beta)$ (see [101, Thm. 11.2b pp. 85]). Moreover, if $\int_0^\infty e^{zt} d\alpha(t)$ converges for $\text{Re}(z) < \omega$ then

$$\alpha(t) = o(e^{-\omega t}) \quad \text{as } t \rightarrow \infty \text{ for } \omega < 0. \quad (1.4)$$

If $\omega \geq 0$, then $\alpha(\infty)$ exists and

$$\alpha(t) - \alpha(\infty) = o(e^{-\omega t}) \quad \text{as } t \rightarrow \infty. \quad (1.5)$$

For a proof see [101, Thms. 2.2a and 2.2b pp. 39–40].

Let $(C_{0,\omega}(\mathbb{R}_0^+))'$ denote the dual space of $C_{0,\omega}(\mathbb{R}_0^+) := \{x : \mathbb{R}_0^+ \rightarrow \mathbb{C} \text{ continuous and } \lim_{t \rightarrow 0} e^{\omega t} x(t) = 0\}$ with $\|x\|_{\infty,\omega} := \sup_{t \in \mathbb{R}_0^+} |e^{\omega t} x(t)|$, where $\omega \in \mathbb{R}$, and let NBV^0 be the space NBV_{loc} with the norm $\|\alpha\|_0 := \int_0^\infty dV_\alpha(t)$.

Lemma 1.2.1. *The Riemann-Stieltjes operator $\Phi : \text{NBV}^0 \rightarrow (C_{0,0}(\mathbb{R}_0^+))'$ defined by*

$$\Phi(\alpha)(f) := \int_0^\infty f(t) d\alpha(t) \quad (f \in C_{0,0}(\mathbb{R}_0^+))$$

is an isometric Banach algebra isomorphism.

Proof. It is clear that $\Phi(\alpha)$ is a linear functional. The inequality

$$\left| \int_0^\infty f(t) d\alpha(t) \right| \leq \|f\|_{\infty,0} \|\alpha\|_0 \quad (1.6)$$

implies that $\Phi(\alpha)$ is bounded. From the Riesz Representation Theorem (see [92, pp. 129]) follows that there is a unique bounded regular complex Borel measure μ on \mathbb{R}_0^+ such that $\Phi(\alpha)(f) = \int_{\mathbb{R}_0^+} f(t) d\mu(t)$ and $\|\Phi(\alpha)\|_{(C_{0,0}([0,\infty)))' } = |\mu|(\mathbb{R}_0^+)$. Now, let $\Psi \in (C_{0,0}([0,\infty)))'$ and $R > 0$. The Hahn-Banach Theorem implies that there exists an extension $\nu_{\mu\Psi}$ to the dual space $(L^\infty([0,\infty)))'$ such that $\|\mu\Psi\|_{(C_{0,0}(\mathbb{R}_0^+))' } = \|\nu_{\mu\Psi}\|_{(L^\infty([0,\infty)))'}$. Define

$$\beta(t) := \langle \chi_{(0,t]}, \nu_{\mu\Psi} \rangle \text{ for } t > 0, \quad \beta(0) := 0,$$

and let $\pi_R = \{0 = t_0 < \dots < t_n = R\}$ be a partition of the interval $[0, R]$. Then

$$\begin{aligned} \sum_{i=1}^n |\beta(t_i) - \beta(t_{i-1})| &= \sum_{i=1}^n |\langle \chi_{(t_{i-1}, t_i]}, \nu_{\mu\Psi} \rangle| \\ &= \sum_{i=1}^n \text{sgn}(\langle \chi_{(t_{i-1}, t_i]}, \nu_{\mu\Psi} \rangle) \langle \chi_{(t_{i-1}, t_i]}, \nu_{\mu\Psi} \rangle \\ &= \sum_{i=1}^n \langle \text{sgn}(\langle \chi_{(t_{i-1}, t_i]}, \nu_{\mu\Psi} \rangle) \chi_{(t_{i-1}, t_i]}, \nu_{\mu\Psi} \rangle \\ &= \left\langle \sum_{i=1}^n \text{sgn}(\langle \chi_{(t_{i-1}, t_i]}, \nu_{\mu\Psi} \rangle) \chi_{(t_{i-1}, t_i]}, \nu_{\mu\Psi} \right\rangle \\ &\leq \|\nu_{\mu\Psi}\|_{(L^\infty)' } \left\| \sum_{i=1}^n \text{sgn}(\langle \chi_{(t_{i-1}, t_i]}, \nu_{\mu\Psi} \rangle) \chi_{(t_{i-1}, t_i]} \right\|_{L^\infty} \\ &= \|\nu_{\mu\Psi}\|_{(L^\infty)' } = \|\mu\Psi\|_{(C_{0,0})' }. \end{aligned}$$

This shows that β is a function of bounded variation on $[0, R]$. By the usual normalization process one can think of β as a normalized function of bounded variation on $[0, R]$. Since the inequality above is independent of R , it follows that

$$\|\beta\|_0 \leq \|\Psi\|_{(C_{0,0}([0,\infty)))' }. \quad (1.7)$$

Now, in order to show that $\Phi(\beta) = \Psi$, an $\frac{\varepsilon}{3}$ -argument will be used. Let $\varepsilon > 0$, and let $f \in C_{0,0}(\mathbb{R}_0^+)$. Then there is $R > 0$ such that $|f(t)| < \varepsilon$ for all $t > R$. Moreover,

$$\left| \int_0^\infty f(t)d\beta(t) - \int_0^R f(t)d\beta(t) \right| \leq \varepsilon \text{Var}_{[R,\infty]}(\beta) \leq \varepsilon \|\beta\|_0.$$

For the second estimate one considers the previous extension ν_{μ_Ψ} , a partition π_R of the interval $[0, R]$, and the step function $f_\varepsilon(t) := f(t_i)$ for $t_{i-1} < t \leq t_i$ and $f_\varepsilon(t) := 0$ else. In this way $f_\varepsilon = \sum_{i=1}^n f(t_i)\chi_{(t_{i-1}, t_i]}$ and $\langle f_\varepsilon, \nu_{\mu_\Psi} \rangle = \sum_{i=1}^n f(t_i)(\beta(t_i) - \beta(t_{i-1}))$. Therefore, if the partition is fine enough,

$$\left| \int_0^R f(t)d\beta(t) - \langle f_\varepsilon, \nu_{\mu_\Psi} \rangle \right| < \varepsilon.$$

The third and final estimate is based on the fact that $\langle g, \mu_\Psi \rangle = \langle g, \nu_{\mu_\Psi} \rangle$ for all $g \in C_{0,0}([0, \infty))$ and that $\|f - f_\varepsilon\|_{L^\infty} < \varepsilon$. In this way one obtains that

$$|\langle f_\varepsilon, \nu_{\mu_\Psi} \rangle - \langle f, \mu_\Psi \rangle| = |\langle f_\varepsilon, \nu_{\mu_\Psi} \rangle - \langle f, \nu_{\mu_\Psi} \rangle| = |\langle f_\varepsilon - f, \nu_{\mu_\Psi} \rangle| \leq \varepsilon \|\nu_{\mu_\Psi}\|_{(L^\infty)'}$$

By combining these three estimates it follows that

$$\left| \int_0^\infty f(t)d\alpha(t) - \langle f, \mu_\Psi \rangle \right| < \varepsilon (\|\nu_{\mu_\Psi}\|_{(L^\infty)'} + \|\beta\|_0 + 1).$$

Since ε and f are arbitrary one obtains that $\Phi(\beta) = \Psi$. Moreover, from (1.6) and (1.7), it follows that

$$\|\Phi(\beta)\|_{(C_{0,0}([0,\infty)))'} = \|\beta\|_0. \quad (1.8)$$

The only thing left to show is that Φ preserves the product between this two Banach algebras. So, one has to show that if $\Lambda_{\mu \star \nu}$ is the linear functional on $C_{0,0}(\mathbb{R}_0^+)$ associated with the measure $\mu \star \nu$ given by $\Lambda_{\mu \star \nu}(f) = \int_{\mathbb{R}_0^+} \int_{\mathbb{R}_0^+} f(t+s)d\mu(t)d\nu(s)$, then there exist $\alpha, \beta \in \text{NBV}_{\text{loc}}$ such that $\Lambda_{\mu \star \nu}(f) = \int_0^\infty f(t)d\gamma(t)$ for all $f \in C_{0,0}(\mathbb{R}_0^+)$, where $\gamma = \alpha \star \beta$. From the previous construction one obtains that there exist $\alpha, \beta \in \text{NBV}_{\text{loc}}$ such that $\Lambda_{\mu \star \nu}(f) = \int_0^\infty \int_0^\infty f(t+s)d\alpha(t)d\beta(s)$. Finally, by using integration by parts and Fubini's Theorem, follows that

$$\begin{aligned} \int_0^\infty \int_0^\infty f(t+s)d\alpha(t)d\beta(s) &= \int_0^\infty \int_s^\infty f(t)d\alpha(t-s)d\beta(s) \\ &= - \int_0^\infty \int_y^\infty \alpha(t-s)df(t)d\beta(s) = - \int_0^\infty \int_0^t \alpha(t-s)d\beta(s)df(t) \\ &= - \int_0^\infty \gamma(t)df(t) = \int_0^\infty f(t)d\gamma(t). \end{aligned}$$

□

Since the map $\varphi_\omega : C_{0,\omega}(\mathbb{R}_0^+) \rightarrow C_{0,0}(\mathbb{R}_0^+)$ defined by $\varphi_\omega(f)(t) = f(t)e^{\omega t}$ is an isometric isomorphism, the map $\varphi_\omega^* : \text{NBV}^0 \rightarrow \text{NBV}^\omega$ defined by $\varphi_\omega^*(\alpha)(t) = \int_0^t e^{-\omega s}d\alpha(s)$ is an isometric isomorphism as well, where

$$\text{NBV}^\omega := \left\{ \alpha \in \text{NBV}_{\text{loc}} : \|\alpha\|_\omega := \int_0^\infty e^{\omega t}dV_\alpha(t) < \infty \right\}.$$

Thus, the following well known representation is obtained.

Theorem 1.2.2 (Riesz Representation). *Let $\omega \in \mathbb{R}$ and let $\Psi \in (C_{0,\omega}(\mathbb{R}_0^+))'$. Then there exist a unique $\alpha_\omega \in NBV^\omega$ such that, for every $f \in C_{0,\omega}(\mathbb{R}_0^+)$,*

$$\langle f, \Psi \rangle = \int_0^\infty f(t) d\alpha_\omega(t). \quad (1.9)$$

Proof. Let $f \in C_{0,\omega}(\mathbb{R}_0^+)$ and let $g := \varphi_\omega^{-1}(f)$. It follows from Lemma 1.2.1 that

$$\begin{aligned} \langle f, \Psi \rangle &= \langle \varphi_\omega(g), \Psi \rangle = \langle g, \Psi \circ \varphi_\omega^{-1} \rangle = \int_0^\infty g(t) d\alpha(t) \\ &= \int_0^\infty f(t) e^{-\omega t} d\alpha(t) = \int_0^\infty f(t) d[\varphi_\omega^*(\alpha)](t), \end{aligned}$$

and the result follows by defining $\alpha_\omega := \varphi_\omega^*(\alpha)$. \square

Remark 1.2.3. In summary, this section shows that the space NBV^ω is a Banach algebra with the Stieltjes-convolution as product and $\|\alpha\|_\omega := \int_0^\infty e^{\omega t} dV_\alpha(t)$. Furthermore, if $(\mathcal{F}^\omega, +, \cdot)$ is the algebra of functions with Laplace-Stieltjes representation $f_\alpha(z) = \int_0^\infty e^{zt} d\alpha(t)$ ($\operatorname{Re}(z) \leq \omega$) for some $\alpha \in NBV^\omega$, then the operator $\Phi : NBV^\omega \rightarrow \mathcal{F}^\omega$ defined by $\Phi(\alpha) = f_\alpha$ is an algebra isomorphism. If $\|f_\alpha\| := \|\alpha\|_\omega$, then \mathcal{F}^ω is a Banach algebra and the inclusion $\mathcal{F}^\omega \subset F^\kappa$ holds for $\omega \geq \kappa$.

As a consequence of Theorem 1.2.2, one obtains directly that the Hille-Phillips Functional Calculus can be expressed in terms of normalized functions of bounded variation instead of measures. The original proof of this fact was given by R.S. Phillips in [83] by using an algebraic approach. M. Kovács shows in Theorem 6 of [61] a constructive version of Phillip's result. In the following sections, Kovács' proof is extended to generators of bi-continuous semigroups as well as C -regularized semigroups.

1.3 The Hille-Phillips Functional Calculus for Generators of Bi-Continuous Semigroups

In order to extend the Hille-Phillips functional calculus given by (1.1) to generators of bi-continuous semigroups¹, it is necessary to consider the τ -Riemann-Stieltjes integral for vector-valued functions.

Definition 1.3.1. Let X be a Banach space with a coherent topology τ and let $\alpha \in NBV[0, R]$. A function $f : [0, R] \rightarrow X$ is τ -Riemann-Stieltjes integrable with respect to α if

$$\int_0^R f(s) d\alpha(s) := \tau\text{-}\lim_{|\pi| \rightarrow 0} \sum_{i=1}^n (\alpha(s_i) - \alpha(s_{i-1})) f(\xi_i)$$

exists for every partition $\pi := \{0 = s_0 < \dots < s_n = R\}$ of $[0, R]$, where $\xi_i \in (s_i, s_{i-1})$ and $|\pi| = \max_{1 \leq i \leq n} \{s_i - s_{i-1}\}$.

¹For a comprehensive introduction to generators of bi-continuous semigroups see [66].

Lemma 1.3.2. *Let $f : [0, R] \rightarrow X$ be τ -continuous, $\alpha \in \text{NBV}[0, R]$, and $B : (X, \tau) \rightarrow (X, \tau)$ be continuous and linear. Then $\int_0^R f(t) d\alpha(t)$ exists, the map $t \rightarrow Bf(t)$ is τ -continuous, and $\int_0^R Bf(t) d\alpha(t) = B \int_0^R f(t) d\alpha(t)$. Moreover, if $\phi \in (X, \tau)'$, then the map $t \rightarrow \langle f(t), \phi \rangle$ is continuous and*

$$\left\langle \int_0^R f(t) d\alpha(t), \phi \right\rangle = \int_0^R \langle f(t), \phi \rangle d\alpha(t). \quad (1.10)$$

Proof. Let $\varepsilon > 0$. Then, for every semi-norm $p \in \mathcal{P}_\tau$ there exist $\delta_{\varepsilon, p} > 0$ such that $p(f(s_1) - f(s_2)) < \varepsilon$ whenever $|s_1 - s_2| < \delta_{\varepsilon, p}$. Let π_1 and π_2 be two partitions of the interval $[0, R]$ such that $|\pi_j| < \frac{\delta_{\varepsilon, p}}{2}$ and let $\pi = \{0 = t_0 < t_1 < \dots < t_n = R\}$ be the partition where the t_i 's are the partitioning points of π_1 and π_2 together. It follows that $S(f, \alpha, \pi_j) = \sum_{i=1}^n \alpha(s_{i,j})(f(t_i) - f(t_{i-1}))$, where the $s_{i,j}$, t_i , and t_{i-1} belong to the same subinterval of π_j . In this way, one obtains that

$$\begin{aligned} p(S(f, \alpha, \pi_1) - S(f, \alpha, \pi_2)) &= p\left(\sum_{i=1}^n \alpha(s_{i,1}) - \alpha(s_{i,2})(f(t_i) - f(t_{i-1}))\right) \\ &\leq \varepsilon \sum_{i=1}^n |\alpha(s_{i,1}) - \alpha(s_{i,2})| \leq \varepsilon V_\alpha(R). \end{aligned}$$

The first statement of the lemma follows by applying Cauchy's convergence criterion. Now, if $\phi \in (X, \tau)'$, then

$$\begin{aligned} \left\langle \int_0^R f(t) d\alpha(t), \phi \right\rangle &= \phi\left(\int_0^R f(t) d\alpha(t)\right) = \phi\left(\tau\text{-}\lim_{|\pi| \rightarrow 0} S(f, \alpha, \pi)\right) \\ &= \lim_{|\pi| \rightarrow 0} \phi(S(f, \alpha, \pi)) = \lim_{|\pi| \rightarrow 0} \phi\left(\sum_{i=0}^n (\alpha(t_i) - \alpha(t_{i-1}))f(s_i)\right) \\ &= \lim_{|\pi| \rightarrow 0} \left(\sum_{i=0}^n (\alpha(t_i) - \alpha(t_{i-1}))\phi(f(s_i))\right) = \int_0^R \phi(f(t)) d\alpha(t) \\ &= \int_0^R \langle f(t), \phi \rangle d\alpha(t). \end{aligned}$$

□

Proposition 1.3.3. *Let T be a bi-continuous semigroup of type (M, ω) on X . If $\alpha \in \text{NBV}^\omega$, then $\lim_{R \rightarrow \infty} \int_0^R T(s)x d\alpha(s)$ exists and the map*

$$x \rightarrow \int_0^\infty T(s)x d\alpha(s) := \lim_{R \rightarrow \infty} \int_0^R T(s)x d\alpha(s)$$

is a bounded linear operator on $(X, \|\cdot\|)$. Furthermore, if B is a continuous linear operator from (X, τ) into (X, τ) , then $B \int_0^\infty T(s)x d\alpha(s) = \int_0^\infty BT(s)x d\alpha(s)$.

Proof. Consider $\alpha \in \text{NBV}^\omega$. By Lemma 1.3.2, $x_R := \int_0^R T(s)x \, d\alpha(s)$ exists for each $R > 0$. Applying Lemma 1.3.2 once again yields

$$\begin{aligned} \|x_R - x_S\| &= \left\| \int_R^S T(s)x \, d\alpha(s) \right\| = \sup_{\phi \in \Phi} \left| \left\langle \int_R^S T(s)x \, d\alpha(s), \phi \right\rangle \right| \\ &= \sup_{\phi \in \Phi} \left| \int_R^S \langle T(s)x, \phi \rangle \, d\alpha(s) \right| \leq \sup_{\phi \in \Phi} \int_R^S |\langle T(s)x, \phi \rangle| \, dV_\alpha(s) \\ &\leq \int_R^S \|T(s)\|_{\mathcal{L}(X)} \|x\| \, dV_\alpha(s) \leq M \|x\| \int_R^S e^{\omega s} \, dV_\alpha(s) \rightarrow 0 \end{aligned}$$

as $R, S \rightarrow \infty$. Therefore, the net $\{x_R\}$ is $\|\cdot\|$ -Cauchy and thus norm-convergent. Since $(X, \tau)' \subseteq (X, \|\cdot\|)'$, it follows that

$$\left\langle \int_0^\infty T(s)x \, d\alpha(s), \phi \right\rangle = \int_0^\infty \langle T(s)x, \phi \rangle \, d\alpha(s) \quad (1.11)$$

for all $\phi \in (X, \tau)'$. By using the norming property of $(X, \tau)'$ and (1.11), the boundedness of the operator $x \rightarrow \int_0^\infty T(s)x \, d\alpha(s)$ follows from

$$\begin{aligned} \left\| \int_0^\infty T(s)x \, d\alpha(s) \right\| &= \sup_{\phi \in \Phi} \left| \left\langle \int_0^\infty T(s)x \, d\alpha(s), \phi \right\rangle \right| = \sup_{\phi \in \Phi} \left| \int_0^\infty \langle T(s)x, \phi \rangle \, d\alpha(s) \right| \\ &\leq M \|x\| \int_0^\infty e^{\omega s} \, dV_\alpha(s) = M \|x\| \|\alpha\|_\omega. \end{aligned}$$

The second part of the statement is a consequence of Lemma 1.3.2. \square

The Hille-Phillips Functional calculus theorem was first described by E. Hille in [52] and was extensively developed by R.S. Phillips in [83, 84, 85]. In the classic work of N. Dunford and J. Schwartz [38], it is shown that the Hille-Phillips Functional Calculus coincides with the Riesz-Dunford Functional Calculus for generators of groups of operators². On the other hand, A.V. Balakrishnan extends the Hille-Phillips Functional Calculus in [5] and allows the study of unbounded operators of the form $f(A)$ where A is the generator of a strongly continuous semigroups and f is the quotient of Laplace-Stieltjes transforms. Similarly, E. Nelson shows in [76] an extension of the Hille-Phillips Functional Calculus by considering distributions in the sense of L. Schwartz instead of normalized functions of bounded variation³.

Theorem 1.3.4 (Hille-Phillips Functional Calculus). *Let A be the generator of a bi-continuous semigroup T of type (M, ω) on X . If $\alpha \in \text{NBV}^\omega$ is such that $f(z) = \int_0^\infty e^{zt} \, d\alpha(t)$ for $\text{Re}(z) \leq \omega$, then $\Psi : \mathcal{F}^\omega \rightarrow \mathcal{L}(X)$ defined by*

$$\Psi(f)x := \int_0^\infty T(t)x \, d\alpha(t) \quad (1.12)$$

is an algebra homomorphism. Moreover, if $f(A) := \Psi(f)$ then $\|f(A)\| \leq M \|\alpha\|_\omega$.

²An operator A is the generator of a strongly continuous group of operators if and only if both A and $-A$ generate strongly continuous semigroups, see [39].

³R.S. Phillips points out in [85] that L. Schwartz developed an extension of the Hille-Phillips functional calculus by using the theory of distributions but that the work was unpublished.

Proof. It will be shown that the product is preserved; the rest of the proof is clear. The proof is divided in three parts. First, let $\phi \in (X, \tau)'$ and let $x \in X$. It follows that

$$\begin{aligned}
\langle \Psi(f_\alpha)\Psi(f_\beta)x, \phi \rangle &= \left\langle \int_0^\infty T(t) \left(\int_0^\infty T(s)x d\beta(s) \right) d\alpha(t), \phi \right\rangle \\
&= \int_0^\infty \left\langle T(t) \left(\int_0^\infty T(s)x d\beta(s) \right), \phi \right\rangle d\alpha(t) \\
&= \int_0^\infty \left\langle \int_0^\infty T(t)T(s)x d\beta(s), \phi \right\rangle d\alpha(t) \\
&= \int_0^\infty \int_0^\infty \langle T(t)T(s)x, \phi \rangle d\beta(s) d\alpha(t) \\
&= \int_0^\infty \int_0^\infty \langle T(t+s)x, \phi \rangle d\beta(s) d\alpha(t) \\
&= \int_0^\infty \int_t^\infty \langle T(u)x, \phi \rangle d\beta(u-t) d\alpha(t).
\end{aligned}$$

Now, suppose that $\omega < 0$. From integration by parts, the property (1.4), Fubini's Theorem, and $\gamma = \alpha * \beta$, it follows that

$$\begin{aligned}
\int_0^\infty \int_t^\infty \langle T(u)x, \phi \rangle d\beta(u-t) d\alpha(t) &= - \int_0^\infty \int_t^\infty \beta(u-t) d\langle T(u)x, \phi \rangle d\alpha(t) \\
&= - \int_0^\infty \int_0^u \beta(u-t) d\alpha(t) d\langle T(u)x, \phi \rangle \\
&= \int_0^\infty -\gamma(u) d\langle T(u)x, \phi \rangle \\
&= \int_0^\infty \langle T(u)x, \phi \rangle d\gamma(u) \\
&= \left\langle \int_0^\infty T(u)x d\gamma(u), \phi \right\rangle.
\end{aligned}$$

Thus, $\langle \Psi(f_\alpha)\Psi(f_\beta)x, \phi \rangle = \langle \int_0^\infty T(u)x d\gamma(u), \phi \rangle$ for $\omega < 0$.

In the case where $\omega \geq 0$, observe that integration by parts and the fact that $\lim_{s \rightarrow \infty} e^{\omega t}(\beta(s-t) - \beta(\infty)) = 0$ (see (1.5)) yields

$$\begin{aligned}
\int_0^\infty \int_t^\infty \langle T(u)x, \phi \rangle d\beta(u-t) d\alpha(t) &= \\
\int_0^\infty \left[\langle T(t)x, \phi \rangle \beta(\infty) - \int_t^\infty (\beta(s-t) - \beta(\infty)) d\langle T(s)x, \phi \rangle \right] d\alpha(t). & \tag{1.13}
\end{aligned}$$

By using Fubini and integration by parts, the second term of the right side of (1.13) becomes

$$\begin{aligned}
& \int_0^\infty \int_t^\infty (\beta(s-t) - \beta(\infty)) d\langle T(s)x, \phi \rangle d\alpha(t) \\
&= \int_0^\infty \int_0^s (\beta(s-t) - \beta(\infty)) d\alpha(t) d\langle T(s)x, \phi \rangle \\
&= \int_0^\infty \int_0^s \beta(s-t) d\alpha(t) d\langle T(s)x, \phi \rangle \\
&\quad - \beta(\infty) \int_0^\infty \alpha(s) d\langle T(s)x, \phi \rangle \\
&= \int_0^\infty \gamma(s) d\langle T(s)x, \phi \rangle - \beta(\infty) \int_0^\infty \alpha(s) d\langle T(s)x, \phi \rangle \\
&= \int_0^\infty (\gamma(s) - \beta(\infty)\alpha(s)) d\langle T(s)x, \phi \rangle \\
&= \lim_{s \rightarrow \infty} (\gamma(s) - \beta(\infty)\alpha(s)) \langle T(s)x, \phi \rangle \\
&\quad - \int_0^\infty \langle T(s)x, \phi \rangle d\gamma(s) - \int_0^\infty \beta(\infty) \langle T(s)x, \phi \rangle d\alpha(s).
\end{aligned}$$

Since $\gamma \in \text{NBV}^\omega$ and by applying (1.5) to γ and α ,

$$\begin{aligned}
& \lim_{s \rightarrow \infty} |(\gamma(s) - \beta(\infty)\alpha(s)) \langle T(s)x, \phi \rangle| \leq \lim_{s \rightarrow \infty} |(\gamma(s) - \gamma(\infty)) \langle T(s)x, \phi \rangle| \\
&\quad + \lim_{s \rightarrow \infty} |\gamma(\infty) - \beta(\infty)\alpha(s)| \langle T(s)x, \phi \rangle| \\
&= \lim_{s \rightarrow \infty} |(\gamma(s) - \gamma(\infty)) \langle T(s)x, \phi \rangle| \\
&\quad + \lim_{s \rightarrow \infty} |\alpha(\infty)\beta(\infty) - \beta(\infty)\alpha(s)| \langle T(s)x, \phi \rangle| = 0.
\end{aligned}$$

In this way, we obtain that

$$\begin{aligned}
& \int_0^\infty \int_t^\infty (\beta(s-t) - \beta(\infty)) d\langle T(s)x, \phi \rangle d\alpha(t) = \\
&\quad - \int_0^\infty \langle T(s)x, \phi \rangle d\gamma(s) - \int_0^\infty \beta(\infty) \langle T(s)x, \phi \rangle d\alpha(s).
\end{aligned} \tag{1.14}$$

Hence, using (1.14) and (1.13), we obtain that for $\omega \geq 0$,

$$\int_0^\infty \int_t^\infty \langle T(u)x, \phi \rangle d\beta(u-t) d\alpha(t) = \int_0^\infty \langle T(u)x, \phi \rangle d\gamma(u). \tag{1.15}$$

Combining the two cases $\omega \geq 0$ and $\omega < 0$, one obtains that for every $\phi \in (X, \tau)'$ and $\omega \in \mathbb{R}$,

$$\langle \Psi(f_\alpha)\Psi(f_\beta)x, \phi \rangle = \left\langle \int_0^\infty T(u)x d\gamma(u), \phi \right\rangle, \tag{1.16}$$

where $\gamma = \alpha * \beta$. Using (1.16), one finally obtains that

$$\begin{aligned} \|\Psi(f_\alpha)\Psi(f_\beta)x - \Psi(f_\gamma)x\| &= \sup_{\phi \in \Phi} |\langle \Psi(f_\alpha)\Psi(f_\beta)x - \Psi(f_\gamma)x, \phi \rangle| \\ &= \sup_{\phi \in \Phi} |\langle \Psi(f_\alpha)\Psi(f_\beta)x, \phi \rangle - \langle \Psi(f_\gamma)x, \phi \rangle| = 0 \end{aligned}$$

□

Corollary 1.3.5. *If A generates a bi-continuous semigroup T of type (M, ω) , then*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x dt \quad \operatorname{Re}(\lambda) \geq \omega. \quad (1.17)$$

Proof. Consider $\alpha_\lambda(t) := \int_0^t e^{-\lambda s} ds \in \mathcal{F}^\omega$ and apply Theorem 1.3.4. □

1.4 The Hille-Phillips Functional Calculus for Generators of C -Regularized Semigroups

The first problem to face when extending the Hille-Phillips Functional Calculus to generators of C -regularized semigroups is the fact that $f(A)$ might not be a bounded operator if A generates a C -regularized semigroup and $f \in \mathcal{F}^\omega$. In order to see this, consider the normalized Heaviside functions

$$H_0(s) := \begin{cases} 0 & \text{if } s = 0 \\ 1 & \text{if } s > 0 \end{cases} \quad \text{and} \quad H_t(s) := \begin{cases} 0 & \text{if } 0 \leq s < t \\ \frac{1}{2} & \text{if } s = t \\ 1 & \text{if } s > t. \end{cases} \quad (1.18)$$

Clearly $H_t \in \text{NBV}^\omega$ for all $\omega \in \mathbb{R}$ and $t \geq 0$. In this way,

$$f_t(z) := e^{tz} = \int_0^\infty e^{sz} dH_t(s) \in \mathcal{F}^\omega \quad (\omega \in \mathbb{R}, t \geq 0). \quad (1.19)$$

Thus, the example given in Section 1.1 by the multiplication operator $Af(s) = sf(s)$ on $C_0(\mathbb{R}, \mathbb{C})$ together with (1.19) shows that $f_t(A)$ is an unbounded operator for all $t \geq 0$.

Thus, regularizing in the same way as for C -regularized semigroups, one should be able to construct a C -regularized Hille-Phillips Functional Calculus. The first constructions of C -regularized Functional Calculi were introduced by R. deLaubenfels in [29] for the Riesz-Dunford Functional Calculus. The main goal of this section is to show such construction for the Hille-Phillips Functional Calculus.

The proof of the following lemma is straight forward and is omitted, cf. Proposition 1.9.4 of [4].

Lemma 1.4.1. *If $f : [0, R] \rightarrow X$ is continuous and $\alpha \in \text{NBV}[0, R]$, then the Riemann-Stieltjes integral $\int_0^R f(t) d\alpha(t)$ exists and, for all $\phi \in (X, \|\cdot\|)'$,*

$$\left\langle \int_0^R f(t) d\alpha(t), \phi \right\rangle = \int_0^R \langle f(t), \phi \rangle d\alpha(t).$$

Proposition 1.4.2. *Let W be a C -regularized semigroup of type (M, ω) on X . If $\alpha \in \text{NBV}^\omega$, then $\lim_{R \rightarrow \infty} \int_0^R W(t)x d\alpha(t)$ exists and the map*

$$x \rightarrow \int_0^\infty W(t)x d\alpha(t) := \lim_{R \rightarrow \infty} \int_0^R W(t)x d\alpha(t) \quad (1.20)$$

is a bounded linear operator on $(X, \|\cdot\|)$. Furthermore, if B is a bounded linear operator on $(X, \|\cdot\|)$, then $B \int_0^\infty W(t)x d\alpha(t) = \int_0^\infty BW(t)x d\alpha(t)$.

Proof. Consider $\alpha \in \text{NBV}^\omega$. From Lemma 1.4.1 follows that $\int_0^R W(t)x d\alpha(t)$ exists for each $R \geq 0$. Let $x_R := \int_0^R W(t)x d\alpha(t)$. Then

$$\begin{aligned} \|x_R - x_T\| &= \left\| \int_R^T W(t)x d\alpha(t) \right\| \leq \int_R^T \|W(t)\|_{\mathcal{L}(X)} \|x\| dV_\alpha(t) \\ &\leq M \|x\| \int_R^T e^{\omega t} dV_\alpha(t) \rightarrow 0 \quad \text{as } R, T \rightarrow \infty. \end{aligned}$$

Therefore, the net $\{x_R\}_{R \geq 0}$ is Cauchy. On the other hand, it follows from the definitions that the map $x \rightarrow \int_0^\infty W(t)x d\alpha(t)$ is linear. Thus,

$$\begin{aligned} \left\| \int_0^\infty W(t)x d\alpha(t) \right\| &\leq \int_0^\infty \|W(t)\|_{\mathcal{L}(X)} \|x\| dV_\alpha(t) \\ &\leq M \|x\| \int_0^\infty e^{\omega t} dV_\alpha(t) = M \|x\| \|\alpha\|_{\text{var}}. \end{aligned}$$

The last part of the statement follows immediately by using the definitions. \square

Theorem 1.4.3 (C -regularized Hille-Phillips Functional Calculus). *Let W be a C -regularized semigroup of type (M, ω) on X generated by A . If $f_\alpha \in \mathcal{F}^\omega$ is such that $f_\alpha(z) = \int_0^\infty e^{zt} d\alpha(t)$ ($\text{Re}(z) \leq \omega$), then the linear map $\Psi : \mathcal{F}^\omega \rightarrow \mathcal{L}(X)$ defined by $\Psi(f_\alpha)x := \int_0^\infty W(t)x d\alpha(t)$ satisfies*

$$\Psi(f_\alpha)\Psi(f_\beta) = C\Psi(f_{\alpha*\beta}) \text{ and } \|\Psi(f_\alpha)\| \leq M\|\alpha\|_\omega.$$

Proof. First of all, let $\phi \in (X, \|\cdot\|)'$ and let $x \in X$. It follows that

$$\begin{aligned} \langle \Psi(f_\alpha)\Psi(f_\beta)x, \phi \rangle &= \left\langle \int_0^\infty W(t) \left(\int_0^\infty W(s)x d\beta(s) \right) d\alpha(t), \phi \right\rangle \\ &= \int_0^\infty \int_0^\infty \langle CW(t+s)x, \phi \rangle d\beta(s) d\alpha(t) \\ &= \int_0^\infty \int_t^\infty \langle CW(u)x, \phi \rangle d\beta(u-t) d\alpha(t). \end{aligned}$$

Now, let us suppose first that $\omega < 0$. Using integration by parts, (1.4), and Fubini's theorem one obtains that

$$\begin{aligned}
& \int_0^\infty \int_t^\infty \langle CW(u)x, \phi \rangle d\beta(u-t)d\alpha(t) = - \int_0^\infty \int_t^\infty \beta(u-t)d\langle CW(u)x, \phi \rangle d\alpha(t) \\
& = - \int_0^\infty \int_0^u \beta(u-t)d\alpha(t)d\langle CW(u)x, \phi \rangle = \int_0^\infty -\gamma(u)d\langle CW(u)x, \phi \rangle \\
& = \int_0^\infty \langle CW(u)x, \phi \rangle d\gamma(u) = \left\langle \int_0^\infty CW(u)x d\gamma(u), \phi \right\rangle,
\end{aligned}$$

where $\gamma = \alpha * \beta$. Now consider the case $\omega \geq 0$. Using integration by parts and the fact that $\lim_{s \rightarrow \infty} e^{\omega t}(\beta(s-t) - \beta(\infty)) = 0$, it follows from property (1.5) that

$$\begin{aligned}
& \int_0^\infty \int_t^\infty \langle CW(u)x, \phi \rangle d\beta(u-t)d\alpha(t) = \\
& \int_0^\infty \left[\langle CW(t)x, \phi \rangle \beta(\infty) - \int_t^\infty (\beta(s-t) - \beta(\infty))d\langle CW(s)x, \phi \rangle \right] d\alpha(t),
\end{aligned} \tag{1.21}$$

By Fubini's theorem and integration by parts, the second term of the right hand side of (1.21) becomes

$$\begin{aligned}
& \int_0^\infty \int_t^\infty (\beta(s-t) - \beta(\infty))d\langle CW(s)x, \phi \rangle d\alpha(t) \\
& = \int_0^\infty \int_0^s (\beta(s-t) - \beta(\infty))d\alpha(t)d\langle CW(s)x, \phi \rangle \\
& = \int_0^\infty \int_0^s \beta(s-t)d\alpha(t)d\langle CW(s)x, \phi \rangle - \beta(\infty) \int_0^\infty \alpha(s)d\langle CW(s)x, \phi \rangle \\
& = \int_0^\infty \gamma(s)d\langle CW(s)x, \phi \rangle - \beta(\infty) \int_0^\infty \alpha(s)d\langle CW(s)x, \phi \rangle \\
& = \int_0^\infty (\gamma(s) - \beta(\infty)\alpha(s))d\langle CW(s)x, \phi \rangle \\
& = \lim_{s \rightarrow \infty} (\gamma(s) - \beta(\infty)\alpha(s))\langle CW(s)x, \phi \rangle - \int_0^\infty \langle CW(s)x, \phi \rangle d\gamma(s) \\
& \quad - \int_0^\infty \beta(\infty)\langle CW(s)x, \phi \rangle d\alpha(s).
\end{aligned}$$

Since $\gamma \in \text{NBV}^\omega$ and by applying (1.5) to γ and α ,

$$\begin{aligned}
& \lim_{s \rightarrow \infty} |(\gamma(s) - \beta(\infty)\alpha(s))\langle CW(s)x, \phi \rangle| \leq \lim_{s \rightarrow \infty} |(\gamma(s) - \gamma(\infty))\langle CW(s)x, \phi \rangle| \\
& \quad + \lim_{s \rightarrow \infty} |\gamma(\infty) - \beta(\infty)\alpha(s))\langle CW(s)x, \phi \rangle| = \lim_{s \rightarrow \infty} |(\gamma(s) - \gamma(\infty))\langle CW(s)x, \phi \rangle| \\
& \quad + \lim_{s \rightarrow \infty} |\alpha(\infty)\beta(\infty) - \beta(\infty)\alpha(s))\langle CW(s)x, \phi \rangle| = 0.
\end{aligned}$$

Thus,

$$\begin{aligned} & \int_0^\infty \int_t^\infty (\beta(s-t) - \beta(\infty)) d\langle CW(s)x, \phi \rangle d\alpha(t) \\ &= - \int_0^\infty \langle CW(s)x, \phi \rangle d\gamma(s) - \int_0^\infty \beta(\infty) \langle CW(s)x, \phi \rangle d\alpha(s). \end{aligned} \quad (1.22)$$

Hence, by combining (1.22) with (1.21), one obtains that

$$\int_0^\infty \int_t^\infty \langle CW(u)x, \phi \rangle d\beta(u-t) d\alpha(t) = \int_0^\infty \langle CW(u)x, \phi \rangle d\gamma(u). \quad (1.23)$$

Therefore,

$$\langle \Psi(f_\alpha)\Psi(f_\beta)x, \phi \rangle = \left\langle C \int_0^\infty W(u)x d\gamma(u), \phi \right\rangle \quad (1.24)$$

for every $\phi \in (X, \|\cdot\|)'$, where $\gamma = \alpha * \beta$. Finally, it follows from (1.24) that

$$\begin{aligned} \|\Psi(f_\alpha)\Psi(f_\beta)x - C\Psi(f_\gamma)x\| &= \sup_{|\phi| \leq 1} |\langle \Psi(f_\alpha)\Psi(f_\beta)x - C\Psi(f_\gamma)x, \phi \rangle| \\ &= \sup_{|\phi| \leq 1} |\langle \Psi(f_\alpha)\Psi(f_\beta)x, \phi \rangle - \langle C\Psi(f_\gamma)x, \phi \rangle| = 0. \end{aligned}$$

□

Remark 1.4.4. Notice that one can think of $\Psi(f)$ as $f(A)C$ for every $f \in F^\omega$.

In this way, Theorem 1.4.3 extends the classical Hille-Phillips functional calculus to a C -regularized version for generators of C -regularized semigroups.

Corollary 1.4.5. *If A generates a C -regularized semigroup W of type (M, ω) , then*

$$R(\lambda, A)Cx = \int_0^\infty e^{-\lambda t} W(t)x dt \quad \operatorname{Re}(\lambda) \geq \omega. \quad (1.25)$$

Proof. Consider $\alpha_\lambda(t) := \int_0^t e^{-\lambda t} dt \in \mathcal{F}^\omega$ and apply Theorem 1.4.3. □

Chapter 2

Approximation Schemes for Operator Semigroups

As previously mentioned, if A generates a strongly continuous semigroup T on a Banach space X , then $T(t)x$ can be thought as the exponential function $t \rightarrow e^{tA}x$. In fact, every strongly continuous semigroup is given by the exponential formula

$$T(t)x := e^{tA}x := \lim_{n \rightarrow \infty} \left(I - \frac{t}{n}A \right)^{-n} x \quad (x \in X, t \geq 0). \quad (2.1)$$

If $r(z) := \frac{1}{1-z}$, then

$$r(z) - e^z = \frac{z^2}{2} + \frac{5z^3}{6} + \dots = o(z^2) \text{ as } z \rightarrow 0$$

and (2.1) becomes

$$T(t)x = \lim_{n \rightarrow \infty} r^n \left(\frac{t}{n}A \right) x \quad (x \in X, t \geq 0). \quad (2.2)$$

R. Hersh and T. Kato in [47], as well as P. Brenner and V. Thomée in [10] noticed that the Hille-Phillips functional calculus provides a powerful tool to study rational approximations of strongly continuous semigroups, such as the one given in (2.2).

R. Hersh and T. Kato remarked:

“Earlier versions of this work...used the Dunford calculus... By using instead the Hille-Phillips calculus... we now get sharper estimates and simpler proofs.”

This chapter shows that the approximation results for strongly continuous semigroups can be fully lifted to the bi-continuous case as well as to the C -regularized case by using the extensions of the Hille-Phillips functional calculus obtained in Chapter 1. The first section describes some rational approximations to the exponential function to be considered here. The second section adapts a method developed in [10] in terms of normalized functions of bounded variation for the estimation of the total variation norm. The third section discusses time discretization for bi-continuous semigroups and the fourth section discusses the corresponding results for C -regularized semigroups. Finally, the last section shows time and space discretization results for C -regularized semigroups by extending a Trotter-Kato type of result to the C -regularized case.

2.1 Rational Approximations of the Exponential Function

Definition 2.1.1. A rational function r is called \mathcal{A} -stable if $|r(z)| \leq 1$ for $\operatorname{Re}(z) \leq 0$. If in addition $r(z) = e^z + o(z)$ as $z \rightarrow 0$, then r is said to be \mathcal{A} -acceptable.

Moreover, r is an approximation to the exponential function of order $q \geq 1$ if

$$r(z) = e^z + O(|z|^{q+1}) \quad \text{as } z \rightarrow 0. \quad (2.3)$$

Notice that an \mathcal{A} -stable rational approximation of order $q \geq 1$ is \mathcal{A} -acceptable.

Definition 2.1.2. An \mathcal{A} -acceptable rational function r is said to satisfy the condition (\star) if the following three conditions hold,

- (a) $|r(i\xi)| < 1$ for $0 \neq \xi \in \mathbb{R}$ and $|r(\infty)| < 1$.
- (b) There exist positive integers M, N , where N is even, $N \geq M + 1$, and a positive number γ such that $r(i\xi) = e^{i\xi + \psi(\xi)}$ with $\psi(\xi) = O(|\xi|^{N+1})$ as $\xi \rightarrow 0$.
- (c) $\text{Re}(\psi(\xi)) \leq -\gamma\xi^N$ for $|\xi| \leq 1$.

Notice that condition (2.3) together with condition (a) imply that both conditions (b) and (c) are satisfied. In order to see this, assume that (2.3) holds, then (b) holds with $M = q$. Furthermore, $\text{Re}(\psi(\xi)) = \gamma_0\xi^N(1 + o(1))$ as $\xi \rightarrow 0$, for some N with $N \geq M + 1$. Therefore, it follows from (a) that $\gamma_0 < 0$ and N is even. Thus, (c) holds, cf. Theorem 1.2 of [96].

In 1892, Henri Padé started the study of what nowadays is known as *Padé approximants* in his dissertation called ‘Sur la représentation approchée d’une fonction par des fraction rationnelles’. However, even though Padé approximants were used by his own advisor C. Hermite to show that the number e is transcendent, it was not until F. Borel presented Padé approximants in his 1901 book on divergent series that they became “well known”. Padé was interested in the approximation of functions by rational functions and he found a closed form for the Padé approximants of the exponential. Here, the closed form is taken as definition.

Definition 2.1.3. The Padé approximation $r_{j,l}$ to the exponential function is a rational function of the form $r_{j,l} = \frac{P_l(z)}{Q_j(z)}$ where

$$P_{j,l}(z) = \sum_{k=0}^l \frac{(l+j-k)!l!}{j!k!(l-k)!} z^k \quad \text{and} \quad Q_{j,l}(z) = \sum_{k=0}^j \frac{(l+j-k)!}{k!(j-k)!} (-z)^k. \quad (2.4)$$

It follows that

$$r_{j,l}(z) - e^z = O(z^{j+l+1}) \quad \text{as } z \rightarrow 0. \quad (2.5)$$

R. Varga shows in [98] that the diagonal Padé approximants to the exponential $r_{j,j}$ are \mathcal{A} -acceptable for every $j \geq 1$. However, as B. Ehle points out in [33], the moduli of all those approximations approach 1 as $|z| \rightarrow \infty$ for $\text{Re}(z) < 0$ which is not consistent with the behavior of e^z . B. Ehle shows in [33] that the subdiagonal Padé approximant $r_{j+1,j}$ as well as the second subdiagonal Padé approximant $r_{j+2,j}$ are \mathcal{A} -acceptable for every $j \geq 0$. Moreover, he shows that the first and second subdiagonal Padé approximations to the exponential function satisfy that the

moduli approach zero as $|z| \rightarrow \infty$ for $\operatorname{Re}(z) < 0$. Finally, G. Wanner, E. Hairer and P. Norsett showed in [105] that a Padé approximation to the exponential function $r_{j,l}$ is A -acceptable if and only if $j - 2 \leq l \leq j$.

The Hermite-Padé approximants, also known as the restricted Padé approximants, are another type of rational approximations of the exponential which are characterized by having denominators of the form $(1 - \gamma z)^k$.

Definition 2.1.4. A Hermite-Padé approximant of the exponential function is a rational function of the form

$$R_k(z) = \frac{(-1)^k}{(1 - \gamma z)^k} \sum_{m=0}^k L_k^{(k-m)}(\gamma^{-1})(\gamma z)^m, \quad (2.6)$$

where $L_k(z) = \sum_{n=0}^k \frac{(-1)^n}{n!} \binom{k}{n} z^n$ is the Laguerre polynomial of degree k .

In [48, 49, 79] it is shown that R_k is of order $k + 1$ for suitable choices of $\gamma \in \mathbb{R}$. Also, these papers contain the following nontrivial result.

Proposition 2.1.5. *The only Hermite-Padé rational approximations of order $k+1$ which are A -acceptable are the ones given by $k = 1, 2, 3$ or 5 where $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{3+\sqrt{3}}{6}$, $\gamma_3 = 1.06857902$ and $\gamma_5 = 0.47326839$.*

In [55], A. Iserles develops a method called *Composite Exponential Approximations* (CEA) which combines rational approximations of the exponential in order to obtain A -acceptable rational functions. The basic idea is to consider the product of rescaled rational approximation of the exponential (which are not necessarily \mathcal{A} -stable) and to show that they are in fact A -acceptable. Moreover, the constructed approximation will improve the order of convergence in many cases. Thus, the (CEA) method allows one to construct higher order \mathcal{A} -stable approximations of the exponential function by using, for instance, lower order Padé approximants. In this section we compile some nontrivial results concerning his construction, for details see [55].

Definition 2.1.6. Let R_1, \dots, R_N be rational approximations to the exponential e^z such that $R_i(z) - e^z = O(|z|^{q+1})$ as $z \rightarrow 0$. for $i \in \{1, \dots, N\}$ and let $d_1, \dots, d_N \in \mathbb{R}^+$ such that $\sum_{i=1}^N d_i = 1$. Then the *Composite Exponential Approximation* (CEA) is defined as

$$R(z; d_1, \dots, d_N) := R_1(d_1 z) R_2(d_2 z) \cdots R_N(d_N z). \quad (2.7)$$

It follows from the definition that $R(z; d_1, \dots, d_N) - e^z = O(|z|)^{q+1}$ as $z \rightarrow 0$.

Theorem 2.1.7. [55] *Let $r_{j,l}$ be the Padé approximant of the exponential function given by (2.4). Then the following holds:*

- (i) *If $R_1 = r_{j,j}$ and $R_2 = r_{j+1,j-1}$, then $R(z; d_1, d_2)$ is L -acceptable (i.e., A -acceptable and tending to zero as $\operatorname{Re}(z) \rightarrow -\infty$) for $d_1 = \frac{d^*}{1+d^*}$ and $d_2 = \frac{1}{1+d^*}$,*

where $d^* = (1 + \frac{1}{j})^{\frac{1}{2j+1}}$. Moreover, $R(z, d_1, d_2)$ is an approximation of the exponential of order $2j + 1$.

(ii) If $R_1 = r_{j,j-1}$, $R_2 = r_{j-1,j}$, $d_1 = \frac{1}{2}$, and $d_2 = \frac{1}{2}$. Then the (CEA) is an A -acceptable approximation of the exponential of order $2j$.

(iii) If $R_1 = r_{j-1,j+1}$, $R_2 = r_{j,j}$, $R_3 = r_{j+1,j-1}$, $d_1 = d_3 = \frac{d^*}{1+2d^*}$, and $d_2 = \frac{1}{1+2d^*}$, where $d^* = \left(\frac{j}{2(j+1)}\right)^{\frac{1}{2j+1}}$. Then the (CEA) is an A -acceptable rational approximation of the exponential of order $2j + 2$.

(iv) Let

$$\begin{aligned} R_1(z, \alpha) &= \frac{(1 - \alpha)P_{j,j-1}(z) + \alpha P_{j,j}(z)}{(1 - \alpha)Q_{j,j-1}(z) + \alpha Q_{j,j}(z)} \\ R_2(z, \alpha) &= \frac{(1 - \alpha)P_{j-1,j}(z) + \alpha P_{j,j}(z)}{(1 - \alpha)Q_{j-1,j}(z) + \alpha Q_{j,j}(z)}. \end{aligned}$$

Then the (CEA) $R_\alpha(z; \frac{1}{2}, \frac{1}{2})$ is an A -acceptable rational approximation of the exponential of order $2j + 1$ for every $\alpha \in [0, 1]$.

Concerning the Hermite approximations of the exponential, the (CEA) method has shown to be fruitful as well by finding an A -acceptable rational function of order 9 by combining two Hermite approximations, see Table II in [55].

Remark 2.1.8. Notice that if each component of the CEA r satisfy the condition (\star) of Definition 2.1.2, then r does.

2.2 Estimating the Total Variation and Determining Functions in NBV^0

In order to show the idea behind the method of approximating a semigroup, consider A to be the generator of a strongly continuous semigroup T . If r is an \mathcal{A} -stable rational approximation of the exponential and if $r \in \mathcal{F}^\omega$ then the Hille-Phillips functional calculus implies that

$$\begin{aligned} \left\| r^n \left(\frac{t}{n} A \right) x - T(t)x \right\| &= \left\| \int_0^\infty T(s)x d\alpha_{n,t}(s) - \int_0^\infty T(s)x dH_t(s) \right\| \\ &\leq \int_0^\infty \|T(s)x\| dV_{\alpha_{n,t}-H_t}(s) \\ &\leq M \|x\| \int_0^\infty e^{\omega t} dV_{\alpha_{n,t}-H_t}(s) \\ &= M \|x\| \|\alpha_{n,t} - H_t\|_\omega. \end{aligned}$$

Thus, one needs to show that $r \in \mathcal{F}^\omega$ and also to estimate the total variation norm $\|\alpha_{n,t} - H_t\|_\omega$. This section shows an adaptation of a method developed in [10] in order to obtain both results.

The *Fourier transform* of $f \in L^1(\mathbb{R})$ is defined by $\widehat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-its} f(s) ds$, and the *Fourier-Stieltjes transform* of a function α of total bounded variation on \mathbb{R} is defined by

$$\mathcal{F}[\alpha](t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-its} d\alpha(s) \quad (t \in \mathbb{R}). \quad (2.8)$$

Notice that NBV^0 can be continuously embedded into $\text{NBV}(\mathbb{R})$ by setting $\alpha \equiv 0$ on \mathbb{R}^- since $\alpha(0) = 0$. Also, if $\alpha(t) = \int_0^t f(s) ds + \alpha(0)$ for some $f \in L^1(\mathbb{R})$, then $\mathcal{F}[\alpha] = \widehat{f}$. By the Riemann-Lebesgue Theorem follows that $\mathcal{F}[\alpha] \in C_{0,0}(\mathbb{R})$ (see Section 1.2 for the definition). Moreover,

$$\|\alpha\|_0 = \text{Var}(\alpha) = \|f\|_{L^1(\mathbb{R})}. \quad (2.9)$$

Lemma 2.2.1. *Let $\alpha \in \text{NBV}^0$. If $\mathcal{F}[\alpha] \in L^2(\mathbb{R})$ then*

$$\alpha(t) = \int_0^t \widehat{f}_0(w) dw, \quad (2.10)$$

where $f_0(t) := \mathcal{F}[\alpha](-t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ist} d\alpha(s)$ ($t \in \mathbb{R}$). If in addition $f'_0 \in L^2(\mathbb{R})$, then $\widehat{f}_0 \in L^1(\mathbb{R}_0^+)$ with

$$\|\alpha\|_0 = \text{Var}(\alpha) = \|\widehat{f}_0\|_{L^1(\mathbb{R}_0^+)} \leq \sqrt{\pi} \|f_0\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|f'_0\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \quad (2.11)$$

Proof. Let $\alpha \in \text{NBV}^0$ be considered as an element in $\text{NBV}(\mathbb{R})$. The inversion formula for the Fourier-Stieltjes transform (see [104]) asserts that

$$\alpha(t) - \alpha(0) = \frac{1}{\sqrt{2\pi}} \lim_{R \rightarrow \infty} \int_{-R}^R \mathcal{F}[\alpha](s) \frac{e^{ist} - 1}{is} ds. \quad (2.12)$$

It follows from Fubini's Theorem that

$$\begin{aligned} \alpha(t) &= \lim_{R \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^R \mathcal{F}[\alpha](s) \frac{e^{ist} - 1}{is} ds = \lim_{R \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^R f_0(-s) \int_0^t e^{iws} dw ds \\ &= \lim_{R \rightarrow \infty} \int_0^t \frac{1}{\sqrt{2\pi}} \int_{-R}^R f_0(-s) e^{iws} ds dw = \lim_{R \rightarrow \infty} \int_0^t \frac{1}{\sqrt{2\pi}} \int_{-R}^R f_0(s) e^{-iws} ds dw. \end{aligned}$$

On the other hand, since $\text{l.i.m.}_{R \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^R f_0(s) e^{-iws} ds = \widehat{f}_0(w)$ (where l.i.m. denotes the limit in the $L^2(\mathbb{R})$ sense), and strong convergence implies weak convergence in $L^2(\mathbb{R})$, one obtains that

$$\begin{aligned} \int_0^t \widehat{f}_0(w) dw &= \langle \widehat{f}_0, \chi_{[0,t]} \rangle = \lim_{R \rightarrow \infty} \left\langle \frac{1}{\sqrt{2\pi}} \int_{-R}^R f_0(s) e^{-i(\cdot)s} ds, \chi_{[0,t]} \right\rangle \\ &= \lim_{R \rightarrow \infty} \int_0^t \frac{1}{\sqrt{2\pi}} \int_{-R}^R f_0(s) e^{-iws} ds dw = \alpha(t). \end{aligned}$$

In particular, $\widehat{f}_0 = 0$ a.e. on \mathbb{R}^- since $\alpha(t) = 0$ for all $t \leq 0$.

Carlson's inequality (see [14]) states that if $g \in L^2(\mathbb{R}_0^+)$ and $h(s) := sg(s) \in L^2(\mathbb{R}_0^+)$, then $g \in L^1(\mathbb{R}_0^+)$ and

$$\|g\|_{L^1(\mathbb{R}_0^+)} \leq \sqrt{\pi} \|g\|_{L^2(\mathbb{R}_0^+)}^{\frac{1}{2}} \|h\|_{L^2(\mathbb{R}_0^+)}^{\frac{1}{2}}. \quad (2.13)$$

If $f_0 \in L^2(\mathbb{R})$ and $f'_0 \in L^2(\mathbb{R})$ then $\widehat{f'_0}$ and $\widehat{f_0}$ are in $L^2(\mathbb{R}_0^+)$ since $\widehat{f'_0}(s) = is\widehat{f_0}(s)$ and $\widehat{f_0} \equiv 0$ a.e. on \mathbb{R}^- . Moreover, Parseval's theorem together with (2.13) yields $\widehat{f_0} \in L^1(\mathbb{R}_0^+)$ and

$$\|\widehat{f_0}\|_{L^1(\mathbb{R}_0^+)} \leq \sqrt{\pi} \|f_0\|_{L^2(\mathbb{R}_0^+)}^{\frac{1}{2}} \|f'_0\|_{L^2(\mathbb{R}_0^+)}^{\frac{1}{2}}. \quad (2.14)$$

Furthermore, since $\alpha(t) = \int_0^t \widehat{f_0}(w)dw$, the result follows from (2.9). \square

Theorem 1.2.2 shows that the estimation obtained in Lemma 2 of [10] is equivalent to (2.11). Let I^0 be the identity on NBV^0 and let I^k denotes k^{th} antiderivative; i.e., $I[\alpha](t) = \int_0^t \alpha(\xi) d\xi$.

Lemma 2.2.2. *Let $\alpha \in \text{NBV}^0$, let $g(t) := \mathcal{F}[\alpha](-t)$ ($t \in \mathbb{R}$), and let $q \in \mathbb{N}$ be fixed. Suppose that for each $k \in \{1, \dots, q\}$ there exist $\beta_k \in \text{NBV}^0$ such that*

$$f_k(t) = \frac{g(t)}{t^k}, \quad (2.15)$$

where $f_k(t) := \mathcal{F}[\beta_k](-t)$. If each f_k satisfies all the conditions of Lemma 2.2.1, then

$$\|\beta_k\|_0 = \text{Var}(\beta_k) = \|I^{k-1}[\alpha]\|_{L^1(\mathbb{R}^+)} \text{ and } \lim_{t \rightarrow \infty} I^{k-1}[\alpha](t) = 0 \quad (2.16)$$

for all $k \in \{1, \dots, q\}$.

Proof. As before, the β_k 's can be continuously embedded into $\text{NBV}(\mathbb{R})$ by setting $\beta_k \equiv 0$ on \mathbb{R}^- . It follows from Lemma 2.2.1 that $\widehat{f_k} \in L^1(\mathbb{R}^+)$ for all $k \in \{1, \dots, q\}$. Now, if $m \in \{0, \dots, k\}$ then

$$\widehat{f_k}^{(m)}(t) = [(-i\widehat{(\cdot)})^m \widehat{f_k}(\cdot)](t) = (-i)^m [\widehat{(\cdot)^m f_k(\cdot)}](t) = (-i)^m \widehat{f_{k-m}}(t).$$

By the Riemann-Lebesgue Theorem one obtains that

$$\lim_{t \rightarrow \pm\infty} \widehat{f_k}^{(m)}(t) = 0, \quad (2.17)$$

for all $m \in \{0, \dots, k\}$. Also, (2.15) yields that $(-it)^k f_k(t) = (-i)^k g(t)$ and

$$\widehat{f_k}^{(k)}(t) = (-i)^k \widehat{g}(t). \quad (2.18)$$

By integrating (2.18) and (2.17) it follows that

$$\widehat{f_k}^{(k-1)}(t) = (-i)^k \int_{-\infty}^t \widehat{g}(s) ds. \quad (2.19)$$

Now, from Lemma 2.2.1 one concludes that $\int_{-\infty}^t \widehat{g}(s) ds = 0$ for all $t \leq 0$. In this way, by integrating (2.18) k -times, one obtains that

$$\widehat{f}_k(t) = (-i)^k I^k[\widehat{g}](t) = (-i)^k I^{k-1}[\alpha](t). \quad (2.20)$$

Thus, by Lemma 2.2.1, $\text{Var}(\beta_k) = \|\widehat{f}_k\|_{L^1(\mathbb{R}_0^+)} = \|I^{k-1}[\alpha]\|_{L^1(\mathbb{R}^+)}$. Finally, the Riemann-Lebesgue Theorem applied to (2.20) yields $\lim_{t \rightarrow \infty} I^{k-1}[\alpha](t) = 0$. \square

Let

$$H_0(s) := \begin{cases} 0 & \text{if } s = 0 \\ 1 & \text{if } s > 0 \end{cases} \quad \text{and} \quad H_t(s) := \begin{cases} 0 & \text{if } 0 \leq s < t \\ \frac{1}{2} & \text{if } s = t \\ 1 & \text{if } s > t. \end{cases} \quad (2.21)$$

Since $H_t \in \text{NBV}^\omega$ for all $\omega \in \mathbb{R}$ and $t \geq 0$, the functions

$$z \rightarrow e^{tz} = \int_0^\infty e^{sz} dH_t(s)$$

are in F_ω ($\omega \in \mathbb{R}$, $t \geq 0$). In particular, the constant functions are in F_ω for all $\omega \in \mathbb{R}$. Furthermore, rational functions that are bounded for $\text{Re}(z) \leq \omega$ are in F_ω . To see this, notice that there are $b_i \in \mathbb{C}$ with $\text{Re}(b_i) > \omega$ and $B_{ij} \in \mathbb{C}$ such that

$$r(z) = B_{00} + \sum_{1 \leq i, j} \frac{B_{ij}}{(b_j - z)^j} \quad (\text{Re}(z) \leq \omega).$$

Since F_ω is an algebra and since $a \rightarrow \frac{1}{b_i - z} = \int_0^\infty e^{sz} d\alpha_i(s) \in F_\omega$, where $\alpha_i(s) = \int_0^s e^{-b_i s} ds$, it follows that $r \in F_\omega$. Moreover if $t > 0$, $n \in \mathbb{N}$, and $r \in F_\omega$ then

$$z \rightarrow r^n \left(\frac{t}{n} z \right) = \int_0^\infty e^{zs} d\alpha_{n,t}(s) \in F_\omega, \quad (2.22)$$

where $\alpha_{n,t}(s) := \alpha^{n*} \left(\frac{t}{n} s \right)$ and α^{n*} denotes the n -th convolution product $\alpha * \dots * \alpha$. By the above, one obtains the following result.

Proposition 2.2.3. *If r is an \mathcal{A} -stable rational function that approximates the exponential function of order q , then*

$$f : z \rightarrow \frac{r^n \left(\frac{t}{n} z \right) - e^{zt}}{z^{k+1}} \in F_0 \quad (2.23)$$

for every $k \in \{0, \dots, q\}$, $n \in \mathbb{N}$ and $t \geq 0$.

The following result summarizes crucial estimates obtained in [10] in terms of functions of bounded variation (compare with [62, Thm. 3.1] and [63, Thm. 1.6]).

Theorem 2.2.4. *If $r(z) = \int_0^\infty e^{sz} d\alpha(s)$ ($\text{Re}(z) \leq 0$) is an \mathcal{A} -stable rational function, then there exists $K > 0$ such that*

$$\|\alpha^{n*}\|_0 \leq K\sqrt{n} \quad \text{for all } n \in \mathbb{N}. \quad (2.24)$$

If, in addition, r is \mathcal{A} -acceptable and satisfies the condition (\star) of Definition 2.1.2 then there is a constant K such that

$$\|\alpha^{n*}\|_0 \leq Kn^{\frac{1}{2}(1-\frac{q+1}{p})} \quad \text{for all } n \in \mathbb{N}. \quad (2.25)$$

Furthermore, if f is as in (2.23), then there exist $\beta_{k,n,t} \in \text{NBV}^0$ and $K > 0$ such that $f(z) = \int_0^\infty e^{zs} d\beta_{k,n,t}(s)$ for $\text{Re}(z) \leq 0$ and

$$\text{Var}(\beta_{k,n,t}) = \|I^{k-1}[\alpha_{n,t} - H_t]\|_{L^1(\mathbb{R}^+)} \leq Kt^k \left(\frac{1}{n}\right)^{\theta_q(k)} \quad (2.26)$$

for every $k \in \{0, \dots, q+1\}$, except for $k = \frac{q-1}{2}$ in which case

$$\text{Var}(\beta_{k,n,t}) = \|I^{k-1}[\alpha_{n,t} - H_t]\|_{L^1(\mathbb{R}^+)} \leq Kt^k \left(\frac{1}{n}\right)^{\theta_q(k)} \ln(n+1), \quad (2.27)$$

where $\theta_q(k) = k\frac{q}{q+1} + \min\left\{0, \frac{k}{q+1} - \frac{1}{2}\right\}$. Furthermore, if in addition r satisfies the condition (\star) of Definition 2.1.2 then θ_q can be replaced by

$$\theta_q^*(k) = k\frac{q}{q+1} + \min\left\{0, \left(k - \frac{1}{2}(q+1)\right)\left(\frac{1}{q+1} - \frac{1}{p}\right)\right\}.$$

Proof. Let $r(z) = \int_0^\infty e^{zs} d\alpha(s)$ ($\text{Re}(z) \leq 0$) for some $\alpha \in \text{NBV}^0$ and let μ be the bounded regular complex Borel measure associated with α . Define $r_{(0)}(t) := \widehat{\mu}(t)$ (the Fourier transform of μ). Then, it follows from [10] that $m(r_{(0)}) = m(\widehat{\mu}) = \|\mu\| = \|\alpha\|_0$. Thus, $m(r_{(0)}^n) = \|\alpha^{*n}\|_0$, and (2.24), (2.25) follow from the proof of Theorem 1 and Theorem 2 of [10] with $\omega = 0$.

On the other hand, from (2.23) it follows that there exists $\beta_{k,n,t} \in \text{NBV}^0$ such that $f(z) = \int_0^\infty e^{zs} d\beta_{k,n,t}(s)$ ($\text{Re}(z) \leq 0$). As above, let μ be the bounded regular complex Borel measure associated with $\beta_{k,n,t}$ and let $f_{(0)} = \widehat{\mu}$ (the Fourier transform of μ). Then, it follows from [10] that $m(f_{(0)}) = m(\widehat{\mu}) = \|\mu\| = \|\beta_{k,n,t}\|_0$. Therefore, (2.26) and (2.27) follow from the proof of Theorem 4 and Remark 3 of [10] with $\omega = 0$. Finally, Lemma 2.2.2 shows that $\text{Var}(\beta_{k,n,t}) = \|I^{k-1}[\alpha_{n,t} - H_t]\|_{L^1(\mathbb{R}_0^+)}$. \square

The estimation (2.24) is sharp. For instance, if $r(z) = \frac{2+z}{2-z} = -1 + \frac{4}{2-z} \in F_0$ (Crank-Nicolson), then it follows from [17, (2.2)] that there are $k_1, k_2 > 0$ such that

$$k_1\sqrt{n} \leq \|\alpha_{CN}^{*n}\|_0 \leq k_2\sqrt{n} \quad \text{for all } n \in \mathbb{N}. \quad (2.28)$$

2.3 Stability and Time Discretization Schemes for Bi-Continuous Semigroups

Theorem 2.3.1. *Let T be a bi-continuous semigroup of type $(M, 0)$ with generator A and let r be an \mathcal{A} -stable rational function. Then there exist a constant $K > 0$ such that*

$$\|r^n(\rho A)\|_{\mathcal{L}(X)} \leq KMn^{\frac{1}{2}}, \quad \rho > 0, \quad n \in \mathbb{N}.$$

If, in addition, r is \mathcal{A} -acceptable and satisfies (\star) , then

$$\|r^n(\rho A)\|_{\mathcal{L}(X)} \leq KMn^{\frac{1}{2}(1-\frac{q+1}{p})}, \quad \rho > 0, \quad n \in \mathbb{N}.$$

Proof. Let $n \in \mathbb{N}$, $\rho > 0$ and $x \in X$. Theorem 1.3.4 and (1.11) yield

$$\begin{aligned} \|r^n(\rho A)x\| &= \sup_{\phi \in \Phi} |\langle r^n(\rho A)x, \phi \rangle| = \sup_{\phi \in \Phi} \left| \left\langle \int_0^\infty T(\rho s)x \, d\alpha^{n*}(s), \phi \right\rangle \right| \\ &= \sup_{\phi \in \Phi} \left| \int_0^\infty \langle T(\rho s)x, \phi \rangle \, d\alpha^{n*}(s) \right| \leq \sup_{\phi \in \Phi} \int_0^\infty |\langle T(\rho s)x, \phi \rangle| \, dV_{\alpha^{n*}}(s) \\ &\leq \int_0^\infty \|T(\rho s)x\| \, dV_{\alpha^{n*}}(s) \leq M\|\alpha^{n*}\|_0 \|x\|. \end{aligned}$$

Now, the statement follows from Theorem 2.2.4. \square

For instance, one can consider the Padé approximants of the exponential function of order $j + l$ given by $r_{j,l}(z) = \frac{P_{j,l}(z)}{Q_{j,l}(z)}$ of Section 2.1. As previously mentioned, G. Wanner, E. Hairer and S. P. Nørsett showed in [48] that $r_{j,l}$ is an \mathcal{A} -stable approximation of the exponential function of order $q = j + l$ if and only if $0 \leq j - l \leq 2$. Moreover, condition (\star) -(c) holds with $p = 2j$, see [49]. Therefore, Theorem 2.3.1 implies that if A is the generator of a bi-continuous semigroup of type $(M, 0)$, then

$$\|r_{j,j-1}^n(\rho A)\|_{\mathcal{L}(X)} \leq C \quad \text{and} \quad \|r_{j,j-2}^n(\rho A)\|_{\mathcal{L}(X)} \leq Cn^{\frac{1}{4j}} \quad (2.29)$$

for $j \in \mathbb{N}_0$ and $\rho > 0$. Similarly, in the case of the restricted Padé approximants R_k of Section 2.1 given by Proposition 2.1.5, one obtains that the so called Calahan scheme given by $R_2(\rho A)$ is stable, and the norm of $R_3^n(\rho A)$ grows as $O(n^{\frac{1}{12}})$. The next result shows the sharpness of the first estimate of Theorem 2.3.1.

Theorem 2.3.2. *Let $X = C_b(\mathbb{R}_0^+)$ and let T be the (bi-continuous) shift semigroup with generator $A = \frac{d}{ds}$. If $r(z) = \frac{2+z}{2-z}$ then there exists $K > 0$ such that*

$$\|r^n(\rho A)\|_{\mathcal{L}(X)} \geq K\sqrt{n}, \quad \rho > 0, \quad n \in \mathbb{N}.$$

Proof. Let Φ be the set of linear functionals $\phi_a : x \rightarrow x(a)$ ($a \in \mathbb{R}_0^+$). Then $\|x\| = \sup_{\phi_a \in \Phi} |\langle x, \phi_a \rangle|$ for each $x \in X$. Now, let $\rho > 0$ and let $x \in X$ be fixed. From Theorem 1.3.4 and (1.11) follows that

$$\begin{aligned} \|r^n(\rho A)x\| &= \sup_{\phi_a \in \Phi} |\langle r^n(\rho A)x, \phi_a \rangle| = \sup_{\phi_a \in \Phi} \left| \left\langle \int_0^\infty T(\rho s)x \, d\alpha^{n*}(s), \phi_a \right\rangle \right| \\ &= \sup_{\phi_a \in \Phi} \left| \int_0^\infty \langle T(\rho s)x, \phi_a \rangle \, d\alpha^{n*}(s) \right| = \sup_{a \in \mathbb{R}_0^+} \left| \int_0^\infty x(a + \rho s) \, d\alpha^{n*}(s) \right|. \end{aligned}$$

Thus, $\|r^n(\rho A)\|_{\mathcal{L}(X)} \leq \|\alpha^{*n}\|_0$. Notice that $\|x_{a,\rho}\| \leq \|x_{0,\rho}\| = \|x\|$ for all $\rho > 0$ and $a \in \mathbb{R}_0^+$, where $x_{a,\rho} := s \rightarrow x(a + \rho s)$. Since NBV^0 and the dual of $C_0(\mathbb{R}_0^+)$ are

isometric isomorphic Banach algebras from Theorem 1.2.2, it follows that

$$\begin{aligned}
\|\alpha^{n*}\|_0 &= \sup_{\substack{x \in C_0(\mathbb{R}_0^+) \\ \|x\| \leq 1}} |\langle x, \alpha^{n*} \rangle| \leq \sup_{\substack{x \in C_b(\mathbb{R}_0^+) \\ \|x\| \leq 1}} |\langle x, \alpha^{n*} \rangle| = \sup_{\substack{x \in C_b(\mathbb{R}_0^+) \\ \|x_{a,\rho}\| \leq 1 \\ a \in \mathbb{R}_0^+}} |\langle x_{a,\rho}, \alpha^{n*} \rangle| \\
&\leq \sup_{\substack{x \in C_b(\mathbb{R}_0^+) \\ \|x_{a,\rho}\| \leq 1 \\ a \in \mathbb{R}_0^+}} \sup_{a \in \mathbb{R}_0^+} |\langle x_{a,\rho}, \alpha^{n*} \rangle| \leq \sup_{\substack{x \in C_b(\mathbb{R}_0^+) \\ \|x\| \leq 1}} \sup_{a \in \mathbb{R}_0^+} |\langle x_{a,\rho}, \alpha^{n*} \rangle| \\
&= \sup_{\substack{x \in C_b(\mathbb{R}_0^+) \\ \|x\| \leq 1}} \|r^n(\rho A)x\| = \|r^n(\rho A)\|_{\mathcal{L}(X)}.
\end{aligned}$$

Therefore $\|r^n(\rho A)\|_{\mathcal{L}(X)} = \|\alpha^{n*}\|_0$ and the result follows by applying (2.28). \square

Theorem 2.3.3. *Let T be a bi-continuous semigroup of type $(M, 0)$ generated by A . If $r(z) = \int_0^\infty e^{zs} d\alpha(s)$ ($\operatorname{Re}(z) \leq 0$) is an \mathcal{A} -stable rational approximation of the exponential function of order $q \geq 1$, then there exists $K > 0$ such that*

$$\left\| r^n \left(\frac{t}{n} A \right) x - T(t)x \right\| \leq MKt^k \left(\frac{1}{n} \right)^{\theta_q(k)} \|A^k x\| \quad (2.30)$$

for every $k \in \{1, \dots, q+1\}$, $k \neq \frac{q-1}{2}$, and $x \in D(A^k)$. If $k = \frac{q-1}{2}$ then

$$\left\| r^n \left(\frac{t}{n} A \right) x - T(t)x \right\| \leq MKt^k \left(\frac{1}{n} \right)^{\theta_q(k)} \ln(n+1) \|A^k x\|. \quad (2.31)$$

If, in addition, r satisfies the condition (\star) , then θ_q can be replaced by θ_q^* .

Proof. Let $x \in D(A^k)$ and let $\phi \in \Phi$. It follows from the proofs of [67, Prop. 11] and [67, Thm. 12] that $t \rightarrow \langle T(t)x, \phi \rangle$ is k -times differentiable and $\frac{d^k}{dt^k} \langle T(t)x, \phi \rangle = \langle T(t)A^k x, \phi \rangle$. Now, by Theorem 1.3.4, (1.11), and integrating by parts k -times,

$$\begin{aligned}
\left\| r^n \left(\frac{t}{n} A \right) x - T(t)x \right\| &= \sup_{\phi \in \Phi} \left| \left\langle r^n \left(\frac{t}{n} A \right) x - T(t)x, \phi \right\rangle \right| \\
&= \sup_{\phi \in \Phi} \left| \left\langle \int_0^\infty T(s)x d\alpha_{n,t}(s) - \int_0^\infty T(s)x dH_t(s), \phi \right\rangle \right| \\
&= \sup_{\phi \in \Phi} \left| \left\langle \int_0^\infty T(s)x d[\alpha_{n,t} - H_t](s), \phi \right\rangle \right| \\
&= \sup_{\phi \in \Phi} \left| \int_0^\infty \langle T(s)x, \phi \rangle d[\alpha_{n,t} - H_t](s) \right| \\
&= \sup_{\phi \in \Phi} \left| \int_0^\infty I^{(k-1)}(\alpha_{n,t}(s) - H_t(s)) \frac{d^k}{ds^k} \langle T(s)x, \phi \rangle ds \right| \\
&= \sup_{\phi \in \Phi} \left| \int_0^\infty I^{(k-1)}(\alpha_{n,t}(s) - H_t(s)) \langle T(s)A^k x, \phi \rangle ds \right| \\
&\leq M \|I^{(k-1)}(\alpha_{n,t} - H_t)\|_{L_1(\mathbb{R}_+)} \|A^k x\|.
\end{aligned}$$

Now, the result follows from Theorem 2.2.4. \square

Whereas Theorem 2.3.3 yields norm-convergence for \mathcal{A} -stable rational approximations to the exponential function of order $q \geq 1$ for sufficiently smooth initial values $x \in D(A^k)$, the Chernoff product formula for bi-continuous semigroups obtained by A. Albanese and E. Mangino in [1] provides convergence in the τ topology for all $x \in X$.

Theorem 2.3.4. *Let A be the generator of a bi-continuous semigroup T of type $(M, 0)$. If $r(z) = \int_0^\infty e^{zs} d\alpha(s)$ ($\operatorname{Re}(z) \leq 0$) is an \mathcal{A} -stable rational approximation to the exponential function of order $q \geq 1$ for which α^{*m} is uniformly bounded in the variation norm, then*

$$T(t)x = \tau\text{-}\lim_{n \rightarrow \infty} r^n \left(\frac{t}{n} A \right) x, \quad (2.32)$$

for all $x \in X$, where the limit is uniform for compact intervals of \mathbb{R}_0^+ .

Proof. It has to be shown that the conditions of the Chernoff product formula (see Thm. 4.1 [1]) hold. Since r is an approximation to the exponential function of order $q \geq 1$, it follows that $r(0) = 1$. Thus, by Theorem 1.3.4, $r(0) = Id$. By assumption, $\|r(hA)^m\| \leq C$ ($h \geq 0, m \in \mathbb{N}$). In order to show uniform bi-equicontinuity of $\{r^m(hA) : h \geq 0\}$ in $m \in \mathbb{N}$, consider a $\|\cdot\|$ -bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ which is τ -convergent to $x \in X$. Let $\varepsilon > 0, p \in \mathcal{P}_\tau, m \in \mathbb{N}$, and choose $R > 0$ such that $p(\int_R^\infty T(s)(x_n - x) d\alpha_h^m(s)) < \varepsilon$. It follows from Theorem 1.3.4 and the local bi-equicontinuity of T that

$$p \left(\int_0^R T(s)(x_n - x) d\alpha_h^{*m}(s) \right) \leq M\varepsilon \|\alpha_h^{*m}\|_0 = M\varepsilon \|\alpha^{*m}\|_0,$$

and the uniform bi-equicontinuity in m follows. Now, from Lemma 7-(b) of [67] and by considering $x_n = n^2 R(n, A)^2 x \in D(A^2)$ one obtains that $\tau\text{-}\lim_{n \rightarrow \infty} x_n = x$ for every $x \in X$; i.e., $D(A^2)$ is bi-dense in X . On the other hand, Theorem 2.3.3 with $k = 2$ and $n = 1$ shows that $\lim_{t \rightarrow 0^+} \frac{V(t)x - T(t)x}{t} = 0$, for all $x \in D(A^2)$. Thus, $\lim_{t \rightarrow 0^+} \frac{V(t)x - x}{t} = Ax$, for all $x \in D(A^2)$. Finally, since $\|\cdot\|$ -convergence implies τ -convergence, one obtains that for all $x \in D(A^2)$,

$$Ax = \tau\text{-}\lim_{t \rightarrow 0^+} \frac{V(t)x - x}{t}.$$

Therefore, all the conditions of [1, Thm. 4.1] are satisfied, and the result follows. \square

In particular, Theorem 2.3.4 shows convergence for all subdiagonal Padé approximants $r_{j,j-1}$ to the exponential function in the τ topology since (2.29) shows that α^{*m} is uniformly bounded. Similarly for the Calahan scheme.

Notice that if the bi-continuous semigroup is of type (M, ω) then all the previously obtained estimates will remain valid by replacing M by $Me^{\omega t}$.

2.4 Stability and Time Discretization Schemes for C -Regularized Semigroups

Theorem 2.4.1. *Let $\{W(t)\}_{t \geq 0}$ be a C -regularized semigroup of type $(M, 0)$ with generator A and let r be an \mathcal{A} -stable rational function. Then there exist a constant $K > 0$ such that*

$$\|r^n(\rho A)C\|_{\mathcal{L}(X)} \leq KMn^{\frac{1}{2}}, \quad \rho > 0, \quad n \in \mathbb{N}.$$

If, in addition, r is \mathcal{A} -acceptable and satisfies (\star) of Definition 2.1.2, then

$$\|r^n(\rho A)C\|_{\mathcal{L}(X)} \leq KMn^{\frac{1}{2}(1-\frac{q+1}{p})}, \quad \rho > 0, \quad n \in \mathbb{N}.$$

Proof. Let $n \in \mathbb{N}$, $\rho > 0$ and $x \in X$. Theorem 1.4.3 yields

$$\|r^n(\rho A)Cx\| \leq \int_0^\infty \|W(\rho s)x\| dV_{\alpha^{n*}}(s) \leq M\|\alpha^{n*}\|_0\|x\|.$$

Now, the statement follows from Theorem 2.2.4. \square

For instance, one can consider the Padé approximants of the exponential function of order $j + l$ given in Section 2.1. Then Theorem 2.4.1 implies that if A is the generator of a C -regularized semigroup of type $(M, 0)$, then

$$\|r_{j,j-1}^n(\rho A)C\|_{\mathcal{L}(X)} \leq K \quad \text{and} \quad \|r_{j,j-2}^n(\rho A)C\|_{\mathcal{L}(X)} \leq Kn^{\frac{1}{4j}} \quad (2.33)$$

for $j \in \mathbb{N}_0$ and $\rho > 0$. Similarly, in the case of the restricted Padé approximants given by Proposition 2.1.5, one obtains that the Calahan scheme given by $r_2(\rho A)C$ is stable, and the norm of $r_3^n(\rho A)C$ grows as $O(n^{\frac{1}{12}})$.

Theorem 2.4.2. *Let W be a C -regularized semigroup of type $(M, 0)$. If r is an \mathcal{A} -stable rational approximation of the exponential function of order q , then there exists $K > 0$ such that*

$$\left\| r^n \left(\frac{t}{n} A \right) Cx - W(t)x \right\| \leq MKt^{k-\theta_q(k)} \left(\frac{t}{n} \right)^{\theta_q(k)} \|A^k x\| \quad (2.34)$$

for every $k \in \{0, \dots, q+1\}$, $k \neq \frac{q-1}{2}$, and $x \in D(A^k)$. If $k = \frac{q-1}{2}$ then

$$\left\| r^n \left(\frac{t}{n} A \right) Cx - W(t)x \right\| \leq MKt^{k-\theta_q(k)} \left(\frac{t}{n} \right)^{\theta_q(k)} \ln(n+1) \|A^k x\|.$$

If in addition r satisfies the condition (\star) , then θ_q can be replaced by θ_q^ .*

Proof. Let $x \in D(A^k)$. It follows from Theorem 1.4.3, Corollary 1.1.9, Equation (2.16), and integratinng by parts k -times that

$$\begin{aligned}
\left\| r^n \left(\frac{t}{n} A \right) Cx - W(t)x \right\| &= \left\| \int_0^\infty W(s)x d[\alpha_{n,t} - H_t](s) \right\| \\
&= \left\| \int_0^\infty I^{(k-1)}(\alpha_{n,t}(s) - H_t(s)) \frac{d^k}{ds^k} [W(s)x] ds \right\| \\
&= \left\| \int_0^\infty I^{(k-1)}(\alpha_{n,t}(s) - H_t(s)) W(s) A^k x ds \right\| \\
&\leq M \|I^{(k-1)}(\alpha_{n,t} - H_t)\|_{L_1(\mathbb{R}_+)} \|A^k x\|.
\end{aligned}$$

Now, the result follows from Corollary 2.2.4. \square

Remark 2.4.3. For $A = \frac{d}{dx}$, and $C = Id$ on $L^\infty(\mathbb{R})$, it is shown in [11] that the estimation given in Theorem 2.4.2 is the best possible.

Whereas Theorem 2.3.4 yields τ -convergence for \mathcal{A} -stable rational approximations with α^{*m} uniformly bounded (in the variation norma) for every $x \in X$ for bi-dense generators, our next result shows that this is not the case for C -regularized semigroups since the generators A of C -regularized semigroups are not necessarily densely defined nor bi-densely defined.

Corollary 2.4.4. *Let A be the generator of a C -regularized semigroup W of type $(M, 0)$. If $r(z) = \int_0^\infty e^{zs} d\alpha(s)$ ($Re(z) \leq 0$) is an \mathcal{A} -stable rational approximation to the exponential function of order $q \geq 1$ for which α^{*m} is uniformly bounded, then*

$$W(t)x = \lim_{n \rightarrow \infty} r^n \left(\frac{t}{n} A \right) Cx, \quad (2.35)$$

for all $x \in \overline{D(A)}$.

Proof. If $x \in D(A)$ then the result follows from Theorem 2.4.2. Let $x \in \overline{D(A)} \setminus D(A)$ and let $t > 0$. Let $\{x_m\}_{m \in \mathbb{N}} \subset D(A)$ be such that $x_m \rightarrow x$ as $m \rightarrow \infty$. The result follows by an $\frac{\varepsilon}{3}$ -argument from the following inequalities and Theorem 2.4.2.

$$\begin{aligned}
\left\| r^n \left(\frac{t}{n} A \right) Cx - W(t)x \right\| &\leq \left\| r^n \left(\frac{t}{n} A \right) Cx - r^n \left(\frac{t}{n} A \right) Cx_m \right\| + \\
&\quad \left\| r^n \left(\frac{t}{n} A \right) Cx_m - W(t)x_m \right\| + \|W(t)x_m - W(t)x\| \\
&\leq K \|x - x_m\| + \left\| r^n \left(\frac{t}{n} A \right) Cx_m - W(t)x_m \right\| \\
&\quad + \|W(t)x_m - W(t)x\|.
\end{aligned}$$

\square

In particular, for strongly continuous semigroups one obtains that

Corollary 2.4.5. *Let A be the generator of a strongly continuous semigroup T of type (M, ω) . If $r(z) = \int_0^\infty e^{zs} d\alpha(s)$ ($\operatorname{Re}(z) \leq 0$) is an \mathcal{A} -stable rational approximation to the exponential function of order $q \geq 1$ for which α^{*m} is uniformly bounded in the variation norm, then*

$$T(t)x = \lim_{n \rightarrow \infty} r^n \left(\frac{t}{n} A \right) x \quad \text{for all } x \in X. \quad (2.36)$$

2.5 Time and Space Discretized Schemes for Semigroups

In order to discuss not only time but also space discretization, the following C -regularized version of the Trotter-Kato Theorem is needed.

Theorem 2.5.1. *Let A be the generator of a C -regularized semigroup $\{W(t)\}_{t \geq 0}$ of type $(M, 0)$ on X . Suppose that for each $n \in \mathbb{N}$ there are operators A_n which generate C_n -regularized semigroup of type $(M, 0)$ on X , and such that $C_n \rightarrow C$. If*

$$\lim_{n \rightarrow \infty} R(\lambda, A_n) C_n x = R(\lambda, A) C x \quad (2.37)$$

for some $\lambda \in \rho_C(A)$ and for all $x \in X$, then

$$\lim_{n \rightarrow \infty} W_n(t)x = W(t)x \quad (2.38)$$

uniformly on $[0, \tau]$ for all $\tau > 0$ and all $x \in \overline{\operatorname{Im}(R(\lambda, A)C)}$.

Proof. Since the Laplace transform of $\{W(t)\}_{t \geq 0}$ is $R(\lambda, A)C$, it is enough to show that the sequence $\{W_n(\cdot)x\}_{n \in \mathbb{N}}$ is equicontinuous for any $x \in \operatorname{Im}(R(\lambda, A)C)$, and then apply [4, Thm. 1.7.5].

In order to show equicontinuity, an $\frac{\varepsilon}{2}$ -argument will be used. Let $y \in X$ be such that $R(\lambda, A)C y = x$, and choose n_0 such that $2M \|R(\lambda, A)C y - R(\lambda, A_n)C_n y\| < \frac{\varepsilon}{2}$ for all $n \geq n_0$. It follows that

$$\begin{aligned} \|W_n(t)x - W_n(s)x\| &= \|W_n(t)R(\lambda, A)C y - W_n(s)R(\lambda, A)C y\| \\ &\leq \|W_n(t)R(\lambda, A_n)C_n y - W_n(s)R(\lambda, A_n)C_n y\| + \frac{\varepsilon}{2} \\ &= \left\| \int_0^t W_n(\tau) A_n R(\lambda, A_n) C_n y d\tau - \int_0^s W_n(\tau) A_n R(\lambda, A_n) C_n y d\tau \right\| + \frac{\varepsilon}{2} \\ &\leq M |t - s| \|A_n R(\lambda, A_n) C_n y\| + \frac{\varepsilon}{2} \\ &= M |t - s| \|\lambda R(\lambda, A_n) C_n y - C_n y\| + \frac{\varepsilon}{2}. \end{aligned}$$

Since $\sup_{n \in \mathbb{N}} \|\lambda R(\lambda, A_n) C_n y - C_n y\| < \infty$, there exists $\delta > 0$ such that $\|W_n(t)x - W_n(s)x\| \leq \varepsilon$ whenever $|t - s| \leq \delta$ for $n \geq n_0$. Finally, since $t \rightarrow W_n(t)x$ is continuous for $n < n_0$, the equicontinuity follows. \square

Remark 2.5.2. Notice that in general $CD(A) \subsetneq \text{Im}(R(\lambda, A)C)$, so Theorem 2.5.1 extends Theorem 1 of [103]. On the other hand, if one assumes that $R(\lambda, A_n)C_n = C_nR(\lambda, A_n)$ for all $x \in X$ and $n \in \mathbb{N}_0$, then the condition (2.37) can be obtained by assuming that $\lim_{n \rightarrow \infty} R(\lambda, A_n) = R(\lambda, A)$, and a simple modification of the proof of Theorem 2.5.1 leads to convergence for all $x \in \overline{\text{Im}(R(\lambda, A))} = \overline{D(A)}$. Thus, Theorem 1 of [72] is a particular case of this result since in the case of m -times integrated semigroups one has that $\overline{C} = R(\lambda, A)^m$. Finally, if one assumes that $\overline{\text{Im}(C)} = X$ then one obtains that $\overline{D(A)} = X$. Thus, convergence on the whole space X is obtained.

Corollary 2.5.3. *Let r be an \mathcal{A} -stable rational approximation of the exponential function of order $q \geq 1$. If the conditions of Theorem 2.5.1 are satisfied, then*

$$\lim_{n \rightarrow \infty} r^n \left(\frac{t}{n} A_n \right) C_n x = W(t)x$$

for all $t \geq 0$ and $x \in \text{Im}(R(\lambda, A)C)$.

Proof. Let $x \in \text{Im}(R(\lambda, A)C)$. Since A_m generates a C_m -regularized semigroup of type $(M, 0)$, it follows from Theorem 2.4.2 with $k = 1$ that

$$\lim_{n \rightarrow \infty} r^n \left(\frac{t}{n} A_m \right) C_m x = W_m(t)x,$$

for all $t \geq 0$ and $m \in \mathbb{N}$. On the other hand, from Theorem 2.5.1 one obtains that

$$\begin{aligned} \left\| r^n \left(\frac{t}{n} A_m \right) C_m x - r^n \left(\frac{t}{n} A \right) C x \right\| &= \left\| \int_0^\infty W_m(s)x - W(s)x d\alpha_{n,t}(s) \right\| \\ &\leq \int_0^\infty \|W_m(s)x - W(s)x\| dV_{\alpha_{n,t}}(s) \rightarrow 0, \end{aligned}$$

as $m \rightarrow \infty$ for all $t \geq 0$ and $n \in \mathbb{N}$. The Corollary follows by Theorem 2.4.2 and Theorem 2.5.1 together with the usual diagonal argument since $x \in D(A)$. \square

Let X_n be Banach spaces with norms $\|\cdot\|_n$. From now on, the following conditions are assumed. For every $n \in \mathbb{N}$ there exist $P_n : X \rightarrow X_n$, and $E_n : X_n \rightarrow X$ in $\mathcal{L}(X)$ such that

- (i) $\|P_n\| \leq N$, $\|E_n\| \leq N'$, where N and N' are independent of n .
- (ii) $\|P_n x\|_n \rightarrow \|x\|$ as $n \rightarrow \infty$ for every $x \in X$.
- (iii) $\|E_n P_n x - x\| \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in X$.
- (iv) $P_n E_n = I_n$, where I_n is the identity operator on X_n .

A sequence $\{x_n\}_{n \in \mathbb{N}} \subset X_n$ converges to $x \in X$ if $\|P_n x - x_n\|_n \rightarrow 0$ as $n \rightarrow \infty$. This type of convergence will be denoted by $x_n \rightarrow x$. Now, a sequence of linear operators $\{A_n\}_{n \in \mathbb{N}}$ converges to an operator A if $D(A) = \{x : P_n x \in D(A_n), \text{ and } A_n P_n x \text{ converges}\}$ and $Ax = \lim_{n \rightarrow \infty} A_n P_n x$ for every $x \in D(A)$. This type of convergence will be denoted by $A_n \rightrightarrows A$ as $n \rightarrow \infty$.

Theorem 2.5.4. *Let $\{W(t)\}_{t \geq 0}$ be a C -regularized semigroup of type $(M, 0)$ generated by A and let $A_n : X_n \rightarrow X_n \in \mathcal{L}(X_n)$ be the generator of a C_n -regularized semigroup $\{W_n(t)\}_{t \geq 0}$ on X_n of type $(M', 0)$ such that $R(\lambda, A_n)C_n x \rightrightarrows R(\lambda, A)Cx$ for all $x \in X$. If r is an \mathcal{A} -stable rational approximation of the exponential function of order q , then*

$$r^n \left(\frac{t}{n} A_n \right) C_n x \rightrightarrows W(t)x \quad \text{as } n \rightarrow \infty \quad (2.39)$$

for all $t \geq 0$ and $x \in \text{Im}(R(\lambda, A)C)$.

Proof. Let $\widetilde{W}_n(t) : X \rightarrow X$ be defined by $\widetilde{W}_n(t) := E_n W_n(t) P_n$ for each $t \geq 0$. It is easy to see that $\{\widetilde{W}_n(t)\}_{t \geq 0}$ is a \widetilde{C}_n -regularized semigroup of type $(NN'M', 0)$, where $\widetilde{C}_n := E_n C_n P_n$. Moreover, the generator of $\{\widetilde{W}_n(t)\}_{t \geq 0}$ is given by $\widetilde{A}_n := E_n A_n P_n$. Let $M'' = \max\{M, NN'M'\}$, then \widetilde{A}_n and A generate regularized semigroups of type $(M'', 0)$. Now, let $x \in X$ and $\lambda > 0$. It follows that

$$\begin{aligned} R(\lambda, \widetilde{A}_n) \widetilde{C}_n x &= \int_0^\infty e^{-\lambda t} \widetilde{W}_n(t) x dt = \int_0^\infty e^{-\lambda t} E_n W_n(t) P_n x dt \\ &= E_n \int_0^\infty e^{-\lambda t} W_n(t) P_n x dt = E_n R(\lambda, A_n) C_n P_n x. \end{aligned}$$

Since $R(\lambda, A_n)C_n \rightrightarrows R(\lambda, A)C$, one obtains that $R(\lambda, \widetilde{A}_n) \widetilde{C}_n \rightarrow R(\lambda, A)C$ as $n \rightarrow \infty$. Similarly, $\widetilde{C}_n \rightarrow C$ as $n \rightarrow \infty$. Therefore, if $x \in \overline{\text{Im}(R(\lambda, A)C)}$, then $\widetilde{W}_n(t)x \rightarrow W(t)x$ as $n \rightarrow \infty$ uniformly on $[0, t]$ for all $t > 0$ by Theorem 2.5.1. Moreover, it follows from Corollary 2.5.3 that

$$\lim_{n \rightarrow \infty} r^n \left(\frac{t}{n} \widetilde{A}_n \right) \widetilde{C}_n x = W(t)x \quad (2.40)$$

for any \mathcal{A} -stable rational approximation of the exponential function of order q . On the other hand, if $\alpha \in \text{NBV}^0$ is such that $r(z) = \int_0^\infty e^{zs} d\alpha(s)$, for $\text{Re}(z) \leq 0$, then

$$\begin{aligned} r^n \left(\frac{t}{n} \widetilde{A}_n \right) \widetilde{C}_n x &= \int_0^\infty \widetilde{W}_n(s) x d\alpha_{n,t}(s) = E_n \int_0^\infty W_n(s) P_n x d\alpha_{n,t}(s) \\ &= E_n r^n \left(\frac{t}{n} A_n \right) C_n P_n x \end{aligned}$$

for all $x \in X$. Thus

$$r^n \left(\frac{t}{n} A_n \right) C_n P_n x = P_n r^n \left(\frac{t}{n} \widetilde{A}_n \right) \widetilde{C}_n x \quad (2.41)$$

for all $x \in X$. Finally, by combining (2.40) and (2.41),

$$\begin{aligned} \left\| r^n \left(\frac{t}{n} A_n \right) C_n P_n x - P_n W(t)x \right\|_n &= \left\| P_n r^n \left(\frac{t}{n} \widetilde{A}_n \right) \widetilde{C}_n x - P_n W(t)x \right\|_n \\ &\leq N \left\| r^n \left(\frac{t}{n} \widetilde{A}_n \right) \widetilde{C}_n x - W(t)x \right\| \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$ for any $x \in \text{Im}(R(\lambda, A)C)$. \square

Corollary 2.5.5. *Let T be a strongly continuous semigroup of type $(M, 0)$ generated by A and let $A_n : X_n \rightarrow X_n \in \mathcal{L}(X_n)$ be the generators of strongly continuous semigroups T_n on X_n of type $(M', 0)$ such that $R(\lambda, A_n)x \rightrightarrows R(\lambda, A)x$ for all $x \in X$. If r is an \mathcal{A} -stable rational approximation of the exponential function of order q , then*

$$r^n \left(\frac{t}{n} A_n \right) x \rightrightarrows T(t)x \quad \text{as } n \rightarrow \infty \quad (2.42)$$

for all $t \geq 0$ and $x \in D(A)$. In addition, if α^{n} is uniformly bounded in the variation norm, then (2.42) holds for every $x \in X$.*

Chapter 3

Applications

The main goal of this chapter is to show the relevance of the results obtained in the previous sections by showing how different techniques based on the results of these sections can be used in order to attack different types of evolution problems. It is shown that if an evolution problem is of convolution type then a direct method (see Section 3.2 below) can be implemented in order to approximate the solutions by using the vector-valued Laplace transform. On the other hand, if the evolution equation is not of convolution type, then one may rewrite the equation in the (ACP) form and applies the methods for either bi-continuous semigroups or C -regularized semigroups described in Chapter 2. For instance, the first section shows that in order to approximate semigroups generated by non-linear flows, one has to first rewrite the problem in the (ACP) form and then apply the results on bi-continuous semigroups since the generator will not be a strongly continuous semigroup in general. The importance of the bi-continuous results is shown in Section 3.2 by obtaining new numerical inversion formulas for the vector-valued Laplace transform. Thus, evolution problems of convolution type can be directly treated by using the Laplace transform methods. The third section shows an interesting case where the two methods can be implemented by considering integrated semigroups. Furthermore, in this case, new integrated rational approximation schemes are developed. Finally, the fourth section outlines possible applications of the numerical inversion of the Laplace transform for a typical convolution equation such as the abstract Volterra integral equations.

As previously mentioned, there are several other important cases of bi-continuous semigroups as well as C -regularized semigroups that one can study by applying the semigroup approximation results of Chapter 2. Among them, various classes of evolution equations that are not of convolution type (e.g. non-autonomous problems) and distributional semigroups are left for further research.

3.1 Semigroups Induced by Non-Linear Flows

A continuous flow on a topological space Ω is a map $\phi : \mathbb{R}_0^+ \times \Omega \rightarrow \Omega$, $\phi_t(v) := \phi(t, v)$, that is jointly continuous and satisfies the semigroup property $\phi_{t+s} = \phi_t \phi_s$, $\phi_0 = Id$. If $X := C_b(\Omega)$, then the semigroup S defined by $S(t) : x \rightarrow x \circ \phi_t$ is called the semigroup induced by the flow ϕ . F. Kühnmeund shows in [66, 67] that these semigroups, considered also by J. R. Dorroh and J. W. Neuberger in [30, 31, 32, 77], are bi-continuous on $C_b(\Omega)$ and

$$\tau\text{-}\lim_{n \rightarrow \infty} \left[\frac{n}{t} R \left(\frac{n}{t}, A \right) \right]^n x = S(t)x \quad (3.1)$$

for all $x \in X$, where the limit of (3.1) is uniform for t in compact sets of \mathbb{R}_0^+ . In the context of semigroups induced by flows, the backward-Euler formula (3.1) was obtained in [31, Thm. 3.2]. Now, if one considers $r_{\text{bE}}(z) = \frac{1}{1-z}$ for $\text{Re}(z) \leq 0$, then r_{bE} is an \mathcal{A} -stable rational approximation to the exponential function of order $q = 1$. Moreover, $\alpha_{\text{bE}}(s) = \int_0^s e^{-\xi} d\xi$, and from Theorem 2.3.4 one obtains that (2.32) coincides with the backward-Euler formula (3.1). Furthermore, if $x \in D(A)$ and $k = 1$, Theorem 2.3.3 provides error estimates in the norm sense for the backward-Euler approximation (3.1); i.e., the τ -limit can be replaced by the $\|\cdot\|$ -limit for $x \in D(A)$. Thus, convergence of the backward-Euler scheme is obtained in the original norm and the error is $O\left(\frac{t}{\sqrt{n}}\right)$ as $n \rightarrow \infty$. However, Theorem 2.3.3 also shows that the backward-Euler scheme is the slowest, in terms of convergence, among the Padé approximants to the exponential function.

In order to illustrate this, consider the bi-continuous semigroup S generated by the operator $Ax(s) = s^{\frac{2}{3}}x'(s)$ with maximal domain, cf. [31, Ex. 4.2]; i.e., $S(t)x(s) = x(\phi_t(s))$ is the semigroup induced by the flow $\phi_t(s) = (s^{\frac{1}{3}} + \frac{t}{3})^3$ solving the non-linear ODE

$$\begin{cases} \phi'(t) &= \phi(t)^{\frac{2}{3}} \\ \phi(0) &= s. \end{cases}$$

Now, in order to approximate the flow $\phi_t(\cdot)$, we define $x(s) := -a$ ($s < -a$), $x(s) = s$ ($|s| \leq a$), and $x(s) := a$ ($s > a$) for $a > 0$. Clearly other choices of x can be taken in order to obtain ϕ ; in particular, x can be chosen to be in C^∞ . Figure 3.1-(a) shows that the backward-Euler scheme for $n = 10$ yields an approximation to $S(t)x$ that is accurate up to one decimal place for $t \in [0, 1]$ and $s \in [0, 4]$. Figure 3.1-(b) shows the approximation error of the scheme $r^n(\frac{t}{n}A)x$ defined via the subdiagonal Padé scheme

$$r_{4,3}(z) = \frac{210 - 90z + 15z^2 - z^3}{840 + 480z + 120z^2 + 16z^3 + z^4} \quad (3.2)$$

with $n = 3$. It follows that the scheme given by $r_{4,3}$ with $n = 3$ is accurate up to ten decimal places for $t \in [0, 1]$ and $v \in [0, 4]$. Thus, these linear Brenner-Thomée type of results of Chapter 2 allow the approximation (time discretization) of non-linear flows defined by nonlinear ODE's with sharp error estimates. A thorough comparison of this approach with classical numerical methods for nonlinear ODE's remains to be done.

3.2 Bi-Continuous Semigroups and Their Importance for the Inversion of the Vector-Valued Laplace Transform

The numerical inversion of the Laplace transform seems to be one of those hard-to-tackle problems in mathematics, see [18]. It has been studied by several different authors using many different techniques. Among the techniques one finds the use

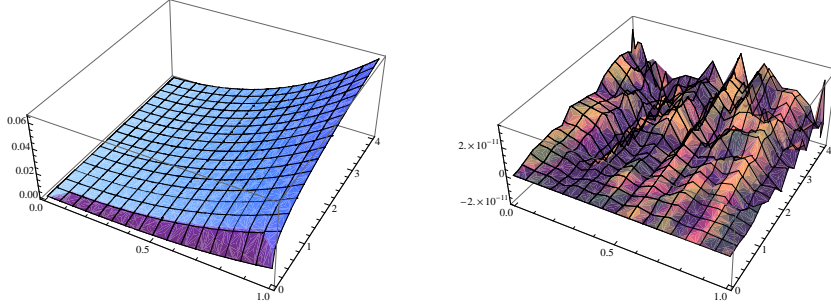


FIGURE 3.1. Comparison between the error of the Backward-Euler scheme with $n = 10$ (left) and the error of the Padé 4,3 scheme with $n = 3$ (right).

of Laguerre polynomials [8, 20, 90, 89, 100, 99], Fourier series representation of the Bromwich integral [34, 22, 21, 35, 88], stochastic processes [43, 94, 95], continued fractions [3], as well as the use of exponential expansions of characteristics [81] and regularized analytic continuation [64, 65], among others. Actually, R. Piessens published a bibliography on numerical methods for the inversion of the Laplace transform [86, 87], and in 2002, P. Valkó proposed a standardized set of functions in order to prove the accuracy of the different inversion formulas due to the unavailability of error estimates at the time, see [97]. Lately, due to the availability of better computers, there have been several articles that compare the different methods, see [2, 23, 37, 45, 70].

This section shows the significance of the results of Chapter 2 (bi-continuous case) for the numerical inversion of the Laplace transform. The main observation, the inequality (3.4) below, is due to F. Neubrander and it is contained in a joint paper with him and K. Ozer (see [58]).

Let \mathcal{X} be a Banach space and let $X := C_b(\mathbb{R}_0^+; \mathcal{X})$. Then, the shift semigroup $T(t)f(s) := f(t + s)$ generated by $D = \frac{d}{ds}$ is bi-continuous with respect to the topology of uniform convergence on compact sets of \mathbb{R}_0^+ . It follows from Theorem 2.3.3 that if r is an \mathcal{A} -stable rational approximation to the exponential function of order $q \geq 1$, then for $f \in D(\frac{d^k}{ds^k})$ there exists $K > 0$ such that

$$\left\| r^n \left(\frac{t}{n} D \right) f - T(t)f \right\|_{\infty} \leq K t^k \left(\frac{1}{n} \right)^{\theta_q(k)} \|f^{(k)}\|_{\infty}. \quad (3.3)$$

(with obvious modifications if r satisfies the condition (\star) of Definition 2.1.2). In particular, by plugging in 0 for the spacial variable and using that $T(t)f(0) = f(t)$, one obtains that

$$\left\| r^n \left(\frac{t}{n} D \right) f(0) - f(t) \right\|_{\mathcal{X}} \leq K t^k \left(\frac{1}{n} \right)^{\theta_q(k)} \|f^{(k)}\|_{\infty}. \quad (3.4)$$

Since $R(\lambda, D)f = \int_0^{\infty} e^{-\lambda t} T(t)f dt = \int_0^{\infty} e^{-\lambda t} f(t + \cdot) dt$, it follows that

$$R(\lambda, D)f(0) = \int_0^{\infty} e^{-\lambda t} f(t) dt = \widehat{f}(\lambda), \quad (3.5)$$

where $\widehat{f}(\lambda)$ denotes the Laplace transform of f . Consequently,

$$\begin{aligned} R(\lambda, D)^{n+1} f(0) &= \frac{(-1)^n}{n!} R(\lambda, D)^{(n)} f(0) \\ &= \frac{(-1)^n}{n!} \int_0^\infty e^{-\lambda t} (-t)^n f(t) dt \\ &= \frac{(-1)^n}{n!} \widehat{f}^{(n)}(\lambda). \end{aligned}$$

Now, let

$$r(z) = B_{0,0} + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq r}} \frac{B_{i,j}}{(b_i - z)^j}$$

be an \mathcal{A} -stable rational approximation to the exponential function of order $q \geq 1$. Then, for each $n \in \mathbb{N}$, there exist constants $C_{n,i,j}$ such that

$$r^n(z) = C_{n,0,0} + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq nr}} \frac{C_{n,i,j}}{(b_i - z)^j}.$$

In particular,

$$\begin{aligned} r^n\left(\frac{t}{n}D\right) f(0) &= C_{n,0,0} f(0) + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq nr}} C_{n,i,j} R^j\left(b_i, \frac{t}{n}D\right) f(0) \\ &= C_{n,0,0} f(0) + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq nr}} C_{n,i,j} \left(\frac{n}{t}\right)^j R^j\left(b_i \frac{n}{t}, D\right) f(0) \\ &= C_{n,0,0} f(0) + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq nr}} C_{n,i,j} \left(\frac{n}{t}\right)^j \frac{(-1)^{j-1}}{(j-1)!} \widehat{f}^{(j-1)}\left(\frac{n}{t}b_i\right). \end{aligned}$$

This yields the following novel inversion formulas for the Laplace transform which have, due to the rapid speed of convergence and the availability of error estimates, significant applications to evolution equations that can be treated directly with Laplace transform methods without having to rewrite the them as (ACP) on a properly chosen state space X (see also [58]).

Theorem 3.2.1 (Laplace Transform Inversion). *Let \widehat{f} be the Laplace transform of $f \in C_b(\mathbb{R}_0^+; \mathcal{X})$ and define $f_0 := \lim_{\lambda \rightarrow \infty} \lambda \widehat{f}(\lambda)$. If r is an \mathcal{A} -stable rational approximation to the exponential function of order $q \geq 1$, then there exist $K > 0$, $b_i \in \mathbb{C}$ with $\operatorname{Re}(b_i) > 0$ and constants $C_{n,i,j} \in \mathbb{C}$ (independent of f) such that*

$$\left\| C_{n,0,0} f_0 + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq nr}} C_{n,i,j} \left(\frac{n}{t}\right)^j \frac{(-1)^{j-1}}{(j-1)!} \widehat{f}^{(j-1)}\left(\frac{n}{t}b_i\right) - f(t) \right\|_{\mathcal{X}} \leq K \frac{t^k}{n^{\theta_q(k)}} \|f^{(k)}\|_{\infty}, \quad (3.6)$$

for every $k \in \{1, \dots, q+1\}$, and $k \neq \frac{q-1}{2}$. If $k = \frac{q-1}{2}$ then

$$\left\| C_{n,0,0}f_0 + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq nr}} C_{n,i,j} \left(\frac{n}{t}\right)^j \frac{(-1)^{j-1}}{(j-1)!} \widehat{f}^{(j-1)} \left(\frac{n}{t}b_i\right) - f(t) \right\|_{\mathcal{X}} \leq \frac{Kt^k \ln(n+1)}{n^{\theta_q(k)}} \|f^{(k)}\|_{\infty}. \quad (3.7)$$

If r satisfies the condition (\star) , then θ_q can be replaced by θ_q^* (where θ_q and θ_q^* are as in Theorem 2.2.4).

Let $r(z) = \int_0^\infty e^{zs} d\alpha(s)$ ($\operatorname{Re}(z) \leq 0$) be an \mathcal{A} -stable rational approximation to the exponential function of order $q \geq 1$ for which $\|\alpha^{*m}\|_0$ is uniformly bounded. Since the shift semigroup is bi-continuous on $L^\infty(\mathbb{R}_0^+, \mathbb{C})$ (see, e.g., [66, Prop. 3.18]), one obtains from Theorem 2.3.4 that

$$f(t) = \tau\text{-}\lim_{n \rightarrow \infty} C_{n,0,0}f_0 + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq nr}} C_{n,i,j} \left(\frac{n}{t}\right)^j \frac{(-1)^{j-1}}{(j-1)!} \widehat{f}^{(j-1)} \left(\frac{n}{t}b_i\right)$$

for all $f \in L^\infty(\mathbb{R}_0^+)$, where the limit is uniform on compact intervals of \mathbb{R}_0^+ and the topology τ is the weak* topology on $L^\infty(\mathbb{R}_0^+, \mathbb{C})$.

Theorem 3.2.2 (Regularized Inversion of the Laplace Transform). *Let \widehat{f} be the Laplace transform of $f \in C_b(\mathbb{R}_0^+; \mathcal{X})$ (or $L^\infty(\mathbb{R}_0^+)$) and let r be an \mathcal{A} -stable rational approximation to the exponential function of order $q \geq 1$ for which α^{*n} is uniformly bounded. If $R(\lambda, D)$ denotes the resolvent operator of D and*

$$L_{n,t,m}[\widehat{f}] := r^n \left(\frac{t}{n}D\right) \left[\left(\frac{n}{t}\right) R\left(\frac{n}{t}, D\right)\right]^m f(0),$$

then

$$\tau\text{-}\lim_{n \rightarrow \infty} L_{n,t,m}[\widehat{f}] = f(t), \quad (3.8)$$

for every $t > 0$ and $m \in \mathbb{N}_0$. Furthermore, the limit is uniform on compact intervals of \mathbb{R}_0^+ (In the L^∞ case, the limit is in the weak*-topology sense).

The proof follows immediately by noticing that $\tau\text{-}\lim_{\lambda \rightarrow \infty} [\lambda R(\lambda, A)]^m x = x$ when A is the generator of a bi-continuous semigroup.

The inversion formula of the Laplace transform due to E. L. Post [90] and D. V. Widder [100] states that if the Laplace transform of f exists for $f \in C_b(\mathbb{R}, \mathbb{C})$ then

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \widehat{f}^{(n)} \left(\frac{n}{t}\right) = f(t), \quad (3.9)$$

where the limit is uniformly on compact intervals of \mathbb{R}_0^+ . On the other hand, since the shift semigroup is bi-continuous on $C_b(\mathbb{R}_0^+, X)$ (or L^∞), it follows from (3.5) that

$$\frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \widehat{f}^{(n)} \left(\frac{n}{t}\right) = \left[\left(\frac{n}{t}\right) R\left(\frac{n}{t}, D\right)\right]^n \left(\frac{n}{t}\right) R\left(\frac{n}{t}, D\right) f(0). \quad (3.10)$$

TABLE 3.1. Order of the error of the schemes provided by the first five Subdiagonal Padé approximants of the exponential for fix f and $t \geq 0$.

k	Backward Euler	Padé 2, 1	Padé 3, 2	Padé 4, 3	Padé 5, 4
1	$n^{-\frac{1}{2}}$	$n^{-\frac{3}{4}}$	$n^{-\frac{5}{6}}$	$n^{-\frac{7}{8}}$	$n^{-\frac{9}{10}}$
2	n^{-1}	$n^{-\frac{3}{2}}$	$n^{-\frac{5}{3}}$	$n^{-\frac{7}{4}}$	$n^{-\frac{9}{5}}$
3	$n^{-\frac{3}{2}}$	$n^{-\frac{9}{4}}$	$n^{-\frac{15}{6}}$	$n^{-\frac{21}{8}}$	$n^{-\frac{27}{10}}$
4	n^{-2}	n^{-3}	$n^{-\frac{10}{3}}$	$n^{-\frac{7}{2}}$	$n^{-\frac{18}{5}}$
5	$n^{-\frac{5}{2}}$	$n^{-\frac{15}{4}}$	$n^{-\frac{25}{6}}$	$n^{-\frac{35}{8}}$	$n^{-\frac{9}{2}}$

Therefore, the Post-Widder inversion formula is obtained by considering r_{bE} and $m = 1$ in Theorem 3.2.2 for $C_b(\mathbb{R}_0^+, X)$ or L^∞ . An equivalent version for the Post-Widder formula holds for $f \in L_{loc}^1(\mathbb{R}_0^+, X)$, see [4].

Among the schemes provided by Theorem 3.2.1 and Theorem 3.2.2, the subdiagonal Padé approximants of the exponential $r_{j,j-1}$ satisfy the condition (\star) and provide schemes of order $q = 2j - 1$ with $p = 2j$. Thus, $\theta_q^*(k) = k^{\frac{2j-1}{2j}}$. Table 3.1 shows the order of the error for a fixed function f at a time $t > 0$ for different k 's associated with the schemes provided by the first five subdiagonal Padé approximants.

Now, in order to implement the different schemes provided by Theorem 3.2.1 and Theorem 3.2.2, the determination of the coefficients $C_{n,i,j}$ is crucial. In the following, we present a series of lemmas that allow the computation of the desired coefficients for schemes associated with rational functions with simple poles. In this way, the determination of explicit formulas for the numerical inversion of the Laplace transform depends on the the knowledge of the poles of the rational function associated with the scheme.

Lemma 3.2.3. *Let $\lambda, \mu \in \mathbb{C}$ be such that $\lambda \neq \mu$ and define $\alpha_{\lambda,\mu} := \frac{-1}{\lambda-\mu}$. Then, for every $z \in \mathbb{C} \setminus \{\lambda, \mu\}$ and $k \in \mathbb{N}$,*

$$\frac{1}{(\lambda - z)^k(\mu - z)} = \frac{(-1)^k \alpha_{\lambda,\mu}^k}{(\mu - z)} + \sum_{i=1}^k \frac{(-1)^{i+1} \alpha_{\lambda,\mu}^i}{(\lambda - z)^{k+1-i}}.$$

The proof of Lemma 3.2.3 follows by simple induction.

Lemma 3.2.4. *Let $r = \frac{P}{Q}$ be an \mathcal{A} -stable rational approximation of the exponential function with l simple poles $b_i \in \mathbb{C}$ such that $\deg(P) < \deg(Q)$, i.e., there exist $B_i \in \mathbb{C}$ such that $r(z) = \sum_{1 \leq i \leq l} \frac{B_i}{(b_i - z)}$. Then, for every $n \in \mathbb{N}$ there exist $B_{n,i,j} \in \mathbb{C}$*

such that

$$r^n(z) = \sum_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}} \frac{B_{n,i,j}}{(b_i - z)^j}.$$

If $\alpha_{i,j} := \frac{-1}{b_i - b_j}$ then the coefficients $B_{n+1,h,w}$ that uniquely determine r^{n+1} are given by

$$B_{n+1,h,w} = \begin{cases} \sum_{\substack{1 \leq p \leq l \\ p \neq h \\ 1 \leq j \leq n}} B_{1,h,1} B_{n,p,j} \alpha_{h,p}^j - B_{1,p,1} B_{n,h,j} \alpha_{p,h}^j & w = 1, \\ B_{1,h,1} B_{n,h,w-1} - \sum_{\substack{1 \leq p \leq l \\ p \neq h \\ w \leq j \leq n}} \alpha_{p,h}^{j+1-w} B_{1,p,1} B_{n,h,j} & 2 \leq w \leq n, \\ B_{n,h,n} B_{1,h,1} & w = n + 1. \end{cases} \quad (3.11)$$

for every $h \in \{1, \dots, l\}$.

Proof. It follows that

$$\begin{aligned} r^{n+1}(z) &= \left(\sum_{p=1}^l \frac{B_{1,p,1}}{(b_p - z)} \right) \left(\sum_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}} \frac{B_{n,i,j}}{(b_i - z)^j} \right) \\ &= \sum_{p=1}^l \left[\sum_{\substack{1 \leq i \leq l \\ i \neq p \\ 1 \leq j \leq n}} \frac{B_{1,p,1} B_{n,i,j}}{(b_p - z)(b_i - z)^j} + \sum_{j=1}^n \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \right] \\ &= \sum_{p=1}^l \left[\sum_{\substack{1 \leq i \leq l \\ i \neq p \\ 1 \leq j \leq n}} \frac{B_{1,p,1} B_{n,i,j}}{(b_p - z)(b_i - z)^j} + \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \right] + \sum_{p=1}^l \frac{B_{1,p,1} B_{n,p,n}}{(b_p - z)^{n+1}}, \end{aligned}$$

and the case $w = n + 1$ follows from the second term of the last equation. Let I be the first term of the last equation, i.e.,

$$I := \sum_{p=1}^l \left[\sum_{\substack{1 \leq i \leq l \\ i \neq p \\ 1 \leq j \leq n}} \frac{B_{1,p,1} B_{n,i,j}}{(b_p - z)(b_i - z)^j} + \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \right].$$

Lemma 3.2.3 yields

$$\begin{aligned}
I &= \sum_{p=1}^l \left[\sum_{\substack{1 \leq i \leq l \\ i \neq p \\ 1 \leq j \leq n}} B_{1,p,1} B_{n,i,j} \left(\frac{(-1)^j \alpha_{i,p}^j}{(b_p - z)} + \sum_{k=1}^j \frac{(-1)^{k+1} \alpha_{i,p}^k}{(b_i - z)^{j+1-k}} \right) + \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \right] \\
&= \sum_{p=1}^l \left[\sum_{\substack{1 \leq i \leq l \\ i \neq p \\ 1 \leq j \leq n}} B_{1,p,1} B_{n,i,j} \left(\frac{(-1)^j \alpha_{i,p}^j}{(b_p - z)} + \sum_{k=1}^j \frac{(-1)^{k+1} \alpha_{i,p}^k}{(b_i - z)^{j+1-k}} \right) \right] \\
&\quad + \sum_{p=1}^l \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \\
&= \sum_{p=1}^l \left[\sum_{\substack{1 \leq i \leq l \\ i \neq p \\ 1 \leq j \leq n}} B_{1,p,1} B_{n,i,j} \left(\sum_{k=1}^j \frac{(-1)^{k+1} \alpha_{i,p}^k}{(b_i - z)^{j+1-k}} \right) \right] \\
&\quad + \sum_{p=1}^l \left[\sum_{\substack{1 \leq i \leq l \\ i \neq p \\ 1 \leq j \leq n}} B_{1,p,1} B_{n,i,j} \frac{(-1)^j \alpha_{i,p}^j}{(b_p - z)} \right] + \sum_{p=1}^l \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \\
&=: \text{II} + \text{III} + \text{IV}.
\end{aligned}$$

Notice that II and III are the only terms that contribute to coefficients of the form $B_{n+1,h,1}$ for any $h \in \{1, \dots, l\}$. It follows that

$$\begin{aligned}
B_{n+1,h,1} &= \sum_{\substack{1 \leq i \leq l \\ i \neq h \\ 1 \leq j \leq n}} B_{1,h,1} B_{n,i,j} (-1)^j \alpha_{i,h}^j + \sum_{\substack{1 \leq p \leq l \\ p \neq h \\ 1 \leq j \leq n}} B_{1,p,1} B_{n,h,j} (-1)^{j+1} \alpha_{h,p}^j \\
&= \sum_{\substack{1 \leq p \leq l \\ p \neq h \\ 1 \leq j \leq n}} B_{1,h,1} B_{n,p,j} (-1)^j \alpha_{p,h}^j + B_{1,p,1} B_{n,h,j} (-1)^{j+1} \alpha_{h,p}^j,
\end{aligned}$$

i.e., the case $w = 1$ follows. Finally, in order to find the terms of the form $B_{n+1,h,w}$ for $2 \leq w \leq n$ we just need to consider the terms II and IV.

It follows by interchanging the order of summation that

$$\begin{aligned}
\text{II} + \text{IV} &= \sum_{p=1}^l \left[\sum_{\substack{1 \leq i \leq l \\ i \neq p \\ 1 \leq j \leq n}} B_{1,p,1} B_{n,i,j} \left(\sum_{k=1}^j \frac{(-1)^{k+1} \alpha_{i,p}^k}{(b_i - z)^{j+1-k}} \right) \right] + \sum_{p=1}^l \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \\
&= \sum_{\substack{1 \leq p \leq l \\ i \neq p \\ 1 \leq i \leq l}} \left[\sum_{j=1}^n \left(\sum_{k=1}^j \frac{B_{1,p,1} B_{n,i,j} (-1)^{k+1} \alpha_{i,p}^k}{(b_i - z)^{j+1-k}} \right) \right] + \sum_{p=1}^l \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \\
&= \sum_{\substack{1 \leq p \leq l \\ i \neq p \\ 1 \leq i \leq l}} \left[\sum_{j=1}^n \sum_{w=1}^j \frac{B_{1,p,1} B_{n,i,j} (-1)^{j+2-w} \alpha_{i,p}^{j+1-w}}{(b_i - z)^w} \right] + \sum_{p=1}^l \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \\
&= \sum_{\substack{1 \leq p \leq l \\ i \neq p \\ 1 \leq i \leq l}} \left[\sum_{w=1}^n \sum_{j=w}^n \frac{B_{1,p,1} B_{n,i,j} (-1)^{j+2-w} \alpha_{i,p}^{j+1-w}}{(b_i - z)^w} \right] + \sum_{p=1}^l \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \\
&= \sum_{i=1}^l \sum_{w=1}^n \left[\sum_{\substack{1 \leq p \leq l \\ i \neq p \\ w \leq j \leq n}} \frac{B_{1,p,1} B_{n,i,j} (-1)^{j+2-w} \alpha_{i,p}^{j+1-w}}{(b_i - z)^w} \right] + \sum_{p=1}^l \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}} \\
&= \sum_{i=1}^l \sum_{w=1}^n \frac{\sum_{\substack{1 \leq p \leq l \\ i \neq p \\ w \leq j \leq n}} B_{1,p,1} B_{n,i,j} (-1)^{j+2-w} \alpha_{i,p}^{j+1-w}}{(b_i - z)^w} + \sum_{p=1}^l \sum_{j=1}^{n-1} \frac{B_{1,p,1} B_{n,p,j}}{(b_p - z)^{j+1}},
\end{aligned}$$

and the statement of the Lemma follows. \square

Lemma 3.2.5. *Let $r = \frac{P}{Q}$ be an \mathcal{A} -stable rational approximation of the exponential function with l simple poles $b_i \in \mathbb{C}$ such that $\deg(P) = \deg(Q)$, i.e., there exist $\lambda, B_i \in \mathbb{C}$ such that $r(z) = \lambda + \sum_{1 \leq i \leq l} \frac{B_i}{(b_i - z)}$ and for every $n \in \mathbb{N}$ there exist $C_{n,i,j} \in \mathbb{C}$ such that*

$$r^n(z) = \lambda^n + \sum_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}} \frac{C_{n,i,j}}{(b_i - z)^j}.$$

If $B_{k,i,j}$ are the coefficients given by Lemma 3.2.4 for $(r(z) - \lambda)^k$ then

$$C_{n,i,j} = \sum_{k=j}^n \binom{n}{k} \lambda^{n-k} B_{k,i,j}. \quad (3.12)$$

Proof. It follows by interchanging the order of summation that

$$\begin{aligned}
r^n(z) &= \left(\lambda + \sum_{i=1}^n \frac{B_{1,i,1}}{(b_i - z)} \right)^n \\
&= \lambda^n + \sum_{k=1}^n \binom{n}{k} \lambda^{n-k} \left[\sum_{i=1}^n \frac{B_{1,i,1}}{(b_i - z)} \right]^k \\
&= \lambda^n + \sum_{k=1}^n \binom{n}{k} \lambda^{n-k} \sum_{i=1}^l \sum_{j=1}^k \frac{B_{k,i,j}}{(b_i - z)^j} \\
&= \lambda^n + \sum_{k=1}^n \sum_{i=1}^l \sum_{j=1}^k \binom{n}{k} \frac{\lambda^{n-k} B_{k,i,j}}{(b_i - z)^j} \\
&= \lambda^n + \sum_{i=1}^l \sum_{k=1}^n \sum_{j=1}^k \binom{n}{k} \frac{\lambda^{n-k} B_{k,i,j}}{(b_i - z)^j} \\
&= \lambda^n + \sum_{i=1}^l \sum_{j=1}^n \sum_{k=j}^n \binom{n}{k} \frac{\lambda^{n-k} B_{k,i,j}}{(b_i - z)^j} \\
&= \lambda^n + \sum_{i=1}^l \sum_{j=1}^n \frac{\sum_{k=j}^n \binom{n}{k} \lambda^{n-k} B_{k,i,j}}{(b_i - z)^j},
\end{aligned}$$

and the statement of the lemma follows. \square

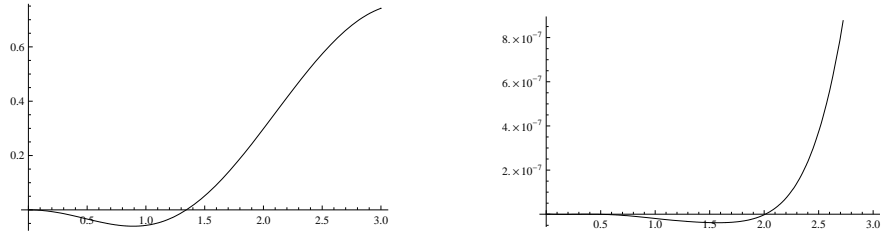
The coefficients for any subdiagonal Padé approximant of the exponential are obtained from Lemma 3.2.4. However, the result is based on the knowledge of their algebraic poles and, as it is presumable, the exact symbolic determination of the algebraic poles of the Padé approximants is a highly non trivial problem if $\deg(Q) \geq 5$. In fact, we do not know if the algebraic poles of $r_{5,4}$ are known. Therefore, the best scheme among the subdiagonal Padé approximants, in terms of speed of convergence, for which this method provides a closed symbolically computable form is $r_{4,3}$. Table 3.2 shows the poles of $r_{4,3}$ obtained from Lemma 3.2.4 and Table 3.3 shows the first coefficients $C_{n,i,j}$ for $n \in \{1, \dots, 5\}$. In general, one can approximate the poles of a subdiagonal Padé scheme by using any software with high-precision algorithm incorporated such as Mathematica[®]. Appendix A shows the implementation of Lemma 3.2.4 trough a simple Mathematica 6 code for the inversion formulas for the Laplace transform provided by any subdiagonal Padé scheme. However, since the calculations involve differences and quotients of big numbers, high errors have to be expected for schemes using the high-precisions algorithms when n increases. Moreover, when computing the coefficients provided by Lemma 3.2.4 in a closed symbolic form, Mathematica takes a considerable amount of time even for small values of n such as $n = 10$ for $r_{4,3}$. Further improvement of the code is needed in order to improve the accuracy and the computational cost.

TABLE 3.2. Poles of $r_{4,3}$

b_1	$4 - \sqrt{-4 + \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} + \sqrt[3]{10(-2+i\sqrt{6})}}$ $- \sqrt{-8 - \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} - \sqrt[3]{10(-2+i\sqrt{6})} - \frac{8}{\sqrt{-4 + \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} + \sqrt[3]{10(-2+i\sqrt{6})}}}}$
b_2	$4 - \sqrt{-4 + \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} + \sqrt[3]{10(-2+i\sqrt{6})}}$ $+ \sqrt{-8 - \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} - \sqrt[3]{10(-2+i\sqrt{6})} - \frac{8}{\sqrt{-4 + \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} + \sqrt[3]{10(-2+i\sqrt{6})}}}}$
b_3	$4 + \sqrt{-4 + \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} + \sqrt[3]{10(-2+i\sqrt{6})}}$ $- \sqrt{-8 - \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} - \sqrt[3]{10(-2+i\sqrt{6})} + \frac{8}{\sqrt{-4 + \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} + \sqrt[3]{10(-2+i\sqrt{6})}}}}$
b_4	$4 + \sqrt{-4 + \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} + \sqrt[3]{10(-2+i\sqrt{6})}}$ $+ \sqrt{-8 - \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} - \sqrt[3]{10(-2+i\sqrt{6})} + \frac{8}{\sqrt{-4 + \frac{10^{2/3}}{\sqrt[3]{-2+i\sqrt{6}}} + \sqrt[3]{10(-2+i\sqrt{6})}}}}$

TABLE 3.3. First Coefficients for Padé 4,3. Notice that $C_{i,j}^n$ denotes $C_{n,i,j}$.

$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$C_{1,1}^1 = 11.302 - 12.472i$	$C_{1,1}^2 = -190.74 + 263.82i$	$C_{1,1}^3 = 4.4589 \times 10^3 - (6.2656 \times 10^3) i$	$C_{1,1}^4 = -115.53 \times 10^3 + (159.38 \times 10^3) i$	$C_{1,1}^5 = 3.1385 \times 10^6 - (4.2075 \times 10^6) i$
$C_{2,1}^1 = 11.302 + 12.472i$	$C_{2,1}^2 = -190.74 - 263.82i$	$C_{2,1}^3 = 4.4589 \times 10^3 + (6.2656 \times 10^3) i$	$C_{2,1}^4 = -115.53 \times 10^3 - (159.38 \times 10^3) i$	$C_{2,1}^5 = 3.1385 \times 10^6 + (4.2075 \times 10^6) i$
$C_{3,1}^1 = -13.302 - 60.072i$	$C_{3,1}^2 = 190.74 - (1.6004 \times 10^3) i$	$C_{3,1}^3 = -4.4589 \times 10^3 - (107.94 \times 10^3) i$	$C_{3,1}^4 = 115.53 \times 10^3 - (5.9355 \times 10^6) i$	$C_{3,1}^5 = -3.1385 \times 10^6 - (386.54 \times 10^6) i$
$C_{4,1}^1 = -13.302 + 60.072i$	$C_{4,1}^2 = 190.74 + (1.6004 \times 10^3) i$	$C_{4,1}^3 = -4.4589 \times 10^3 + (107.94 \times 10^3) i$	$C_{4,1}^4 = 115.53 \times 10^3 + (5.9355 \times 10^6) i$	$C_{4,1}^5 = -3.1385 \times 10^6 + (386.54 \times 10^6) i$
	$C_{1,2}^2 = -27.818 - 281.9i$	$C_{1,2}^3 = 1.702 \times 10^3 + (8.0407 \times 10^3) i$	$C_{1,2}^4 = -53.611 \times 10^3 - (218.88 \times 10^3) i$	$C_{1,2}^5 = 1.5214 \times 10^6 + (6.0288 \times 10^6) i$
	$C_{2,2}^2 = -27.818 + 281.9i$	$C_{2,2}^3 = 1.702 \times 10^3 - (8.0407 \times 10^3) i$	$C_{2,2}^4 = -53.611 \times 10^3 + (218.88 \times 10^3) i$	$C_{2,2}^5 = 1.5214 \times 10^6 - (6.0288 \times 10^6) i$
	$C_{3,2}^2 = -3.431 \times 10^3 + (1.598 \times 10^3) i$	$C_{3,2}^3 = -148.01 \times 10^3 + (14.744 \times 10^3) i$	$C_{3,2}^4 = -9.8289 \times 10^6 + (1.9663 \times 10^6) i$	$C_{3,2}^5 = -592.28 \times 10^6 - (78.803 \times 10^6) i$
	$C_{4,2}^2 = -3.431 \times 10^3 - (1.598 \times 10^3) i$	$C_{4,2}^3 = -148.01 \times 10^3 - (14.744 \times 10^3) i$	$C_{4,2}^4 = -9.8289 \times 10^6 - (1.9663 \times 10^6) i$	$C_{4,2}^5 = -592.28 \times 10^6 - (78.803 \times 10^6) i$
		$C_{1,3}^3 = -3.8301 \times 10^3 - (2.8389 \times 10^3) i$	$C_{1,3}^4 = 159.36 \times 10^3 + (92.86 \times 10^3) i$	$C_{1,3}^5 = -5.1888 \times 10^6 - (2.7083 \times 10^6) i$
		$C_{2,3}^3 = -3.8301 \times 10^3 + (2.8389 \times 10^3) i$	$C_{2,3}^4 = 159.36 \times 10^3 - (92.86 \times 10^3) i$	$C_{2,3}^5 = -5.1888 \times 10^6 + (2.7083 \times 10^6) i$
		$C_{3,3}^3 = 141.65 \times 10^3 + (184.89 \times 10^3) i$	$C_{3,3}^4 = 3.806 \times 10^6 + (11.594 \times 10^6) i$	$C_{3,3}^5 = 309.14 \times 10^6 + (805.23 \times 10^6) i$
		$C_{4,3}^3 = 141.65 \times 10^3 - (184.89 \times 10^3) i$	$C_{4,3}^4 = 3.806 \times 10^6 - (11.594 \times 10^6) i$	$C_{4,3}^5 = 309.14 \times 10^6 - (805.23 \times 10^6) i$
			$C_{1,4}^4 = -78.693 \times 10^3 + (15.684 \times 10^3) i$	$C_{1,4}^5 = 3.6989 \times 10^6 - (1.1725 \times 10^6) i$
			$C_{2,4}^4 = -78.693 \times 10^3 - (15.684 \times 10^3) i$	$C_{2,4}^5 = 3.6989 \times 10^6 + (1.1725 \times 10^6) i$
			$C_{3,4}^4 = 9.2225 \times 10^6 - (10.968 \times 10^6) i$	$C_{3,4}^5 = 807.27 \times 10^6 - (478.55 \times 10^6) i$
			$C_{4,4}^4 = 9.2225 \times 10^6 + (10.968 \times 10^6) i$	$C_{4,4}^5 = 807.27 \times 10^6 + (478.55 \times 10^6) i$
				$C_{1,5}^5 = -693.75 \times 10^3 + (1.1587 \times 10^6) i$
				$C_{2,5}^5 = -693.75 \times 10^3 - (1.1587 \times 10^6) i$
				$C_{3,5}^5 = -781.56 \times 10^6 - (408.12 \times 10^6) i$
				$C_{4,5}^5 = -781.56 \times 10^6 + (408.12 \times 10^6) i$



(a) Error backward-Euler with $n = 3$ (b) Error Padé $r_{4,3}$ with $n = 3$

FIGURE 3.2. Error comparison between Backward-Euler and Padé $r_{4,3}$ for the Curtiss-Hirschfelder equation

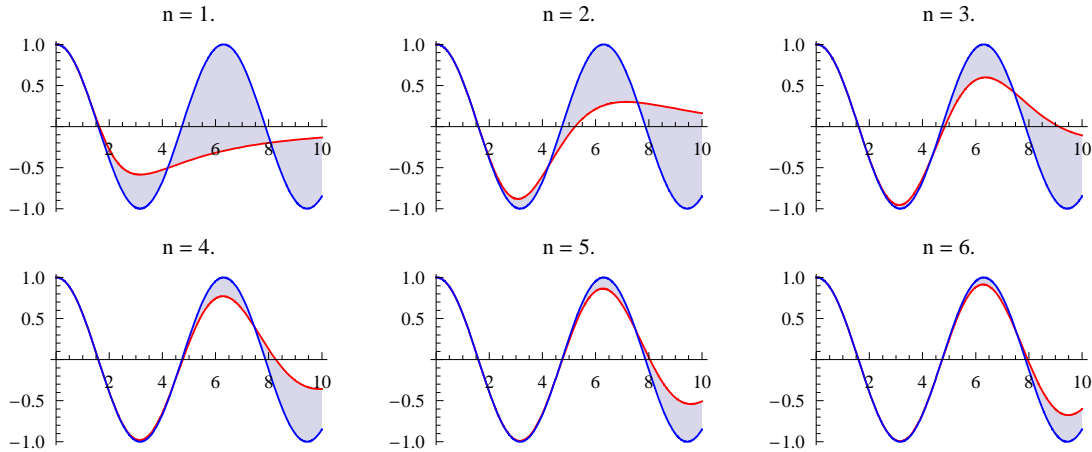


FIGURE 3.3. Approximation of the Curtiss-Hirschfelder differential equation by using the scheme associated with the Subdiagonal Padé approximant $r_{2,1}$

Example 3.2.6. The Stiff ODE proposed by C.F. Curtiss and J.O. Hirshfelder

$$y'(t) = -50(y - \cos(t)), \quad y(0) = 1,$$

with solution $y(t) = \frac{2500}{2501} \cos(t) + \frac{50}{2501} \sin(t) + \frac{1}{2501} e^{-50t}$ ($t \geq 0$) is known to be an excellent test for computational algorithms, cf. [54]. Figure 3.2-(a) shows the error of the backward-Euler approximation to the solution of the Curtiss-Hirschfelder equation for $t \in [0, 3]$ and $n = 3$. The backward-Euler approximation to the solution of the Curtiss-Hirschfelder equation is accurate up to zero decimal places for $t \in [0, 3]$ and $n = 3$. Figure 3.2-(b) shows that the error of the subdiagonal Padé scheme provided by $r_{4,3}$ yields an accuracy of at least five decimal places for $t \in [0, 3]$ with $n = 3$. In other words, if the b_i 's are the four roots of $840 + 480z + 120z^2 + 16z^3 + z^4$, then by using \hat{f} , $\frac{d}{d\lambda} \hat{f}$ and $\frac{d^2}{d\lambda^2} \hat{f}$ in (3.6) by means of evaluating them at just 12 points of the form $\frac{n}{t} b_i$ for each $t \in [0, 3]$, one ensures an error less than 2.5×10^{-6} on the interval $[0, 3]$.

Finally, the composite exponential approximation methods of Section 2.2 provide a scheme of order 8 for which one obtains a closed form by using Lemma 3.2.5. In

this case,

$$r_{\text{cea-8}}(z) = -1 + \frac{p_{\text{cea-8}}(z)}{q_{\text{cea-8}}(z)},$$

where $p_{\text{cea-8}}(z) = -124z^6 - 62880z^4 - 5184000z^2 - 45158400$ and $q_{\text{cea-8}}(z) = z^7 - 62z^6 + 1800z^5 - 31440z^4 + 355200z^3 - 2592000z^2 + 11289600z - 22579200$.

3.3 Integrated Semigroups and the Second Order Abstract Cauchy Problem

If A is an operator on a Banach space X and m is a non-negative integer, then A is the generator of a m -times integrated semigroup if there exists $\omega \geq 0$ and a strongly continuous function $S : \mathbb{R}_0^+ \rightarrow \mathcal{L}(X)$ such that $(\omega, \infty) \subset \rho(A)$ and

$$R(\lambda, A) = \lambda^m \int_0^\infty e^{-\lambda t} S(t) dt, \quad (\lambda > \omega). \quad (3.13)$$

In this case S is called a m -times integrated semigroup generated by A . If there exists $M, \omega \geq 0$ such that $\|\int_0^t S(s) ds\| \leq M e^{\omega t}$ then one says that the m -times integrated semigroup S is of type (M, ω) . If $m = 1$, one also uses the notion of once integrated semigroup. The following result can be found in [28, Thm. 2.5].

Theorem 3.3.1. *An operator A is the generator of a m -times integrated semigroup $\{S(t)\}_{t \geq 0}$ of type (M, ω) on a Banach Space X if and only if $W(t) = \frac{d^m}{dt^m} S(t) R(\mu, A)^m$ is a $R(\mu, A)^m$ -regularized semigroup of type (M, ω) on X . In this way,*

$$S(t) R(\mu, A)^m x = I^m [W](t) x. \quad (3.14)$$

Formally, (3.13) suggests that an m -times integrated semigroup is of the form $S(t) = I^m [T](t)$, where T is a semigroup on X . Since one can approximate strongly continuous semigroups and bi-continuous semigroups (with error estimates) by \mathcal{A} -stable rational functions which approximate the exponential, it is natural to ask if there is an integrated version of such schemes for integrated semigroups. V. Cachia shows in [12] that this is indeed the case for the backward Euler approximation of once integrated semigroups; i.e., for $r(z) = \frac{1}{1-z}$ and $m = 1$. However, his method does not provide error estimates for the approximation schemes. The following theorem complements Cachia's result by proving error estimates of integrated schemes for m -times integrated semigroups for smooth initial value. In this way, Cachia's convergence result can be extended with error estimates for sufficiently smooth initial data to m -times integrated semigroups and also for any \mathcal{A} -stable rational approximation of the exponential.

Theorem 3.3.2. *Let A be the generator of an m -times integrated semigroup S of type $(M, 0)$. If r is an \mathcal{A} -stable rational approximation of the exponential function of order q , then there exists $K > 0$ such that*

$$\left\| I^m \left[r^n \left(\frac{(\cdot)}{n} A \right) \right] (t) y - S(t) y \right\| \leq MK \frac{t^{k+m-\theta_q(k)}}{k+m} \left(\frac{t}{n} \right)^{\theta_q(k)} \|A^{k+m} y\|, \quad (3.15)$$

for every $k \in \{0, \dots, q+1\}$, $k \neq \frac{q-1}{2}$, and $y \in D(A^{k+m})$. If $k = \frac{q-1}{2}$ then one obtains an extra $\ln(n+1)$ in (3.15).

Proof. Without loss of generality we can assume that $\mu = 0$ in Theorem 3.3.1 by translating A if necessary. Let $y \in D(A^{k+m})$. Then there exist $x \in X$ such that $A^m y = x$, and the result follows by integrating (2.34) applied to $R(0, A)^m x$ and (3.14). \square

In particular, one can approximate solutions of the second order Cauchy problem associated with generators of cosine functions. Let A be a closed operator on X and assume that A generates a cosine function (see [4] for definitions and the basic properties of cosine functions and their generators). Let $x \in X$, $y \in X$, and consider the second order Cauchy problem

$$\text{ACP}^2(x, y) \begin{cases} u''(t) = Au(t) & (t \geq 0) \\ u(0) = x, \\ u'(0) = y. \end{cases}$$

By transforming the problem $\text{ACP}^2(x, y)$ into a first order system one obtains that $\text{ACP}^2(x, y)$ is equivalent to the Cauchy problem on the Banach space $X \times X$, with norm $\|(x, y)\|_{X \times X} = \|x\|_X + \|y\|_X$, given by

$$\begin{cases} v'(t) = \mathcal{A}v(t) & (t \geq 0) \\ v(0) = (x, y), \end{cases}$$

where $D(\mathcal{A}) := D(A) \times X$, and

$$\mathcal{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Moreover, if $\lambda^2 \in \rho(A)$ then $\lambda \in \rho(\mathcal{A})$ and

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} \lambda R(\lambda^2, A) & R(\lambda^2, A) \\ AR(\lambda^2, A) & \lambda R(\lambda^2, A) \end{pmatrix}. \quad (3.16)$$

One knows that the operator A generates a cosine function on X is equivalent for the operator \mathcal{A} to generate a once integrated semigroup $\{S(t)\}_{t \geq 0}$ on $X \times X$, see [4, Thm. 3.14.7]. Thus, \mathcal{A} generates a $R(\mu, \mathcal{A})$ -regularized semigroup of type (M, ω) for any $\mu \in \rho(\mathcal{A})$. Now, without loss of generality one can assume that $\omega = 0$, otherwise $\mathcal{A} - \omega$ generates a $R(\mu, \mathcal{A} - \omega)$ -regularized semigroup of type $(M, 0)$. Theorem 2.4.2 yields approximations to the solutions of the Cauchy problem

$$\begin{cases} v'(t) = \mathcal{A}v(t) & (t \geq 0) \\ v(0) = R(\mu, \mathcal{A})(z, w), \end{cases}$$

for any $(z, w) \in D(\mathcal{A})$.

If, in particular, one considers the rational function $r(z) = \frac{1}{1-z}$ for $\text{Re}(z) \leq 0$, then r is an approximation of order $q = 1$ of the exponential function. Now, if $x \in D(A^2)$

and $y \in D(A)$, then there exist $(z, w) \in D(\mathcal{A})$ such that $R(\mu, \mathcal{A})(z, w) = (x, y)$, since $\text{Im}(R(\mu, \mathcal{A})) = D(\mathcal{A})$. Therefore, by considering $k = 1$ in Theorem 2.4.2 one obtains that there exists $K > 0$ such that the solution $u(\cdot, x, y)$ of the second order problem $P^2(x, y)$ can be approximated by $r^n(\frac{t}{n}\mathcal{A})$, which corresponds to the Backward-Euler scheme, in the following way

$$\left\| \left(\frac{n}{t}\right)^n V^n \left(\frac{n}{t}, \mathcal{A}\right) (x, y) - u(t, x, y) \right\| \leq MK \frac{t}{\sqrt{n}} \|\mathcal{A}(z, w)\|, \quad (3.17)$$

where

$$\begin{aligned} V^n(\lambda, \mathcal{A})(x, y) &= \sum_{j=1}^m \binom{2m-1}{2j-1} \lambda^{2j-1} A^{m-j} R(\lambda^2, A)^n x \\ &\quad + \sum_{j=0}^{m-1} \binom{2m-1}{2j} \lambda^{2j} A^{m-j-1} R(\lambda^2, A)^n y, \end{aligned}$$

for n odd; i.e. $n = 2m - 1$, and

$$\begin{aligned} V^n(\lambda, \mathcal{A})(x, y) &= \sum_{j=0}^m \binom{2m}{2j} \lambda^{2j} A^{m-j} R(\lambda^2, A)^n x \\ &\quad + \sum_{j=1}^m \binom{2m}{2j-1} \lambda^{2j-1} A^{m-j} R(\lambda^2, A)^n y, \end{aligned}$$

for n even; i.e. $n = 2m$.

Notice that by knowing an explicit form of the backward-Euler scheme, one can derive formulas for any other \mathcal{A} -stable rational function considered in Theorem 2.4.2 by decomposing r into partial fractions. Thus, the Crank-Nicholson scheme giving by $r(z) = \frac{2+z}{2-z}$ for $\text{Re}(z) \leq 0$ which is an approximation of order $q = 2$ can be obtained. In the case of the second order problem, with $k = 3$, the Crank-Nicholson scheme approximates the solution u in the following way

$$\left\| U^n \left(\frac{n}{t}, \mathcal{A}\right) (x, y) - u(t, x, y) \right\| \leq MK \frac{t}{n^2} \|\mathcal{A}^3(z, w)\|, \quad (3.18)$$

where $U^n \left(\frac{n}{t}, \mathcal{A}\right) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \left(\frac{4n}{t}\right)^k V^k \left(\frac{2n}{t}, \mathcal{A}\right)$.

In the particular case of generators of once integrated semigroups, it is known that there exists a Banach space V such that $D(A) \hookrightarrow V \hookrightarrow X$ and such that the part \mathcal{A} in $V \times X$ generates a strongly continuous semigroup, see Theorem 3.14.11 of [4]. However, even though one obtains the same estimates than (3.17) and (3.18) when applying the results obtained by P. Brenner and V. Thomée for strongly continuous semigroups, the constant K depends also on the phase space V which is an abstract space. Theorem 2.4.2 shows that the constant K of (3.17) and (3.18) depends only on the \mathcal{A} -stable rational function r and it is actually independent of the phase space.

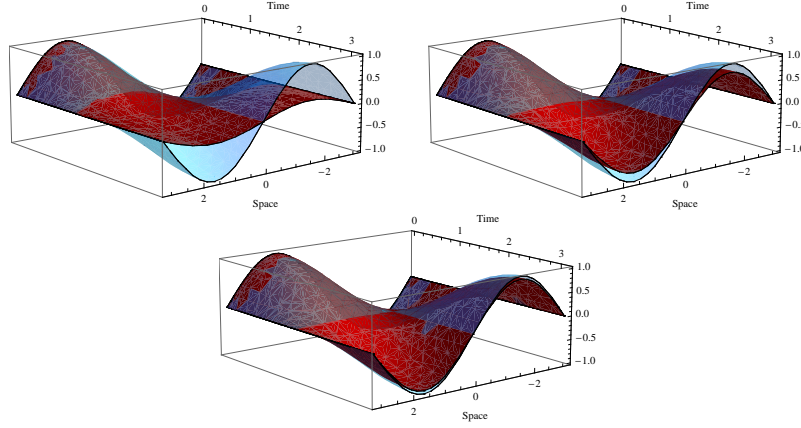


FIGURE 3.4. Backward Euler scheme for the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial s^2}$ for $n = 3$ (top left), $n = 25$ (top right), and $n = 75$ (bottom). The red surface is the approximation and the cyan surface is the solution $u(t, s) = \cos(t) \sin(s)$.

Finally, in order to illustrate the backward-Euler method, one can consider the one dimensional wave equation on the Banach space of bounded continuous functions $C_b(\mathbb{R})$ with the uniform norm. If $A = \frac{d^2}{ds^2}$ and $X = C_b(\mathbb{R})$, then \mathcal{A} generates a once integrated semigroup (see [50]). The following figures show the Backward-Euler approximation (3.17), with time $t \in [0, \pi]$, for the second order problem associated with A and initial conditions given by $x = \sin$, and $y = 0$ with solution $u(t, s) = \sin(s) \cos(t)$.

3.4 Approximating Solutions of Volterra Integral Equations of Scalar Type

During the last thirty years, the theory of abstract Volterra equations has been developed by several different authors. The range of applications of the theory is wide, see [91] for a comprehensive treatment of abstract Volterra integral equations. This section indicates the particular form of the results obtained in Section 3.2 when applied to approximate solutions of Volterra integral equations of scalar type. However, since the generation theorems for both scalar and nonscalar type of Volterra integral equations heavily rely on Laplace transform theory, one would expect similar results for the nonscalar case. We follow [91] for the first results and definitions concerning the theory.

Let X be a complex Banach space and let A be a closed linear operator in X with dense domain $D(A)$. Let $a \in L^1_{loc}(\mathbb{R}_0^+)$ be a non-zero scalar kernel. We consider the Volterra equation

$$u(t) = g(t) + \int_0^t a(t-s)Au(s)ds, \quad (t \in J) \quad (3.19)$$

where $g \in C(J; X)$ and $J = [0, T]$. We denote by X_A the Banach space consisting of the domain $D(A)$ of A provided with the graph norm $\|x\|_A := \|x\| + \|Ax\|$,

where $\|\cdot\|$ denotes the norm of X . We say that u is a strong solution of (3.19) if $u \in C(J; X_A)$ and (3.19) holds on J . Furthermore, we say that (3.19) is well-posed if (i) for each $x \in D(A)$ there is a unique strong solution $u(t; x)$ on \mathbb{R}_0^+ of

$$u(t) = x + (a * Au)(t), \quad t \geq 0,$$

and (ii) if $\{x_n\}_{n \in \mathbb{N}} \subset D(A)$ is such that $x_n \rightarrow 0$ as $n \rightarrow \infty$, then $u(t, x_n) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on compact intervals.

Definition 3.4.1. A Volterra resolvent operator is a map V from $[0, \infty)$ into $\mathcal{L}(X)$ that satisfies the following conditions.

- (i) V is strongly continuous on \mathbb{R}_0^+ and $V(0) = I$;
- (ii) V commutes with A . i.e., $V(t)D(A) \subseteq D(A)$ and $V(t)Ax = AV(t)x$ for all $x \in D(A)$ and $t \geq 0$;
- (iii) V satisfies that

$$V(t)x = x + \int_0^t a(t-s)AV(s)x ds \quad \text{for all } x \in D(A) \text{ and } t \geq 0. \quad (3.20)$$

Notice that if (3.19) is well-posed, then $V(t)x := u(t, x)$ is a Volterra resolvent operator. Furthermore, the reciprocal is also true.

Proposition 3.4.2. *The problem (3.19) is well-posed if and only if (3.19) admits a resolvent operator. Moreover,*

$$V(t)x = x + A \int_0^t a(t-s)V(s)x ds \quad \text{for all } x \in X \text{ and } t \geq 0. \quad (3.21)$$

See [91, Prop. 1.1] for a proof. Similarly than before, we say that a Volterra resolvent operator is of type (M, ω) if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|V(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$.

Theorem 3.4.3. *Let A be a closed linear operator in X with dense domain $D(A)$ and let $a \in L_{loc}^1(\mathbb{R}_0^+)$ be such that $\int_0^\infty e^{-\omega t}|a(t)|dt < \infty$. Then (3.19) admits a Volterra resolvent operator V of type (M, ω) if and only if the following conditions hold.*

- (i) $\widehat{a}(\lambda) \neq 0$ and $\frac{1}{\widehat{a}(\lambda)} \in \rho(A)$ for all $\lambda > 0$.
- (ii) $H(\lambda) = \frac{1}{\lambda}(I - \widehat{a}(\lambda)A)^{-1}$ satisfies that

$$|H^{(n)}(\lambda)| \leq Mn!(\lambda - \omega)^{-(n+1)}, \quad \text{for all } \lambda > \omega \text{ and } n \in \mathbb{N}.$$

For a proof, see [91, Thm. 1.3]. Now, the generation Theorem 3.4.3 is based on a purely Laplace transform approach by showing that those conditions guarantee the existence of \widehat{V} and

$$\widehat{V}(\lambda) = H(\lambda) \text{ for all } \lambda > \omega. \quad (3.22)$$

In this way, the generation Theorem 3.4.3 together with Theorem 3.2.1 provide a direct application for Volterra integral equations of scalar type.

Theorem 3.4.4. *Let r be an \mathfrak{A} -stable rational approximation to the exponential of order $q \geq 1$. If (3.19) admits a Volterra resolvent operator V of type $(M, 0)$ and $u(t) := V(t)x$ is $(q + 1)$ -times differentiable, then there exist $K > 0$, $b_i \in \mathbb{C}$ with $\operatorname{Re}(b_i) > 0$ and constants $C_{n,i,j} \in \mathbb{C}$ (independent of u) such that*

$$\left\| C_{n,0,0}u_0 + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq nr}} C_{n,i,j} \left(\frac{n}{t}\right)^j \frac{(-1)^{j-1}}{(j-1)!} \widehat{u}^{(j-1)} \left(\frac{n}{t}b_i\right) - u(t) \right\|_X \leq K \frac{t^{q+1}}{n^q} \|u^{(q+1)}\|_\infty. \quad (3.23)$$

Notice that the availability of the inversions for bounded (exponentially bounded) continuous functions via the bi-continuous shift semigroup are crucial since without them one would have to assume that the resolvent operators are uniformly continuous in order to be able to apply the (strongly continuous) shift on $C_{\text{bu}}([0, \infty), X)$.

Example 3.4.5. : The well-posed abstract Cauchy problem. Let $a(t) := 1$ for $t > 0$ and $a(0) = 0$. Clearly $a \in \text{NBV}_{\text{loc}}$ and if $g(t) := x$ for a fix $x \in D(A)$, then (3.19) becomes the well-posed abstract Cauchy problem $u'(t) = Au(t)$, $u(0) = x \in D(A)$. Theorem 3.4.4 shows that

$$u(t) = \lim_{n \rightarrow \infty} C_{n,0,0}u_0 + \sum_{\substack{1 \leq i \leq s \\ 1 \leq j \leq nr}} C_{n,i,j} \left(\frac{n}{t}\right)^j \frac{(-1)^{j-1}}{(j-1)!} \widehat{u}^{(j-1)} \left(\frac{n}{t}b_i\right),$$

where the limit is uniform on compact intervals of \mathbb{R}_0^+ .

Finally, we would like to point out that the assumption of Theorem 3.4.4 concerning the $(q + 1)$ -times differentiability of the solution u is not trivial and further research will be necessary in order to overcome this situation.

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Appendix: A Mathematica Code for the Numerical Inversion of the Laplace Transform

This Mathematica 6 code implements the results described in Section 3.2 for inverting the Laplace transform for the one dimensional case by using the subdiagonal Padé approximants.

```
(*Numerical Inversion of the Laplace Transform*)
deg := 0; (*degree of p(x)*)
n := 5; (*Power of r^n*)
m := 1000; (*decimal places of precision*)
degq := deg + 1;
pd := PadeApproximant[Exp[x], {x, 0, {deg, deg + 1}}];
e[x_] = Numerator[pd];
f[x_] = Denominator[pd];
laconstante = Derivative[degq][f][x]/(degq!);
p[x_] = e[x]/laconstante;
q[x_] = f[x]/laconstante;
Asubi := Array[A, deg + 1];
Productsubi := Array[produ, deg + 1];
B = x /. Solve[q[x] == 0];
(*Numerical Precision of the roots replace
Solve by NSolve[q[x]==0,m]*)
const = Denominator[Factor[q[x]]];
r[x_] = ExpandDenominator[
ExpandNumerator[Together[Pade[Exp[x], {x, 0, deg, deg+ 1}]]]
];
For[i = 1, i < deg + 2, i++,
  Var = 1;
  For[j = 1, j < deg + 2, j++,
    If[i != j,
      Var = Var*(B[[j]] - B[[i]])
    ]
  ];
  produ[i] = Var;
];
For[i = 1, i < deg + 2, i++,
  A[i] = p[B[[i]]]/produ[i]
];
CoeffA := Array[Coef, {n, degq, n}];
For[i = 1, i <= n, i++,
  For[j = 1, j <= degq, j++,
    For[k = 1, k <= n, k++,
```

```

    Coef[i, j, k] = 0]
  ]
];
For[j = 1, j <= degq, j++, Coef[1, j, 1] = (-1)^deg*A[j]];
For[i = 1, i <= n - 1, i++,
  For[k = 1, k <= i, k++,
    For[j = 1, j <= degq, j++,
      For[l = 1, l <= degq, l++,
        If[l != j,
          For[m = 1, m <= k, m++,
            variable = ((-1)^(m + 1)*((-1/(B[[j]] - B[[l]])))^m*
              Coef[i, j, k]*Coef[1, l, 1]);
            Coef[i + 1, j, k + 1 - m] =
              Coef[i + 1, j, k + 1 - m] + variable
              (*variable=0*)
          ];
          Coef[i + 1, l, 1] =
            Coef[i + 1, l,
              1] + ((-1)^(k)*((-1/(B[[j]] - B[[l]])))^k)*Coef[i, j, k]*
              Coef[1, l, 1]),
          Coef[i + 1, j, k + 1] =
            Coef[i + 1, j, k + 1] + (Coef[i, j, k]*Coef[1, l, 1])
        ]
      ]
    ]
  ]
];
fhat[s_] := (((50*s)/(1 + s^2)) + 1)/(s + 50);
(*Laplace Transform of Curtiss-Hirshfelder Stiff ODE*)
For[i = 0, i <= n, i++, Derivada[i] = Derivative[i][fhat][y]];
Approx[t_, numero_] :=
  Sum[Coef[numero, j,
    k]*((numero/t)^k)*((-1)^(k - 1))/
    Factorial[k - 1]*(Derivada[k - 1] /.
    y -> ((numero/t)*B[[j]])), {k, 1, numero}, {j, 1, degq}];

```

Vita

Patricio Gabriel Jara was born in February of 1975, in Santiago, Chile. He finished his undergraduate studies at the Universidad de Santiago de Chile in December of 1999. He earned a Master of Science Degree in mathematics from the Universidad de Santiago de Chile in September 2001. In August of 2002 he came to Louisiana State University and earned a Master of Science in mathematics in 2004. Currently, he is a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2008.