METHOD OF RIEMANN SURFACES IN MODELLING OF CAVITATING FLOW

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To Alextina and Sophia
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Abstract

This dissertation is concerned with the applications of the Riemann-Hilbert problem on a hyperelliptic Riemann surface to problems on supercavitating flows of a liquid around objects. For a two-dimensional steady irrotational flow of liquid it is possible to introduce a complex potential \( w(z) \) which allows to apply the powerful methods of complex analysis to the solution of fluid mechanics problems. In this work problems on supercavitating flows of a liquid around one or two wedges have been stated. The Tulin single-spiral-vortex model is employed as a cavity closure condition. The flow domain is transformed into an auxiliary domain with known boundaries using the conformal mapping method. After that the problems have been reduced to the solution of Riemann-Hilbert problems on elliptic or hyperelliptic Riemann surfaces. The final step is to solve a system of transcendental equations which is accomplished numerically. The numerical results are presented. To the best of the author’s knowledge no numerical results were available for non-linear problems on supercavitating flows in multiply connected domains before.
Chapter 1

Introduction

It is easy to see that when an object moves in a stream of liquid, the flow separates and a wake appears behind the object. This is especially true for sharp-cornered (not streamlined) objects. Under special conditions the wake becomes a vapor-filled area or a cavity. The bounding streamlines of the cavity are free streamlines; their location is initially unknown and needs to be found as a part of the solution. Two conditions are necessary for the cavitation to appear. Firstly, the cavitation appears in the areas of liquid where the pressure of liquid approaches its vapor pressure. Secondly, the cavitation bubbles need a surface on which they can grow and nucleate. Cavitation is generally an undesirable phenomenon in engineering, because the cavity periodically collapses and the resulting shock wave may cause damage to the cavitating body, loss of efficiency, corrosion, vibration and noise. In some cases, however, the cavitation is inevitable. If a cavity length becomes large compared to the dimensions of the cavitating object, and the cavity extends beyond the object, this flow regime is called a supercavitation. Supercavitation has its benefits: the bubble of gas reduces the drag on the object allowing it to achieve higher speeds. Since the vapor or gas that fills the cavity has low density compared to the fluid it is reasonable to assume that the pressure $p_c$ inside of the cavity is constant. One of the important parameters of the cavitation is a cavitation number $\sigma$ defined as

$$
\sigma = \frac{p_\infty - p_c}{\frac{1}{2} \rho v_\infty^2}.
$$(1.1)

Here and further, $p_\infty$ and $v_\infty$ are the pressure and the velocity of the stream of liquid at infinity, $\rho$ is the density of the liquid and $p_c$ and $v_c$ are the pressure inside of the cavity and the speed on the cavity boundary. The cavitation number is a basic parameter of the flow and must be assigned a priori. We can assume [BZ57]:

- $p = p_\infty$ on the free boundary of the flow which is not the boundary of the cavity (such as boundaries of the jet, etc);
- $p = p_c$ on the boundary of the cavity;
- $p \geq p_c$ inside of the liquid.

Because of the last condition the gradient of the pressure on the cavity boundary should be directed away from the cavity into the liquid and, hence, the cavity must be convex [BZ57].
Most of the available cavitation models assume inviscid, irrotational and incompressible flows. Experiments have shown that these are indeed reasonable assumptions. Some comparison with experimental results has been given in [TH80]. The flow of the ideal liquid is governed by the Euler equations:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho},
\]

where \( \mathbf{v} \) is the velocity, \( p \) is the pressure and it is assumed that there is no body forces acting in the fluid.

Additionally a continuity equation for an incompressible liquid must be satisfied:

\[
\nabla \cdot \mathbf{v} = 0.
\]

If the flow is irrotational, i.e. \( \nabla \times \mathbf{v} = 0 \) at all points of liquid, then there is a velocity potential \( \varphi \), such that

\[\mathbf{v} = \nabla \varphi.\]

Hence, the continuity equation (1.3) is satisfied if the function \( \varphi \) satisfies the Laplace equation:

\[\nabla^2 \varphi = 0.\]

Using the velocity potential we can integrate the equation (1.2) to obtain the Bernoulli equation:

\[
\frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + \frac{1}{2} |\mathbf{v}|^2 = F(t).
\]

In what follows we consider a steady plane (2-dimensional) flow of liquid. Hence, the term \( \partial \varphi/\partial t \) can be dropped in the last equation, and the right-hand side is constant on the streamlines. Since the flow is irrotational, the right-hand side is constant everywhere in the flow domain:

\[
\frac{p}{\rho} + \frac{1}{2} |\mathbf{v}|^2 = \text{constant everywhere.}
\]

Observe that from the equation (1.4) and the fact that the pressure is constant on the boundary of the cavity it follows that the absolute value of the velocity \( \mathbf{v} \) is constant on the free boundaries of the cavity.

In the case of a two-dimensional steady and incompressible flow we can define a stream function \( \psi \), which is a complementary function to \( \varphi \):

\[u_1 = \frac{\partial \psi}{\partial y}, \quad u_2 = -\frac{\partial \psi}{\partial x},\]

where \( u_1 \) and \( u_2 \) are the components of the velocity vector \( \mathbf{v} = \{u_1, u_2\} \). The streamlines of the flow are defined by the condition \( \psi(x,y) = \text{const.} \)

It is easy to see that the velocity potential \( \varphi \) and the stream function \( \psi \) satisfy the Cauchy-Riemann conditions:

\[
\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}.
\]
and so we can define an analytic function \( w(z) \) in the flow domain which is called a complex potential of the flow:

\[
    w(z) = \varphi(x, y) + i\psi(x, y) \tag{1.5}
\]

This allows us to use the powerful machinery of complex analysis to solve the cavitation problems. The complex potential \( w(z) \) has the following important property:

\[
    w'(z) = u_1 - iu_2. \tag{1.5}
\]

Thus, the derivative of the complex potential is the conjugate of the velocity at the point \( z \) of the liquid. One of the useful functions in the study of cavitation is the logarithmic hodograph variable which is defined as

\[
    \omega(z) = \ln \left( \frac{|w'(z)|}{v_\infty} \right) = \ln \left( \frac{|w'(z)|}{v_\infty} \right) + i \arg w'(z). \tag{1.6}
\]

This function has several convenient properties. Since \( |w'(z)| \) defines the absolute value of the velocity at the point \( z \) and \( -\arg w'(z) \) defines the direction of the velocity vector at the point \( z \), it follows that the real part of the function (1.6) is known on the free boundary of the flow and the imaginary part of the function (1.6) is known on the solid boundary.

Consider a supercavitating object in the stream of liquid (fig. 1.1). Observe that at the front stagnation point the stream of liquid separates and the velocity must vanish. There are two essential questions to ask about a supercavitating flow: where does the cavity begin and how does it end? Sometimes the detachment of the cavity is dictated by the geometry of the cavitating object; this type of detachment is called an abrupt detachment (fig. 1.1). If the cavitating object is smooth, the point of detachment can not be prescribed a priori and needs to be found as a part of the solution. Different types of the conditions have been proposed to find the detachment point of a cavity. According to the Villat-Armstrong criterion, the detachment point is the point of minimum pressure on the wall of the supercavitating object [BZ57], [Bre95], [FM04]. This is equivalent to the assumption that the detachment position is determined by the non-singular behavior of the mathematical solution. This criterion is also called the smooth
detachment criterion; the curvature of the cavity and the object are equal at the detachment point. Another criterion is the laminar separation criterion. For more details the reader is referred to [FM04]. In this work we will always assume the abrupt detachment of the cavity.

Another important issue to address is the cavity closure point. At the cavity closure point two free streamlines which are the boundaries of the cavity must join and, hence, the velocity must vanish just as at the front stagnation point. The following statement is known as the Brillouin paradox [BZ57]:

**Theorem 1.0.1.** For the cavity of the finite size the speed is constant on the boundary of the cavity and the stagnation point can not exist on the boundary of the cavity.

In another words, the cavity can not be closed in the ideal liquid model. This paradox appears because of the assumption that the flow is steady at the tail part of the cavity. Though this assumption is mostly correct for the flow in the front part of the cavity, the flow at the tail part of the cavity is almost always turbulent and unsteady. However, steady models are still appropriate provided they are interpreted as describing the mean flow [RF67]. It is therefore necessary to make some artificial assumptions in order to model the flow at the tail part of the cavity. A number of the cavity closure models have been devised for this purpose, each of them has its own advantages and disadvantages.

Let us discuss some of the most common models in more details.

The first and the simplest model belongs to Kirchhoff (1869). In this model a semi-infinite wake exists behind a body, the pressure inside of the cavity is equal to the pressure at infinity $p_\infty$, and the speed of the liquid on the boundary of the cavity is the same as the speed at infinity $v_\infty$ (fig. 1.2a). Since the drag on the cavitating body is non-zero, this model has been initially proposed by Kirchhoff to avoid the D’Alambert paradox. This flow corresponds to the zero cavitation number $\sigma = 0$. The Kirchhoff model leads to relatively simple mathematical problems. The disadvantage of this model is in the unrealistically thick semi-infinite wake behind the cavitating object. Another disadvantage is that the pressure in the cavity is equal to the free streamline pressure while in real flows the cavity pressure is usually lower than the free streamline pressure.

The Kirchhoff model has been modified by Joukowski and Roshko to include the non-zero cavitation number. In this model the cavity terminates on two parallel plates
where the speed of the stream changes from $v_c$ to $v_\infty$ (fig. 1.2b). This model leads to simple mathematical problems but it assumes the existence of the artificial plates at the tail-part of the cavity. Another modification of the “open-wake” model has been proposed by T. Wu [Wu62], [WW63] (fig. 1.2c). In this model the cavitational flow is approximately described by an infinitely long wake which consists of the near-wake of constant underpressure $p_c$ and a far-wake trailing downstream. The pressure increases continuously from $p_c$ back to its free-stream value $p_\infty$ along the far-wake.

Ryabushinski proposed the scheme in which the cavity terminates on the “image body” (fig. 1.3a). The advantage of this model is in the simplicity of geometry and mathematical solution. The disadvantage is that the streamlines downstream are essentially the images of those upstream. There is a modification of this model in which the free boundaries of the cavity terminate on the short vertical plate (fig. 1.3b). This model was originally used by Geurst [Geu61]. In both schemes the critical point at the tail of the cavity is removed from the surface of the liquid onto the solid boundary so that the Brillouin paradox is no longer valid.

The reentrant jet model proposed by Efros and Gilbarg [Efr46], [Gil60], [GiRo46] has been widely used in the literature (fig. 1.4). In this model the cavity ends in the reentrant jet which flows back into the cavity. The flow can be mathematically imagined as the flow on the two-sheeted Riemann surface made from two copies of the extended complex plane with a cut along the cavitating object. The banks of the cuts on two sheets of the surface are joined criss-cross. The reentrant jet flows from the inside of the cavity onto the second sheet of the Riemann surface. The critical (stagnation) point in this model is removed from the boundary of the cavity into the body of the liquid. Disadvantages of this model are that the liquid is unrealistically removed from the flow and that the solution depends on the choice of the direction of the reentrant jet. It is necessary to note that the reentrant jet does indeed appear in the real cavitating flows though it does not have a nice organized structure of the jet in the Efros-Gilbarg model, but more resembles a turbulent mass of liquid flowing back into the cavity [Bre95].
In 1964 Tulin proposed two cavity closure models for a non-zero cavitation number \( \sigma \neq 0 \) [Tul64]. In both of these schemes the cavity terminates in two spirals from which emerge the boundary streamlines of the wake.

In the single-spiral-vortex model the complex potential has the following singularity at the centers of the spirals \( C^\pm \):

\[
\ln\left(\frac{dw}{dz}\right) \sim -M (w - w_0)^{-1/2}, \quad z \to C^\pm , \tag{1.7}
\]

where \( M > 0 \) is a real parameter, \( w_0 \) is the value of the complex potential at the points \( C^\pm \). It needs to be noted, that in the original Tulin’s paper [Tul64] the parameter \( M \) has been taken to be \( M = -1 \). This allows to make a circulation \( \Gamma \) around any curve enclosing the supercavitating object and the cavity to be zero. The singularity in the form (1.7) has been proposed by Terent’ev [Ter76]. In this case the circulation \( \Gamma \) is no longer zero, and the presence of additional parameter \( M \) makes it possible to satisfy the condition

\[
\oint_L dz = 0, \tag{1.8}
\]

which means that the object together with the cavity can be enclosed by one continuous closed curve \( L \). The original Tulin single-spiral-vortex model also possessed a wake behind the cavity which is absent in its modification.

The streamlines of the flow cover the plane multiple times near the points \( C^\pm \). The flow can be thought of as a flow on the half of the Riemann surface of the logarithm \( \ln((z - C^+)(z - C^-)) \). This surface is obtained by taking infinitely many copies of the extended complex plane with a cut along the segment \( C^+ C^- \). The left bank of the cut on the first sheet of the surface is glued with the right bank of the cut on the second sheet of the surface, the left bank of the cut on the second sheet is glued with the right bank of the cut on the third sheet and so on. The part of this surface near the point \( C^\pm \) is shown on the fig. 1.6. The streamline which is sufficiently close to the boundary of the cavity reaches the cut \( C^+ C^- \) and moves on the second sheet of the surface, makes a turn around the point \( C^+ \) or \( C^- \) and moves onto the third sheet and so on. After making the finite number of turns the streamline spirals back and returns to the first sheet of the surface. The boundary streamline of the cavity (denoted with “0” on the fig. 1.7) spans all the sheets of the Riemann surface. It consist of two boundaries of the cavity and the line which emerges from the inside of the cavity. The schematic picture of the flow at the tail part of the cavity is shown on the fig. 1.7.

Consider the Tulin double-spiral-vortex model. In this model the boundary streamlines of the cavity terminate in spirals at the points \( C^+ \) and \( C^- \) and from those points
Figure 1.6: Neighborhood of the point $C^\pm$ on the logarithmic surface $\ln((z - C^+)(z - C^-))$.

Figure 1.7: The flow at the tail part of the cavity for the Tulin single-spiral-vortex model.
emerge the boundary streamlines of the wake. The flow near the points $C^\pm$ can be thought of as the flow on the previously mentioned Riemann surface of the function \( \ln((z - C^+)(z - C^-)) \). The spirals first curl inward passing from one sheet of the surface onto another sheet. At the centers $C^\pm$ of the spirals the velocity jumps from $v_c$ to $v_\infty$ and the streamlines spiral outward and return to the flow domain. Thus, as in the previous model the flow domain in the vicinity of the points $C^\pm$ is covered by the streamlines multiple times. The complex potential in this model has the following singularities at the points $C^\pm$:

\[
\ln(\frac{dw}{dz}) \sim iM \ln(w - w_0), \quad z \to C^\pm,
\]

where $M$ is a real parameter and $w_0$ is the value of the complex potential at the points $C^\pm$. This model turned out to be very convenient for studying supercavitating flows with multiple free surfaces. The main advantage of this model is that the flow domain is always simply connected, so that the auxiliary domain used in the conformal mapping method is also simply connected which significantly simplifies the solution of the mathematical problems.

Note that most of these schemes give similar results for symmetric flows of liquid. However, for non-symmetric flows an additional parameter is present in the solution such as, for instance, the direction of the reentrant jet in the Efros-Gilbarg model. This parameter is usually fixed using a zero-circulation condition or some other artificial condition which can not be explained from the mechanical viewpoint. This parameter does not appear in the Tulin single- and double-spiral-vortex models. Thus, the solution of the cavitation problem is determined exclusively by the valid physical conditions.

Let us briefly mention some of the important results in the cavitation problems.

The book [BZ57] contains the basic methods of complex analysis in the ideal fluid flow including the hodograph method, the method of conformal mapping and the method of continuation by symmetry. The cavitation flows have been studied in the book mostly for the Kirchhoff (zero-cavitation-number) cavity closure model. The book [Bre95] describes different stages and types of cavitation, including the cavitation inception, bubble cavitation, vortex, cloud and sheet cavitation and supercavitation. Large attention is also paid to the cavity detachment and cavity closure conditions. The main ideas of the complex variable approach to the cavitating flows have been described in [Gur79]. The solution for a supercavitating or a partially cavitating flow around a hydrofoil under the Tulin single-spiral-vortex cavity closure condition has been presented. The book by Rozhdestvenskii [Roz77] contains a number of useful classical results about supercavitating flows around a hydrofoil with different cavity closure conditions. The models considered include the Kirchhoff model, the Efros-Gilbarg model, the Tulin single- and double-spiral-vortex models. The problems have been solved using the method of conformal mappings and the Riemann-Hilbert boundary value problem on the complex plane.

The Tulin single-spiral-vortex model has been extensively studied by A.G. Terent’ev and his collaborators. A number of problems have been published in the book [Ter81], including problems for a symmetrical supercavitating flow around a wedge in the infinite plane, in the channel with rigid walls or in the jet. The solutions of these problems have been found in the closed form using the conformal mapping from the semicircle or a rectangle onto the flow domain. It is necessary to note that due to the symmetry of
the problem the flow domain is essentially simply connected in all of these cases. The problem for a hydrofoil in the infinite plane of liquid has been studied using the conformal mapping from the first quadrant onto the flow domain. The problem for a hydrofoil under a free surface has been also considered in the book. The solution involves a conformal mapping from a rectangle onto the flow domain. The resulting formulas have been given in terms of elliptic theta-functions. The numerical results are provided for small angles of attack. The solutions to the problems for a partially cavitating hydrofoil or a cascade of hydrofoils has been also given in the book. The linearized problem for a cavitating cascade of hydrofoils has been also presented in [Bre95]. A cascade of bluff hydrofoils with the Tulin double-spiral-vortex cavity closure condition has been studied in [WI91]. The solution has been given in explicit form, the numerical results have been presented. The problem for a supercavitating flow around a curvilinear wedge has been studied in [Ter81]. The cavity detachment points are determined from the smooth detachment criterion. The solution presented exploits the method of Levi-Civita. The unknown function has been sought as a sum of two functions, where one of the functions is the solution for a wedge with straight sides while another function is in the form of Taylor series with unknown coefficients. Two ways to approximately find these coefficients have been considered. The first is the method of iterations and another is the method of collocations, which means that certain conditions have been satisfied only in a finite number of points on the boundary of the auxiliary domain. The cavitating flow around an arc of a circle has been studied using this method, and the numerical results have been presented. The flow around a circular arc with the Kirchhoff cavity closure model has been studied in [Kuf52]. A number of problems for a thin partially cavitating profile and for an object vertically plunged into the liquid with a ventilated cavity behind it have been studied in the book [GT84]. Several types of cavitating flows around a flexible shell have been studied in the recent paper [TZ06].

The linearized problem for two supercavitating hydrofoils under a free surface with the Tulin double-spiral-vortex cavity closure condition has been studied in [GS67]. The solution has been reduced by the method of conformal mapping to the Riemann-Hilbert boundary value problem on the complex plane. The numerical results have been given for different hydrofoil configurations. The mutual influence of two slender wedges has been considered in [GS65]. The linearized problem has been reduced to the Riemann-Hilbert problem on the plane which is solved explicitly. The numerical results have been given for the main parameters of the flow. The single-spiral-vortex model has been used in [LaS65] to study a supercavitating flow around a hydrofoil in an infinite body of liquid. The problem for a supercavitating hydrofoil with the double-spiral-vortex cavity closure condition has been solved in [LaS67]. Both solutions are non-linear and have been obtained using the Riemann-Hilbert technique on the complex plane. The paper [LaS68] is dedicated to the study of a supercavitating flow past a curvilinear body in an infinite flow domain under the Tulin single-spiral-vortex cavity closure condition. The solution has been obtained using a semi-inverse method for a hydrofoil design; the designer may control the maximal pressures and the general body shape, but the exact pressure distribution and body shape need to be found as a part of the solution. The supercavitating flow around a symmetric wedge with gravity acting parallel to the axis of symmetry has been studied in [LeS65]. The problem has been reduced to the
nonlinear integral equation with a constraint which is solved numerically by the method of successive approximations.

A cavitation flow past a polygonal obstacle with the Tulin double-spiral-vortex cavity closure model in the infinite body of liquid has been studied by Bassanini and Elcrat in [BS88]. The problem has been reduced to the Riemann-Hilbert problem on the complex plane and has been solved in the closed form. The numerical results have been given for a symmetric flow. In the paper [BS93] the results of the paper [BS88] have been combined with a boundary layer computation. The same authors proposed a method for computation of a steady cavitation flow past a three-dimensional object with a polygonal cross-section [BS89]. The Tulin double-spiral-vortex model is used to solve the problem in each of the cross sections; after that the solutions for the cross sections are matched together using asymptotic expansions.

Antipov and Silvestrov [AS07] have studied a supercavitating flow in a channel past two hydrofoils with the Tulin single-spiral-vortex cavity closure model. The method employed for the solution uses the conformal mapping together with the Riemann-Hilbert boundary value problem on the hyperelliptic Riemann surface. The flow past a wedge in an infinite body of liquid has been considered by the same authors in [AS08]. The flow branches at the lower side of the wedge. The cavity appears behind the wedge and along the upper side of the wedge. The Tulin single-spiral-vortex model is taken as the cavity closure condition for both cavities. The numerical results have been given for a particular case when a flow separates at the vertex of the wedge. The double cavity flow past a wedge has also been studied in [CC58]. The Kirchhoff model is used for a cavity closure behind the wedge and the Efros-Gilbarg reentrant jet model is used for the modelling of a subsidiary cavity on the upper side of the wedge. The supercavitating flow for an arbitrary number $n + 1$ of hydrofoils has been studied in [AS09]. The method of the conformal mapping from the $(n + 1)$-connected circular domain onto the flow domain has been used for the solution of the problem. The problem has been reduced to two Riemann-Hilbert problems of the theory of symmetric automorphic functions. The numerical results have been provided for a single wedge.

Below follows the brief outline of the main steps in the solution of the cavitation problems in this work.

The boundaries of the cavity are free streamlines. Hence, the flow domain in which the problem for a complex potential is stated is a priori unknown. To overcome this difficulty the following approach known as the method of conformal mapping is used: take an auxiliary domain of the same connectivity as the flow domain. The choice of the auxiliary domain depends on us and can be made so that to simplify the mathematical solution. The only condition made on the auxiliary domain is that the conformal mapping from this domain onto the flow domain must exist and be unique. For example, we can take the auxiliary domain to be the exterior of several circles or several cuts parallel to the same line. It has been proved [Cou50], [Neh82] that in the last case the cuts can be taken along the same line for simply-, doubly- and triply-connected domains. Some of the positions of the end points, however, can not be prescribed arbitrarily for doubly- and triply-connected domains and become the parameters to be determined during the solution of the problem. For domains of higher connectivity, in general, it is not true that all of the cuts lie on the same straight line. Observe also that the preimages of
the “characteristic” points of the flow (such as, for example, a cavity closure point, end points of the sides of a wedge or a hydrofoil, a vertex of a wedge) can not be prescribed arbitrarily and are the parameters to be determined from the solution.

After the domain for the auxiliary variable $\zeta$ has been fixed, the problem is reduced to finding the conformal mapping $z = f(\zeta)$. This can be done due to the fact that cavitation problems are overdefined. Two separate boundary value problems can be stated for the functions $dw/dz$ and $dw/d\zeta$. These problems can be reduced to the Riemann-Hilbert problems on the complex plane (for a simply-connected flow domain), elliptic or hyperelliptic Riemann surfaces (for a doubly- or a triply-connected flow domains). After solving those problems, the derivative of the conformal mapping can be found from the formula:

$$\frac{dz}{d\zeta} = f'(\zeta) = \frac{dw}{d\zeta} : \frac{dw}{dz}. \tag{1.10}$$

The final step to restore the conformal mapping is to find a number of parameters (such as the parameters of the conformal mapping, the preimages of the “characteristic” points on the flow domain, the parameters in the solution of the Riemann-Hilbert problems). For these parameters a system of transcendental equations can be stated. The solution of this system is notoriously complicated and is done numerically.

The main goal of this work is to study supercavitating flows with the Tulin single-spiral-vortex cavity closure condition (1.7). It needs to be noted that while this model has been widely studied in the literature, almost no numerical results have been obtained for nonlinear problems in multiply connected domains. The reason for this is both in the complexity of the theoretical solution and in the fact that the resulting transcendental systems of equations present challenges for numerical solution.

This work is organized as follows.

In Chapter 2 we present the necessary theoretical background on the Riemann-Hilbert problem on the Riemann surface. The important definitions in the theory of Riemann surfaces of algebraic functions are given, the Weierstrass kernel is presented as an analogue of the Cauchy kernel on the hyperelliptic Riemann surface, the inhomogeneous Riemann-Hilbert problem is stated and the main steps of the solution are illustrated.

In Chapter 3 the problem for a supercavitating wedge under a free surface is considered. The Tulin single-spiral-vortex model is used as a cavity closure condition. The method of conformal mapping allows to transform the flow domain with free boundaries into some fixed auxiliary domain with known boundaries. Two Riemann-Hilbert problems on the elliptic Riemann surface are solved to restore the conformal mapping and to find the complex potential $w(z)$ of the flow. The problem is finally reduced to the solution of the system of transcendental equation. The method of numerical solution of this system is presented and the numerical results are given for different geometric configurations of the wedge and different cavitation numbers.

The main goal of Chapter 4 is to investigate how the choice of the cavity closure model affects the flow around a supercavitating wedge. Towards this goal a supercavitating flow around a wedge in a jet is considered for the Tulin single- and double-spiral-vortex models. The problems are solved using the method of conformal mapping in combination with the method of the Riemann-Hilbert problem. For the double-spiral-vortex model
two Riemann-Hilbert problems on the complex plane are stated and solved. For the single-spiral-vortex model the Riemann-Hilbert problems on the elliptic Riemann surface are stated and solved. Both problems are reduced to the systems of transcendental equations which are solved numerically. The results show that the flow in the front part of the cavity is mostly not affected by the choice of the model. However, the flow at the tail part of the cavity depends strongly on the cavity closure model.

In Chapter 5 a supercavitating flow around a flexible hydrofoil or a wedge with flexible sides is considered. The solution of the fluid mechanics problem is found by using the methods of conformal mapping and the Riemann-Hilbert problem. The hydrofoil and the sides of the wedge are treated as elastic plates, and the equations of bending of elastic plates are employed for their modelling. The problem is reduced to the system of functional equations incorporating both fluid mechanics and elasticity aspects of the problem. A numerical iterative procedure is developed for the solution of the system. The numerical results are presented.

In Chapter 6 a flow around two wedges in an infinite plane, under a free surface of liquid or a symmetric flow in a jet is considered. The problems are reduced to the Riemann-Hilbert problems on an elliptic or hyperelliptic Riemann surfaces. The solution is given in the closed form using the previously mentioned methods. This chapter is more theoretical in nature, however, the numerical results for a symmetric flow around two wedges in a jet are given.
Chapter 2

Riemann-Hilbert Problems on Riemann Surfaces

This chapter presents an introduction into the theory of the Riemann-Hilbert problem on a Riemann surface which is used by us in the subsequent chapters. The theory of the Riemann-Hilbert problem on the complex plane is well established and can be found in the classical books [Gak90], [Mus77], [Chi77]. The theory of the Riemann-Hilbert problem on a Riemann surface has been developed in [Zve71], [Chi80]. In this chapter the basic definitions and concepts related to Riemann surfaces are reviewed. The inhomogeneous Riemann-Hilbert problem is stated and the solution of this problem is presented for the case of a hyperelliptic Riemann surface. Our main reference for this chapter is the paper [Zve71].

2.1 Riemann Surfaces

Definition 2.1.1. A manifold \( \mathcal{R} \) is called a complex analytic manifold or a Riemann surface [Spr57], [AS60], [FK81], [HC68] if

(i) there is a collection \( \{U_i, \Phi_i\}_{i \in I} \), where, for the index set \( I \), \( \{U_i\}_{i \in I} \) is an open covering of \( \mathcal{R} \) and \( \Phi_i \) is a homeomorphism of \( U_i \) onto an open set in the complex \( z \)-plane \( (z = x + iy) \);

(ii) if \( U_i \cap U_j \neq \emptyset \), then \( \Phi_j(\Phi^{-1}_i) \) is a conformal sense-preserving mapping of \( \Phi_i(U_i \cap U_j) \) onto \( \Phi_j(U_i \cap U_j) \); that is, \( w(z) = \Phi_j(\Phi^{-1}_i)(z) \) is an analytic function of \( z \) in \( \Phi_i(U_i \cap U_j) \).

We will be concerned exclusively with the Riemann surfaces of algebraic functions.

Definition 2.1.2. An analytic function \( u = u(z) \) is called an algebraic function if it satisfies the functional equation

\[
F(z, u) = a_0(z)u^n + a_1(z)u^{n-1} + \ldots + a_n(z) = 0, \quad a_0(z) \neq 0, \quad (2.1)
\]

where \( a_i(z) \) are polynomials in \( z \) with complex numbers as coefficients.

The Riemann surface \( \mathcal{R} \subset \mathbb{C}^2 \) of the \((n\text{-valued})\) function \( u = u(z) \) is given by the equation \( F(z, u) = 0 \). In what follows we will assume that the polynomial \( F(z, u) \) is
irreducible. We consider the surface $\mathcal{R}$ as composed of several copies of the extended complex plane $\mathbb{C}$ which are connected together according to some rule. We denote a point on the Riemann surface by two coordinates $p = (z, u) \in \mathcal{R}$, where $u$ is one of the solutions of the equation (2.1).

Definition 2.1.3. The point $p_0 = (z_0, u_0) \in \mathcal{R}$ is called a singular point of the Riemann surface $\mathcal{R}$ [Dub01] if

$$\begin{align*}
F'_z(z_0, u_0) &= 0, \\
F'_u(z_0, u_0) &= 0.
\end{align*}$$

Otherwise, the point is called regular. The Riemann surface $\mathcal{R}$ which does not contain singular points is called a regular Riemann surface.

One of the simplest examples of the Riemann surfaces of algebraic functions is the surface of the function $u = \sqrt{z}$. This function has two branches $u_1 = \sqrt{re^{i\varphi/2}}$ and $u_2 = \sqrt{re^{i(\varphi+2\pi)/2}}$, where $(r, \varphi)$ are the polar coordinates. If we start with the first branch $u_1$ and go around the origin along any closed contour we arrive at the second branch $u_2$. If we go around this contour again we return to the first branch $u_1$. Thus, $u = \sqrt{z}$ is two-valued on the complex plane. The function $u = \sqrt{z}$ can be made single-valued by narrowing the domain of the function, for example, by making a cut in the complex plane from the origin to infinity along the real axis. By taking two copies of the extended complex plane with this cut and joining the banks of the cut criss-cross we obtain the Riemann surface of the function $u = \sqrt{z}$ (fig. 2.1). The function $u = \sqrt{z}$ is single-valued on this surface.

The Riemann surface given by the equation (2.1) has $n$ sheets, i.e. let $\pi : \mathcal{R} \to \mathbb{C}$ be a projection of the Riemann surface $\mathcal{R}$ onto the complex $z$-plane given by the formula

$$\pi(z, u) = z.$$  

Then for almost all $z$ the full preimage $\pi^{-1}(z)$ consists of $n$ different points of the surface $\mathcal{R}$

$$(z, u_1(z)), (z, u_2(z)), \ldots , (z, u_n(z)),$$

where $u_1(z), u_2(z), \ldots, u_n(z)$ are $n$ roots of the equation (2.1) for a given $z$. For some values of $z$ some of the points of the preimage may coincide. These points are called the branch points of the surface $\mathcal{R}$ and can be found from the system

$$\begin{align*}
F(z, u) &= 0, \\
F'_u(z, u) &= 0.
\end{align*}$$
A regular surface $\mathcal{R}$ has a finite number of branch points.

It can be shown that any compact Riemann surface is homeomorphic to a sphere with $g$ handles [Spr57], [HC68]. The number of handles $g$ is called the genus of the surface. On the Riemann surface $\mathcal{R}$ of the genus $g$ we can find $2g$ simple oriented closed curves (called canonical cross-sections) $a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g$, such that the surface $\mathcal{R}$ cut along these curves becomes simply connected. These curves can be chosen so that only the curves $a_k$ and $b_k$ are intersecting at the single point and the curve $a_k$ crosses the curve $b_k$ from left to right.

Suppose the algebraic Riemann surface $\mathcal{R}$ has the branch points $p_j = (z_j, u_j)$, $j = 1, 2, \ldots, m$, and $r_j$ ($r_j \geq 2$) sheets of the Riemann surface join at the branch point $p_j$. Then the genus of the Riemann surface $\mathcal{R}$ can be found from the Riemann-Hurwitz formula [Dub01], [HC68]:

$$g = \frac{1}{2} \sum_{j=1}^{m} (r_j - 1) - n + 1.$$

In what follows we consider only the Riemann surfaces $\mathcal{R}$ which are defined by the equation

$$u^2 = p(z) = \prod_{j=1}^{n} (z - z_j). \quad (2.2)$$

We assume that all the roots $z_j$ of the polynomial $p(z)$ are simple roots. This guarantees that the Riemann surface $\mathcal{R}$ does not have singular points. The points $z_1, z_2, \ldots, z_n$ are the branch points of the surface, and if $n$ is odd then $z = \infty$ is also a branch point. The Riemann surface of the function (2.2) consists of two sheets (copies of the extended complex plane $\bar{\mathbb{C}}$) with the cuts joining the branch points of the surface pairwise. The banks of the cuts are connected criss-cross, similarly to the fig. 2.1. If $n = 1$ or $n = 2$ we obtain a rational Riemann surface which is homeomorphic to a sphere; if $n = 3$ or $n = 4$ we obtain an elliptic Riemann surface which is homeomorphic to a torus. Finally, if $n > 4$ we obtain a hyperelliptic Riemann surface.

In the neighborhood of any point of the Riemann surface $\mathcal{R}$ which is not a branch point, the local parameter $\xi$ can be taken to be $\xi = z$, in the neighborhood of the branch point $z_j$ the local parameter can be taken to be $\xi = \sqrt{z - z_j}$. Finally, if $n$ is odd, the local parameter at infinity can be taken to be $\xi = 1/\sqrt{z}$, and if $n$ is even then $\xi = 1/z$.

One of the ways [Zve71] to select the canonical cross-sections on the hyperelliptic Riemann surface of genus $g$ is shown on the fig. 2.2. The parts of the canonical cross-sections which lie on the upper (lower) sheet of the hyperelliptic Riemann surface are shown with solid (dashed) lines. For computational purposes it is often convenient to deform parts of the canonical sections $a_j$ and $b_j$ into the straight segments joining the branch points of the surface.

### 2.2 Meromorphic Functions and Differentials on Riemann Surface

**Definition 2.2.1.** A function $f$ is meromorphic on the Riemann surface if it is holomorphic in the neighborhood of any point of the surface $\mathcal{R}$ except for a finite number
of points $p_1, \ldots, p_m$, i.e. can be locally represented as $f = f(\xi)$, where $\xi$ is a local parameter and $f(\xi)$ is a holomorphic function. At the points $p_1, \ldots, p_m$ the function $f$ has poles of orders $n_1, \ldots, n_m$ respectively, i.e. in a neighborhood of the point $p_i$ the function $f$ can be written in the form $f = \xi^{-n_i} \tilde{f}_i(\xi_i)$, where $\xi_i$ is the local parameter in the neighborhood of the point $p_i$, $\xi_i(p_i) = 0$, $\tilde{f}_i(\xi_i)$ is holomorphic in the neighborhood of $\xi_i = 0$ and $\tilde{f}_i(\xi_i) \neq 0$.

Meromorphic functions on the Riemann surface $\mathcal{R}$ generate a field whose structure carries all the information about the surface $\mathcal{R}$ itself. It can be shown that the definition 2.2.1 for the Riemann surface of the algebraic function (2.1) is equivalent to the following:

**Definition 2.2.2.** A function $f = f(z,u)$ is a meromorphic function on the Riemann surface $\mathcal{R}$ defined by the equation (2.1) if it is a rational function of $z$ and $u$, i.e. it has a form

$$f(z,u) = \frac{P(z,u)}{Q(z,u)},$$

where $P(z,u)$, $Q(z,u)$ are polynomials in $z$ and $u$, and $Q(z,u)$ is not identically zero.

For instance, the functions $z$ and $u$ are analytic and single-valued on the hyperelliptic Riemann surface $\mathcal{R}$ given by the equation (2.2). If $n = 2g + 1$ is odd, then at the infinity point of the surface $\mathcal{R}$ the function $z$ has the pole of the second order and the function $u$ has the pole of the order $2g + 1$. If $n = 2g + 2$ is even, then at the two infinity points of the surface $\mathcal{R}$ the function $z$ has simple poles, and the function $u$ has two poles of the order $g + 1$.

It can be shown [Dub01]:

**Theorem 2.2.1.** Any meromorphic function on the Riemann surface $\mathcal{R}$ has the same number of zeros and poles (counting multiplicity).

Let $\xi = x + iy$ be a local parameter in some region $G$ of the Riemann surface $\mathcal{R}$.

**Definition 2.2.3.** A first-order differential $d\omega$ on $G$ is an expression defined in $G$ which in any local parameter $(x,y)$ has the form

$$d\omega = p(x,y)dx + q(x,y)dy,$$
where $p$ and $q$ are complex-valued functions of $(x, y)$ and which transform into

$$d\omega = \tilde{p}(u, v)du + \tilde{q}(u, v)dv,$$

where $p, q, \tilde{p}, \tilde{q}$ satisfy the conditions

$$\tilde{p}(u, v) = p(x(u, v), y(u, v))\frac{dx}{du} + q(x(u, v), y(u, v))\frac{dy}{du},$$

$$\tilde{q}(u, v) = p(x(u, v), y(u, v))\frac{dx}{dv} + q(x(u, v), y(u, v))\frac{dy}{dv},$$

when local parameters are changed.

First-order differentials on the Riemann surface can be written locally as

$$d\omega = f(\xi)d\xi + g(\xi)d\bar{\xi}.$$

**Definition 2.2.4.** A differential $d\omega$ is holomorphic if it can be written in local coordinates as $d\omega = f(\xi)d\xi$, where $f(\xi)$ is an analytic function of the local parameter $\xi$.

Many standard theorems of complex analysis, such as the Cauchy Theorem and the Liouville Theorem, are valid on compact Riemann surfaces.

It can be shown [Spr57]:

**Theorem 2.2.2.** The dimension of the vector space of holomorphic differentials on a compact Riemann surface is equal to the genus $g$ of the surface.

The holomorphic differentials on $R$ are called the abelian differentials of the first kind. In particular, in the case of hyperelliptic Riemann surface $R$ given by the equation (2.2) the differentials

$$dw_k(z) = \frac{z^{k-1}dz}{u(z)}, \ k = 1, 2, \ldots, g,$$  \ (2.3)

constitute the basis of the vector space of abelian differentials of the first kind.

**Definition 2.2.5.** The numbers

$$A_j = \oint_{a_j} d\omega, \ B_j = \oint_{b_j} d\omega$$

are called $A$- and $B$-periods of the differential $d\omega$.

**Definition 2.2.6.** The basis $d\tilde{w}_1, d\tilde{w}_2, \ldots, d\tilde{w}_g$ of the vector space of abelian differentials of the first kind which satisfies the conditions

$$\oint_{a_j} d\tilde{w}_k = \delta_{jk}, \ j, k = 1, \ldots, g,$$

is called a normalized (canonical) basis for the abelian differentials of the first kind with respect to the canonical cross-sections $a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g$.

We have the following important property of the normalized basis [Spr57], [Zve71]:
Theorem 2.2.3. Let $d\tilde{w}_1, d\tilde{w}_2, \ldots, d\tilde{w}_g$ be a normalized basis of abelian differentials of the first kind. Consider

$$
B_{jk} = \oint_{B_j} d\tilde{w}_k, \; j, k = 1, \ldots, g.
$$

Then the matrix $\|B_{jk}\|$ is symmetric, and $\text{Im}\|B_{jk}\|$ is positively definite.

A meromorphic (abelian) differential on the Riemann surface $\mathcal{R}$ may have poles at a finite number of points of the surface $\mathcal{R}$. Let the differential $d\omega$ have a pole of order $k$ at the point $p_0 \in \mathcal{R}$, i.e. it can be written in terms of the local parameter $\xi$, $\xi(p_0) = 0$, as

$$
d\omega = \left(\frac{c_{-k}}{\xi^k} + \ldots + \frac{c_{-1}}{\xi} + O(1)\right) d\xi.
$$

The order of the pole does not depend on the choice of the local parameter.

Definition 2.2.7. The coefficient $c_{-1}$ is called a residue $\text{Res}_{p_0} d\omega$ of the differential $d\omega$ at the point $p_0$.

The residue $\text{Res}_{p_0} d\omega$ does not depend on the choice of the local parameter and can be found from the formula

$$
\text{Res}_{p_0} d\omega = c_{-1} = \frac{1}{2\pi i} \oint_C d\omega,
$$

where $C$ is a contour surrounding the point $p_0$ which does not contain any other singularities of $d\omega$.

The following theorem holds [Dub01]:

Theorem 2.2.4 (Residue theorem). The sum of the residues of the meromorphic differential $d\omega$ on the Riemann surface $\mathcal{R}$ taken along all the poles of this differential is equal to zero.

2.3 Formulation of the Riemann-Hilbert Problem on the Hyperelliptic Riemann Surface $\mathcal{R}$

The Cauchy kernel

$$
\frac{d\tau}{\tau - z}
$$

(2.4)

plays a fundamental role in the theory of boundary value problems for analytic functions on the complex plane.

Let us summarize the most important properties of the kernel (2.4):

1. The kernel (2.4) is a meromorphic function of the variable $z$ with the single simple pole at the point $z = \tau$ and a zero at the infinity point $z = \infty$.

2. The kernel (2.4) is a meromorphic (abelian) differential with respect to the variable $\tau$ with two simple poles at the points $\tau = z$ and $\tau = \infty$ and the residues at those points equal to $+1$ and $-1$ correspondingly.
The integrals with the Cauchy kernel have important properties such as the Cauchy integral theorem and the Sokhotski-Plemelj formulas [Gak90]. To obtain the analogues of these theorems on compact Riemann surfaces we need, firstly, to find an analogue of the Cauchy kernel.

It can be shown [Zve71] that on the Riemann surface of genus $g > 0$ a meromorphic function cannot have a single simple pole, and thus, there is no analogue of the Cauchy kernel which satisfies both properties 1 and 2. Hence, to find an analogue $A(\tau, z)d\tau$ of the Cauchy kernel we drop the property 1 and require that

$$A(\tau, z)d\tau = \frac{d\tau}{\tau - z} + \{\text{regular terms}\}, \quad \tau \to z. \quad (2.5)$$

Then there are infinitely many analogues of the Cauchy kernel on any compact Riemann surface. More details on the construction of different types of analogues of the Cauchy kernel can be found in [Zve71]. On the hyperelliptic Riemann surface $\mathcal{R}$ defined by the equation (2.2), we will work with the Weierstrass kernel given by the formula:

$$K(\tau, z)d\tau = \frac{1}{2} \left(1 + \frac{u(z)}{u(\tau)}\right) \frac{d\tau}{\tau - z}, \quad (\tau, u(\tau)), (z, u(z)) \in \mathcal{R}. \quad (2.6)$$

This kernel has the property (2.5). It also has additional pole (poles) at infinity point (points) of the hyperelliptic Riemann surface $\mathcal{R}$. Here and further, when there is no confusion, we will write only the affixes $z$ and $\tau$ of the points $(z, u(z))$ and $(\tau, u(\tau))$ on the Riemann surface $\mathcal{R}$. For example, in the formula (2.6) under $K(\tau, z)d\tau$ we understand $K((\tau, u(\tau)), (z, u(z)))d\tau$. This slight abuse of the notation is made for the sake of brevity.

Let $L$ be a piecewise smooth contour on $\mathcal{R}$ and let $\varphi(\tau)$ be a Hölder continuous function defined on $L$. The integral of the Cauchy type with the kernel (2.6)

$$\Phi(z) = \frac{1}{2\pi i} \int_L \varphi(\tau) K(\tau, z)d\tau$$

is discontinuous along the curve $L$ and analytic in the remaining part $\mathcal{R}\setminus L$. In particular, as a consequence of the property (2.5), the following Sokhotski-Plemelj formulas hold:

$$\Phi^\pm(t) = \pm \frac{1}{2\pi i} \varphi(t) + \frac{1}{2\pi i} \int_L \varphi(\tau) K(\tau, t)d\tau, \quad t \in L, \quad (2.7)$$

which connect the limiting values $\Phi^+(t)$ and $\Phi^-(t)$ from the left- and the right-hand side of the curve $L$ with respect to the given direction on $L$ (fig. 2.3).

Let $p_1, p_2, \ldots, p_k$ be points on a Riemann surface $\mathcal{R}$, and let the integers $n_1, n_2, \ldots, n_k$ be multiplicities of these points. A symbol $\Delta = p_1^{n_1}p_2^{n_2} \cdots p_k^{n_k}$ (which is independent of the order of these points) is called a divisor and the number $\text{ord} \Delta = \sum_{j=1}^{k} n_k$ is called the order of the divisor $\Delta$. The divisor $\Delta$ is called entire if each $n_\kappa \geq 0$, otherwise it is called fractional. Divisors may be multiplied or divided, and then the multiplicities of the corresponding points are added or subtracted. We say that a divisor $\Delta_1$ is a multiple of a divisor $\Delta_2$ (written as $\Delta_2|\Delta_1$) if the divisor $\Delta_1 : \Delta_2$ is entire. A function $\phi(p)$ or a differential $d\psi(p)$ having a finite number of poles and zeros (of integer multiplicities) can
be obviously associated with a unique divisor \((\phi)\) or \((d\psi)\), respectively, formed from its zeros and poles with the appropriate multiplicities. An order of a function (differential) is the order of its divisor. The order of a functions meromorphic everywhere on \(\mathcal{R}\) is zero. The order of an abelian differential on \(\mathcal{R}\) is \(2g - 2\).

A symbol \(\Delta = p_1^{n_1}p_2^{n_2} \cdots p_k^{n_k}\), where \(n_1, n_2, \ldots, n_k\) are real numbers, is called a quasi-divisor.

Figure 2.3: Limiting values of the function \(\Phi(z)\).

Suppose that a function \(\phi(p)\) is defined (possibly not single-valued) in a neighborhood of every point of a divisor \(\Delta\). Let \(\xi\) be the local parameter in a neighborhood of a point \(p_j\) in the divisor \(\Delta\) and \(p_j = p_j(\xi)\) be the corresponding parametric mapping, where \(p_j = p_j(0)\). Then the function \(\phi(p)\) is a pseudo-multiple of \(\Delta\) \((\Delta \wr (\phi))\) if

\[
\frac{\phi(p_j(\xi))}{\xi^{n_j}} = O(|\ln \xi|), \quad \xi \to 0, \quad j = 1, 2, \ldots, k.
\]

The function \(\phi(p)\) is a quasi-multiple of \(\Delta\) \((\Delta \parallel (\phi))\) if all the functions

\[
\frac{\phi(p_j(\xi))}{\xi^{n_j}}, \quad j = 1, 2, \ldots, k,
\]

have a singularity of integrable order in \(\xi\). Analogous definitions can be given for differentials.

Let \(L\) be a composite piecewise smooth contour given on \(\mathcal{R}\), that is, a closed set consisting of a finite number of smooth oriented (closed or open) curves which may have a finite number of common points. Define a divisor \(\Lambda = t_1t_2 \cdots t_r\) so that \(L \setminus \Lambda\) consists of a finite number of connected components \(L_j\) \((j = 1, 2, \ldots, N)\) each component being a smooth Jordan arc homeomorphic to the interval \((0, 1)\) of the real line. Thus, each \(L_j\) has two ends which are contained among the points of \(\Lambda\). It may happen that two end points of the arc \(L_j\) coincide. On each arc \(L_j\) there is a given Hölder continuous function \(G_j(t)\) which is finite, does not vanish and can be Hölder continuously continued to the end points, and the limiting values are finite and non-zero. We write

\[
G(t) = \sum_{j=1}^{N} \alpha(t, L_j)G_j(t), \quad t \in L \setminus \Lambda,
\]

where \(\alpha(t, L_j) = \delta_{jk}\) when \(t \in L_k\).
Similarly, on each arc $L_j$ define a finite Hölder continuous function $g_j(t)$, which can be Hölder continuously continued to the end points and the limiting values are finite:

$$g(t) = \sum_{j=1}^{N} \alpha(t, L_j)g_j(t), \ t \in L \setminus \Lambda.$$ 

Finally, on $\mathcal{R} \setminus L$ there is a given divisor $\mathcal{D}$ of some order $n$. This information is sufficient to formulate the inhomogeneous Riemann-Hilbert boundary value problem on $\mathcal{R}$.

**Formulation** (Inhomogeneous Riemann-Hilbert boundary value problem on $\mathcal{R}$). Find all the functions $\Phi(p)$ meromorphic on $\mathcal{R} \setminus L$ that are multiples of $\mathcal{D}$ and admit a Hölder-continuous continuation to $L \setminus \Lambda$ with the boundary values satisfying the condition:

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \ \mathcal{D} \mid (\Phi). \quad (2.8)$$

At the points of the divisor $\Lambda$ the function $\Phi(p)$ may have at most integrable singularities.

Observe, that we may look for other solutions of the problem besides the function $\Phi(p)$ having integrable singularities at the points of the divisor $\Lambda$. For instance, we may require the solution $\Phi(p)$ to be bounded at all or some of the points of the divisor $\Lambda$. For more general formulation of the problem the reader is referred to [Zve71].

### 2.4 Canonical Function of the Riemann-Hilbert Problem (2.8)

Let us follow [Gak90] and introduce a definition:

**Definition 2.4.1.** A canonical function of the Riemann-Hilbert problem (2.8) is a sectionally analytic function $X(z)$ which satisfies the homogeneous boundary condition

$$X^+(t) = G(t)X^-(t), \ t \in L \setminus \Lambda, \quad (2.9)$$

has at most integrable singularities at the points of the divisor $\Lambda$ and may have at most a finite number of zeros and poles on $\mathcal{R} \setminus L$.

This function is not unique. For convenience, we would like the function $X(z)$ to have the least possible number of zeros and poles in $\mathcal{R} \setminus L$ of the least possible order.

On each arc $L_j$ of the contour $L$ we can select a single-valued branch of the function $\ln G_j(t)$. Then we can write

$$\ln G(t) = \sum_{j=1}^{N} \alpha(t, L_j) \ln G_j(t).$$

Introduce a piecewise analytic function

$$X_0(z) = e^{\Gamma(z)}, \quad \Gamma(z) = \frac{1}{2\pi i} \int_{L} \ln G(\tau)K(\tau, z) d\tau, \quad (2.10)$$
where $K(\tau, z)d\tau$ is the Weierstrass kernel (2.6) and the integral is taken to be the sum over all arcs $L_j$. By applying the Sokhotski-Plemelj formulas (2.7) to the integral (2.10) we can see that the function (2.10) satisfies the homogeneous condition (2.9).

Investigate the behavior of $X_0(z)$ in the neighborhoods of the points of the divisor $\Lambda$. It follows from the property (2.5) that the standard asymptotics which holds for the integrals of the Cauchy type on the complex plane is valid in the local coordinates on the Riemann surface $\mathcal{R}$. Suppose the arc $L_j$ starts at the point $t_k$ of the divisor $\Lambda$ and ends at the point $t_l$. Denote $G_j(t_k + 0)$ and $G_j(t_l - 0)$ the limiting values of $G_j(t)$ at these points. Let $\xi$ be a local parameter in a neighborhood of a point $t_k$ such that $z = t_k$ corresponds to $\xi = 0$. Then we obtain the following asymptotic representation for $X_0(z)$ in the neighborhood of $t_k$:

$$X_0(z) = X_{0k}(\xi) \exp \left\{ \frac{1}{2\pi i} \left( \sum_j '' \ln G_j(t_k - 0) - \sum_j ' \ln G_j(t_k + 0) \right) \ln \xi \right\},$$

where $|X_{0k}(\xi)|$ is bounded as $\xi \to 0$, and the sum $\sum' (\sum'')$ is extended over all arcs $L_j$ that start (end) at the point $t_k$. Denote

$$\kappa_k = \frac{1}{2\pi} \left( \sum_j '' \arg G_j(t_k - 0) - \sum_j ' \arg G_j(t_k + 0) \right),$$

then it follows that the functions

$$\frac{X_0(z(\xi))}{\xi^{\kappa_k}}, \quad \frac{\xi^{\kappa_k}}{X_0(z(\xi))}$$

are bounded as $\xi \to 0$. Additionally, $X_0(z)$ is finite and non-zero everywhere outside $L$ except for the infinity point (points) of the surface $\mathcal{R}$ where it has essential singularities. Thus, the zeros and the poles of the function $X_0(z)$ form a quasi-divisor $(X_0) = t_1^{\kappa_1}t_2^{\kappa_2} \ldots t_r^{\kappa_r}$. The numbers $\kappa_k$ depend on the choice of the branches of $\ln G_j(t)$ and when these branches are changed the values $\kappa_k$ may change by integer numbers. But since the arcs $L_j$ start and end only at the points of the divisor $\Lambda$, the numbers $\kappa_k$ change in such a way that the sums

$$\sum_{k=1}^r \kappa_k, \quad \kappa = \sum_{k=1}^r [\kappa_k],$$

remain unchanged, where $[\ldots]$ denotes the integer part. The number $\kappa$ is called the index of the coefficient $G(t)$ of the problem (2.8).

The function $X_0(z)$, defined by the formula (2.10), is inconvenient for our purposes for two main reasons. Firstly, this function may have zeros and infinities of generally non-integrable order on the line $L$. Secondly, this function has essential singularity at the infinity point (points) of the surface $\mathcal{R}$.

Let us deal with the first problem. We call the point $t_k$ of the divisor $\Lambda$ (non-) singular if the number $\kappa_k$ calculated by the formula (2.11) is (not) an integer. Introduce the numbers $\kappa'_k$ such that

$$\kappa'_k = \begin{cases} \kappa_k, & \text{if the point } t_k \text{ is singular;} \\ [\kappa_k] + 1, & \text{if the point } t_k \text{ is non-singular.} \end{cases}$$
Then consider the function
\[ X_0(z) \exp \left( - \sum_{k=1}^{r} \kappa_k' \int_{r_k}^{t_k} K(\tau, z) \, d\tau \right) = \exp \left( \frac{1}{2\pi i} \int_L \ln G(\tau) K(\tau, z) \, d\tau - \sum_{k=1}^{r} \kappa_k' \int_{r_k}^{t_k} K(\tau, z) \, d\tau \right), \] (2.12)

where \( r_k \) are some arbitrarily fixed points on \( \mathcal{R} \setminus L \) and the contours joining \( r_k \) and \( t_k \) are smooth simple curves which do not have any common points with the curve \( L \) except for the point \( t_k \). This function is bounded at the singular points \( t_k \) and has integrable singularities at the non-singular points \( t_k \). It has poles or zeros of integer order at the points \( r_k \in \mathcal{R} \setminus L \). Since the numbers \( \kappa_k' \) are integers, this function is continuous through the curves joining the points \( r_k \) and \( t_k \).

Finally, to eliminate the essential singularity of the function (2.12) at infinity consider
\[ X(z) = \exp \left( \frac{1}{2\pi i} \int_L \ln G(\tau) K(\tau, z) \, d\tau - \sum_{j=1}^{r} \kappa_j' \int_{r_j}^{t_j} K(\tau, z) \, d\tau - \sum_{j=1}^{g} \left[ \int_{p_j}^{q_j} K(\tau, z) \, d\tau - m_j \oint_{a_j} K(\tau, z) \, d\tau - n_j \oint_{b_j} K(\tau, z) \, d\tau \right] \right). \] (2.13)

Here \( p_j \) are arbitrarily fixed points of the Riemann surface \( \mathcal{R} \setminus L \) which do not lie on the canonical cross-sections \( a_j \) and \( b_j \) of the surface. The points \( q_j \) and the integers \( m_j \) and \( n_j \) need to be fixed so that the resulting function (2.13) does not have essential singularity at the infinity point (points). The paths of integration \( p_jq_j \) do not intersect the canonical cross-sections \( a_j \) and \( b_j \). Taking asymptotic expansion as \( z \to \infty \) we see that to eliminate essential singularity at infinity the following conditions must be satisfied:
\[ \sum_{j=1}^{g} \int_{p_j}^{q_j} dw_k(\tau) = \frac{1}{2\pi i} \int_L \ln G(\tau) dw_k(\tau) - \sum_{j=1}^{r} \kappa_j' \int_{r_j}^{t_j} dw_k(\tau) + \sum_{j=1}^{g} (m_j A_{kj} + n_j B_{kj}), \] (2.14)

where the abelian differentials \( dw_k(z) \) are defined by the formulas (2.3) and
\[ A_{kj} = \oint_{a_j} dw_k(\tau), \quad B_{kj} = \oint_{b_j} dw_k(\tau) \]
are the \( A \) - and \( B \)-periods of the differentials (2.3). The problem (2.14) is known as the Jacobi inversion problem.

### 2.5 Jacobi Inversion Problem

**Formulation.** Find \( g \) points \( q_1, q_2, \ldots, q_g \) on the Riemann surface \( \mathcal{R} \) and \( 2g \) integers \( m_1, m_2, \ldots, m_g, n_1, n_2, \ldots, n_g \) satisfying the conditions (2.14).
This problem is always solvable [Dub01], [Zve71]. The equation (2.14) can be rewritten in terms of the normalized basis of abelian differentials \( d\tilde{w}_1(z), d\tilde{w}_2(z), \ldots, d\tilde{w}_g(z) \) corresponding to the basis (2.3):

\[
\sum_{j=1}^{g} \int_{p_j}^{q_j} d\tilde{w}_k(\tau) = \frac{1}{2\pi i} \int_L \ln G(\tau) d\tilde{w}_k(\tau) - \sum_{j=1}^{r} \kappa'_j \int_{r_j}^{L} d\tilde{w}_k(\tau) + m_k + \sum_{j=1}^{g} n_j B_{kj},
\]

where \( B_{kj} \) is the symmetric matrix with a positive-definite imaginary part.

The last problem can be further rewritten as

\[
\sum_{j=1}^{g} \tilde{w}_\nu(q_j) \equiv e_\nu - k_\nu \pmod{\text{periods}}, \quad \nu = 1, 2, \ldots, g, \tag{2.15}
\]

where

\[
\tilde{w}_\nu(q) = \int_{p_j}^{q} d\tilde{w}_\nu(\tau), \quad \nu = 1, 2, \ldots, g,
\]

are the abelian integrals of the first kind corresponding to the abelian differentials \( d\tilde{w}_\nu(z) \), and [Zve71]

\[
\frac{1}{2\pi i} \int_L \ln G(\tau) d\tilde{w}_k(\tau) - \sum_{j=1}^{r} \kappa'_j \int_{r_j}^{L} d\tilde{w}_k(\tau) = e_\nu - k_\nu,
\]

\[
k_\nu = \frac{1}{2} \sum_{\mu=1}^{h} B_{\mu\nu} - \frac{\nu}{2}, \quad \nu = 1, 2, \ldots, g.
\]

The solution of the problem (2.15) involves the use of the Riemann theta functions.

**Definition 2.5.1.** The theta function is the transcendental entire function of the \( g \) complex variables \( u_1, u_2, \ldots, u_g \), defined as the sum of the following \( g \)-fold series:

\[
\theta(u_1, u_2, \ldots, u_g) = \theta(u_\nu) =
\sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \ldots \sum_{m_g=-\infty}^{\infty} \exp \left( \pi i \sum_{\nu=1}^{g} m_{\nu} u_{\nu} + 2\pi i \sum_{\mu=1}^{g} \sum_{\nu=1}^{g} \beta_{\mu\nu} m_{\mu} m_{\nu} \right), \tag{2.16}
\]

where all the summation indices \( m_1, m_2, \ldots, m_g \) range independently from \(-\infty\) to \( \infty \), and the matrix \( ||\beta_{\mu\nu}|| \) of the numbers \( \beta_{\mu\nu} \) (the parameters of \( \theta \)-function) is assumed to be symmetric (\( \beta_{\mu\nu} = \beta_{\nu\mu} \)) with a positive-definite imaginary part. The last condition guarantees the convergence of the series (2.16) for all the values of \( u_1, u_2, \ldots, u_g \).

The theta function has the following periodicity properties

\[
\theta(u_1, \ldots, u_\nu + 1, \ldots, u_g) = \theta(u_1, \ldots, u_\nu, \ldots, u_g),
\]

\[
\theta(u_1 + \beta_1, u_2 + \beta_2, \ldots, u_g + \beta_g) = e^{-\pi i \beta_\nu} \cdot \theta(u_1, u_2, \ldots, u_g).
\]

This function is even:

\[
\theta(-u_1, -u_2, \ldots, -u_g) = \theta(u_1, u_2, \ldots, u_g).
\]
Definition 2.5.2. The Riemann theta function is obtained by the substitution of $u_1 = \tilde{w}_1(z) - e_1$, $u_2 = \tilde{w}_2(z) - e_2$, $\ldots$, $u_g = \tilde{w}_g(z) - e_g$ and $\beta_{\mu \nu} = \mathcal{B}_{\mu \nu}$ into (2.16):

$$
\theta(\tilde{w}_1(z) - e_1, \tilde{w}_2(z) - e_2, \ldots, \tilde{w}_g(z) - e_g) = \theta(\tilde{w}_\nu(z) - e_\nu) = \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \ldots \sum_{m_g = -\infty}^{\infty} \exp \left( \pi i \sum_{\mu=1}^{g} \sum_{\nu=1}^{g} \mathcal{B}_{\mu \nu} m_\mu m_\nu + 2\pi i \sum_{\nu=1}^{g} m_\nu (\tilde{w}_\nu(z) - e_\nu) \right), \tag{2.17}
$$

It has been proved [Dub01], [Zve71]:

Theorem 2.5.1. If the Riemann theta function is not identically zero (non-trivial), then it must have on $\mathcal{R}$ exactly $g$ zeros which form the solution of the Jacobi inversion problem.

If the Riemann theta function is identically zero one needs to consider partial derivatives by $\tilde{w}_j$ of the function (2.17) until a non-trivial function is found. Such a function can be found in a finite number of steps [Zve71]. Denote $F(z)$ to be the Riemann theta function (2.17) (if it is non-trivial) or its non-trivial partial derivative of the least order.

For the hyperelliptic Riemann surfaces $\mathcal{R}$ we can obtain the solution of the Jacobi inversion problem by solving the following system of algebraic equations

$$
\sum_{\nu=1}^{g} z_\nu^j = \sum_{\nu=1}^{g} \oint_{a_\nu} \tau^j d\tilde{w}_\nu(\tau) - \sum \text{res}_{z=\infty} \{z^j d \ln F\} \tag{2.18}
$$

where $z_\nu$ are the affixes of the points $q_\nu = (z_\nu, u(z_\nu))$ which solve the Jacobi inversion problem, and the last sum is taken along all the infinity points of the Riemann surface $\mathcal{R}$. The system (2.18) is symmetric and does not change with any permutation of the points $z_\nu$. This allows to further reduce this system to the solution of one algebraic equation of the degree $g$ [AS02].

2.6 Solution of the Inhomogeneous Riemann-Hilbert Problem

Consider the function $X(z)$ defined by the formula (2.13) where the points $q_\nu$ solve the Jacobi inversion problem (2.14). This function satisfies the homogeneous boundary condition (2.9). It is finite and Hölder continuous on the curve $L$ except for the points $t_k$ of the divisor $\Lambda$ where it may have at most integrable singularities. In $\mathcal{R} \setminus L$ the function (2.13) may have only a finite number of poles and zeros of integer order.

Factorize the coefficient $G(t)$ of the inhomogeneous problem (2.8) as

$$
G(t) = X^+(t) \cdot [X^-(t)]^{-1}, \quad t \in L,
$$

and substitute it into (2.8):

$$
\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{g(t)}{X^+(t)}, \quad t \in L. \tag{2.19}
$$
The problem (2.19) is known as a “jump” problem and can be solved using the Sokhotski-Plemelj formulas (2.7):

\[ \Phi(z) = X(z) \left( \frac{1}{2\pi i} \int_{L} \frac{g(\tau)}{X(\tau)} K(\tau, z) d\tau + R_1(z) + R_2(z) u(z) \right), \quad (2.20) \]

where the functions \( R_1(z) \) and \( R_2(z) \) are rational functions of \( z \) and need to be chosen so that the solution \( \Phi(z) \) satisfies the condition \( \mathcal{D}|(\Phi) \). This is not always possible and, hence, the problem (2.8) does not always have a solution. The solution of the problem also may not be unique. The precise conditions of the existence and uniqueness of the solution of the problem (2.8) in the general case of a compact Riemann surface \( \mathcal{R} \) is given in [Zve71].
Chapter 3

Motion of a Yawed Supercavitating Wedge Beneath a Free Surface

The problem that is studied in this chapter concerns a yawed wedge under a free surface, moving at a uniform speed. The model involves a trailing cavity whose boundary is a dividing streamline through the vertex of the wedge. The cavity closure mechanism is described according to the Tulin-Terent’ev single-spiral-vortex model. A closed-form solution to the governing nonlinear boundary-value problem is found by the method of conformal mappings. The doubly connected flow domain is treated as the image by this map of the exterior of two slits in a parametric plane. The mapping function is constructed through the solution to two boundary-value problems of the theory of analytic functions, the Hilbert problem for two slits in a plane and the Riemann-Hilbert problem on an elliptic surface. Numerical results for the shape of the cavity and the free surface, the yaw angle, the drag and lift coefficients, and the circulation are reported.

3.1 Mathematical Formulation

The work is concerned with the motion of a wedge $DAB$ under a free fluid surface (fig. 3.1). Far away from the wedge, the calm free surface is described by $x_2 = h$. At time $t$, the vertex $A$ is located at the point $x_1 = -v_\infty t$, $x_2 = 0$, and the wedge is moving with uniform speed, $v_\infty$, in the negative direction. The speed $v_\infty$ is much smaller than the sound speed, and the fluid may be considered as an incompressible liquid. The geometry of the wedge is described by four parameters, $\lambda_1$, $\lambda_2$, $\alpha_0$, and $\beta_0$, where $\lambda_1$ and $\lambda_2$ are the lengths of the sides $AB$ and $AD$, and $\alpha_0$ and $\beta_0$ are the angles the upper and lower faces of the wedge initially form with the $x_1$-axis, respectively. The fluid is assumed to be inviscid and irrotational. Gravity is neglected. It is also assumed that the wedge may move about the $z$-axis passing through the point $A$ and orthogonal to the 2d flow domain. The angle of yaw, $\delta$, is to be determined from the condition that the front stagnation point coincides with the vertex $A$. This means that the actual location of the wedge ends is described by the angles $\alpha = \alpha_0 + \delta$ and $\beta = \beta_0 + \delta$. The dividing streamline through the point $A$ traverses the upper and lower faces of the wedge, breaks away at the rear points $B$ and $D$ and forms the upper and lower boundaries, $BC^+$ and $DC^-$, of a cavity behind the wedge.
Figure 3.1: The flow domain \( \tilde{D} \) and the parametric domain \( D \).

For the problem of concern, it is desirable to employ a coordinate system that moves with the wedge. Thus we introduce

\[
x = x_1 + v_\infty t, \quad y = x_2, \quad z = x + iy,
\]

and define the complex velocity potential \( w(z) = \varphi(z) + i\psi(z), z \in \tilde{D} \), and the complex conjugate velocity, \( dw/dz = u_1 - iu_2 \). Here \( \varphi \) is the velocity potential, \( \psi \) is the stream function, \( \tilde{D} \) is the flow domain, and \( v = (u_1, u_2) \) is the velocity vector. The function \( w(z) \) is analytic in the flow domain \( \tilde{D} \) and it satisfies the following boundary conditions:

\[
\text{Im} \ w(z) = C_j, \quad z \in L_j, \quad j = 0, 1,
\]

\[
\left| \frac{dw}{dz} \right| = \begin{cases} 
  v_\infty, & z \in L_0, \\
  v_c, & z \in BC^+ \cup DC^-,
\end{cases}
\]

\[
\arg \frac{dw}{dz} = \begin{cases} 
  -\alpha, & z \in AB, \\
  \pi - \beta, & z \in AD,
\end{cases}
\]

where \( C_0 \) and \( C_1 \) are real constants not necessarily the same, \( L_0 = E^-E^+ \) is the free surface, and the contour \( L_1 \) consists of the boundary of the cavity \( BC^+ \cup DC^- \) and the faces of the wedge \( DAB \). The first condition in (3.2) means that the free surface \( L_0 \) and the contour \( L_1 \) are some streamlines \( \psi(x, y) = C_0 \) and \( \psi(x, y) = C_1 \), respectively. The second condition is due to the fact that the speeds of the motion along the free surface, \( v_\infty \), and the cavity boundary, \( v_c = \sqrt{\sigma + 1}v_\infty \), are constant and prescribed. Here \( \sigma \) is the cavitation number

\[
\sigma = \frac{p_\infty - p_c}{\frac{1}{2}\rho v_\infty^2},
\]

\( \rho \) is the density of the liquid, \( p_c \) is the pressure inside the cavity, and \( p_\infty \) is the liquid pressure far away from the wedge. The last condition in (3.2) means that the flow is tangential to the wedge faces (the walls of the wedge are assumed to be rigid).
There are two singular points, A and C, in the model. The former point is the front stagnation point, and \( dw/dz = 0 \) at \( z = A \). The second point C is a point where the upper and lower streamlines attempt to close the cavity. This point is unknown a priori and will be recovered from the solution. According to the single-spiral-vortex model in the Terent’ev interpretation [Ter81], the two branches of the dividing streamline reach the two vortices behind the foil, \( C^+ \) and \( C^- \), and then pass to a half of an infinitely sheeted Riemann surface of the logarithmic function with the branch points \( C^+ \) and \( C^- \). After that the same streamline emerges from the infinite sheet of the Riemann surface and returns to the first, physical, sheet. The streamlines that are close to the boundary of the cavity traverse first around the cavity and then pass to the Riemann surface. After they have traversed a finite number of sheets of the Riemann surface the streamlines return to the physical sheet.

At a neighborhood of the point C we assume that

\[
\log \frac{dw}{dz} = O((w - w(C))^{-1/2}), \quad z \to C. \tag{3.4}
\]

It will later be shown that condition (3.4) leads to the Tulin-Terent’ev condition

\[
\log \frac{dw}{dz} \sim -K((w - w(C))^{-1/2}), \quad z \to C, \quad -\pi \leq \arg[w(z) - w(C)] \leq \pi. \tag{3.5}
\]

Here \( K \) is a positive constant, and the branch of the square root is chosen such that \([w(z) - w(C)]^{1/2} > 0\) when \( \arg[w(z) - w(C)] = 0 \).

The flow domain \( \tilde{D} \) is doubly connected, and the solution to the nonlinear problem (3.2) is very much aided by mapping the entire boundary of the flow into the exterior of two cuts, \( l_1 = [0, 1] \) and \( l_0 = [m, \infty) \), \( m \) is a parameter to be fixed, \( 1 < m < \infty \) (fig. 3.1). Let \( z = f(\zeta) \) be a conformal map of a parametric \( \zeta \)-plane cut along the segments \( l_1 \) and \( l_0 \) onto the flow domain \( \tilde{D} \) such that the cuts \( l_1 \) and \( l_0 \) are mapped onto the cavity boundary \( L_1 \) and the free surface \( L_0 \), respectively. Let some boundary points \( a, b, c, \) and \( d \) of the cut \( l_1 \) fall into the points \( A, B, C, \) and \( D \), respectively, and a point \( e_\infty \in l_0 \) fall into the infinite point of the flow domain. Such a map always exists and it is defined up to one real parameter. Choose \( e_\infty = \infty \). Define next the derivative \( df/d\zeta \) of the conformal mapping through the derivative \( dw/d\zeta \) and the logarithmic hodograph variable by

\[
\frac{df}{d\zeta} = \omega_0(\zeta) e^{-\omega_1(\zeta)}, \tag{3.6}
\]

where

\[
\omega_0(\zeta) = \frac{dw}{v_\infty d\zeta}, \quad \omega_1(\zeta) = \ln \frac{dw}{v_\infty dz}. \tag{3.7}
\]

Our next step is to show that these two functions, \( \omega_0(\zeta) \) and \( \omega_1(\zeta) \), provide the solutions to two boundary-value problems of the theory of analytic functions, a Hilbert problem on a plane and a Riemann-Hilbert problem on an elliptic Riemann surface.


3.2 Hilbert Problem for the Function $\omega_0(\zeta)$

Because of the first boundary condition (3.2), the imaginary part of the function $\omega_0(\zeta)$ vanishes at the banks of the cuts $l_0$ and $l_1$:

$$\text{Im} \, \omega_0(\xi) = 0, \quad \xi \in l_0 \cup l_1. \quad (3.8)$$

In the domain $D = \mathbb{C} \setminus (l_0 \cup l_1)$, the function $\omega_0(\zeta)$ is analytic. At the the point $\zeta = a$ this function has a simple zero [AS07], [AS08]. Since the infinite point of the parametric plane is mapped into the infinite point of the physical plane, and $\frac{dw}{dz} \to v_\infty$ as $z \to \infty$, it follows from (3.6) and (3.7) that

$$\omega_0(\zeta) \sim \frac{df}{d\zeta} = O(\zeta^{-1/2}), \quad \zeta \to \infty. \quad (3.9)$$

The most general form of the function analytical in the domain $D$, decaying at infinity as (3.9) and vanishing at the point $\zeta = a$ is [Sed65], [Che64]

$$\omega_0(\zeta) = N\tilde{\omega}_0(\zeta), \quad \tilde{\omega}_0(\zeta) = \frac{i(\zeta - a)}{p^{1/2}(\zeta)}, \quad (3.10)$$

where $N$ is an arbitrary real constant, and $p(\zeta) = \zeta(1 - \zeta)(\zeta - m)$. The function $p^{1/2}(\zeta)$ is analytic in the $\zeta$-plane cut along the lines $l_1$ and $l_0$. Its single branch is fixed by the condition $p^{1/2}(\zeta) = i\sqrt{|p(\xi)|}$ as $\zeta = \xi + i0$, $\xi > m$. This branch has the following properties:

$$p^{1/2}(\zeta) = \pm i\sqrt{|p(\xi)|}, \quad \zeta = \xi \pm i0, \quad m < \xi < +\infty,$$

$$p^{1/2}(\zeta) = -\sqrt{|p(\xi)|}, \quad \zeta = \xi, \quad 1 < \xi < m,$$

$$p^{1/2}(\zeta) = \mp i\sqrt{|p(\xi)|}, \quad \zeta = \xi \pm i0, \quad 0 < \xi < 1,$$

$$p^{1/2}(\zeta) = \sqrt{|p(\xi)|}, \quad \zeta = \xi, \quad -\infty < \xi < 0. \quad (3.11)$$

At the point $c$, the preimages $bc$, $dc$, and $cq$ of the three branches of the same streamline, $BC^+$, $DC^-$, and $CQ$, respectively, meet, and the function $\omega_0(\zeta)$ has the following asymptotics [AS08]:

$$\omega_0(\zeta) \sim K_c(\zeta - c), \quad \zeta \to c \quad K_c = \text{const.} \quad (3.12)$$

Thus, the function $\omega_0(\zeta)$ must have a simple zero at the point $\zeta = c \neq a$. It is evident that the only way to meet this requirement is to put $c = \bar{a}$. Now, to fix the constant $N$, we use the conservation of mass law to a closed contour $E^−EAE_0$ between two streamlines $E^−E_0$ and $EA$. Here $E^− = -\infty + ih$, $E = -\infty + i0$, $A = 0$, and $E_0$ is a finite point on the free surface. This results in the following relation:

$$v_\infty h = \text{Im} \int_{E_0}^{E} \frac{dw}{d\zeta} d\zeta, \quad (3.13)$$

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$e_0$ is the preimage of the point $E_0$ or, equivalently,

$$N = h \left( \operatorname{Im} \int_{1}^{m} \tilde{\omega}_0(\xi) d\xi \right)^{-1}. \quad (3.14)$$

Here we used the fact that the function $\tilde{\omega}_0(\xi)$ is real on the banks of the cuts $l_0$ and $l_1$ (it is pure imaginary on the segment $[1, m]$). Thus, we have found that $a = \bar{c}$ and that the function $\omega_0(\zeta)$ is defined up to two real parameters, $a$ and $m$.

### 3.3 Riemann-Hilbert Problem on an Elliptic Surface for the Function $\omega_1(\zeta)$

The function $\omega_1(\zeta)$ is analytic in the domain $D$, and because of its definition (3.7) and the boundary conditions (3.2) we have

$$\operatorname{Re} \omega_1(\zeta) = \begin{cases} \sigma', & \zeta \in bcd, \\ 0, & \zeta \in l_0, \end{cases} \quad \sigma' = \log \sqrt{\sigma + 1},$$

$$\operatorname{Im} \omega_1(\zeta) = \begin{cases} -\alpha, & \zeta \in ab, \\ \pi - \beta, & \zeta \in da. \end{cases} \quad (3.15)$$

As $z$ approaches the infinite point, $dw/dz \to v_{\infty}$, and therefore, $\omega_1(\zeta) \to 0$ as $\zeta \to \infty$. At the stagnation point $z = A$, the velocity vanishes, and hence the function $\omega_1(\zeta)$ has a logarithmic singularity at the point $\zeta = a$. From (3.4), in a neighborhood of the point $c$, the function $\omega_1(\zeta)$ has the following singularity:

$$\omega_1(\zeta) = O([w(z) - w(C)]^{-1/2}), \quad z \to C. \quad (3.16)$$

This implies [AS07] that the function $\omega_1(\zeta)$ is infinite at the point $\zeta = c$,

$$\omega_1(\zeta) = O((\zeta - c)^{-1}), \quad \zeta \to c. \quad (3.17)$$

We shall identify the point $\zeta = c$ as a pole of the function $\omega_1(\zeta)$. However, this definition does not coincide with the classical one since $\zeta = c$ is a boundary point of the contour $l_1$ where the function is not analytic. To determine the function $\omega_1(\zeta)$, we reduce the problem (3.15) to a Riemann-Hilbert problem on an elliptic surface.

Let $\mathcal{R}$ be a genus-1 Riemann surface defined by the algebraic equation

$$u^2 = p(\zeta), \quad p(\zeta) = \zeta(1 - \zeta)(\zeta - m). \quad (3.18)$$

The surface is formed by gluing two copies $\mathbb{C}_1$ and $\mathbb{C}_2$ of the extended complex $\zeta$-plane $\mathbb{C} \cup \{\infty\}$ cut along the segments $l_1$ and $l_0$. The upper sides $l_j^+$ of the cuts $l_j \subset \mathbb{C}_1$ are glued to the lower sides $l_j^-$ of the cuts $l_j \subset \mathbb{C}_2$, and the sides $l_j^- \subset \mathbb{C}_1$ are glued to $l_j^+ \subset \mathbb{C}_2$ ($j = 0, 1$). The function $u(\zeta)$ is single-valued on $\mathcal{R}$:

$$u = \begin{cases} p^{1/2}(\zeta), & (\zeta, u) \in \mathbb{C}_1, \\ -p^{1/2}(\zeta), & (\zeta, u) \in \mathbb{C}_2. \end{cases} \quad (3.19)$$
Here \( p^{1/2}(\zeta) \) is the same branch as the one fixed in Section 3. The pairs \((\zeta, p^{1/2}(\zeta))\) and \((\zeta, -p^{1/2}(\zeta))\) correspond to the points with affix \( \zeta \) lying on the upper and lower sheets, \( \mathbb{C}_1 \) and \( \mathbb{C}_2 \), respectively. The sides of the contour \( \mathcal{L} = l_0 \cup l_1 \) form the symmetry line for the elliptic surface \( \mathcal{R} \) which splits the surface into two symmetric halves. This fact is expressed by the relation between two symmetric points \((\zeta, u) \in \mathbb{C}_1 \) and \((\zeta, u) \in \mathbb{C}_2\): \((\zeta, u) = (\zeta, -u(\zeta)).\)

Introduce next the following auxiliary function:

\[
\Phi(\zeta, u) = \begin{cases} 
-i\omega_1(\zeta), & (\zeta, u) \in \mathbb{C}_1, \\
i\omega_1(\zeta), & (\zeta, u) \in \mathbb{C}_2.
\end{cases}
\] (3.20)

It is directly verified that this function satisfies the symmetry condition

\[
\overline{\Phi(\zeta, u_*)} = \Phi(\zeta, u), \quad (\zeta, u) \in \mathcal{R}.
\] (3.21)

On the symmetry line \( \mathcal{L} \), we can define the boundary values of the function \( \Phi(\zeta, u) \),

\[
\Phi^+(\xi, v) = -i\omega_1(\xi) = -i \text{Re}\omega_1(\xi) + \text{Im}\omega_1(\xi),
\]

\[
\Phi^-(\xi, v) = i\omega(\xi) = i \text{Re}\omega_1(\xi) + \text{Im}\omega_1(\xi), \quad (\xi, v) \in \mathcal{L}, \quad v = u(\xi),
\] (3.22)

where \( \Phi^+(\xi, v) \) (or \( \Phi^-(\xi, v) \)) is the limiting value of the function \( \Phi(\zeta, u) \) on the contour \( \mathcal{L} \) from the upper (lower) sheet of the surface. The boundary conditions (3.15) therefore imply that the function \( \Phi(\zeta, u) \) is the solution to the following Riemann-Hilbert problem.

**Formulation.** Find all functions \( \Phi(\zeta, u) \) analytic in \( \mathcal{R} \setminus \mathcal{L} \), Hölder-continuous up to the boundary \( \mathcal{L} \) apart from the singular points \( a, b, c, \) and \( d \) with the boundary values satisfying the relation

\[
\Phi^+(\xi, v) = G(\xi, v)\Phi^-(\xi, v) + g(\xi, v), \quad (\xi, v) \in \mathcal{L},
\] (3.23)

and the symmetry condition \( \overline{\Phi(\zeta, u_*)} = \Phi(\zeta, u) \). Here

\[
G(\xi, v) = \begin{cases} 
-1, & (\xi, v) \in dab, \\
1, & (\xi, v) \in bcd \cup l_0,
\end{cases}
\]

\[
g(\xi, v) = \begin{cases} 
-2\alpha, & (\xi, v) \in ab, \\
2(\pi - \beta), & (\xi, v) \in da, \\
-2i\sigma', & (\xi, v) \in bcd, \\
0, & (\xi, v) \in l_0.
\end{cases}
\] (3.24)

The function \( \Phi(\zeta, u) \) has a logarithmic singularity at the point \((a, u(a))\), and a simple pole at the point \((c, u(c))\). It is bounded at the points \((b, u(b))\) and \((d, u(d))\), and \( \Phi(\zeta, u) \to 0 \) as \( \zeta \to \infty \).

### 3.3.1 Factorization of the Function \( G \)

We first find a piece-wise meromorphic function \( X(\zeta, u) \) which is symmetric on the surface, \( X(\zeta, u) = \overline{X(\zeta, -u(\zeta))}, \quad (\zeta, u) \in \mathcal{R} \setminus \mathcal{L} \), discontinuous through the contour \( dab \in \mathcal{R} \), and whose boundary values on the contour are linked by

\[
X^+(\xi, v) = -X^-(\xi, v), \quad (\xi, v) \in dab.
\] (3.25)
By using the Weierstrass kernel, an analogue of the Cauchy kernel for the elliptic surface $\mathcal{R}$,

$$dW = \frac{1}{2} \left( 1 + \frac{u}{v} \right) \frac{d\xi}{\xi - \zeta}, \quad (3.26)$$

we can find a function which meets the boundary condition (3.25),

$$\chi(\zeta, u) = \exp \left\{ \frac{1}{4} \int_{dab} \left( 1 + \frac{u}{v} \right) \frac{d\xi}{\xi - \zeta} \right\}. \quad (3.27)$$

The integration in (3.27) is readily carried out in terms of the Legendre elliptic integral of the third type $\Pi$ [GrRy00],

$$\chi(\zeta, u) = \left( \frac{\zeta - b}{\zeta - d} \right)^{1/4} \times \exp \left\{ \frac{u(\zeta)}{2i\zeta \sqrt{m}} \left[ \Pi \left( \sin^{-1} \sqrt{b}, \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}} \right) - \Pi \left( \sin^{-1} \sqrt{d}, \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}} \right) \right] \right\}. \quad (3.28)$$

Here $[(\zeta - b)/(\zeta - d)]^{1/4}$ is a fixed branch of the multi-valued function.

Analysis of formula (3.26) as $\zeta \to \infty$ shows that the function $\chi(\zeta, u)$ has an essential singularity at infinity. To quench the singularity, we consider the function

$$\chi_0(\zeta, u) = \chi_1(\zeta, u)\chi_1(\zeta, u^*) \quad (3.29)$$

where

$$\chi_1(\zeta, u) = \exp \left\{ -\frac{1}{2} \left( \int_{\gamma} +m_a \oint_a +m_b \oint_b \right) \left( 1 + \frac{u}{v} \right) \frac{d\xi}{\xi - \zeta} \right\}. \quad (3.30)$$

The contour $\gamma$ is a continuous curve whose starting and terminal points are $\eta_0 = (\eta_0, u(\eta_0))$ and $\zeta_0 = (\zeta_0, u(\zeta_0))$, respectively. The point $\eta_0$ is an arbitrary fixed point lying on the upper sheet, whilst the point $\zeta_0$ can lie on either sheet. The point $\zeta_0$ and the integers $m_a$ and $m_b$ are not fixed and are to be recovered from a condition which guarantees the boundedness of the solution to the problem (3.25) at infinity. The contour $\gamma$ does not cross the $a$- and $b$-cross-sections. In the case $\zeta_0 \in \mathbb{C}_2$, it passes through the point $\zeta = 0$, a branch point of the surface $\mathcal{R}$, and consists of two parts, $(\eta_0, 0) \subset \mathbb{C}_1$ and $(0, \zeta_0) \subset \mathbb{C}_2$. If it turns out that the point $\zeta_0$ lies on the upper sheet, then the contour $\gamma$ can be taken as the straight line joining these points provided it does not cross the segment $[1, +\infty)$.

The system of canonical cross-sections $\{a, b\}$ of the surface $\mathcal{R}$ is chosen as follows. The contour $a$ lies on both sheets of the surface and coincides with the banks of the semi-infinite cut $l_0$ (fig. 3.2). The loop $b$ consists of the segments $[m, 1] \subset \mathbb{C}_1$ and $[1, m] \subset \mathbb{C}_2$. The positive direction on the loop $a$ is chosen such that the upper sheet is on the left. The loop $a$ intersects the loop $b$ at the branch point $\zeta = m$ from left to the right (the positive direction on the cross-section $b$ is chosen according to the left-hand traffic rule).
Consider now the function \( X(\zeta, u) = \chi(\zeta, u) \chi_0(\zeta, u) \). By using the Cauchy theorem, we may simplify the function \( X(\zeta, u) \),

\[
X(\zeta, u) = \exp \left\{ \frac{1}{4} \int_{dab} \left( 1 + \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} \right. \\
\left. - \frac{1}{2} \int_{\gamma} \left( 1 + \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} - \frac{1}{2} \int_{\gamma} \left( 1 - \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} - 2m_a \int_{a^+} \frac{u(\zeta)}{u(\xi)} \frac{d\xi}{\xi - \zeta} \right\},
\]

where \( a^+ = l_0^+ \) is the upper bank of the cut \( l_0 \). By the Sokhotski-Plemelj formulas, this function satisfies the boundary condition (3.25). The second and the third integrals in (3.31) are discontinuous through the contours \( \gamma \) and \( \gamma_* \), respectively, where the contour \( \gamma_* \) is symmetric to \( \gamma \) with respect to the line \( L \). The fourth integral is discontinuous through the contour \( a^+ \). The jumps are multiples of \( 2\pi i \), and therefore the function \( X(\zeta, u) \) is continuous through the contours \( \gamma, \gamma_* \), and \( a^+ \). By analyzing the integrals in (3.31) at the points \( b, d, \eta_0 \), and \( \zeta_0 \) we find that the function \( X(\zeta, u) \) vanishes at the point \( b \), has an integrable singularity at the point \( d \), a simple zero at the point \( \eta_0 \) and a simple pole at the point \( \zeta_0 \):

\[
X(\zeta, u) = O((\zeta - b)^{1/2}), \quad \zeta \to b, \quad X(\zeta, u) = O((\zeta - d)^{-1/2}), \quad \zeta \to d,
\]

\[
X(\zeta, u) \sim A_0(\zeta - \eta_0), \quad (\zeta, u) \to \eta_0, \quad X(\zeta, u) \sim A_1(\zeta - \zeta_0)^{-1}, \quad (\zeta, u) \to \zeta_0. \tag{3.32}
\]

Here \( A_0 \) and \( A_1 \) are nonzero constants. We emphasize that at the conjugate points, \((\bar{b}, u(\bar{b})), (\bar{d}, u(\bar{d})), (\bar{\eta}_0, u(\bar{\eta}_0))\), and \((\bar{\zeta}_0, u(\bar{\zeta}_0))\), the function \( X(\zeta, u) \) is bounded and, in general, does not vanish.
3.3.2 Jacobi Inversion Problem

Now, the function $X(\zeta, u)$ must be bounded at infinity. By expanding the integrals in (3.31) in a neighborhood of the two infinite points of the surface $\mathcal{R}$, we find that the necessary and sufficient condition for the boundedness of the function $X(\zeta, u)$ at the infinite points of the surface, $(\infty, \infty)_1$ and $(\infty, \infty)_2$, is

$$
\frac{1}{4} \int_{dab} \frac{d\xi}{p^{1/2}(\xi)} - \frac{1}{2} \int_{\gamma} \frac{d\xi}{u(\xi)} + \frac{1}{2} \int_{\gamma} \frac{d\xi}{u(\xi)} - 2m_a \int_{a^+} \frac{d\xi}{p^{1/2}(\xi)} = 0.
$$

(3.33)

This nonlinear equation with respect to $\zeta_0$ and $m_a$ can be reduced to a Jacobi inversion problem. Indeed, since

$$
2 \int_{a^+} \frac{d\xi}{p^{1/2}(\xi)} = \oint_a \frac{d\xi}{u(\xi)} = A, \quad \Re A = 0,
$$

$$
\oint_b \frac{d\xi}{u(\xi)} = B, \quad \Im B = 0,
$$

(3.34)

and the left-hand side in (3.33) is pure imaginary, we have the following complex equation

$$
\zeta_0 \int_{0} d\xi \frac{d\xi}{u(\xi)} + m_a A + m_b B = g_0,
$$

(3.35)

where

$$
g_0 = \frac{1}{4} \int_{dab} \frac{d\xi}{p^{1/2}(\xi)} + \int_{0} \frac{d\xi}{p^{1/2}(\xi)}.
$$

(3.36)

This is the classical genus-1 Jacobi inversion problem. Notice that the imaginary part of this equation coincides with the condition (3.33) which guarantees the boundedness of the function $X(\zeta, u)$ at infinity. The integrals in (3.34) are the $A$- and $B$- periods of the abelian integral in the left-hand side in (3.35) and can be expressed through the Legendre complete elliptic integrals [Low50]

$$
A = -2i \int_{m}^{\infty} \frac{d\xi}{\sqrt{\xi(\xi - 1)(\xi - m)}} = -4ikK(k),
$$

$$
B = 2 \int_{1}^{m} \frac{d\xi}{\sqrt{\xi(\xi - 1)(\xi - m)}} = 4kK'(k),
$$

(3.37)

Here $k = m^{-1/2}$. To solve the problem (3.35), we make the substitution $\zeta = \tau^2$ and notice that

$$
p^{1/2}(\zeta) = -\frac{i\tau}{k}p_0^{1/2}(\tau), \quad p_0^{1/2}(\tau) = [(1 - \tau^2)(1 - k^2\tau^2)]^{1/2},
$$

(3.38)
where the function $p_0^{1/2}(\tau)$ is single-valued in the $\tau$-plane cut along the segments $[-1, 1]$ and the segment joining the points $\pm 1/k$ and passing through the infinite point. The branch of the function is fixed by the condition $p_0^{1/2}(0^+) = 1$.

Assume first that $\zeta_0 \in \mathbb{C}_1$. The Jacobi inversion problem becomes then

$$
\sqrt{\zeta_0} \int_0^\tau \frac{d\tau}{p_0^{1/2}(\tau)} = 2m_a K + 2m_b K' - \frac{ig_0}{2k},
$$

(3.39)

where $\sqrt{\zeta_0}$ is the value of an arbitrary fixed branch of the function $\zeta^{1/2}$ at $\zeta = \zeta_0$. This immediately defines

$$
\sqrt{\zeta_0} = -(-1)^m \text{sn} \frac{ig_0}{2k}, \quad \zeta_0 = \text{sn}^2 \frac{ig_0}{2k}.
$$

(3.40)

Since the affix $\zeta_0$ is fixed we can evaluate

$$
I_\pm = \frac{1}{4} \int_{dab} \frac{d\xi}{p^{1/2}(\xi)} + \int_0^{\zeta_0} \frac{d\xi}{p^{1/2}(\xi)} \pm \int_0^{\zeta_0} \frac{d\xi}{p^{1/2}(\xi)},
$$

(3.41)

and from (3.39) find the numbers $m_a$ and $m_b$,

$$
m_a = -\text{Im} I_-, \quad m_b = \text{Re} I_-.
$$

(3.42)

If both the numbers are integers, then the problem is solved, and the point $\zeta_0$ lies on the upper sheet. Otherwise, it falls on the lower sheet. In this case, similarly, we get

$$
\zeta_0 = \text{sn}^2 \frac{ig_0}{2k}, \quad m_a = -\text{Im} I_+, \quad m_b = \text{Re} I_+.
$$

(3.43)

### 3.3.3 Solution to the Riemann-Hilbert Problem

Having the solution to the Jacobi inversion problem we are now equipped with a bounded at infinity solution of the factorization problem (3.25). With it, we can represent the term $g(\xi, v)[X^+(\xi, v)]^{-1}$ as

$$
\Psi^+(\xi, v) - \Psi^-(\xi, v) = \frac{g(\xi, v)}{X^+(\xi, v)}, \quad (\xi, v) \in l_0 \subset \mathcal{R},
$$

(3.44)

By the Sokhotski-Plemelj formulas, the function $\Psi(\zeta, u)$ has the form

$$
\Psi(\zeta, u) = -\frac{\alpha}{2\pi i} \int_{ab} \frac{(1 + u/v)d\xi}{X^+(\xi, v)(\xi - \zeta)} + \frac{\pi - \beta}{2\pi i} \int_{da} \frac{(1 + u/v)d\xi}{X^+(\xi, v)(\xi - \zeta)} - \frac{\sigma'}{2\pi} \int_{bcd} \frac{(1 + u/v)d\xi}{X(\xi, v)(\xi - \zeta)}.
$$

(3.45)
Clearly, the function $X(\zeta, u)\Psi(\zeta, u)$ satisfies the Riemann-Hilbert boundary condition (3.23). The general solution can be written as follows:

$$\Phi(\zeta, u) = X(\zeta, u)[\Psi(\zeta, u) + \Omega(\zeta, u)],$$

(3.46)

where $\Omega(\zeta, u)$ is a rational function on the surface $\mathcal{R}$. To define this function, we sum up the properties of the functions in (3.46).

(i) Because of the simple pole of the function $\Phi(\zeta, u)$ at the point $\zeta = c$ the function $\Omega(\zeta, u)$ must have a simple pole at the point $\zeta = c$ and be bounded at the point $\zeta = \bar{c} = a$. Then since the function $\Psi(\zeta, u)$ has a logarithmic singularity at the point $\zeta = a$, the same property is valid for the solution to the Riemann-Hilbert problem (3.23) as it is required.

(ii) The function $X(\zeta, u)$ has a simple zero at the point $\eta_0$. Therefore, the function $\Omega(\zeta, u)$ has to have a simple pole at the point $\eta_0$.

(iii) The function $X(\zeta, u)$ has a simple pole at the point $\zeta_0$. Therefore, the function $\Psi(\zeta, u) + \Omega(\zeta, u)$ has to have a simple zero at the point $\zeta_0$.

(iv) The function $X(\zeta, u)$ has a square-root singularity at the point $\zeta = d$. Thus, the function $\Psi(\zeta, u) + \Omega(\zeta, u)$ has to have a simple zero at the point $\zeta = d$.

(v) The function $\Phi(\zeta, u)$ vanishes as $\zeta = \infty$.

(vi) The function $\Phi(\zeta, u)$ must be symmetric, $\Phi(\zeta, u) = \Phi(\zeta^*, u^*)$.

The general form of the function $\Omega(\zeta, u)$ which meets the conditions (i), (ii), and (vi) is given by

$$\Omega(\zeta, u) = iM_0\frac{u(\zeta) + u(c)}{\zeta - c} + (M_1 + iM_2)\frac{u(\zeta) + u(\eta_0)}{\zeta - \eta_0} - (M_1 - iM_2)\frac{u(\zeta) - u(\eta_0)}{\zeta - \bar{\eta}_0},$$

(3.47)

where $M_j$ ($j = 0, 1, 2$) are real constants to be fixed. It is directly verified that the function $\Psi(\zeta, u)$ is symmetric, and the condition (vi) is also met by the function $\Phi(\zeta, u)$. To satisfy the conditions (iii) and (iv) we put

$$\Psi(\zeta_0, u(\zeta_0)) + \Omega(\zeta_0, u(\zeta_0)) = 0,$$

$$\Psi(d, u(d)) + \Omega(d, u(d)) = 0.$$ 

(3.48)

We emphasize that the former relation is a complex equation whilst the last one is a real equation. Next, we derive the principal term of the expansion of the function $\Phi(\zeta, u)$ at infinity. Using the condition (v) we find

$$M_0 = \Psi_0 - 2M_2,$$

(3.49)

where $\Psi_0$ is real and

$$\Psi_0 = \frac{\alpha}{2\pi} \int_{ab} \frac{d\xi}{vX^+(\xi, v)} - \frac{\pi - \beta}{2\pi} \int_{da} \frac{d\xi}{vX^+(\xi, v)} + \frac{i\sigma'}{2\pi} \int_{bcd} \frac{d\xi}{vX(\xi, v)}.$$ 

(3.50)

We now determine the real constants $M_1$ and $M_2$ and the yaw angle $\delta$. Let

$$\rho_0(\zeta) = \frac{u(\zeta) + u(\eta_0)}{\zeta - \eta_0}, \quad \rho_1(\zeta) = \frac{u(\zeta) - u(\eta_0)}{\zeta - \bar{\eta}_0}, \quad \rho_2(\zeta) = \frac{u(\zeta) + u(c)}{\zeta - c},$$
\[ \Psi_1 = \Psi(\zeta_0, u(\zeta_0)), \quad \Psi_2 = \Psi(d, u(d)). \]  

Since the position of the wedge is described by the angles \( \alpha = \alpha_0 + \delta \) and \( \beta = \beta_0 + \delta \) it will be convenient to represent the constants \( M_j \) and \( \Psi_j \) \( (j = 0, 1, 2) \) in the form

\[ M_j = M_j^0 + \delta M_j^1, \quad \Psi_j = \Psi_j^0 + \delta \Psi_j^1, \]  

(3.52)

where \( \Psi_j^0 = \Psi_j|_{\alpha=\alpha_0, \beta=\beta_0} \), and the constants \( \Psi_j^1 \) coincide with \( \Psi_j \) if \( \alpha \) and \( \pi - \beta \) are replaced by 1 and \( -1 \), respectively. On using equations (3.48), it is a matter of simple algebra to show that

\[ M_{\nu}^1 = \frac{1}{\Delta} \{ [\Psi_{\nu}^\nu \Im \rho_2(\zeta_0) - \Re \Psi_{\nu}^\nu] \Re \mu_1 - [\Psi_{\nu}^\nu \Re \rho_2(\zeta_0) + \Im \Psi_{\nu}^\nu] \Im \mu_1 \}, \]

\[ M_{\nu}^2 = -\frac{1}{\Delta} \{ [\Psi_{\nu}^\nu \Re \rho_2(\zeta_0) + \Im \Psi_{\nu}^\nu] \Re \mu_2 + [\Psi_{\nu}^\nu \Im \rho_2(\zeta_0) - \Re \Psi_{\nu}^\nu] \Im \mu_2 \}, \quad \nu = 0, 1, \]

\[ M_{\nu}^3 = \Psi_{\nu}^\nu + \imath \rho_2(d) M_{\nu}^0 + \{ \rho_0(d) - \rho_1(d) \} M_{\nu}^1 + i \{ \rho_0(d) + \rho_1(d) \} M_{\nu}^2, \quad \nu = 0, 1. \]  

(3.53)

where

\[ \Delta = \Re \mu_1 \Re \mu_2 + \Im \mu_1 \Im \mu_2, \]

\[ \mu_1 = \rho_0(\zeta_0) + \rho_1(\zeta_0) - 2 \rho_2(\zeta_0), \quad \mu_2 = \rho_0(\zeta_0) - \rho_1(\zeta_0), \]

\[ \Delta_{\nu} = \Psi_{\nu}^\nu + \imath \rho_2(d) M_{\nu}^0 + \{ \rho_0(d) - \rho_1(d) \} M_{\nu}^1 + i \{ \rho_0(d) + \rho_1(d) \} M_{\nu}^2, \quad \nu = 0, 1. \]

Analysis of the right-hand side in the third formula in (3.53) shows that it is real.

### 3.4 Numerical Results

#### 3.4.1 Parameters \( a, b, d, \) and \( m \)

In the preceding sections, the expression of the derivative \( df/d\zeta \) of the conformal mapping \( z = f(\zeta) \) was obtained by formula (3.6) in terms of the functions \( \omega_0(\zeta) \) and \( \omega_1(\zeta) = \imath \Phi(\zeta, u), (\zeta, u) \in C_1 \) defined by (3.10) and (3.46), respectively. There are four parameters, \( a, b, d, \) and \( m \), left to be fixed. In general, the conformal mapping found does not satisfy the single-valuedness condition

\[ \int_{L_1^*} dz = 0, \]  

(3.55)

and the points \( a, b, \) and \( d \) of the parametric domain are not necessarily the images of the points \( A, B, \) and \( D \), respectively. Here \( L_1^* \) is a closed contour in the flow domain such that \( L_1^* \supset L_1 \). To ensure this we require

\[ \int_{L_1^*} \tilde{\omega}_0(\zeta)e^{-\omega_1(\zeta)}d\zeta = 0, \]

\[ \lambda_1 \sin \alpha - N \Omega_1 = 0, \] 

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\[
\lambda_2 \sin \beta - N \Omega_2 = 0, \tag{3.56}
\]

where
\[
\Omega_1 = \text{Im} \int_{ab} \tilde{\omega}_0(\zeta)e^{-\omega_l(\zeta)}d\zeta, \quad \Omega_2 = \text{Im} \int_{da} \tilde{\omega}_0(\zeta)e^{-\omega_l(\zeta)}d\zeta, \tag{3.57}
\]

and \(l_1^*\) is a closed contour which does not cross the cut \(l_0\) and \(l_1^* \supset l_1\). For numerical purposes, \(l_1^*\) can be chosen as follows
\[
l_1^* = \left\{ \zeta : \left| \zeta - \frac{1}{2} \right| = r^* \right\}, \quad \frac{1}{2} < r^* < m - \frac{1}{2}. \tag{3.58}\]

The first equation in (3.56) is complex, and the other two are real. Thus, equations (3.56) constitute a system of four real nonlinear equations for the unknown parameters \(a, b, d,\) and \(m\). Without loss of generality we may accept that \(a \in l_1^\ast\), \(a < b < 1\), and \(0 < d < a\). Based on the results [AS08] for a supercavitating wedge in a plane we assume that \(b \in l_1^{\ast}\) and \(d \in l_1^{\ast}\). For all the parameters of the problem chosen for the tests this assumption is confirmed numerically. The unknown parameters \(a, b, d,\) and \(m\) have to meet the following constraints: \(0 < d < a < b < 1\) and \(m > 1\). To eliminate the constraints we introduce a new vector \(x = \{x_0, x_1, x_2, x_3\}\) [Tre80] whose components are
\[
x_0 = \ln(m - 1), \quad x_1 = \ln \frac{d}{a - d}, \quad x_2 = \ln \frac{a - d}{b - a}, \quad x_3 = \ln \frac{b - a}{1 - b},
\]
\[-\infty < x_j < +\infty, \quad j = 0, 1, 2, 3. \tag{3.59}\]

The nonlinear system (3.56) written in the form \(\mathbf{r}(\mathbf{x}) = 0\) for the new unknown parameters is solved by the Newton method
\[
\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - [F(\mathbf{x}^{(n)})]^{-1}\mathbf{r}(\mathbf{x}^{(n)}), \quad n = 0, 1, \ldots, \tag{3.60}
\]

where \([F(\mathbf{x})]^{-1}\) is the inverse of the Jacobian matrix \(F = (\partial r_i/\partial x_j), i, j = 0, \ldots, 3,\) evaluated approximately
\[
\frac{\partial r_i}{\partial x_j} \approx r_i(x_0, \ldots, x_j + \delta x_j, \ldots, x_3) - r_i(x_0, \ldots, x_j, \ldots, x_3)/\delta x_j. \tag{3.61}\]

The parameters of the conformal mapping are recovered through the solution to the system \(\mathbf{r}(\mathbf{x}) = 0\) by the formulas
\[
m = 1 + e^{x_0}, \quad a = \frac{x_2 x_3 (1 + x_1^* + x_2^*)}{\Delta^*}, \quad b = \frac{x_3 (1 + x_2 + x_1^* x_2^* + x_2 x_3^*)}{\Delta^*}, \quad d = \frac{x_1^* x_2^* x_3^*}{\Delta^*},
\]
\[
\Delta^* = 1 + x_3^* (1 + x_2^* + x_1^* x_2^*), \quad x_j^* = e^{x_j}, \quad j = 1, 2, 3. \tag{3.62}\]

The idea to work with the variables \(x_j\), not with the original unknowns \(a, b, d,\) and \(m\), is fruitful for two reasons. Firstly, we get rid of the constraints and can apply the classical Newton scheme, and, secondly, the variations of an approximate solution \(\mathbf{x}^{(n)}\) in a neighborhood of the exact solution \(\mathbf{x}\) is bigger than the variations of the corresponding approximate values of the parameters \(a, b, d,\) and \(m\).
Table 3.1: The values of the parameters $a$, $b$, $d$, $m$, and the yaw angle $\delta$ in the case (3.63) for some values of depth $h$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$a$</th>
<th>$b$</th>
<th>$d$</th>
<th>$m - 1$</th>
<th>$\delta$</th>
</tr>
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<td>0.676380</td>
<td>0.379709</td>
<td>1.566769</td>
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<td>0.753061</td>
<td>0.450134</td>
<td>0.181660</td>
<td>0.002286</td>
</tr>
<tr>
<td>10</td>
<td>0.710496</td>
<td>0.847700</td>
<td>0.528498</td>
<td>0.182847 $\cdot 10^{-4}$</td>
<td>0.016236</td>
</tr>
<tr>
<td>5</td>
<td>0.783349</td>
<td>0.915387</td>
<td>0.576899</td>
<td>0.138606 $\cdot 10^{-2}$</td>
<td>0.04620</td>
</tr>
<tr>
<td>4</td>
<td>0.803352</td>
<td>0.932955</td>
<td>0.587015</td>
<td>0.488768 $\cdot 10^{-3}$</td>
<td>0.060882</td>
</tr>
<tr>
<td>3</td>
<td>0.829184</td>
<td>0.952958</td>
<td>0.601417</td>
<td>0.106904 $\cdot 10^{-3}$</td>
<td>0.082173</td>
</tr>
<tr>
<td>2</td>
<td>0.824997</td>
<td>0.967054</td>
<td>0.528133</td>
<td>0.619607 $\cdot 10^{-3}$</td>
<td>0.146737</td>
</tr>
<tr>
<td>1.5</td>
<td>0.820221</td>
<td>0.969388</td>
<td>0.500073</td>
<td>0.238153 $\cdot 10^{-3}$</td>
<td>0.165417</td>
</tr>
</tbody>
</table>

Computations implemented for the parameters

\[
\alpha_0 = \frac{\pi}{3}, \quad \beta_0 = \frac{2\pi}{3}, \quad \lambda_1 = 1, \quad \lambda_2 = 1, \quad \sigma = 0.5, \quad (3.63)
\]

and for different values of the depth $h$ show that indeed $a, b, d \in l_1^+$ ($c = \bar{a} \in l_1^-$). The effect of the free boundary on the location of the parameters of the conformal mapping is substantial even for big depths (table 3.1).

For the parameters chosen it is found that $a \to 0.5$, $b \to 0.64848$, $d \to 1-b$, $m \to \infty$, and $\delta \to 0$ when $h \to \infty$. This is consistent with the corresponding result for a wedge in a plane [AS08]. When $h$ is decreasing then, first, the four parameters, $a$, $b$, $c$, and $d$ are moving to the right. But then, for small $h$, the parameters $a$, $c$, and $d$ are moving back while the parameter $b$ continues approaching the branch point $\zeta = 1$ of the surface. As for the parameter $m$, when $h$ is increasing it grows as well. However, the rate of growth is different. It turns out that when the wedge is close to the free surface and the depth is in the range $h \in (1.5, 4)$, the parameter $m-1$ is very small: $m-1 \in (0.2 \cdot 10^{-5}, 0.5 \cdot 10^{-3})$. Also, when a symmetric wedge is approaching the free surface, the angle of yaw $\delta$ is increasing and becomes noticeable for small $h$ (table 3.1).

### 3.4.2 Cavity Shape, Free surface, Drag, Lift, and Circulation

To restore the shape of the cavity, we integrate the function $df/d\zeta$ over the contours $b\tau$ ($\tau \in bc$) and $d\tau$ ($\tau \in dc$) to obtain the upper and lower boundary, respectively,

\[
z(\tau) = B + \int_{b\tau} \frac{df}{d\zeta} d\zeta, \quad \tau \in bc \quad (z \in BC^+),
\]

\[
z(\tau) = D + \int_{d\tau} \frac{df}{d\zeta} d\zeta, \quad \tau \in dc \quad (z \in DC^-).
\quad (3.64)
Similarly, for the free surface,

\[ z(\tau) = B + \int_{b\tau} \frac{df}{d\zeta} d\zeta, \quad \tau \in \lambda_0, \quad z \in E^{-}E^{+}. \]  

(3.65)

The fig. 3.3 shows the cavity shape and the free boundary for the parameters (3.63) when \( m = 1.00001 \). In this case \( h = 2.11527, \quad a = \bar{c} = 0.83010, \quad b = 0.96627, \) and \( d = 0.54931 \). It is seen that the presence of the free boundary breaks the symmetry of the cavity. On the other hand, the supercavitating wedge creates waves on the free surface. Their amplitude becomes higher when the wedge approaches the boundary (fig. 3.3 and fig. 3.4) or when the cavitation number decreases (fig. 3.4). It is found that for big depths, the cavity is practically symmetric as it should be [AS08]. When \( h \) is decreasing, the corresponding parameters \( b, \bar{a} = \bar{c} \), and \( d \) are moving to the left end of the segment \([0, 1]\). After \( h \) has passed a certain critical point, and the wedge is close enough to the free surface, the parameter \( d \) is moving back to the left. The cavity grows and the yaw angle increases when the cavitation number \( \sigma \) decreases.

We have also computed the solution for different angles \( \alpha_0 (\beta_0 = \pi - \alpha_0) \) (fig. 3.5). It turns out that the cavity and the amplitude of the waves on the free surface grow when the angle \( \alpha_0 \) grows. The length of the cavity and the amplitude of the waves attain their maximum for \( \alpha_0 = \frac{\pi}{2} \), for a hydrofoil orthogonal to the free surface at rest. The same tendency is observed for the yaw angle: \( \delta = 0.0106 \) for \( \alpha_0 = \frac{\pi}{6} \), \( \delta = 0.0284 \) for \( \alpha_0 = \frac{\pi}{4} \), \( \delta = 0.0462 \) for \( \alpha_0 = \frac{\pi}{3} \), and \( \delta = 0.0665 \) for \( \alpha_0 = \frac{\pi}{2} \).

In fig. 3.6, the cavity shape and the free surface for the symmetric wedge \( \lambda_1 = \lambda_2 = 1 \) are compared with the corresponding profiles for two nonsymmetric wedges, \( \lambda_2 = 2 \) and \( \lambda_2 = 3 \), while \( \lambda_1 = 1 \).
Figure 3.4: The cavity shape and the free surface when \( \alpha_0 = \frac{\pi}{3}, \beta_0 = \frac{2\pi}{3}, \lambda_1 = \lambda_2 = 1, \) and \( h = 5 \) for some values of the cavitation number \( \sigma: \sigma = 1 \) (1), \( \sigma = 0.5 \) (2), \( \sigma = 0.3 \) (3).

Fig. 3.7 shows the effect on the length, \( l_c \), of the upper and lower boundaries of the cavity of changing the depth \( h \). It is seen that when the wedge approaches the surface both the lengths decrease. The upper boundary becomes longer than the lower one for small depths.

The spiral shape of the cavity at a neighborhood of the upper and lower vortices \( C^+ \) and \( C^- \) is shown in fig. 3.8a and fig. 3.8c, respectively. A closer neighborhood of the same points is given in fig. 3.8b and fig. 3.8d. By approaching the lower vortex \( C^- \), the spiral structure of the cavity becomes evident (fig. 3.9).

To clarify how the flow behaves near the centers of the vortices, we plot the upper branch of the streamlines \( \psi(x, y) = C_0, C_0 = \frac{h v_{\infty}}{20} \) (1), \( h v_{\infty}/50 \) (2), \( h v_{\infty}/100 \) (3), \( h v_{\infty}/1000 \) (4), and \( h v_{\infty}/10000 \) (5) close to the streamline \( \psi(x, y) = 0 \) (6) which defines the cavity boundary (fig. 3.10). The preimages of these streamlines are shown in fig. 3.11. It is seen that the preimage of the streamline \( \psi = 0 \) is orthogonal to the slit \([0, 1]\) while the others are not. For small values of the constant \( C_0 \), the streamline \( \psi(x, y) = C_0 \) first spirals and then proceeds to sheets of the Riemann surface of a logarithmic function with the branch points \( C^+ \) and \( C^- \). The number of sheets used for modeling the flow is infinite for \( C_0 = 0 \) and decreases as \( C_0 \) increases. For example, for \( C_0 = \frac{h v_{\infty}}{100} \) (line 3 in fig. 3.10), the flow does not leave the physical plane.

Next we determine the drag, \( X \), and the lift, \( Y \), by

\[
X + iY = -i \int_{DAB} (p - p_c) dz, \tag{3.66}
\]

where \( p \) is the water pressure in \( \tilde{D} \) and \( p_c \) is the vapor pressure in the cavity. By using
the Bernoulli law, \( p - p_c = \frac{1}{2} \rho (v_c^2 - V^2) \), \( V = |v| \), and since
\[
V^2 = v_\infty^2 e^{2 \text{Re} \omega_1(\zeta)},
\]
we obtain
\[
X + iY = -\frac{i \rho}{2} \int_{\text{dab}} (v_c^2 - v_\infty^2 e^{2 \text{Re} \omega_1(\zeta)}) e^{-\omega_1(\zeta)} \omega_0(\zeta) d\zeta.
\]

The drag and lift coefficients, \( C_X \) and \( C_Y \), related to the velocity at infinity \( v_\infty \) and the length \( \lambda = \lambda_1 \sin \alpha + \lambda_2 \sin \beta \), are
\[
C_X + iC_Y = \frac{2}{\rho v_\infty^2 \lambda} (X + iY).
\]

These coefficients can be computed by the formula
\[
C_X + iC_Y = -\frac{i}{\lambda} \int_{\text{dab}} [\sigma + 1 - e^{2 \text{Re} \omega_1(\zeta)}] e^{-\omega_1(\zeta)} \omega_0(\zeta) d\zeta.
\]

Table 3.2 shows the effect on the drag and lift coefficients of varying the depth \( h \). It is seen that the drag coefficient is decreasing and the lift coefficient is increasing when the wedge is approaching the free boundary.

Fig. 3.10 and fig. 3.11 show the effect on the drag and lift coefficients of changing the cavitation number. It is seen that the drag coefficient is a linearly increasing function of the cavitation number \( \sigma \) while lift first increases, attains its maximum and then decreases. The variation of the lift coefficient is small in comparison with that of the
Table 3.2: The values of the drag and lift coefficients $C_X$ and $C_Y$, and the circulation $\Gamma$ in the case (3.63) for some values of depth $h$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$C_X$</th>
<th>$C_Y$</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>2.043086</td>
<td>-0.004891</td>
<td>-0.003292</td>
</tr>
<tr>
<td>100</td>
<td>2.012256</td>
<td>-0.004823</td>
<td>-0.004020</td>
</tr>
<tr>
<td>75</td>
<td>2.004884</td>
<td>-0.004707</td>
<td>-0.004202</td>
</tr>
<tr>
<td>50</td>
<td>1.998008</td>
<td>-0.004248</td>
<td>-0.006300</td>
</tr>
<tr>
<td>25</td>
<td>1.992799</td>
<td>-0.000513</td>
<td>-0.025780</td>
</tr>
<tr>
<td>10</td>
<td>1.991986</td>
<td>0.026710</td>
<td>-0.325814</td>
</tr>
<tr>
<td>5</td>
<td>1.987338</td>
<td>0.098340</td>
<td>-1.225218</td>
</tr>
<tr>
<td>4</td>
<td>1.983682</td>
<td>0.114038</td>
<td>-1.404919</td>
</tr>
<tr>
<td>3</td>
<td>1.975224</td>
<td>0.15548</td>
<td>-2.204043</td>
</tr>
<tr>
<td>2</td>
<td>1.947963</td>
<td>0.279897</td>
<td>-3.510621</td>
</tr>
<tr>
<td>1.5</td>
<td>1.938851</td>
<td>0.315563</td>
<td>-3.571451</td>
</tr>
</tbody>
</table>

Table 3.3: The values of the coefficients $C_X$, $C_Y$, and $C_n$ in the case (3.63) when $h = 5$ for some values of the cavitation number $\sigma$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>0.175</th>
<th>0.2</th>
<th>0.25</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_Y$</td>
<td>0.114</td>
<td>0.124</td>
<td>0.154</td>
<td>0.123</td>
<td>0.113</td>
<td>0.098</td>
<td>0.088</td>
<td>0.083</td>
<td>0.075</td>
<td>0.071</td>
<td>0.067</td>
</tr>
<tr>
<td>$C_n$</td>
<td>1.518</td>
<td>1.550</td>
<td>1.605</td>
<td>1.711</td>
<td>1.819</td>
<td>1.990</td>
<td>2.094</td>
<td>2.239</td>
<td>2.373</td>
<td>2.553</td>
<td>2.700</td>
</tr>
</tbody>
</table>
Figure 3.6: The cavity shape and the free surface for a nonsymmetric wedge when \( \alpha_0 = \frac{\pi}{3}, \beta_0 = \frac{2\pi}{3}, \lambda_0 = 1, \sigma = 0.3, h = 5 \), and \( \lambda_1 = 1 \) for some values of \( \lambda_2 \): \( \lambda_2 = 1 \) (1), \( \lambda_2 = 2 \) (2), and \( \lambda_2 = 3 \) (3).

Drag coefficient, and the value \( C_n = \sqrt{C_X^2 + C_Y^2} \) monotonically increases (see table 3.3) as it does for a supercavitating plate in a plane [Gur79].

Next, we determine the circulation of the velocity around the closed contour \( L_1 = ABCDA \),

\[
\Gamma = \int_{l_1^*} \frac{dw}{d\zeta} d\zeta = N v_\infty \int_{l_1^*} \tilde{\omega}_0(\zeta) d\zeta,
\]

where \( l_1^* \) is the preimage of the contour \( L_1 \). In table 3.2, we present the values of the circulation \( \Gamma/v_\infty \) for some values of the depth \( h \).

Finally, we compute the singularity factor \( K \) in the Terent’ev formula (3.5),

\[
K = -\lim_{\zeta \to c} \omega_1(\zeta) [w(z) - w(C)]^{1/2}.
\]

Since

\[
\left. \frac{dw}{d\zeta} \right|_{\zeta = c} = 0, \quad \left. \frac{d^2w}{d\zeta^2} \right|_{\zeta = c} = \frac{v_\infty N}{\sqrt{|p(c)|}},
\]

by expanding the function \( w(z(\zeta)) \) in a Taylor series, we obtain

\[
w(z) - w(C) \sim \frac{v_\infty N}{2\sqrt{|p(c)|}} (\zeta - c)^2, \quad z \to C \quad (\zeta \to c).
\]

Numerical results for different data show that \( N < 0 \). Because of the chosen branch of the function \( [w(z) - w(C)]^{1/2} \), we have \( \arg[-(\zeta - c)^2] \in [-\pi, \pi] \). Now, \( \arg(\zeta - c) \in [-\pi, 0] \), and therefore, \( [N(\zeta - c)^2]^{1/2} = i\sqrt{|N|}(\zeta - c) \). This implies

\[
[w(z) - w(C)]^{1/2} \sim i \sqrt{\frac{v_\infty |N|}{2\sqrt{|p(c)|}}}(\zeta - c), \quad \zeta \to c.
\]
Figure 3.7: The length of the upper (the solid line) and lower (the broken line) boundaries of the cavity vs the depth $h$ when $\alpha_0 = \frac{\pi}{3}$, $\beta_0 = \frac{2\pi}{3}$, $\lambda_1 = \lambda_2 = 1$, and $\sigma = 0.5$.

The resulting formula for the factor $K$ comes from (3.46) and (3.47),

$$K = -M_0 |X(c, p^{1/2}(c))| \sqrt{2v_\infty |N| \sqrt{|p(c)|}}. \quad (3.76)$$

Numerical tests implemented for different values of the parameters of the problem show that this constant is indeed positive. For example, for $v_\infty = 1$, $h = 5$ and the data (3.63), we have $K = 0.502650$ ($M_0 = -0.30993$ and $N = -7.32526$). Notice that the other parameters in this case are $a = 0.783349$, $b = 0.915387$, $d = 0.576899$, and $m = 1.00138606$.

### 3.4.3 Numerical Aspects of the Algorithm

In order to recover the mapping parameters, the shape of the cavity, the free surface and drag and lift, we need to compute some integrals, regular and singular. The first quadratures come from the solution to the factorization problem (3.25). Its solution for $(\zeta, u) \in \mathcal{R} \setminus \mathcal{L}$ is given by formula (3.31) and the boundary value of this function on the contour $bcd$ becomes $X^+(\xi, v) = X(\xi, v)$. The first integral in the representation of $X(\xi, v)$ is singular and is understood in the sense of the principal value. To evaluate the exponent of this integral for $\xi \in t_1^+$, it is convenient to transform it into the form

$$\exp \left\{ \frac{1}{4} \int_{dab} \left( 1 + \frac{u(\xi)}{u(\tau)} \right) \frac{d\tau}{\tau - \xi} \right\}$$
The cavity shape in the vicinity of the upper (a, b) and the lower (c, d) vortices ($\alpha_0 = \frac{2\pi}{3}$, $\beta_0 = \frac{2\pi}{3}$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\sigma = 0.5$, and $h = 5$).

$$\left| \frac{\xi - b}{\xi - d} \right|^{1/2} \exp \left\{ \frac{1}{4} \int_{dab} \left( -1 + \frac{u(\xi)}{u(\tau)} \right) \frac{d\tau}{\tau - \xi} \right\}, \quad \xi \in l_1^+, \quad dab \subset l_1^+. \quad (3.77)$$

The integral in the right-hand side is regular. This quadrature, as the other integrals in the representation of the function $X(\zeta, u)$, is evaluated by the Gauss quadrature rule.

We recall that the function $X(\zeta, u)$ is bounded at infinity if and only if the point $\zeta_0$ and the integer $n_a$ satisfy the Jacobi inversion problem (3.35) (the function $X(\zeta, u)$ is independent of the second integer $n_b$). It turns out that for all the parameters $\alpha_0$, $\beta_0$, $\lambda_1$, $\lambda_2$, $\sigma$, and $h$ and the started point $\eta_0 \in C_1$ we tested, the point $\zeta_0$ always falls in the upper sheet, and $m_a = m_b = 0$. For example, for the parameters (3.63), $h = 5$, and $\eta_0 = (i, p^{1/2}(i))$, the affix of the point $\zeta_0 \in C_1$ is $\zeta_0 = 0.51516 + i0.85739$.

Double integrals become involved in the representation (3.45) of the function $\Psi(\zeta, u)$ and therefore of the function $\Phi(\zeta)$. The boundary value $\Phi^+(\xi, v)$ is recovered by the Sokhotski-Plemelj formula and it is given by

$$\Phi^+(\xi, v) = \frac{1}{2} a(\xi, v) + X^+(\xi, v) \left[ \Psi(\xi, v) + \Omega(\xi, v) \right], \quad (\xi, v) \in l_1. \quad (3.78)$$

The principal values of the singular integrals in (3.78) are computed by using the following formula

$$\int_{\delta_1}^{\delta_2} \frac{[1 + u(\xi)/u(\tau)]d\tau}{X^+(\tau, u(\tau)) (\tau - \xi)} = \frac{\pi}{n} \sum_{j=1}^{n} H(\tau_j, \xi), \quad \delta_1 < \xi < \delta_2, \quad (3.79)$$

where $\tau_j = \frac{1}{n} (\delta_2 + \delta_1) + \frac{j}{n} (\delta_2 - \delta_1) \cos \left( \frac{j - \frac{1}{2}}{n} \pi \right),$

$$H(\tau, \xi) = \frac{1}{\tau - \xi} \left[ \left( 1 + \frac{u(\xi)}{u(\tau)} \right) \frac{\sqrt{(\delta_2 - \tau)(\tau - \delta_1)}}{X^+(\tau, u(\tau))} - \frac{2\sqrt{(\delta_2 - \xi)(\xi - \delta_1)}}{X^+(\xi, u(\xi))} \right]. \quad (3.80)$$
Figure 3.9: The cavity shape in the vicinity of the lower vortex $C^{-}$: a closer look ($\alpha_0 = \frac{\pi}{3}$, $\beta_0 = \frac{2\pi}{3}$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\sigma = 0.5$, and $h = 5$).

It is derived from the order $n$ Gauss quadrature rule and the fact that the principal value of the Cauchy integral of the function $[(\delta_2 - \tau)(\tau - \delta_1)]^{-1/2}$ over the segment $(\delta_1, \delta_2)$ is equal to 0.

The formulas (3.56), (3.57), (3.64), and (3.65) require the integration of the function $\omega_0(\zeta)e^{-\omega_1(\zeta)}$. This gives triple integrals. For their evaluation, the Gauss quadrature formulas appear to be efficient. To plot the shape of the cavity and the free surface, it becomes useful to use the following preimages of points on the curves $BC$, $DC$, and $E^{-}E^{+}$:

$$
\xi_j^{++} = b + (1 - b) \sin j s_0 \in b_1, \quad z \in Bz_B,
$$
$$
\xi_j^{+-} = c + (1 - c) \cos j s_0 \in 1, \quad z \in z_B C, \quad j = 1, 2, \ldots, k_0,
$$
$$
\xi_j^{--} = c - c \cos j s_1 \in 0, \quad z \in z_D C,
$$
$$
\xi_j^{-+} = d - d \sin j s_1 \in d 0, \quad z \in Dz_D, \quad j = 1, 2, \ldots, k_1,
$$
$$
\eta_j^{+} = m + r(1 - \sin j s_2) \in e_0^{+} m \subset l_0^{+},
$$
$$
\eta_j^{-} = m + r(1 - \cos j s_2) \in me_0^{+} c \subset l_0^{-}, \quad j = 1, 2, \ldots, k_2,
$$

(3.81)

where $r > 0$, $s_i = \pi/(2k_i)$, $i = 0, 1, 2$; $2k_0$, $2k_1$, and $2k_2$ are the numbers of the points taken to reconstruct the curves $BC$, $DC$, and $E^{-}E^{+}$, respectively. The points $z_B$ and $z_D$ are the images of the branch points $\zeta = 1$ and $\zeta = 0$ of the surface $\mathcal{R}$, respectively. The points $e_0^{+}$ and $e_0^{-}$ are the preimages of the ending points $E_0^{-}$ and $E_0^{+}$ of a piece of the free boundary $E^{-}E^{+}$ we want to reconstruct.

**Summary**

In this chapter, a method of conformal mappings has been developed for a model problem on the uniform motion of a supercavitating yawed wedge beneath a free surface.
To describe the motion, we have used the Tulin-Terent’ev single-spiral-vortex model. By contrast with the double-spiral-vortex model which would lead to a boundary-value problem for a simply connected domain, the model employed requires studying the motion in a doubly connected domain. We have shown that the derivative of the conformal mapping $df/dζ$ from the exterior of two slits, $[0, 1]$ and $[m, ∞)$, in a parametric plane into the flow domain can be expressed through the solutions to two boundary-value problems of the theory of analytic functions. The first problem is a standard Hilbert problem for two segments on a plane. The second one is a Riemann-Hilbert problem on a Riemann (elliptic) surface. We have managed to solve both the problems in a closed form. The formula for $df/dζ$ we have derived possesses four unknown parameters. These parameters have been recovered from a system of four nonlinear equations by the Newton method.

We have implemented the method numerically. One of the key steps of the procedure is the solution of the parameter problem. Based on the previous analysis [AS08] for the case of a wedge in a plane, as the first approximation, we have chosen the parameters $b$ and $d$ to be located on the upper side of the slit $[0, 1]$, the side where the parameter $a$ is placed. Since we have proved that $c = \bar{a}$, this choice allows the points $a$, $b$, and $d$ slide along the upper side of the slit and never be on the lower side while the point $c$ is always on the lower side of the slit and $|c| = |a|$. We have also tried the other three possibilities, (i) $b \in [0, 1]^{-}$, $d \in [0, 1]^{-}$, (ii) $b \in [0, 1]^{-}$, $d \in [0, 1]^{+}$, and (iii) $b \in [0, 1]^{+}$, $d \in [0, 1]^{-}$ (while $a = \bar{c} \in [0, 1]^{+}$). It turns out that for different values of the physical parameters the system of transcendental equations does not have solution in these three cases. Comparing this method with the one that maps a doubly connected circular domain (an annulus or the exterior of two circles) onto the flow domain we note that the method we have presented derives a closed-form solution and requires the computation.

Figure 3.10: The streamlines $\psi(x, y) = hv∞/I$ close to the center of the upper vortex when $\alpha_0 = \pi/3$, $\beta_0 = 2\pi/3$, $\lambda_1 = 1$, $\lambda_2 = 1$, $h = 5$, and $\sigma = 0.5$: $I = 20$ (1), $I = 50$ (2), $I = 100$ (3), $I = 1000$ (4), $I = 10000$ (5), and $I = ∞$ (6).
Figure 3.11: The preimages of the streamlines $\psi(x, y) = hv_\infty/I$ close to the center of the upper vortex when $\alpha_0 = \pi/3$, $\beta_0 = 2\pi/3$, $\lambda_1 = 1$, $\lambda_2 = 1$, $h = 5$, and $\sigma = 0.5$: $I = 20$ (1), $I = 50$ (2), $I = 100$ (3), $I = 1000$ (4), $I = 10000$ (5), and $I = \infty$ (6).

of certain singular and regular integrals along some real segments. The method of automorphic functions recently developed for an $(n + 1)$-connected flow domain and numerically implemented for a simply connected case [AS09] if applied to the present problem, would derive the solution in a series form and would need to compute singular and regular integrals over arcs and their images. It may happen that for triply connected flow domains the unknown parameters can be on any side of the cuts, and then to map the exterior of a circular domain onto the flow domain will be a better approach.

It has been shown that the free surface affects the angle of yaw, the circulation integral, the lift and drag coefficients and breaks the symmetry of the cavity even if the wedge is symmetric. When the wedge approaches the free surface the angle of yaw, the circulation and the lift increase while the drag decreases. When the wedge approaches the free surface the upper boundary becomes noticeably longer than the lower one. We have also found that when $h$ decreases to 0 the parameter $m$ decreases to 1. However, the decrease rate for $h$ and $m$ is different. For example, in the case (3.63), for $h = 100$, $h = 10$, and $h = 2$, we have the following values of the parameter $m$: $m = 2.56677$, $m = 1 + 0.18285 \cdot 10^{-1}$, and $m = 1 + 0.61961 \cdot 10^{-5}$, respectively. The numerical scheme is stable even when $m$ is very close to another branch point $\zeta = 1$. The ability of the present method to deal with small $m - 1$ makes it applicable for wedges moving at small depths (for the data (3.63) the method is stable for $h \geq 1.5$).

This method can also be employed for the Tulin-Terent’ev model for a wedge moving in a jet or a wind tunnel. The analysis of the problem for a wedge in a jet and the comparison of the numerical results produced by the Tulin-Terent’ev single-spiral-vortex model and the Tulin double-spiral-vortex model will be described in the next chapter.
Figure 3.12: The drag coefficient vs the cavitation number $\sigma$ for $\alpha_0 = \frac{\pi}{3}$, $\beta_0 = \frac{2\pi}{3}$, $\lambda_1 = 1$, $\lambda_2 = 1$, and $h = 5$.

Figure 3.13: The lift coefficient vs the cavitation number $\sigma$ for $\alpha_0 = \frac{\pi}{3}$, $\beta_0 = \frac{2\pi}{3}$, $\lambda_1 = 1$, $\lambda_2 = 1$, and $h = 5$. 
Chapter 4

Single- and Double-spiral-vortex Models for a Supercavitating Nonsymmetric Wedge in a Jet

The problem of determining the free surface of a jet incident on a rigid wedge and the boundary of a cavity behind the wedge is considered. The single- and double-spiral-vortex models by Tulin are used to describe the flow at the rear part of the cavity. The location of the wedge in the jet and the sides lengths are arbitrary. This circumstance makes the flow domain doubly connected for the single-spiral-vortex model while it is simple connected for the double-spiral-vortex model. Both models are solved in closed form by the method of conformal mappings. The maps are expressed through the solutions to certain Riemann-Hilbert problems. For the former model, this problem is formulated on a genus-1 Riemann surface. The double-vortex model requires the solution to a standard Riemann-Hilbert problem on a plane. By comparative analysis of the numerical results for the two models, it is found that the drag and lift are practically the same while the jet surface, the cavity boundary at the rear part and the deflection angle of the jet at infinity are different. Also, the problem of determining the parameters for the conformal mapping in the single-vortex model has two solutions. It is shown that one of the solutions leads to a nonphysical shape of the cavity and needs to be disregarded.

4.1 Double-spiral-vortex Model

4.1.1 Formulation

The flow is two-dimensional, incompressible, irrotational, and the gravity is neglected. The vertex, $A$, of the wedge, $DAB$, is fixed and is chosen to be the origin of the plane $z = x_1 + ix_2$ (fig. 4.1). Far away from the wedge, as $x_1 \to -\infty$, the upper and the lower free surfaces of the jet are described by the equations $x_2 = h_1$ and $x_2 = -h + h_1$, respectively. As $x_1 \to -\infty$, the velocity of the flow is also prescribed, $v = (v_\infty, 0)$. The upper and the lower sides of the wedge have lengths $\lambda_1$ and $\lambda_2$, and the angles they form
Figure 4.1: The double-spiral-vortex model domain.

with the $x_1$-axis are $\alpha_0$ and $\beta_0$, respectively. A motion with the following features is to be considered:

(i) The wedge may move about the $x_3$-axis orthogonal to the flow plane. The angle of yaw, $\delta$, is to be determined from the condition that the vertex $A$ is the only stagnation point of the flow.

(ii) The sides of the wedge are straight and rigid. The flow branches at the point $A$, and the velocity vector is tangent to the faces of the wedge,

$$\text{arg} \frac{dw}{dz} = \begin{cases} -\alpha, & z \in AB, \\ \pi - \beta, & z \in AD, \end{cases}$$

where $\alpha = \alpha_0 + \delta$ and $\beta = \beta_0 + \delta$. These two angles define the actual position of the wedge when the flow becomes steady-state. The derivative $\frac{dw}{dz} = u_1 + iu_2$ is the complex velocity, $u_1$ and $u_2$ are the components of the velocity vector $v$, and $w(z) = \varphi(z) + i\psi(z)$ is a complex potential of the flow.

(iii) Behind the wedge, there is a cavity formed by two branches, $ABC_2$ and $ADC_1$, of the same streamline. The cavity pressure, $p_c$, is constant and prescribed. The flow separates smoothly from the points $B$ and $D$. The free streamlines $ABC_2$ and $ADC_1$ form two spirals at the ending points $C_2$ and $C_1$. The speed on the boundary of the cavity is constant, $V = v_c$, where $v_c = \sqrt{\sigma + 1}v_\infty$, $\sigma$ is the cavitation number, $\sigma = 2(p_\infty - p_c)(\rho v_\infty^2)^{-1}$, $\rho$ is the density of the liquid, and $p_\infty$ is the pressure as $x_1 \to -\infty$. At the centers of the spiral vortices, $C_1$ and $C_2$, the logarithm of the complex velocity has the following singularity [Tul64]:

$$\ln \frac{dw}{dz} = O\{\ln[w - w(C_j)]\}, \quad z \to C_j, \quad j = 1, 2.$$  \hspace{1cm} (4.2)

At the points $C_j$, the speed is discontinuous. First, the streamlines spiral at speed $v_c$, then the speed jumps to $v = v_\infty$, and the streamlines spiral backwards and continue in the direction of the infinite point $+\infty + ix_2$ ($x_2$ is finite) forming a wake. Thus, we have

$$\left| \frac{dw}{dz} \right| = \begin{cases} v_c, & z \in BC_2 \cup DC_1, \\ v_\infty, & z \in C_2E_2 \cup C_1E_1, \end{cases}$$
Figure 4.2: The ζ- and w-planes.

\[ \text{Im } w(z) = \psi_0, \quad z \in ABC_2E_2 \cup ADC_1E_1, \quad \psi_0 = \text{const.} \quad (4.3) \]

(iv) The boundary of the free surfaces of the jet is formed by two streamlines, \( PE_2 \) and \( PE_1 \), and the speed on the free surfaces is assumed to be constant \( v = v_\infty \). Thus,

\[ \left| \frac{dw}{dz} \right| = v_\infty, \quad z \in PE_1 \cup PE_2, \]

\[ \text{Im } w(z) = \begin{cases} -\psi_1, & z \in PE_1, \\ \psi_2, & z \in PE_2, \end{cases} \psi_j = \text{const, } j = 1, 2. \quad (4.4) \]

(v) The complex potential \( w(z) \) has the same values at the centers of the double spirals, the points \( C_1 \) and \( C_2 \) [LaS67] or, equivalently,

\[ \text{Re } w(C_1) = \text{Re } w(C_2). \quad (4.5) \]

(vi) The width of the jet is finite as \( x_1 \to +\infty \). This condition means that

\[ \arg \left| \frac{dw}{dz} \right|_{z=E_1} = \arg \left| \frac{dw}{dz} \right|_{z=E_2}. \quad (4.6) \]

4.1.2 Conformal Mapping

The double-spiral-vortex model of supercavitating flow of a jet past \( n \) finite obstacles is flow in a simply connected domain regardless of the number \( n \). Therefore, there exists a function \( z = f(\zeta) \) which maps conformally a half-plane into the flow domain. We denote the preimages of the points \( A, B, C_j, D, E_j, \) and \( P \) by \( a, b, c_j, d, e_j, \) and \( p \), respectively (fig. 4.2a). Three real parameters can be fixed arbitrarily, and we choose \( a = 0, d = -1, \) and \( p = \infty \).

To derive the expression of the mapping function \( f(\zeta) \), we represent its derivative in the form

\[ \frac{df}{d\zeta} = \frac{\omega_0(\zeta)}{v_\infty} e^{-\omega_1(\zeta)}, \quad (4.7) \]

where

\[ \omega_0(\zeta) = \frac{dw}{d\zeta}, \quad \omega_1(\zeta) = \ln \left| \frac{dw}{v_\infty dz} \right|. \quad (4.8) \]
The standard Schwarz-Christoffel formula is employed to recover the function $\omega_0(\zeta)$,

$$\omega_0(\zeta) = \frac{q_1 \zeta}{(\zeta - e_1)(\zeta - e_2)}. \quad (4.9)$$

By integrating this expression, we find the complex potential $w(z(\zeta))$

$$w = \frac{q_1}{e_1 - e_2} [e_1 \ln(\zeta - e_1) - e_2 \ln(\zeta - e_2)] + q_2. \quad (4.10)$$

Here $\ln(\zeta - e_j)$ are the branches fixed by the condition $0 \leq \arg(\zeta - e_j) < \pi$, and $q_1$ and $q_2$ are some constants. To fix these constants, in addition to the parametric $\zeta$-plane, consider the $w$-plane (fig. 4.2b). Since $w(0) = 0$ we may find $q_2$,

$$q_2 = -\frac{q_1}{e_1 - e_2} [e_1 \ln(-e_1) - e_2(\ln e_2 + i\pi)]. \quad (4.11)$$

Determine now the constant $q_1$. Notice that as a point $\zeta$ traverses around the point $\zeta = e_j$ ($j = 1, 2$) along a path in the upper half-plane (fig. 4.2a), the variation of the function $\ln(\zeta - e_j)$ is $i\pi$ while the corresponding variation of $w$ is $-i\psi_j$ (fig. 4.2b). Consequently, $q_1 = -(\psi_1 + \psi_2)/\pi$, $e_1 = -\psi_1 e_2 / \psi_2$. The use of the conservation of mass law defines the constants $\psi_1$ and $\psi_2$: $\psi_1 = v_\infty(h - h_1)$, $\psi_2 = v_\infty h_1$. Thus, the function $\omega_0(\zeta)$ is defined by the expression

$$\omega_0(\zeta) = -\frac{v_\infty h \zeta}{\pi[\zeta - (1 - h/h_1)e_2]|(\zeta - e_2)} \quad (4.12)$$

which possesses one unknown real parameter $e_2$.

We turn now to the determination of the function $\omega_1(\zeta)$. On referring to the boundary conditions (4.1), (4.3), and (4.4), we see from (4.8) that

$$\text{Re} \omega_1(\xi) = 0, \quad \xi \in pe_1 \cup e_1c_1 \cup c_2e_2 \cup e_2p,$$

$$\text{Re} \omega_1(\xi) = \frac{1}{2} \ln(1 + \sigma), \quad \xi \in c_1d \cup bc_2,$$

$$\text{Im} \omega_1(\xi) = -\alpha, \quad \xi \in ab; \quad \text{Im} \omega_1(\xi) = \pi - \beta, \quad \xi \in da. \quad (4.13)$$

It will be convenient to introduce an auxiliary function, $\Phi(\zeta)$, defined in the whole $\zeta$-plane by

$$\Phi(\zeta) = \begin{cases} -i\omega_1(\zeta), & \text{Im} \zeta > 0, \\ i\omega_1(\zeta), & \text{Im} \zeta < 0. \end{cases} \quad (4.14)$$

From the boundary conditions (4.13), we see that the function $\Phi(\zeta)$ represents the solution to the following Riemann-Hilbert problem for symmetric functions:

**Formulation.** Find all functions $\Phi(\zeta)$ analytic in the upper and lower half-planes, Hölder-continuous up to the real axis except for the points $a = 0$, $b$, $d = -1$, $c_1$, and $c_2$ and whose one-sided limits, $\Phi^+(\xi)$ and $\Phi^-(\xi)$, satisfy the following boundary condition:

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + g(\xi), \quad -\infty < \xi < +\infty, \quad (4.15)$$

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The function $\Phi(\zeta)$ is symmetric, $\Phi(\zeta) = \Phi(\bar{\zeta})$, bounded at the points $b$ and $d = -1$ and may have logarithmic singularities at the points $a = 0$, $c_1$, and $c_2$. At the point $p = \infty$, it vanishes.

To factorize the coefficient $G(\xi)$, we use the function $\chi(\zeta) = \sqrt{(\zeta - b)(\zeta + 1)}$, single valued in the $\zeta$-plane cut along the segment $[-1, b]$. The branch is fixed by the condition $\chi(\xi) > 0$, $\xi > b$. In the class of functions bounded at the points $b$, $d = -1$, and $p = \infty$, the solution is unique and given by

$$\Phi(\zeta) = \frac{\chi(\zeta)}{2\pi i} \int^{c_2}_{c_1} \frac{g(\xi)d\xi}{\chi^+(\xi)(\xi - \zeta)^2}. \tag{4.17}$$

It vanishes at the point $p$ if and only if

$$\int^{c_2}_{c_1} g(\xi)d\xi = 0. \tag{4.18}$$

By computing the integral in (4.18), we obtain the following real condition for the unknown parameters of the mapping

$$-\ln(1 + \sigma) \ln \frac{2\chi^+(c_2) + 2c_2 - b + 1}{2\chi^+(c_1) - 2c_1 + b - 1} + 2\alpha\rho^- + 2(\pi - \beta)\rho^+ = 0, \tag{4.19}$$

where

$$\rho^\pm = \frac{\pi}{2} \pm \sin^{-1} \frac{1 - b}{1 + b}. \tag{4.20}$$

The singular integral (4.17) can be evaluated explicitly [AZ09b]. The final formula for the function $\Phi(z)$ becomes

$$\Phi(z) = \frac{\ln(1 + \sigma)}{2\pi} \left( \ln \frac{\rho_1 - \hat{\zeta}}{\rho_1 + \hat{\zeta}} - \ln \frac{\rho_2 - 1/\hat{\zeta}}{\rho_2 + 1/\hat{\zeta}} \right) + \alpha \frac{i(\pi + \alpha - \beta)}{\pi} \ln \frac{\sqrt{b} + i\hat{\zeta}}{\sqrt{b} - i\hat{\zeta}}. \tag{4.21}$$

Here

$$\hat{\zeta} = \sqrt{\frac{\zeta - b}{\zeta + 1}}, \quad \rho_1 = \sqrt{\frac{b - c_1}{-c_1 - 1}}, \quad \rho_2 = \sqrt{\frac{c_2 + 1}{c_2 - b}},$$

$$\text{arg}(\rho_1 \pm \hat{\zeta}), \text{arg}(\rho_2 \pm 1/\hat{\zeta}), \text{arg}(\sqrt{b} \pm i\hat{\zeta}) \in [-\pi, \pi], \tag{4.22}$$

$\rho_j > 0$ ($j = 1, 2$), and the single branch of the function $\hat{\zeta}$ has the following boundary values as $\zeta = \xi \pm i0$:

$$\hat{\zeta} = \begin{cases} |\hat{\zeta}|, & \xi < -1 \text{ or } \xi > b, \\ \pm i|\hat{\zeta}|, & -1 < \xi < b, \end{cases} \tag{4.23}$$
4.1.3 Definition of the Parameters and Numerical Results

The derivative of the conformal mapping (4.7) has been expressed through the functions $\omega_0(\zeta)$ and $\omega_1(\zeta) = i\Phi(\zeta)$, $\text{Im} \, \zeta > 0$, given by (4.12) and (4.21). It will be convenient to rewrite its expression in the form

$$\frac{df}{d\zeta} = hF(\zeta), \quad F(\zeta) = -\frac{\zeta e^{-\omega_1(\zeta)}}{\pi(\zeta - e_1)(\zeta - e_2)}. \quad (4.24)$$

The function $F(\zeta)$ has 5 unknown real parameters, $e_2, c_1, c_2$, and $b$, the preimages of the points $E_2, C_1, C_2$, and $B$, and the yaw angle $\delta$. The parameter $e_1$ is expressed through the unknown $e_2$ by

$$e_1 = \frac{l-1}{l}e_2, \quad l = h_1/h \in (0, 1). \quad (4.25)$$

For the definition of these five parameters, we have the condition (4.19), the following two geometric conditions:

$$\text{Im} \int_0^b F(\zeta)d\zeta = \lambda_1^0, \quad \text{Im} \int_{-1}^0 F(\zeta)d\zeta = \lambda_2^0, \quad \lambda_j^0 = \frac{\lambda_j}{h}, \quad (4.26)$$

and the relations

$$\ln \frac{c_1 - e_1}{e_2 - e_1} = \frac{e_2}{e_1} \ln \frac{e_2 - c_1}{e_2 - c_2},$$

$$\text{Im} \omega_1(e_1) = \text{Im} \omega_1(e_2). \quad (4.27)$$

The last two conditions follow from equations (4.5) and (4.6) of the model. Notice that equation (4.19) and the second equation in (4.27) are linear with respect to the parameter $\delta$. This makes it possible to express this parameter from one of these two equations say, (4.19), through the other four parameter, $b, c_1, c_2$, and $e_2$. For the solution of the system of the four nonlinear equations (4.26) and (4.27) we use a scheme based on the Newton iterative method [AZ09a].

Since the derivative of the conformal mapping has been found, it is possible to reconstruct the free boundary which consists of the jet surface, the cavity and the wake profile. By integrating the function $df/d\zeta$, we obtain the lower and upper boundary of the jet,

$$z(\tau) = D + \int_{\tau}^{df/d\zeta} d\zeta, \quad \tau \in p e_1 (z \in PE_1),$$

$$z(\tau) = B + \int_{\tau}^{df/d\zeta} d\zeta, \quad \tau \in p e_2 (z \in PE_2). \quad (4.28)$$

For the cavity and wake boundaries, we have similar formulas. For the lower part of the cavity boundary $\tau \in d c_1 (z \in DC_1)$ and for the upper one, $\tau \in b c_2 (z \in BC_2)$. Fig. 4.3 shows the cavity shape and the profile of the wake and the jet when $\lambda_1^0 = \lambda_2^0 = 0.1$, $l = \frac{5}{8}$, and $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$ for the values 0.4, 0.5, and 1 of the cavitation number $\sigma$. The parameters of the conformal mapping for $\sigma = 1$ have the following values: $e_1 = -1.82018, c_1 = -1.74692, b = 1.18188, c_2 = 2.62667, \text{and } e_2 = 3.03363$. It is seen
Figure 4.3: The cavity, wake, and jet profiles when \( \lambda_1 = \lambda_2 = 1, h = 16, h_1 = 10, \alpha_0 = \frac{\pi}{3}, \) and \( \beta_0 = \frac{2\pi}{3} \) for some values of the cavitation parameter \( \sigma: \sigma = 0.4 \) (1), \( \sigma = 0.5 \) (2), and \( \sigma = 1 \) (3).

that the cavity size and the width of the wake behind the cavity increase when the the cavitation number decreases. When \( \alpha_0 + \beta_0 \) and the angle of attack are not small while the cavitation number is small, the model reminisces the Joukowsky open wake model. In this case it is worth to replace equation (4.5) (the zero circulation condition) by the condition \( h_w = 0 \), where \( h_w \) is the thickness of the wake at infinity. This guarantees the closure of the wake at infinity [Tul64].

We proceed now to compute the drag and lift coefficients

\[
C_X + IC_Y = \frac{2(X + iY)}{\rho v^2 \lambda^o h},
\]

where \( \lambda^o = \lambda^o_1 \sin \alpha + \lambda^o_2 \sin \beta \), \( X \) and \( Y \) are drag and lift, respectively, which by Bernoulli’s law can be represented in the form

\[
X + iY = -\frac{i\rho}{2} \int_{DAB} (v_c^2 - V^2)dz,
\]

where \( V = |dw/dz| \). We have finally

\[
C_X + IC_Y = -\frac{i}{\lambda^o} \int_{dab} [\sigma + 1 - e^{2\text{Re}\omega_1(\zeta)}]F(\zeta)d\zeta.
\]

For the parameters \( \alpha_0 = \pi - \beta_0 = \frac{\pi}{3}, \lambda^o_1 = 0.05, \lambda^o_2 = 0.1 \) and \( l = 0.5 \), the drag and lift coefficients increase when the cavitation number \( \sigma \) increases (fig. 4.4).

Our scheme applied to a single hydrofoil for small \( l = h_1/h \) is consistent with the results by Larock and Street in [LaS67] for the coefficient \( C_D + iC_L = \lambda^o (\lambda^o_1 + \lambda^o_2)^{-1}(C_X + \)
Figure 4.4: The drag and lift coefficients, $C_X$ and $C_Y$, when $\lambda_1 = \lambda_2 = 1$, $h = 20$, $h_1 = 10$, $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$ vs the parameter $\sigma$: the single-spiral-vortex model (–) and the double-spiral-vortex model (– -).

$iC_Y$ obtained for a foil beneath a free surface ($h_1 = 1$, $h = \infty$). For $h = 1000$ and $h_1 = 1$, the angle of attack $5.66^\circ$ and $\sigma = 0.096$, the coefficient $C_D + iC_L$ obtained from our jet-solution is $0.0190037 + i0.191522$, and the one reported by Larock and Street is $0.019 + i0.191$.

### 4.2 Single-spiral-vortex Model

#### 4.2.1 Description of the Model

The first two assumptions, (i) and (ii), of the single-spiral-vortex model are the same as for the double-spiral-vortex model described in Section 2.1. We write down the other assumptions of the model which distinguish this model from the double-spiral-vortex model.

(iii) The closure cavity mechanism for the single-spiral-vortex model is different from (4.2) and it is described by Terent’ev in [Ter76]

$$\log \frac{dw}{dz} \sim -K((w - w(C))^{-1/2}), \quad z \to C, \quad -\pi \leq \arg[w(z) - w(C)] \leq \pi. \quad (4.32)$$

Here $K$ is a positive constant, and the branch of the square root is chosen such that $[w(z) - w(C)]^{1/2} > 0$ when $\arg[w(z) - w(C)] = 0$. According to the Terent’ev (1981) interpretation of the Tulin single-spiral-vortex model, the two branches of the dividing streamline at the centers of the vortices behind the foil, $C_1$ and $C_2$, pass to a half of an infinitely sheeted Riemann surface of the logarithmic function with the branch points $C_1$ and $C_2$. After that the same streamline emerges from the infinite sheet of the
Riemann surface and returns to a point C of the first, physical, sheet. In contrast to the double-spiral-vortex model, the speed is continuous at the rear part of the cavity (fig. 4.5).

On the boundary of the cavity, the complex potential \( w(z) \) satisfies the following boundary conditions:

\[
\text{Im } w(z) = K_0, \quad z \in L_1,
\]

\[
\left| \frac{dw}{dz} \right| = \begin{cases} 
    v_\infty, & z \in L_0, \\
    v_c, & z \in BC^+ \cup DC^-,
\end{cases}
\]

where \( K_0 \) is a real constant, and the contour \( L_1 \) consists of the boundary of the cavity \( BC_2 \cup DC_1 \) and the faces of the wedge \( DAB \).

(iv) On the jet surface \( L_0 = E_1^-E_2^- \cup E_1^+E_2^+ \),

\[
\text{Im } w(z) = K_1^\pm, \quad z \in E_1^\pm E_2^\pm,
\]

where \( K_1^+ \) and \( K_1^- \) are some real constants.

(v) By contrast with the double-spiral-vortex model, the flow domain, \( \tilde{D} \), is not simply connected but doubly connected. To assure that the flow is single-valued, it is required that

\[
\int_{L_*} dz = 0,
\]

Here \( L_* \) is a closed contour in the flow domain exterior to the contour \( L_1 \).

As for the double-spiral-vortex model, we use the conformal mapping technique. Let \( z = f(\zeta) \) map the exterior of two cuts, \( l_1 = [0, 1] \) and \( l_0 = [m, \infty) \) onto the physical domain \( \tilde{D} \) (fig. 4.6). Here \( m \in (1, +\infty) \) is a parameter to be fixed. Denote the preimages of the points \( A, B, C, D, E_1^\pm \), and \( E_2^\pm \) by \( a, b, c, d, e_1 \), and \( e_2 \), respectively. Since such a map is defined up to one real parameter and since \( e_1 \neq e_2 \), we choose \( e_1 = \bar{e}_2 \). Clearly, two cases need to be considered, \( e_1 = e_0 + i0 \) and \( e_1 = e_0 - i0 \), where \( e_0 = |e_1| = |e_2|, e_0 \in (m, +\infty) \).

As before, the derivative \( df/d\zeta \) is conveniently represented in terms of two functions, \( \omega_0(\zeta) \) and \( \omega_1(\zeta) \), by (4.7).
4.2.2 Function $\omega_0(\zeta)$

The function $\omega_0(\zeta)$ is analytic in the exterior of the cuts $l_0$ and $l_1$. At infinity, the function $f(\zeta)$ decays as $K\zeta^{-1/2}$, $K = \text{const}$. This implies $\omega_0(\zeta) = O(\zeta^{-3/2})$, $\zeta \to \infty$. At the preimages of the points $E_j^\pm$, it has a logarithmic singularity, $f(\zeta) \sim h\pi^{-1}(-1)^{j-1}\ln(\zeta - e_0)$, $j = 1, 2$. Since $\frac{dw}{dz} \sim v_\infty$, $\zeta \to e_1$, we obtain $\omega_0(\zeta) \sim hv_\infty[\pi(\zeta - e_0)]^{-1}$, $\zeta \to e_1$. It has been shown in [AS08] that the function $\frac{dw}{d\zeta}$ has to vanish at the stagnation point and the point where the branched streamline emerges from the Riemann surface of flow. In our case this means that $\omega_0(\zeta)$ has simple zeros at the points $a$ and $c$. Because of the first condition in (4.33) and equation (4.34), $\text{Im}\omega_0(\zeta) = 0$ on $l_0$ and $l_1$. All these conditions can be written as a homogeneous Riemann-Hilbert problem. By solving it we find that $a = \bar{c}$. Without loss of generality, we assume that $a \in l_1^+$ and then $c \in l_1^-$. The most general form of the function $\omega_0(\zeta)$ with such properties is

$$\omega_0(\zeta) = hv_\infty\omega_0^*(\zeta),$$

where

$$\omega_0^*(\zeta) = \frac{p^{1/2}(e_1)}{\pi p^{1/2}(\zeta)} \left( \frac{1}{\zeta - e_0} - \frac{1}{a - e_0} \right).$$

Here $p(\zeta) = \zeta(1 - \zeta)(\zeta - m)$ and $p^{1/2}(\zeta)$ is the branch fixed by the condition $p^{1/2}(\xi) > 0$ if $\xi < 0$. At the banks of the cuts $l_0$ and $l_1$, $\zeta = \xi \pm i0$, it has the properties $p^{1/2}(\zeta) = \pm i|p^{1/2}(\xi)|$, $0 < \xi < 1$, and $p^{1/2}(\zeta) = \pm i|p^{1/2}(\xi)|$, $m < \xi < +\infty$. If $1 < \xi < m$, then the function $p^{1/2}(\xi)$ is negative.

The function $\omega_0(\zeta)$ has three real parameters, $a$, $e_0$, and $m$ to be determined. By conservation of mass, we can write down the first real condition for them,

$$\text{Im}\int_{a}^{e_*} \omega_0(\zeta)d\zeta = h_1v_\infty,$$  

(4.38)

where $e_*$ is the preimage of a point $E_*$ in the upper boundary of the jet. This condition can be transformed into the form

$$\text{Im}\int_{1}^{m} \omega_0^*(\zeta)d\zeta = \begin{cases} 
  l - 1, & e_1 \in l_0^-, \\
  l, & e_1 \in l_0^+.
\end{cases}$$  

(4.39)
4.2.3 Function $\omega_1(\zeta)$

From the conditions (4.1) and (4.32) to (4.34) we conclude that the function $\omega_1(\zeta)$ satisfies the boundary conditions

\[
\begin{align*}
\text{Re} \omega_1(\zeta) &= \begin{cases} 
\log \sqrt{\sigma + 1}, & \zeta \in bcd, \\
0, & \zeta \in l_1,
\end{cases} \\
\text{Im} \omega_1(\zeta) &= \begin{cases} 
-\alpha, & \zeta \in ab, \\
\pi - \beta, & \zeta \in da,
\end{cases}
\end{align*}
\]

and as $\zeta \to c$, $\omega_1(\zeta) = O(1/(z-c))$. The function $\omega_1(\zeta)$ has a logarithmic singularity at the point $a$ and it is bounded at the points $b$ and $d$. At infinity, the function $\omega_1(\zeta)$ is bounded and it vanishes at the point $\zeta = e_1$.

Apart from the conditions at $\zeta = \infty$ and $\zeta = e_1$, these conditions are the same as those for the function $\omega_1(\zeta)$ in the double-spiral-vortex model for a wedge beneath a free surface [AZ09a]. Therefore, the function $\omega_1(\zeta)$ can be determined in a similar manner through the solution to a Riemann-Hilbert problem on a two-sheeted genus-1 Riemann surface, $\mathcal{R}$, of the algebraic function $u = p^{1/2}(\zeta)$, $\zeta \in \mathbb{C}_1$, and $u = -p^{1/2}(\zeta)$, $\zeta \in \mathbb{C}_2$. Here $\mathbb{C}_1$ and $\mathbb{C}_2$ are two replicas of the extended $\zeta$-plane with the cuts $l_0$ and $l_1$. We write down only the final formulas for the solution. Let $\Phi(\zeta, u) = -i\omega_1(\zeta)$ on the upper sheet $\mathbb{C}_1$ and $\Phi(\zeta, u) = i\omega_1(\zeta)$ on the lower sheet $\mathbb{C}_2$. Then

\[
\Phi(\zeta, u) = X(\zeta, u)[\Psi(\zeta, u) + \Omega(\zeta, u)], \quad (\zeta, u) \in \mathcal{R},
\]

where

\[
\begin{align*}
\Psi(\zeta, u) &= -\frac{\alpha}{2\pi i} \int_{ab} \frac{(1 + u/v) d\xi}{X^+(\xi, v)(\xi - \zeta)} \\
&\quad + \frac{\pi - \beta}{2\pi i} \int_{da} \frac{(1 + u/v) d\xi}{X^+(\xi, v)(\xi - \zeta)} - \frac{\ln(\sigma + 1)}{4\pi} \int_{bcd} \frac{(1 + u/v) d\xi}{X(\xi, v)(\xi - \zeta)}, \quad v = u(\xi).
\end{align*}
\]

The function $\Omega(\zeta, u)$ is a rational function on the surface $\mathcal{R}$ given by

\[
\begin{align*}
\Omega(\zeta, u) &= iM_0 \frac{u(\zeta) + u(c)}{\zeta - c} + (M_1 + iM_2) \frac{u(\zeta) + u(\eta_0)}{\zeta - \eta_0} - (M_1 - iM_2) \frac{u(\zeta) - u(\eta_0)}{\zeta - \eta_0} + M_3,
\end{align*}
\]

where $M_j$ ($j = 0, 1, 2, 3$) are real constants to be fixed.

As for the function $X(\zeta, u)$, it is a piece-wise meromorphic function, symmetric on the surface, $X(\zeta, u) = X(\zeta, -u(\zeta))$, $(\zeta, u) \in \mathcal{R} \setminus \mathcal{L}$, $\mathcal{L} = l_0 \cup l_1$, discontinuous through the contour $dab \in \mathcal{R}$, and whose one-sided limits satisfy the boundary condition $X^+(\xi, v) = -X^-(\xi, v)$, $(\xi, v) \in dab$. This function is defined by singular integrals

\[
X(\zeta, u) = \exp \left\{ \frac{1}{4} \int_{dab} \left( 1 + \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} \right\}
\]

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\[-\frac{1}{2} \int \left( 1 + \frac{u(\zeta)}{u(\xi)} \right) \frac{d\zeta}{\zeta - \xi} - \frac{1}{2} \int \left( 1 - \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\zeta - \xi} - 2n_a \int_{l_0^+} \frac{u(\zeta)}{u(\xi) \xi - \zeta} \right\}, \quad (4.44)\]

where \( \gamma \) is a continuous curve whose starting and terminal points are \( \eta_0 = (\eta_0, u(\eta_0)) \) and \( \zeta_0 = (\zeta_0, u(\zeta_0)) \), respectively. The point \( \eta_0 \) is an arbitrary fixed point lying on the upper sheet \( \mathbb{C}_1 \), whilst the point \( \zeta_0 \) can lie on either sheet. The affix \( \zeta_0 \) of the starting point is defined by

\[ \zeta_0 = \text{sn} \frac{i \eta_0}{2k}, \quad (4.45) \]

where

\[ k = m^{-1/2}, \quad g_0 = \frac{1}{4} \int_{d}^{a} \frac{d\xi}{p^{1/2}(\xi)} + \int_{0}^{\eta_0} \frac{d\xi}{p^{1/2}(\xi)}. \quad (4.46) \]

Denote

\[ I_{\pm} = \frac{1}{4} \int_{d}^{a} \frac{d\xi}{p^{1/2}(\xi)} + \int_{0}^{\eta_0} \frac{d\xi}{p^{1/2}(\xi)} \pm \int_{0}^{\zeta_0} \frac{d\xi}{p^{1/2}(\xi)}. \quad (4.47) \]

If it turns out that both the numbers

\[ -\frac{\text{Im} I_-}{4kK} \quad \text{and} \quad \frac{\text{Re} I_-}{4kK'}, \quad (4.48) \]

are integers, then the point \( \zeta_0 \in \mathbb{C}_1 \) and \( n_a = -\text{Im} I_- (4kK)^{-1} \). Otherwise, the point \( \zeta_0 \) falls on the lower sheet \( \mathbb{C}_2 \) and \( n_a = -\text{Im} I_+ (4kK)^{-1} \). Here \( K = K(k) \) is the complete elliptic integral of the first kind, and \( K' = K(\sqrt{1 - k^2}) \).

The curve \( \gamma \) does not cross the contour \( l_0 \). In the case \( \zeta_0 \in \mathbb{C}_2 \), it passes through the point \( \zeta = 0 \) and consists of two parts, \( \eta_0 \in \mathbb{C}_1 \) and \( 0 \in \mathbb{C}_2 \). If the point \( \zeta_0 \) lies on the upper sheet, then the contour \( \gamma \) can be chosen as the straight line joining the points \( \eta_0 \) and \( \zeta_0 \) provided it does not cross the contour \( l_0 \). We notice that in all the numerical tests implemented the point \( \zeta_0 \in \mathbb{C}_1 \).

The solution (4.41) possesses 10 unknown real constants. They are \( M_0, M_1, M_2, \) and \( M_3 \) (the coefficients in the representation of the rational function \( \Omega(\zeta, u) \)), the angle of yaw \( \delta \), and the points \( a, b, d, c_0, \) and \( m \). To fix these unknowns we have the same number of equations, linear and nonlinear. The first equation (4.39) links the three parameters \( a, b, d, c_0, \) and \( m \). Write down the other equations. Due to the simple pole of the function \( X(\zeta, u) \) at the point \( \zeta_0 \), the function \( \omega_1(\zeta) \) has an inadmissible pole at this point. It becomes a removable singularity if the following complex condition holds

\[ \Psi(\zeta_0, u(\zeta_0)) + \Omega(\zeta_0, u(\zeta_0)) = 0. \quad (4.49) \]

To guarantee a smooth attachment of the jet breaking away from the wedge at the point \( z = D \), we require

\[ \Psi(d, u(d)) + \Omega(d, u(d)) = 0. \quad (4.50) \]

Notice that at the point \( \zeta = b \) the solution is automatically bounded.
Since the function $\omega_1(\zeta)$ vanishes at the point $\zeta = e_1$, we impose the following condition
\[ \Psi(e_1, u(e_1)) + \Omega(e_1, u(e_1)) = 0. \] (4.51)
Next, we wish the function $\omega_1(\zeta)$ being bounded at the infinite point. By analyzing the principal term in (4.41) at infinity, we have
\[ M_0 = \Psi_0 - 2M_2, \] (4.52)
where $\Psi_0$ is a real constant given by
\[ \Psi_0 = \frac{\alpha}{2\pi} \int_{ab} \frac{d\xi}{vX^+(\xi, v)} - \frac{\pi - \beta}{2\pi} \int_{da} \frac{d\xi}{vX^+(\xi, v)} + \frac{i \ln(\sigma + 1)}{4\pi} \int_{bcd} \frac{d\xi}{vX(\xi, v)}. \] (4.53)
We also add the standard geometrical conditions
\[ \lambda_1^0 \sin \alpha - \Omega_1 = 0, \quad \lambda_2^0 \sin \beta - \Omega_2 = 0, \] (4.54)
where
\[ \Omega_1 = \text{Im} \int_{ab} \omega_{0}^*(\zeta) e^{-\omega_1(\zeta)} d\zeta, \quad \Omega_2 = \text{Im} \int_{da} \omega_{0}^*(\zeta) e^{-\omega_1(\zeta)} d\zeta. \] (4.55)
The final two real equations come from the requirement for the mapping $z = f(\zeta)$ to satisfy the single-valuedness condition (4.35) or, equivalently, the following condition
\[ \int_{l_1^*} \omega_0^*(\zeta) e^{-\omega_1(\zeta)} d\zeta = 0, \] (4.56)
where $l_1^*$ is a closed contour around the cut $l_1$ which does not cross the cut $l_0$.

Our next step is to determine the real constants $M_0, \ldots, M_3$ and the angle of yaw $\delta = \alpha - \alpha_0$ explicitly from the linear equations (4.49) to (4.52). This can most conveniently be done by splitting the unknowns $M_j$ as follows
\[ M_j = M_j^0 + \delta M_j^1, \quad j = 0, \ldots, 3. \] (4.57)
We shall use, for brevity, the notations
\[ \rho_0(\zeta) = \frac{u(\zeta) + u(\eta_0)}{\zeta - \eta_0}, \quad \rho_1(\zeta) = \frac{u(\zeta) - \overline{u(\eta_0)}}{\zeta - \eta_0}, \quad \rho_2(\zeta) = \frac{u(\zeta) + u(c)}{\zeta - c}, \]
\[ \Psi_1 = \Psi(\zeta_0, u(\zeta_0)), \quad \Psi_2 = \Psi(d, u(d)), \quad \Psi_3 = \Psi(e_1, u(e_1)), \]
\[ \Psi_j = \Psi_j^0 + \delta \Psi_j^1, \quad j = 0, \ldots, 3. \] (4.58)
where $\Psi_j^0 = \Psi_j|_{\alpha = \alpha_0, \beta = \beta_0}$, and the constants $\Psi_j^1$ coincide with $\Psi_j$ if $\alpha$ and $\pi - \beta$ are replaced by 1 and $-1$, respectively. By applying the conditions (4.49) to (4.52) we express the angle of yaw through the constants $M_j^\nu$ as follows: $\delta = -\frac{\Delta_0}{\Delta_1}$, where
\[ \Delta_\nu = \Psi_{j_2}^\nu + i\rho_2(d)M_{j_1}^\nu + [\rho_0(d) - \rho_1(d)]M_{j_1}^\nu + i[\rho_0(d) + \rho_1(d)]M_{j_2}^\nu + M_{j_3}^\nu, \quad \nu = 0, 1. \] (4.59)
The coefficients \( M_j^\nu \) themselves are determined by

\[
M_0^\nu = \Psi_0^\nu - 2M_2^\nu,
\]

\[
M_3^\nu = -\Psi_3^\nu - iM_0^\nu \rho_2(e_1) - (M_1^\nu + iM_2^\nu)\rho_0(e_1) + (M_1^\nu - iM_2^\nu)\rho_1(e_1),
\]

\[
M_1^\nu = \frac{1}{\Delta}(GL_1^\nu \text{Re} \mu_1 - GL_2^\nu \text{Im} \mu_1),
\]

\[
M_2^\nu = -\frac{1}{\Delta}(GL_2^\nu \text{Re} \mu_2 + GL_1^\nu \text{Im} \mu_2),
\]

(4.60)

where

\[
\Delta = \text{Re} \mu_1 \text{Re} \mu_2 + \text{Im} \mu_1 \text{Im} \mu_2,
\]

\[
\mu_1 = \rho_0(\zeta_0) + \rho_1(\zeta_0) - 2\rho_2(\zeta_0) - \rho_0(e_1) - \rho_1(e_1) + 2\rho_2(e_1),
\]

\[
\mu_2 = \rho_0(\zeta_0) - \rho_1(\zeta_0) - \rho_0(e_1) + \rho_1(e_1),
\]

\[
GL_1^\nu = \text{Im}[\rho_2(\zeta_0) - \rho_2(e_1)]\Psi_0^\nu - \text{Re} \Psi_1^\nu + \Psi_3^\nu,
\]

\[
GL_2^\nu = \text{Re} \rho_2(\zeta_0)\Psi_0^\nu + \text{Im} \Psi_1^\nu, \quad \nu = 0, 1.
\]

(4.61)

The other unknown parameters of the conformal mapping, \( a, b, d, e_1, \) and \( m, \) can be found from a system of three real and one complex transcendental equations (4.39), (4.54), and (4.56).

### 4.2.4 Comparative Analysis of the Single- and Double-spiral-vortex Models

The nonlinear system (4.39), (4.54), and (4.56) of five real equations is solved numerically by a technique based on the Newton method similarly to the system of four nonlinear equations associated with the problem for a wedge beneath a free surface [AZ09a]. The main feature of the system (4.39), (4.54), and (4.56) is the presence of certain constraints for the unknown parameters. Indeed, we have chosen \( a \in l_1^+ \), have proved that \( c = \bar{a} \in l_2^+ \), and \( 1 < m < \infty \) by the definition. Therefore, \( d \in l_1^+ \), \( b \in l_1^+ \) and \( 0 < d < a, \ a < b < 1 \). All numerical tests implemented show that in fact, \( d \in l_1^+ \) and \( b \in l_1^+ \). It turns out that there are two sets of parameters of the conformal mapping, \( \{a, b, d, e_1, m\} \) and \( \{a, b, d, \bar{e}_1, m\} \), which satisfy the system of nonlinear equations. However, the set of parameters with \( e_1 = e_0 - i0 \) produces a nonphysical solution: the two branches of the free streamline which define the cavity intersect each other, and the Brillouin condition is therefore violated (fig. 4.7).

For all the problem parameters tested, the physical solution corresponds to the case when \( e_1 = e_0 + i0 \in l_0^+ \) and therefore \( e_2 = \bar{e}_1 \in l_0^+ \). The values of the parameters of the conformal mapping and the angle of yaw for some values of the cavitation number \( \sigma \) when

\[
\alpha_0 = \frac{\pi}{3}, \quad \beta_0 = \frac{2\pi}{3}, \quad \lambda_1 = 1, \quad \lambda_2 = 2, \quad h = 20, \quad h_1 = 10,
\]

(4.62)

are given in table 4.1. It is seen that the angle of yaw increases when the cavitation number increases.
Figure 4.7: The nonphysical second solution: the cavity shape.

Table 4.1: The values of the parameters \( a, b, d, e_0, m, \) and the yaw angle \( \delta \) for the parameters (4.62) and some values of the cavitation number \( \sigma \).

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( a )</th>
<th>( b )</th>
<th>( d )</th>
<th>( e_0 )</th>
<th>( m - 1 )</th>
<th>( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.991565</td>
<td>0.996192</td>
<td>0.971464</td>
<td>1.0084579</td>
<td>1.395441·10^{-6}</td>
<td>0.129164</td>
</tr>
<tr>
<td>0.4</td>
<td>0.991764</td>
<td>0.996345</td>
<td>0.971563</td>
<td>1.008370</td>
<td>6.504862·10^{-6}</td>
<td>0.131272</td>
</tr>
<tr>
<td>0.6</td>
<td>0.960173</td>
<td>0.982899</td>
<td>0.862127</td>
<td>1.043103</td>
<td>1.561740·10^{-3}</td>
<td>0.134618</td>
</tr>
<tr>
<td>0.8</td>
<td>0.915786</td>
<td>0.965239</td>
<td>0.713255</td>
<td>1.100234</td>
<td>7.614774·10^{-6}</td>
<td>0.137920</td>
</tr>
<tr>
<td>1.0</td>
<td>0.870908</td>
<td>0.948976</td>
<td>0.570754</td>
<td>1.170520</td>
<td>1.958512·10^{-2}</td>
<td>0.141443</td>
</tr>
</tbody>
</table>

To restore the shape of the cavity, we integrate the function \( df/d\zeta \) over the contours \( b\tau (\tau \in bc) \) and \( d\tau (\tau \in dc) \) as was described in the case of the double-spiral-vortex model in Section 2.3. We have reconstructed the shape of the cavity behind the wedge and the jet for a symmetric wedge for different widths \( h \) of the jet or equivalently for different values of the parameter \( \lambda_1^0 = \lambda_2^0 \) (fig. 4.8). The numerical results show that when \( h \) grows and the cavitation number is fixed the length of the cavity grows as well.

The jet boundary, the cavity shape and the streamline which splits at the vertex of the wedge and then emerges at the rear part of the cavity are shown in fig. 4.9 for some cavitation numbers in the nonsymmetric case. The amplitude of the wave on the surface of the jet and the cavity length increase when the cavitation number decreases.

A flow map with several streamlines plotted is given in fig. 4.10. As in the case of a wedge beneath a free surface [AZ09a], the streamline \( \psi = 0 \) and those which are very close to it (they are not shown) spiral at the points \( C_1 \) and \( C_2 \), and they are discontinuous at the point \( C \). The other streamlines are continuous at the rear part of the cavity. The points \( C_1 \) and \( C_2 \) are determined as the images of the limit points \( c^+ \) and \( c^- \) (\( \zeta \) approaches the point \( c \in l_1^c \) from the right and the left, respectively). However, the position of the point \( C \) cannot be determined in a similar manner. We identify it as the point of intersection of the streamline returning from the infinite sheet of the Riemann surface of the model and the cut line joining the centers of the vortices \( C_1 \) and \( C_2 \).

In fig. 4.11, we present the cavity and jet profiles predicted according to the single-spiral-vortex model (a solid line) and the double-spiral-vortex model (a broken line).
Figure 4.8: The cavity shape and the jet surface when $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$, $\lambda_1 = \lambda_2 = 1$, $l = 0.5$, $\sigma = 0.5$ for some values of $h$: $h = 10$ (1), $h = 16$ (2), $h = 20$ (3), $h = 30$ (4), and $h = 50$ (5).

Figure 4.9: The cavity shape and the jet surface for $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$, $\lambda_1 = \lambda_2 = 1$, $l = \frac{5}{8}$ when $\sigma = 1$ (1), $\sigma = 0.5$ (2), and $\sigma = 0.4$ (3).
Figure 4.10: The streamlines $\psi(x, y) = s$ for some values of the constant $s$ when $\alpha_0 = \frac{\pi}{3}$, $\beta_0 = \frac{2\pi}{3}$, $\sigma = 0.5$, $l = \frac{5}{8}$, and $\lambda_1 = \lambda_2 = 1$.

Figure 4.11: The cavity shape and the free surface for a nonsymmetric wedge when $\alpha_0 = \frac{\pi}{3}$, $\beta_0 = \frac{2\pi}{3}$, $\sigma = 0.3$, $h = 5$, and $\lambda_1 = 1$ for some values of $\lambda_2$: $\lambda_2 = 1$ (1), $\lambda_2 = 2$ (2), and $\lambda_2 = 3$ (3).
Table 4.2: The angle of deflection $\epsilon$ of the jet at infinity for $\alpha_0 = \pi - \beta_0 = \pi/3$, $\lambda_1 = 1$, $\lambda_2 = 2$, $h = 20$, $h_1 = 10$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Single-spiral-vortex model</th>
<th>Double-spiral-vortex model</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>-0.01524</td>
<td>-0.01661</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.02294</td>
<td>-0.01811</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.02745</td>
<td>-0.01954</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.03392</td>
<td>-0.02160</td>
</tr>
</tbody>
</table>

Table 4.3: Circulation $(v_{\infty})^{-1}\Gamma$ for the single-spiral-vortex model: $\alpha_0 = \pi/3$, $\beta_0 = 2\pi/3$, $\lambda_1 = 1$, $\lambda_2 = 2$, $h = 20$, $h_1 = 10$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(v_{\infty})^{-1}\Gamma$</td>
<td>-10.454608</td>
<td>-5.668399</td>
<td>-1.094609</td>
<td>-0.873162</td>
<td>-0.792022</td>
</tr>
</tbody>
</table>

The shapes of the cavity computed according to the two models, are different only at the rear part of the cavity. The length of the cavity is smaller for the double model, however the separation point between the cavity and the wake is hardly noticeable. Also, the jet is wider for the double model.

The solid lines in fig. 4.4 correspond to the drag and lift coefficients $C_X$ and $C_Y$ computed in the framework of the single-spiral-vortex model. It is seen that the curves for the single- and double-spiral-vortex model (the broken lines) are very close to each other. In the nonsymmetric case, as $x_1 \to +\infty$, the speed $V \to v_{\infty}$. The velocity vector $v$ however does not tend to $(v_{\infty}, 0)$. This is because of the jet deflexion. In table 4.2, we give some values of the angle of deflection $\epsilon$ at infinity for both the models. It is small and of the same order for both the models.

Finally, we determine the circulation of the velocity around the closed contour $L_1 = ABCDA$ for the single-spiral-vortex model

$$\Gamma = \int_{i_1} \frac{d\omega}{d\zeta} d\zeta = hv_{\infty} \int_{i_1} \tilde{\omega}_0(\zeta) d\zeta.$$  \hspace{1cm} (4.63)

It is seen from table 4.3 that for a nonsymmetric wedge, the absolute values of the circulation, $|\Gamma|$, decreases when the cavitation number $\sigma$ increases. As $h$ increases and $h_1$ is fixed, $\Gamma/v_{\infty}$ decreases: for $h_1 = 10$, $\lambda_1 = \lambda_2 = 1$, $\sigma = 0.5$, $\alpha_0 = \pi - \beta_0 = \pi/3$ and for $h = 30, 80, 150$ we have $\Gamma/v_{\infty} = -0.4845, -0.3520$ and $-0.3288$, respectively that is consistent with the results in [AZ09a] for a wedge beneath a free surface. Because of the condition (4.5), the corresponding integral around the contour $C_1DABC_2$ for the double-spiral-vortex model is zero.

**Summary**

The main contribution of this chapter is the comparative analysis of the two nonlinear models by Tulin, the single- and double-spiral-vortex models applied to the problem for a jet past a yawed nonsymmetric wedge.
By solving certain Riemann-Hilbert problems we have derived the conformal mapping from a parametric half-plane onto the flow domain for the double-spiral-vortex model and from a plane cut along two segments, [0, 1] and [m, ∞), onto the physical domain for the single-spiral-vortex model. The former case is simpler since the Riemann-Hilbert problem is set on the complex plane while it is formulated on a genus-1 Riemann surface in the case of the single-spiral-vortex model. In both models, the final step of the method is the solution of an associated system of transcendental equations for the unknown parameters of the conformal mapping. We have solved these systems by the Newton type method. It turns out that the nonlinear system in the double-spiral-vortex model has a unique solution. For the single-spiral-vortex model, we have found two sets of parameters. However, one of them violates the Brillouin condition which requires the free streamlines do not intersect each other. The second solution obeys all the conditions of the model and is therefore physical.

The numerical results for the drag and lift coefficients computed according to the single- and double-spiral-vortex models are very close. What is different is the shape of the rear part of the cavity, its length, and also the profile of the jet. In general, the amplitude of the waves on the jet are higher in the double-spiral-vortex model. One of assumptions of the double-spiral-vortex model used for numerical computations is that the complex potential is the same at the centers of the upper and lower vortices. This condition leads to a non-zero thickness of the wake at infinity. We have not analyzed the model when this condition is replaced by the one which closes the wake at infinity.
Chapter 5

A Flexible Wedge or a Hydrofoil in a Stream of Liquid

In this chapter we will state and solve a problem of a flexible hydrofoil or a wedge with flexible sides in a stream of liquid. The Tulin single-spiral-vortex model is used as a cavity closure condition. A conformal mapping from the exterior of the unit circle onto the flow domain is employed. Observe that it is also possible to use conformal mappings from other simply-connected auxiliary domains, e.g. a half-plane, the first quadrant of a complex plane, a complex plane with a cut. The solutions for a rigid hydrofoil or a wedge with rigid sides using these auxiliary domains can be found in [Roz77], [Gur79], [Ter81]. The problem for an arbitrary number $n + 1$ of rigid hydrofoils has been solved in [AS09] using the conformal mapping from the exterior of $n + 1$ circles in the complex domain and the Riemann-Hilbert problem for symmetric automorphic functions. In this chapter, two Riemann-Hilbert problems are stated and solved for the functions $dw/dζ$ and $ω(ζ)$. The solution contains an unknown function $α(z)$ which is an angle between the tangent line to the hydrofoil and the positive direction of the $x$-axis. This angle can be found by solving the equation of bending of an elastic plate. The solution to the last equation depends in its turn on the pressure which the flow of liquid exerts on the plate. Thus, we obtain a fluid-structure interaction problem. The iterative numerical procedure is described for the solution of the problem, and the numerical results are presented.

5.1 A Flexible Supercavitating Hydrofoil in a Stream of Liquid

5.1.1 Statement of a Problem for a Hydrofoil

Consider a potential flow of liquid past a flexible elastic hydrofoil $DB$ which can bend as a result of the liquid pressure (fig. 5.1). Here and further assume that the thickness $h$ of the hydrofoil is small compared to its length $λ$. The Young modulus $E$ and the Poisson ratio $ν$ of the hydrofoil is given. Assume that the end-point $D$ of the hydrofoil is fixed and the tangent line to the hydrofoil at the point $D$ makes an angle $α_0$ with a positive
direction of $x$-axis. The flow of liquid at infinity is parallel to $x$-axis and has a velocity $v = (v_\infty, 0)$. The stagnation point $A$ where the stream of liquid separates around the hydrofoil is a priori unknown and needs to be found as a part of the solution. The Tulin single-spiral-vortex model [Tul64] is employed at the cavity closure point. The speed $v_c$ on the boundary of the cavity is prescribed and, hence, the cavitation number is

$$\sigma = \frac{v_c^2}{v_\infty^2} - 1.$$  

The fluid flow is described by the complex potential $w(z) = \varphi(x,y) + i\psi(x,y)$, which satisfies the following boundary value problem

\begin{align}
\text{Im} w(z) &= \psi_0, \quad z \in ABCDA, \\
\left| \frac{dw}{dz} \right| &= v_c, \quad z \in BCD, \\
\arg \frac{dw}{dz} &= \begin{cases} 
-\alpha(z), & z \in AB, \\
\pi - \alpha(z), & z \in DA,
\end{cases}
\end{align}  

where $\alpha(z)$ is an angle which the tangent line to the hydrofoil at the point $z$ makes with the positive direction of $x$-axis. This angle depends on the elastic properties of the hydrofoil and the loading applied to the hydrofoil and is initially unknown.

\subsection{Conformal Mapping}

Consider a conformal mapping $z = f(\zeta)$ from the exterior of the unit circle $l$ onto the flow domain (fig. 5.2). Without loss of generality we can assume that the point $\zeta = \infty$ of the auxiliary domain is mapped into the infinity point $z = \infty$ of the flow domain. Additionally, we may fix one of the preimages $a, b, c$ and $d$ of the points $A, B, C$ and $D$ arbitrarily, say $a = -1$. Locations of the other points $b, c, d$ need to be found as a result of the solution, with the only condition that the points $a, b, c$ and $d$ follow each other in the clockwise direction (fig. 5.2).

To find a conformal mapping $z = f(\zeta)$ we will state and solve two Riemann-Hilbert problems on a complex plane for the functions $dw/d\zeta$ and $dw/dz$. The derivative of the conformal mapping can be found then from the formula:

$$\frac{dz}{d\zeta} = \frac{dw}{d\zeta} \cdot \frac{dw}{dz}.  \quad (5.4)$$
5.1.3 Function $dw/d\zeta$

Parametrize a unit circle $l$ as $\zeta = e^{is}$, $0 \leq s \leq 2\pi$, where $s$ is an arc length. Then the following is true:

$$\frac{dw}{ds} = \frac{dw}{d\zeta} \cdot \frac{d\zeta}{ds} = i e^{is} \frac{dw}{d\zeta} = i\zeta \frac{dw}{d\zeta}.$$ 

According to (5.1) the function $w(z)$ is constant along the boundary of the wedge and the cavity $ABCDA$. Then it follows that

$$\text{Im} \frac{dw}{ds} = 0$$

on the unit circle $l$. Hence, using the last two equations we can obtain

$$\text{Re} \left( \zeta \frac{dw}{d\zeta} \right) = 0, \quad \zeta \in l.$$

The last condition can be further rewritten as

$$\zeta \frac{dw}{d\zeta} + \overline{\zeta} \frac{dw}{d\zeta} = 0, \quad \zeta \in l.$$

Using the symmetry with respect to the unit circle we can write $\overline{\zeta} = \frac{1}{\zeta}$. Then it follows that

$$\frac{dw}{d\zeta} = -\frac{1}{\zeta^2} \frac{dw}{d\zeta}, \quad \zeta \in l. \quad (5.5)$$

At the preimages $\zeta = a$ and $\zeta = c$ of the front stagnation point and of the cavity closure point the function $\frac{dw}{d\zeta}$ must be equal to zero:

$$\frac{dw(a)}{d\zeta} = 0, \quad \frac{dw(c)}{d\zeta} = 0. \quad (5.6)$$

Introduce a function

$$\Phi(\zeta) = \frac{dw}{d\zeta}, \quad |\zeta| > 1. \quad (5.7)$$
Extend this function onto the whole complex plane by the symmetry
\[ \Phi(\zeta) = \overline{\Phi(\bar{\zeta})}, \quad |\zeta| < 1. \] (5.8)

Then the equation (5.5) can be stated as the following Riemann-Hilbert problem for the function \( \Phi(\zeta) \):

**Formulation.** Find a piecewise analytic function \( \Phi(\zeta) \) which satisfies the boundary condition
\[ \Phi^+(\xi) = -\frac{1}{\xi^2} \Phi^-(\xi), \quad \xi \in l, \] (5.9)
the symmetry condition (5.8), and has zeros at the points \( \zeta = a \) and \( \zeta = c \).

Further we use the approach proposed in [AS09]. Choose the kernel
\[ K(\zeta, \xi) = \frac{1}{\xi - \zeta} - \frac{1}{\xi - \zeta_*}, \]
where \( \zeta_* \) is an arbitrarily fixed point in the exterior of the unit circle \( l \). Consider next the functions
\[ \Gamma_k(\zeta) = -\frac{1}{4\pi i} \int_l \ln_k \xi \left( \frac{1}{\xi - \zeta} - \frac{1}{\xi - \zeta_*} \right) d\xi, \quad k = 0, 1, \] (5.10)
where the branches \( \ln_k \zeta \) of the logarithm function are chosen in the following way:

- \( \ln_0 \zeta \) is a branch of the logarithm defined by a cut \( \gamma_0 \) through the points \( \zeta = 0, \zeta = a \) and \( \zeta = \infty \) (fig. 5.3a);
- \( \ln_1 \zeta \) is a branch of the logarithm defined by a cut \( \gamma_1 \) through the points \( \zeta = 0, \zeta = c \) and \( \zeta = \infty \) (fig. 5.3b).

It is easy to see that
\[ \ln^+_k \xi - \ln^-_k \xi = 2\pi i, \] (5.11)
where \( \xi \) lies on the bank of the cut, and the signs “+” or “−” denote the values of the logarithm on the “+” or “−” side of the cut as shown on the fig. 5.3.

Assume that \( |\zeta| > 1 \) and consider
\[ \Gamma_0(\zeta) = -\frac{1}{4\pi i} \int_{\gamma_0^+ \cup \gamma_0^- \cup l} \ln_0 \xi \left( \frac{1}{\xi - \zeta} - \frac{1}{\xi - \zeta_*} \right) d\xi + \]
\[ \frac{1}{4\pi i} \int_{\gamma_0^+ \cup \gamma_0^-} \ln_0 \xi \left( \frac{1}{\xi - \zeta} - \frac{1}{\xi - \zeta_*} \right) d\xi, \] (5.12)
where under the curves \( \gamma_0^+ \) we understand “+” sides of the cut \( \gamma_0 \) from the point \( \zeta = a \) to \( \zeta = \infty \). Applying the residue theorem to the first integral we can see that
\[ -\frac{1}{4\pi i} \int_{\gamma_0^+ \cup \gamma_0^- \cup l} \ln_0 \xi \left( \frac{1}{\xi - \zeta} - \frac{1}{\xi - \zeta_*} \right) d\xi = \frac{1}{2} \ln \left( \frac{\zeta_*}{\zeta} \right). \]
The second integral in (5.12) can be computed directly using the property (5.11):

\[
\frac{1}{4\pi i} \int_{\gamma_0^- \cup \gamma_0^+} \ln \xi \left( \frac{1}{\xi - \zeta} - \frac{1}{\xi - \zeta^*} \right) d\xi = \frac{1}{2} \ln \left( \frac{a - \zeta}{a - \zeta^*} \right).
\]

Then

\[
\Gamma_0(\zeta) = \frac{1}{2} \ln \left( \frac{\zeta_*(a - \zeta)}{\zeta(a - \zeta_*)} \right), \quad |\zeta| > 1.
\]

Similarly, for \(|\zeta| < 1\) we obtain:

\[
\Gamma_0(\zeta) = \frac{1}{2} \ln \left( \frac{\zeta_*(a - \zeta)}{\zeta(a - \zeta_*)} \right), \quad |\zeta| < 1.
\]

The same formulas are valid for the function \(\Gamma_1(\zeta)\) if we substitute \(c\) for \(a\). Let \(\zeta_* \to \infty\), then

\[
\Gamma_0(\zeta) = \begin{cases} 
\frac{1}{2} \ln \left( \frac{\zeta - a}{\zeta} \right), & |\zeta| > 1, \\
\frac{1}{2} \ln(\zeta - a), & |\zeta| < 1,
\end{cases} \quad \Gamma_1(\zeta) = \begin{cases} 
\frac{1}{2} \ln \left( \frac{\zeta - c}{\zeta} \right), & |\zeta| > 1, \\
\frac{1}{2} \ln(\zeta - c), & |\zeta| < 1.
\end{cases}
\]

Since we are looking for symmetric solutions to the Riemann-Hilbert problem, then the canonical function of the problem (5.9) can be taken in the form

\[
\chi \Phi(\zeta) = \chi_0(\zeta) \exp \left\{ \Gamma_0(\zeta) + \Gamma_0(1/\zeta) + \Gamma_1(\zeta) + \Gamma_1(1/\zeta) \right\},
\]

where

\[
\chi_0(\zeta) = \begin{cases} 
i, & |\zeta| > 1, \\
-i, & |\zeta| < 1.
\end{cases}
\]

The last formula can be simplified:

\[
\chi \Phi(\zeta) = \frac{i}{\sqrt{ac}} \frac{(\zeta - a)(\zeta - c)}{\zeta^2}.
\]

Observe that the function \(z = f(\zeta)\) is a one-to-one mapping of the exterior of the unit circle \(D\) onto the flow domain \(\tilde{D}\) and the point \(\zeta = \infty\) is mapped into \(z = \infty\). Then the function \(dz/d\zeta\) is bounded in \(D\) everywhere including the infinity point \(\zeta = \infty\). The
function $dw/dζ$ is bounded in $D$ as a conjugate to the complex velocity. Hence, the function $Φ(ζ) = \frac{dw}{dz} = \frac{dw}{dζ} \cdot \frac{dζ}{dz}$ is bounded everywhere in $D$ including the infinity point $ζ = ∞$. Thus, we obtain that the ratio $Φ(ζ)/χΦ(ζ)$ is also bounded everywhere and, hence, all the solutions to the problem (5.9) are of the form

$$Φ(ζ) = NχΦ(ζ) = \frac{iN(ζ - a)(ζ - c)}{\sqrt{ac}} \zeta^2,$$

(5.14)

where $N$ is a real constant.

5.1.4 Function $dw/dz$

As before, introduce a logarithmic hodograph variable:

$$ω(ζ) = \ln\left(\frac{1}{v_∞} \frac{dw}{dz}\right), \ |ζ| > 1.$$  

(5.15)

From the conditions (5.2), (5.3) it follows that the function (5.15) is a solution to the following boundary value problem:

**Formulation.** Find all analytic functions $ω(ζ)$ in the domain $D$ which satisfy the following boundary conditions

$$\text{Re} ω(ξ) = \frac{1}{2} \ln(\sigma + 1), \ ξ ∈ bcd,$$

(5.16)

$$\text{Im} ω(ξ) = \begin{cases} \quad -α(f(ξ)), & ξ ∈ ad, \\ \quad π - α(f(ξ)), & ξ ∈ da, \end{cases}$$

(5.17)

have a simple pole at the point $ζ = c$ and have a zero at the infinity point.

To reduce this problem to the Riemann-Hilbert problem on the complex plane continue the function $ω(ζ)$ by the symmetry:

$$ω(ζ) = \overline{ω(1/ζ)}.$$  

(5.18)

From the condition (5.18) we obtain that

$$ω^+(ξ) = ω^-(ξ), \ ξ ∈ l,$$

(5.19)

where “+” and “−” signs denote the limiting values of the function $ω(ζ)$ with respect to the chosen (clockwise) direction of the contour $l$, i.e. from the outside and from the inside of the unit circle $l$ correspondingly.

Substituting the relation (5.19) into the conditions (5.16), (5.17) we obtain the following Riemann-Hilbert boundary value problem:

**Formulation.** Find all piecewise analytic functions $ω(ζ)$ in the complex plane which satisfy the following boundary conditions

$$ω^+(ξ) + ω^-(ξ) = \ln(σ + 1), \ ξ ∈ bcd,$$

(5.20)

$$ω^+(ξ) - ω^-(ξ) = \begin{cases} \quad -2iα(f(ξ)), & ξ ∈ ab, \\ \quad 2i\{π - α(f(ξ))\}, & ξ ∈ da, \end{cases}$$

(5.21)

satisfy the symmetry condition (5.18), have a simple pole at the point $ζ = c$ and have a zero at the infinity point.
Observe that the conditions (5.20), (5.21) contain the function $\alpha(f(\xi))$ which depends on the unknown conformal mapping $z = f(\zeta)$. We temporarily will treat the function $\alpha(f(\xi))$ as given.

Our first step to solve the Riemann-Hilbert problem (5.20), (5.21) is to find a canonical function $\chi_\omega(\zeta)$, which is a piecewise meromorphic function satisfying homogeneous conditions (5.20), (5.21) and the symmetry condition (5.18). Thus we need to find a symmetric function $\chi_\omega(\zeta)$ which satisfies the condition

$$\chi_\omega^+(\xi) = -\chi_\omega^-(\xi), \quad \xi \in bcd. \quad (5.22)$$

Consider the function

$$\Delta(\zeta) = \frac{1}{4\pi i} \int_{bcd} \frac{\ln(-1)d\xi}{\xi - \zeta} = \frac{1}{4} \int_{bcd} \frac{d\xi}{\xi - \zeta} = \frac{1}{4} \ln \left( \frac{\zeta - d}{\zeta - b} \right).$$

Then

$$\exp\{\Delta(\zeta) + \Delta(1/\zeta)\} = \sqrt[4]{\frac{b}{d}} \sqrt[4]{\frac{\zeta - d}{\zeta - b}},$$

where the branch of the square root is determined by the cut along the circular arc $bcd$. This function is symmetric and satisfies the condition (5.22), so we could choose this function to be a canonical function. However, the solution of the inhomogeneous problem (5.20), (5.21) will be simplified if the canonical function (5.22) is bounded at both points $\zeta = b$ and $\zeta = d$ and has a simple pole at the point $\zeta = c$. This function is given by the formula

$$\chi_\omega(\zeta) = \sqrt[4]{\frac{c \zeta - b}{b \zeta - c}} \exp \left\{ \Delta(\zeta) + \Delta(1/\zeta) \right\},$$

$$= \sqrt[4]{\frac{c^2}{bd}} \sqrt[4]{\frac{(\zeta - b)(\zeta - d)}{\zeta - c}}, \quad (5.23)$$

where the branch of the square root is determined by the cut along the circular arc $bcd$ and the condition:

$$\sqrt{(\zeta - b)(\zeta - d)} \sim \zeta, \quad \zeta \to \infty.$$

Using the canonical function (5.23) we can solve the inhomogeneous problem. Define a new function

$$\Psi(\zeta) = \omega(\zeta)/\chi_\omega(\zeta).$$

This function satisfies the following boundary condition

$$\Psi^+(\xi) - \Psi^-(\xi) = g(\xi)/\chi_\omega^+(\xi), \quad \xi \in l, \quad (5.24)$$

where

$$g(\xi) = \begin{cases} 
\ln(\sigma + 1), & \xi \in bcd, \\
-2i\alpha(f(\xi)), & \xi \in ab, \\
2i\{\pi - \alpha(f(\xi))\}, & \xi \in da.
\end{cases}$$
The problem (5.24) is a problem of finding a sectionally analytic function from the given jump on the contour \( l \). This problem can be solved in terms of the integral of the Cauchy type. Consider a function

\[
\Psi_0(\zeta) = \frac{1}{4\pi i} \left( \ln(\sigma + 1) \int_{bcda} \frac{d\xi}{\chi^+(\xi)(\xi - \zeta)} - 2i \int_{ab} \frac{\alpha(f(\xi))d\xi}{\chi^+(\xi)(\xi - \zeta)} + 2i \int_{da} \frac{\{\pi - \alpha(f(\xi))\}d\xi}{\chi^+(\xi)(\xi - \zeta)} \right).
\]

To preserve the symmetry take the function \( \Psi(\zeta) \) in the form:

\[
\Psi(\zeta) = \Psi_0(\zeta) + \frac{1}{\Psi_0(1/\zeta)} + R(\zeta),
\]

where \( R(\zeta) \) is a rational function satisfying the symmetry condition (5.18). Since the function \( \Psi(\zeta) \) must be bounded everywhere in the complex plane except for maybe logarithmic singularities at the points \( \zeta = a, \zeta = b, \zeta = d \), it follows that \( R(\zeta) \) is bounded everywhere. Then \( R(\zeta) \) must be a real constant. Finally, the function \( \Psi(\zeta) \) has a form

\[
\Psi(\zeta) = \frac{1}{4\pi i} \left( \ln(\sigma + 1) \int_{bcda} \frac{\xi + \zeta}{\chi^+(\xi)(\xi - \zeta)} d\xi - 2i \int_{ab} \frac{\xi + \zeta \alpha(f(\xi))d\xi}{\chi^+(\xi)(\xi - \zeta)} + 2i \int_{da} \frac{\{\pi - \alpha(f(\xi))\}d\xi}{\chi^+(\xi)(\xi - \zeta)} \right) + N_0,
\]

where \( N_0 \) is a real constant.

Observe that because of the equation (5.15) and since the velocity of the flow at infinity is equal to \( v_\infty \) it follows that

\[
\Psi(\zeta) \rightarrow 0 \text{ as } \zeta \rightarrow \infty. \tag{5.26}
\]

It is easy to see from the conditions (5.16), (5.17) that the function \( \chi^+(\xi) \) is pure real on \( dac \) and pure imaginary on \( bcd \). Consider the asymptotic expansion of the expression (5.25) as \( \zeta \rightarrow \infty \). Then from the condition (5.26) it follows that

\[
N_0 = 0,
\]

\[
\ln(\sigma + 1) \int_{bcda} \frac{d\xi}{\chi^+(\xi)} - 2i \int_{ab} \frac{\alpha(f(\xi))d\xi}{\chi^+(\xi)} + 2i \int_{da} \frac{\{\pi - \alpha(f(\xi))\}d\xi}{\chi^+(\xi)} = 0. \tag{5.27}
\]

Finally, the solution to the initial inhomogeneous problem (5.16), (5.17) has the form:

\[
\omega(\zeta) = \chi_\omega(\zeta) \left( \ln(\sigma + 1) \int_{bcda} \frac{\xi + \zeta}{\chi^+(\xi)(\xi - \zeta)} d\xi - \frac{1}{2\pi} \int_{ab} \frac{\xi + \zeta \alpha(f(\xi))d\xi}{\chi^+(\xi)} \right) + \frac{1}{2\pi} \int_{da} \frac{\{\pi - \alpha(f(\xi))\}d\xi}{\chi^+(\xi)}.
\]
5.1.5 Unknown Parameters and Additional Conditions

Our main goal is to restore the unknown conformal mapping \( z = f(\zeta) \) from the formula:

\[
\frac{df}{d\zeta} = \frac{1}{v_{\infty}} \frac{dw}{d\zeta} e^{-\omega(\zeta)},
\]  

(5.29)

which follows from the equation (5.4). The function (5.29) contains four unknown parameters, namely, \( b, c, d \) and \( N \). To find these parameters we have previously stated the condition (5.27) which fixes the velocity at infinity. Two additional conditions can be obtained from the condition of single-valuedness of the conformal mapping:

\[
\int_{l^*} \frac{df}{d\zeta} d\zeta = 0,
\]  

(5.30)

where \( l^* \) is a closed contour enclosing the unit circle \( l \). Finally, the constant \( N \) is fixed by the given length of the hydrofoil:

\[
\int_{dab} \left| \frac{df}{d\zeta} \right| |d\zeta| = \lambda.
\]  

(5.31)

The last condition allows to determine only the absolute value of the constant \( N \). The sign of the constant can be found, for example, by considering the real or the imaginary part of the integral \( \int_{dab} \frac{df}{d\zeta} d\zeta \).

Thus, we have the system of four nonlinear transcendental equations with four unknowns.

5.1.6 Bending of Thin Plates

Up to this point we have formally followed the method of the solution of the problem for a rigid hydrofoil. However, the derivative of the conformal mapping \( df/d\zeta \), given by the formula (5.29), contains the function \( \alpha(f(\zeta)) \) which represents the angle between the tangent line to the hydrofoil and the positive direction of \( x \)-axis. This function depends on the conformal mapping \( z = f(\zeta) \). Thus, our next step is to establish a connection between the elastic properties of the hydrofoil and the fluid flow around the hydrofoil.

To describe the hydrofoil reaction to the liquid pressure we will use the equations of bending of thin elastic plates. These equations can be found, for example, in [TW59].

Assume that the forces act on the hydrofoil only in the normal direction to the hydrofoil and that the deflections of the hydrofoil are small. Direct the \( s \)-axis of the coordinate system connected with the hydrofoil along the hydrofoil length, \( r \)-axis along the hydrofoil width and \( z' \)-axis in a direction perpendicular to the hydrofoil as shown on the fig. 5.4. Denote the intensity of the loading on the plate as \( q(s, r) \), so that the loading acting on the element of the hydrofoil with dimensions \( ds \) and \( dr \) is equal to \( q(s, r)dsdr \). Denote the deflections of the hydrofoil as \( u(s, r) \). Assume that the deflections of the points of the hydrofoil are normal to the initial position of the hydrofoil. Then it is known that the deflections \( u(s, r) \) must satisfy the equation:

\[
\Delta\Delta u = \frac{q}{D_0},
\]  

(5.32)
where $\Delta$ is a Laplace operator and the coefficient $D_0$ can be found from the formula

$$D_0 = \frac{Eh^3}{12(1-\nu^2)}.$$

where $E$ is the Young modulus of the plate, $\nu$ is the Poisson ratio and $h$ is the thickness of the plate.

Since the flow is two-dimensional, we can assume that the functions $q(s, r)$ and $u(s, r)$ do not depend on the coordinate $r$, so everywhere further we will write simply $q(s)$ and $u(s)$ omitting the coordinate $r$. Thus, the equation (5.32) is reduced to the ordinary differential equation

$$\frac{d^4 u}{ds^4} = \frac{q(s)}{D_0}. \quad (5.33)$$

Assume that the end $D$ of the hydrofoil is clamped, which leads to the conditions:

$$u|_{s=0} = 0, \quad \frac{du}{ds}|_{s=0} = 0. \quad (5.34)$$

For the second end $B$ two types of boundary conditions are considered:

- the end $B$ is free:
  $$\frac{d^2 u}{ds^2}|_{s=\lambda} = 0, \quad \frac{d^3 u}{ds^3}|_{s=\lambda} = 0, \quad (5.35)$$

- the end $B$ is clamped:
  $$u|_{s=\lambda} = 0, \quad \frac{du}{ds}|_{s=\lambda} = 0. \quad (5.36)$$

The intensity of the loading $q(s)$ acting on the hydrofoil is, in fact, the difference between a pressure of the stream of liquid on the plate and the pressure inside of the
cavity. Then we can use the Bernoulli equation to connect the complex potential \( w(z) \) of the flow with the intensity of loading \( q(s) \) acting on the hydrofoil:

\[
\frac{q(s) + p_c}{\rho} + \frac{1}{2}|v|^2 = \frac{p_\infty}{\rho} + \frac{1}{2}v_\infty^2.
\]

Then

\[
q(s) = p_\infty - p_c - \frac{\rho}{2}(|v|^2 - v_\infty^2),
\]

where

\[
|v|^2 = \frac{dw}{dz} \frac{d\bar{w}}{d\bar{z}} = v_\infty^2 e^{2Re(\zeta)}.
\]

Hence, using the formula (1.1) we obtain

\[
q(s) = \frac{\rho v_\infty^2}{2} \left(1 + \sigma - e^{2Re(\zeta)}\right). \tag{5.37}
\]

Since we assume that the deflections are small, it follows that the change in the length of the hydrofoil is insignificant, and we can treat the coordinate \( s \) as a length of the hydrofoil from the point \( D \) to a given point on the hydrofoil. Thus, in the auxiliary coordinates

\[
s = \int_{\zeta}^{\zeta_d} \left| \frac{df}{d\zeta} \right| |d\zeta|, \quad \zeta \in dab, \tag{5.38}
\]

where the integral is taken along the arc of the unit circle \( l \).

We can integrate the equation (5.33) together with the boundary conditions (5.34) and (5.35) or (5.36) using the change of the order of integration three times to obtain

\[
u(s) = \frac{1}{6D} \int_0^s (s - t)^3 q(t)dt + \frac{C_1}{6} s^3 + \frac{C_2}{2} s^2, \tag{5.39}
\]

where the integral is taken along the hydrofoil, and \( C_1, C_2 \) are constants. If the end \( B \) is free then

\[
C_1 = -\frac{1}{D} \int_0^\lambda q(t)dt, \tag{5.40}
\]

\[
C_2 = \frac{1}{D} \int_0^\lambda t q(t)dt.
\]

If the end \( B \) of the hydrofoil is clamped, then

\[
C_1 = -\frac{1}{D \lambda^3} \int_0^\lambda (\lambda + 2t)(\lambda - t)^2 q(t)dt, \tag{5.41}
\]

\[
C_2 = \frac{1}{D \lambda^2} \int_0^\lambda t(\lambda - t)^2 q(t)dt,
\]

where all the integrals are taken along the length of the hydrofoil.

Finally, find the dependence between the angle \( \alpha(f(\xi)) \) which the tangent line to the hydrofoil makes with the positive direction of \( x \)-axis and the deflections of the hydrofoil \( u(s) \). The position of the point with the distance \( s \) from the point \( D \) on the hydrofoil
DB in the flow domain $\tilde{D}$ is given by the formula

$$z(s) = se^{i\alpha_0} + u(s)e^{i(\alpha_0 - \pi/2)}.$$  

Then the angle $\alpha(f(\zeta))$ can be found from the formula

$$\alpha(f(\zeta)) = \arctan \frac{y'(s)}{x'(s)} = \arctan \left( \frac{\sin \alpha_0 - u'(s) \cos \alpha_0}{\cos \alpha_0 + u'(s) \sin \alpha_0} \right),$$  

(5.42)

where

$$u'(s) = \frac{1}{2D} \int_0^s (\lambda - t)^2 q(t) \, dt + \frac{C_1}{2} s^2 + C_2 s.$$  

(5.43)

Thus, the solution to the fluid mechanics problem (5.14), (5.28) depends on the derivative of the deflections $u'(s)$ (5.43) which in turn depends on the pressure (5.37) which the liquid exerts on the hydrofoil. Thus, we obtain a fluid-structure interaction problem. This problem will be solved numerically using an iterative procedure.

### 5.1.7 Numerical Iterative Procedure

To solve the fluid-structure interaction problem stated in the previous section we develop the following iterative procedure. First of all, compute the complex potential $w(z)$ for a rigid hydrofoil using the formulas (5.14), (5.28) and satisfying the conditions (5.27), (5.30), (5.31). This can be achieved by substituting $\alpha(f(\zeta)) = \alpha_0 = \text{const}$. Now, using the solution for a rigid hydrofoil we can compute the pressure $q(s)$ on the hydrofoil using the formulas (5.37). Observe that we obtain the pressure in the formula (5.37) as a function of the auxiliary variable $\zeta$ which is convenient for the following computations. Once we know the pressure distribution on the hydrofoil we can compute the deflections and, hence, the angle $\alpha(f(\zeta))$ which the tangent line to the hydrofoil makes with the positive direction of $x$-axis using the formulas (5.39), (5.41), (5.40), (5.42) and (5.43). Again, we obtain the angle $\alpha(f(\zeta))$ as a function of the auxiliary variable $\zeta$. After this we can substitute these new values of $\alpha$ into the formulas (5.14), (5.28), (5.27), (5.30), (5.31) and repeat the process.

The iterative procedure is stopped if the maximum of the absolute value of the difference between the deflections in two consecutive iterations, computed using the formula (5.39), becomes less than a certain value of error $\epsilon$. 

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The computations have shown that this iterative procedure converges relatively fast (less than ten iterations with the precision $\epsilon = 10^{-6}$) even for a relatively large deflections of the hydrofoil (approximately 10% of the hydrofoil length).

### 5.1.8 Numerical Results

The system (5.27), (5.30), (5.31) will be solved using the Newton method. Observe that the constant $N$ can be found from the equation (5.31), and, thus, we have a system with three unknowns $b, c, d$ located on the unit circle. Each of the unknowns $b, c, d$ can be defined by its polar angle $\theta_b, \theta_c, \theta_d$ correspondingly. To make the values of $\theta_b, \theta_c, \theta_d$ unique, we can assume

$$\theta_a - 2\pi = -\pi < \theta_d < \theta_c < \theta_b < \theta_a = \pi. \quad (5.44)$$

Since the Newton method applies to the solution of the systems with unconstrained variables, we can remove the constrains (5.44) by introducing new variables:

$$t_b = \ln \frac{\pi - \theta_b}{\theta_b - \theta_c}, \quad t_c = \ln \frac{\theta_b - \theta_c}{\theta_c - \theta_d}, \quad t_d = \ln \frac{\theta_c - \theta_d}{\theta_d + \pi}. \quad (5.45)$$

The system (5.27), (5.30) can be solved with respect to the variables (5.45). After that we can restore the initial variables $\theta_b, \theta_c, \theta_d$ from the following formulas:

$$\theta_b = \pi \frac{-e^{tb} e^{tc} e^{td} + e^{tc} e^{td} + e^{td} + 1}{e^{tb} e^{tc} e^{td} + e^{tc} e^{td} + e^{td} + 1},$$

$$\theta_c = \pi \frac{-e^{tb} e^{tc} e^{td} - e^{tc} e^{td} + e^{td} + 1}{e^{tb} e^{tc} e^{td} + e^{tc} e^{td} + e^{td} + 1},$$

$$\theta_d = \pi \frac{-e^{tb} e^{tc} e^{td} - e^{tc} e^{td} - e^{td} + 1}{e^{tb} e^{tc} e^{td} + e^{tc} e^{td} + e^{td} + 1}.$$

Computations are implemented for the following values of the parameters:

$$\alpha_0 = \pi/2, \quad \lambda = 1, \quad \upsilon_\infty = 1, \quad \rho = 1, \quad D_0 = 1.66667$$

and varying values of the cavitation number $\sigma$. The results of the computations are presented in the table 5.1 for the hydrofoil with the free end $B$, and in table 5.2 for the hydrofoil with the fixed end $B$.

The cavity and the hydrofoil profiles for the cases of the free or the fixed end $B$ are shown on the fig. 5.5 and fig. 5.6 for four different cavitation numbers $\sigma = 0.3, \sigma = 0.5, \sigma = 0.7$ and $\sigma = 1.0$. The dependence of the length of the hydrofoil on the cavitation number $\sigma$ is shown on the fig. 5.7. Obviously, the cavity length tends to infinity as $\sigma \to 0$.

However, while the cavitation number strongly affects the length of the cavity, the hydrofoil profile and the maximal deflections do not change much. The hydrofoil profile for the fixed end $D$ and the free end $B$ and different cavitation numbers is shown on the fig. 5.8. The hydrofoil profile with the smallest deflections corresponds to $\sigma = 0.1$, with the largest deflections to $\sigma = 1.0$. Other profiles on the fig. 5.8 correspond to cavitation...
Table 5.1: The values of the parameters $\theta_\beta$, $\theta_c$, $\theta_d$ and $N$ for the varying cavitation number $\sigma$ for the case of the fixed end $D$ and the free end $B$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\theta_\beta$</th>
<th>$\theta_c$</th>
<th>$\theta_d$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-3.094860</td>
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</tr>
<tr>
<td>0.3</td>
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<td>-2.978045</td>
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</tr>
<tr>
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<td>-4.105134</td>
</tr>
<tr>
<td>0.6</td>
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<td>-2.914626</td>
<td>-3.152434</td>
</tr>
<tr>
<td>0.7</td>
<td>2.860985</td>
<td>2.581583 \cdot 10^{-3}</td>
<td>-2.886218</td>
<td>-2.546994</td>
</tr>
<tr>
<td>0.8</td>
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<td>3.463181 \cdot 10^{-3}</td>
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</tr>
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<td>0.9</td>
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<td>4.500059 \cdot 10^{-3}</td>
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<tr>
<td>1.0</td>
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<td>5.695594 \cdot 10^{-3}</td>
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<td>-1.614419</td>
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</tbody>
</table>

Table 5.2: The values of the parameters $\theta_\beta$, $\theta_c$, $\theta_d$ and $N$ for the varying cavitation number $\sigma$ for the case of the fixed ends $B$ and $D$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\theta_\beta$</th>
<th>$\theta_c$</th>
<th>$\theta_d$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.093943</td>
<td>1.789199 \cdot 10^{-7}</td>
<td>-3.093957</td>
<td>-64.753408</td>
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<tr>
<td>0.2</td>
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<td>-18.498254</td>
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</tr>
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</tr>
<tr>
<td>0.5</td>
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<td>-2.940074</td>
<td>-4.187789</td>
</tr>
<tr>
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<td>-3.220357</td>
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<tr>
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<td>-2.185950</td>
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<tr>
<td>0.9</td>
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<td>-1.884259</td>
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<td>2.020013 \cdot 10^{-5}</td>
<td>-2.800520</td>
<td>-1.658398</td>
</tr>
</tbody>
</table>

Figure 5.5: Cavity and hydrofoil profile in the case of fixed end $D$ and free end $B$ for the following cavitation numbers: (1) $\sigma = 0.3$, (2) $\sigma = 0.5$, (3) $\sigma = 0.7$, (4) $\sigma = 1.0$.  

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Figure 5.6: Cavity and hydrofoil profile in the case of fixed ends $B$ and $D$ for the following cavitation numbers: (1) $\sigma = 0.3$, (2) $\sigma = 0.5$, (3) $\sigma = 0.7$, (4) $\sigma = 1.0$.

Figure 5.7: Dependence of the length of the cavity on the cavitation number $\sigma$ for the cases of (1) fixed end $D$ and free end $B$, (2) fixed ends $B$ and $D$, (3) rigid hydrofoil.
numbers $\sigma = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$ correspondingly. The dependence of the maximal deflections of the hydrofoil on the cavitation number is shown on the fig. 5.9. The line “1” corresponds to the hydrofoil with the fixed end $D$ and free end $B$, the line “2” to the hydrofoil with both ends $B$ and $D$ fixed. The maximal deflections for the first case occur at the end $B$ of the hydrofoil, for the second case at the middle of the hydrofoil.

The effect of the boundary conditions on the hydrofoil ends on the flow around the hydrofoil is presented on the fig. 5.10. It is shown that in the case when the end $D$ is fixed and the end $B$ is free the cavity becomes smaller compared to the case of the rigid hydrofoil. This is due to the fact that bending of the hydrofoil makes it more streamlined and reduces the size of the cavity. On the other hand, bending of the hydrofoil results in the increase in the cavity size compared to the rigid hydrofoil if both ends $B$ and $D$ of the hydrofoil are fixed.

Consider the drag and lift coefficients $C_X$ and $C_Y$ defined by the formula

$$C_X + iC_Y = -\frac{i}{\lambda u_\infty} \int_{dab} (\sigma + 1 - e^{2Re \omega(\zeta)})e^{-\omega(\zeta)} \frac{dw}{d\zeta} d\zeta.$$
Figure 5.10: Comparison of the cavity profile for the cases of (1) fixed ends $B$ and $D$, (2) rigid hydrofoil, (3) fixed end $D$ and free end $B$.

Figure 5.11: Dependence of the drag and lift coefficients $C_X$ and $C_Y$ on the cavitation number $\sigma$ for the hydrofoil with (1) fixed end $D$ and free end $B$, (2) both ends $B$ and $D$ are fixed, (3) rigid hydrofoil $BD$. The difference between the drag coefficient $C_X$ for the last two cases is very small due to the small deflections of the hydrofoil and small difference in the cavity size. However, for the larger deflections of the hydrofoil it is expected that the coefficient $C_X$ is larger for the second case compared to the rigid hydrofoil. Thus, the flexibility of the hydrofoil results in the reduction of the drag if one of the ends is free and in the increase of the drag if both ends are fixed. The flexibility of the hydrofoil also accounts for the appearance of the non-zero lift coefficient $C_Y$ if the end $D$ is fixed and end $B$ is free (fig. 5.11). In the cases when both ends $B$ and $D$ are fixed or the hydrofoil is rigid the lift coefficient $C_Y$ is zero due to the symmetry of the construction.

Finally, the dependence of the circulation $\Gamma$ on the cavitation number $\sigma$ for the case of the fixed end $D$ and the free end $B$ is shown on the fig. 5.12. The circulation in the cases of the flexible hydrofoil with both ends $B$ and $D$ fixed and in the case of the rigid
Figure 5.12: Dependence of the circulation $\Gamma$ on the cavitation number $\sigma$ for the hydrofoil with the fixed end $D$ and free end $B$.

![Diagram of a hydrofoil with fixed end D and free end B.]

Figure 5.13: A wedge with flexible sides in the stream of liquid.

The circulation is zero due to the symmetry of the construction. Observe, that the circulation can be found explicitly as

$$\Gamma = \int_{l^*} \frac{dw}{d\zeta} d\zeta = \frac{2\pi N}{\sqrt{ac}} (a + c).$$

Also observe that the dependence of the quantities shown on the fig. 5.9 and fig. 5.11 on the cavitation number $\sigma$ in the interval $0.1 \div 1.0$ appears to be linear, though it is difficult to prove it analytically.

## 5.2 A Wedge with Flexible Sides in a Stream of Liquid

Consider the flow of ideal liquid past a wedge with flexible sides of the lengths $\lambda_1$ and $\lambda_2$ (fig. 5.13). Assume that the angles which the tangent line to the sides $AB$ and $DA$ of the wedge makes with the positive direction of $x$-axis are $\alpha_0$ and $\beta_0$ correspondingly. Assume also that the stream of liquid separates at the point $A$, i.e. the wedge can rotate around the point $A$, and $\delta$ is a priori unknown angle of rotation.

The presented above solution for a flexible hydrofoil can be almost completely repeated for the case of a wedge with flexible sides with the exception of the equation (5.28) being replaced by

$$\omega(\zeta) = \chi \omega(\zeta) \left( \frac{\ln(\kappa + 1)}{4\pi i} \int_{abcd} \frac{d\xi}{\xi - \zeta} \frac{\xi}{\xi^+ \chi(\xi)} - \frac{1}{2\pi} \int_{ab} \frac{\xi + \zeta \alpha(f(\xi))d\xi}{\xi - \zeta} \frac{\xi}{\xi^+ \chi(\xi)} + \right)$$
Table 5.3: The values of the parameters $\theta_b$, $\theta_c$, $\theta_d$ and $N$ for the varying cavitation number $\sigma$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\theta_b$</th>
<th>$\theta_c$</th>
<th>$\theta_d$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
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<td>92.872788</td>
</tr>
<tr>
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</tr>
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</tr>
<tr>
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<td>0</td>
<td>-2.613149</td>
<td>2.402760</td>
</tr>
</tbody>
</table>

\[
\frac{1}{2\pi} \int \frac{\xi + \zeta \{ \pi - \beta(f(\xi)) \}}{\xi \chi(\xi)} d\xi, \tag{5.46}
\]

where $\alpha(f(\xi))$ and $\beta(f(\xi))$ are the angles which the tangent lines to the sides $AB$ and $DA$ make with the positive direction of $x$-axis.

A change also occurs in the number of unknown variables and additional conditions. In this case we have five unknown parameters, namely, $b$, $c$, $d$, $N$ and $\delta$. To find these parameters we have the condition:

\[
\ln(\sigma + 1) \int_{bc} \frac{d\xi}{\xi \chi(\xi)} - 2i \int_{ab} \frac{\alpha(f(\xi)) d\xi}{\xi \chi(\xi)} + 2i \int_{da} \frac{\{ \pi - \beta(f(\xi)) \}}{\xi \chi(\xi)} d\xi = 0, \tag{5.47}
\]

which is an analogue of the condition (5.27). The condition (5.30) is necessary for this case as well. Additionally, we have two geometric conditions, specifying the lengths of the sides of the wedge:

\[
\int_{ab} \left| \frac{df}{d\zeta} \right| d\zeta = \lambda_1, \quad \int_{da} \left| \frac{df}{d\zeta} \right| d\zeta = \lambda_2. \tag{5.48}
\]

One of these conditions can be used to find the parameter $N$ similarly to the case of the hydrofoil. To find the functions $\alpha(f(\xi))$ and $\beta(f(\xi))$ we need to solve the problem of the bending of the plate for each of the sides of the wedge.

Consider a symmetric wedge with the sides of the lengths $\lambda_1 = \lambda_2 = 1$ and the angles $\alpha_0 = \pi/3$ and $\beta_0 = 2\pi/3$. The pressure and the velocity at infinity are taken to be $p_\infty = 1$ and $v_\infty = 1$. The density is $\rho = 1$ and the material parameter $D_0 = 1.66667$. Both ends $B$ and $D$ of the sides of the wedge are free. The points $b$, $c$ and $d$ are identified by the polar angles $\theta_b$, $\theta_c$ and $\theta_d$ as in the case of hydrofoil. The results of the computations are presented in the table 5.3. Observe that due to the symmetry of the problem $\theta_b = -\theta_d$, $\theta_c = 0$ and $\delta = 0$.

The cavity and the wedge profiles are shown on the fig. 5.14 for four different cavitation numbers $\sigma = 0.3$, $\sigma = 0.5$, $\sigma = 0.7$ and $\sigma = 1.0$. The comparison of the cavity profiles for the case of the wedge with flexible sides (1) and with rigid sides (2)
Figure 5.14: Cavity and wedge profiles for different cavitation numbers: (1) $\sigma = 0.3$, (2) $\sigma = 0.5$, (3) $\sigma = 0.7$, (4) $\sigma = 1.0$.

Figure 5.15: Comparison of the cavity profile for a wedge with (1) flexible or (2) rigid sides for the cavitation number $\sigma = 0.5$.

Figure 5.16: Dependence of the cavity length on the cavitation number for (1) wedge with flexible sides and (2) rigid wedge.
are shown on the fig. 5.15. It is clear, that taking into account the flexibility of the wedge significantly reduces the size of the cavity. Comparison of the cavity lengths for the flexible and the rigid wedge can be seen on the fig. 5.16. It can be seen that the flexibility of the wedge affects the flow stronger for the small cavitation numbers $\sigma$, while for a relatively large cavitation numbers the flexibility is insignificant.

Profiles of the upper side of the wedge for different cavitation numbers $\sigma$ in the system of coordinates connected with a side of the wedge are shown on the fig. 5.17. The lower side of the wedge is not shown due to the symmetry. The cavitation number $\sigma = 0.1$ corresponds to the profile with the smallest deflections, the cavitation number $\sigma = 1.0$ to the profile with the largest deflections. Other profiles correspond to the cavitation numbers $\sigma = 0.2 - 0.9$. It can be noted that again the cavitation number strongly affects the size of the cavity, but does not affect much the deflections of the wedge. Dependence of the maximal deflections of the wedge on the cavitation number is shown on the fig. 5.18. Observe, that this dependence appears to be linear. The dependence of the drag coefficient $C_X$ on the cavitation number $\sigma$ is shown on the fig. 5.19. Again taking into account the flexibility of the sides of the wedge reduces the drag due to the fact that the wedge becomes more streamlined. Finally, the fig. 5.20 shows the cavity and the wedge profiles for the nonsymmetric wedge with the sides $\lambda_1 = 1$,
Figure 5.19: Dependence of the drag on the cavitation number for (1) a wedge with flexible sides and (2) a rigid wedge.

Figure 5.20: Cavity and wedge profile for a nonsymmetric wedge with sides $\lambda_1 = 1$ and $\lambda_2 = 2$ for the cavitation number $\sigma = 0.5$. 
\( \lambda_2 = 2 \) for the cavitation number \( \sigma = 0.5 \). The solution of the system for this case is \( \theta_b = 2.825751, \theta_c = -7.127682 \cdot 10^{-3}, \theta_d = -2.627697, \delta = -2.537413 \cdot 10^{-2} \) and \( N = 6.240652 \). The drag and lift coefficients are \( C_X + iC_Y = 1.084639 + 0.196775i \), and the circulation is \( \Gamma = -4.448129 \cdot 10^{-2} \).
Chapter 6

Interaction of Two Supercavitating Wedges in a Stream of Liquid

In this chapter the mutual influence of two supercavitating wedges in a stream of liquid is considered. The Tulin single-spiral-vortex model is taken as a cavity closure condition. In the Section 1 a flow around two wedges in the infinite domain of liquid is considered. In the Section 2 the influence of the free surface on two supercavitating wedges is studied. Finally, the symmetric flow around two wedges in a jet of liquid is considered in the Section 3. All of the problems are solved in a closed form by using the conformal mapping method in combination with the Riemann-Hilbert technique. Numerical results are presented for the symmetric flow around two wedges in a jet.

6.1 Two Wedges in an Infinite Domain Filled with Liquid

6.1.1 Statement of the Problem

Consider a flow around two wedges $B_1A_1D_1$ and $B_2A_2D_2$ (fig. 6.1) with the sides of the lengths $\lambda_{11}$, $\lambda_{12}$ and $\lambda_{21}$, $\lambda_{22}$ correspondingly. The upper sides of the wedges make the angles $\alpha_{01}$ and $\alpha_{02}$ with the positive direction of the real axis, and the lower sides make the angles $\beta_{01}$ and $\beta_{02}$ (fig. 6.1). Far away from the wedges the flow of liquid is uniform with a velocity $v = \{v_\infty, 0\}$. The stream of liquid breaks away from the points $B_j$ and $D_j$ and the cavities form behind the wedges. The pressures $p_1$ and $p_2$ inside of the cavities (or, equivalently, the speeds $v_1$ and $v_2$ on the boundaries of the cavities) are prescribed and, thus, the cavitation numbers $\sigma_1$ and $\sigma_2$ are given for each of the cavities. Assume that the wedges can rotate around their vertices $A_1$, $A_2$ so that the flow separates at the points $A_1$, $A_2$. The angles of rotation $\delta_1$, $\delta_2$ are initially unknown and need to be found from the solution.

Under these assumptions the problem of the fluid mechanics can be reduced to finding the complex potential $w(z) = \varphi(z) + i\psi(z)$ in the flow domain $\mathcal{D}$ satisfying the following conditions:

$$\text{Im } w(z) = \psi_j, \quad z \in A_jB_jC_jD_jA_j, \quad j = 1, 2,$$  \hspace{1cm} (6.1)
Figure 6.1: Two supercavitating wedges in a stream of liquid.

Figure 6.2: The auxiliary domain $\mathcal{D}$ of the variable $\zeta$.

\[
\left| \frac{dw}{dz} \right| = v_j, \quad z \in B_j C_j D_j, \quad j = 1, 2, \quad (6.2)
\]

\[
\arg \frac{dw}{dz} = \begin{cases} 
-\alpha_j, & z \in A_j B_j, \\
\pi - \beta_j, & z \in D_j A_j,
\end{cases} \quad (6.3)
\]

where $\psi_j$ are real constants; $\alpha_j = \alpha_{0j} + \delta_j$, $\beta_j = \beta_{0j} + \delta_j$, $j = 1, 2$; the pressure $p_j$ and the velocity $v_j$ are related to the pressure $p_\infty$ and the velocity $v_\infty$ at infinity through the Bernoulli equations:

\[
\frac{1}{2} v_j^2 + \frac{p_j}{\rho} = \frac{1}{2} v_\infty^2 + \frac{p_\infty}{\rho}.
\]

6.1.2 Conformal Mapping

To solve the problem (6.1)-(6.3) we consider a conformal mapping $z = f(\zeta)$ from the auxiliary domain $\mathcal{D}$ onto the flow domain $\tilde{\mathcal{D}}$. The flow domain $\tilde{\mathcal{D}}$ is doubly connected, hence, we need to choose a doubly connected auxiliary domain $\mathcal{D}$. Take the exterior of two cuts $l_1 = [k, 1]$ and $l_2 = [-1, -k]$, $0 < k < 1$ (fig. 6.2) as the auxiliary domain $\mathcal{D}$. The cuts $l_1$ and $l_2$ are the preimages of the wedges and the boundaries of the cavities.
The points $a_j, b_j, c_j$ and $d_j$ are the preimages of the points $A_j, B_j, C_j$ and $D_j$ on the physical domain $\tilde{D}$. Without loss of generality we can assume that the point $\zeta = e$ on the real axis is the preimage of the infinity point $z = \infty$ on the flow domain. None of these points can be prescribed a priori; they must be found as a result of the solution. The parameter $k$ of the conformal mapping also needs to be found during the solution of the problem.

As before two Riemann-Hilbert problems will be stated and solved for the functions $dw/d\zeta$ and $dw/dz$. The derivative of the conformal mapping can be found after that from the formula

$$\frac{df}{d\zeta} = \frac{dw}{d\zeta} : \frac{dw}{dz}. \quad (6.4)$$

Firstly, consider the function $dw/d\zeta$. From the condition (6.1) we can see that

$$\text{Im} \frac{dw}{d\zeta} = 0, \quad \zeta \in l_1 \cup l_2. \quad (6.5)$$

At the points $\zeta = a_j$ and $\zeta = c_j$ the function $dw/d\zeta$ has simple zeros if these points do not coincide with the end points of the cuts. Additionally, this function must have the pole of the second order at the point $\zeta = e$ and behave as $O(\zeta^{-2})$ as $\zeta \to \infty$. The last two conditions follow from the fact that $z = f(\zeta)$ maps the point $\zeta = e$ to the point $z = \infty$ and the point $\zeta = \infty$ to some finite point of the physical domain $\tilde{D}$. This function also must have singularity of the orders $1/2$ at the end points of the cuts $l_1$ and $l_2$. This is due to the fact that the curves $A_jB_jC_j$ and $A_jD_jC_j$ are smooth.

Consider the function $p^{1/2}(\zeta) = \sqrt{(\zeta^2 - 1)(\zeta^2 - k^2)}$. Choose the branch of this function according to the condition

$$p^{1/2}(\zeta) \sim O(\zeta^2) \text{ as } \zeta \to \infty.$$
Then the condition (6.5) means that the function $\Xi(\zeta, u)$ is continuous through the cuts $l_1$ and $l_2$:

$$\Xi^+(\xi, v) = \Xi^-(\xi, v), \quad (\xi, v) \in l_1 \cup l_2,$$

where “±” signs denote the limiting values of the function $\Xi(\zeta, u)$ on the cuts $l_1$ and $l_2$ from the upper $C_1$ or the lower $C_2$ sheet of the Riemann surface $R$ correspondingly.

Thus, we need to find a symmetric function continuous on the whole Riemann surface $R$, which has simple zeros at the points $\zeta = a_j$ and $\zeta = c_j$, poles of the second order at the points with the affixes $\zeta = e$ on the upper sheet $C_1$ and the lower sheet $C_2$ of the surface, and zeros of the second order at the infinity points of the Riemann surface $R$. Finally, the function $\Xi(\zeta, u)$ has simple poles (in the local parameter) at the end points $\zeta = \pm k$, $\zeta = \pm 1$ of the cuts $l_1$ and $l_2$. This function must be a rational function on the Riemann surface $R$. We will look for the solution in the form

$$\frac{dw}{d\zeta} = \frac{i(N_1 + N_2\zeta + N_3\zeta^2 + iN_4p^{1/2}(\zeta))}{(\zeta - e)^2p^{1/2}(\zeta)},$$

where $N_1, N_2, N_3$ and $N_4$ are real constants.

This function satisfies all the necessary conditions with the exception of the zeros at the points $\zeta = a_j$ and $\zeta = c_j$. To satisfy these conditions we must have

$$N_1 + N_2a_j + N_3a_j^2 + iN_4p^{1/2}(a_j) = 0, \quad j = 1, 2,$$

$$N_1 + N_2c_j + N_3c_j^2 + iN_4p^{1/2}(c_j) = 0, \quad j = 1, 2.$$

This is a homogeneous linear system of four equations with four unknowns. It has a non-trivial solution if and only if its discriminant is equal to zero. Thus, the following condition must be satisfied:

$$\begin{vmatrix}
1 & a_1 & a_1^2 & ip^{1/2}(a_1) \\
1 & a_2 & a_2^2 & ip^{1/2}(a_2) \\
1 & c_1 & c_1^2 & ip^{1/2}(c_1) \\
1 & c_2 & c_2^2 & ip^{1/2}(c_2)
\end{vmatrix} = 0. \quad (6.6)$$

If this condition is satisfied, then we can find the constants $N_1, N_2, N_3$ and $N_4$ explicitly:

$$N_1 = N\gamma_1, \quad N_2 = N\gamma_2, \quad N_3 = N\gamma_3, \quad N_4 = N\gamma_4, \quad (6.7)$$
where $N$ is a real constant and

$$
\begin{align*}
\gamma_1 &= i\{p^{1/2}(c_1) a_1 a_2 (a_2 - a_1) + p^{1/2}(a_1) a_2 c_1 (c_1 - a_2) - p^{1/2}(a_2) a_1 c_1 (c_1 - a_1)\}, \\
\gamma_2 &= i\{p^{1/2}(a_2) (c_2^2 - a_1^2) - p^{1/2}(c_1) (a_2^2 - a_1^2) - p^{1/2}(a_1) (c_1^2 - a_2^2)\}, \\
\gamma_3 &= i\{p^{1/2}(c_1) (a_2 - a_1) - p^{1/2}(a_2) (c_1 - a_1) + p^{1/2}(a_1) (c_1 - a_2)\}, \\
\gamma_4 &= -(a_2 - a_1) (c_1 - a_1) (c_1 - a_2).
\end{align*}
$$

Then the function

$$
\frac{dw}{d\zeta} = \frac{iN(\gamma_1 + \gamma_2 \zeta + \gamma_3 \zeta^2 + i\gamma_4 p^{1/2}(\zeta))}{(\zeta - e)^2 p^{1/2}(\zeta)}.
$$

satisfies all the necessary conditions.

### 6.1.3 Function $dw/dz$

As in previous chapters we consider the logarithmic hodograph variable:

$$
\omega(\zeta) = \log \left( \frac{1}{v_\infty} \frac{dw}{dz} \right).
$$

Then from the equations (6.2) and (6.3) we obtain that

$$
\text{Re} \omega(\zeta) = \sigma'_j, \quad \sigma'_j = \log(v_j/v_\infty), \quad \zeta \in b_j c_j d_j, \quad j = 1, 2, \quad (6.10)
$$

$$
\text{Im} \omega(\zeta) = \left\{ \begin{array}{ll}
-\alpha_j, & \zeta \in a_j b_j, \\
\pi - \beta_j, & \zeta \in d_j a_j, \quad j = 1, 2.
\end{array} \right. \quad (6.11)
$$

Due to the Tulin single-spiral-vortex model for the cavity closure the function $\omega(\zeta)$ must have the following singularities at the preimages $\zeta = c_j$ of the cavity closure points:

$$
\omega(\zeta) = O((\zeta - c_j)^{-1}), \quad \zeta \to c_j.
$$

Introduce a new function

$$
\Phi(\zeta) = \left\{ \begin{array}{ll}
-\frac{i\omega(\zeta)}{\omega(\zeta)}, & (\zeta, u) \in \mathbb{C}_1, \\
\frac{i\omega(\zeta)}{\omega(\zeta)}, & (\zeta, u) \in \mathbb{C}_2.
\end{array} \right.
$$

For this function we obtain the following Riemann-Hilbert boundary value problem:

**Formulation.** Find all the functions $\Phi(\zeta, u)$ analytic in $\mathcal{R} \setminus (l_1 \cup l_2)$, Hölder continuous up to the boundary $l_1 \cup l_2$ with the limiting values satisfying the following boundary condition:

$$
\Phi^+(\xi, v) = G(\xi, v) \Phi^-(\xi, v) + g(\xi, v), \quad (\xi, v) \in l_1 \cup l_2,
$$

where

$$
G(\xi, v) = \left\{ \begin{array}{ll}
1, & (\xi, v) \in b_j c_j d_j, \quad j = 1, 2, \\
-1, & (\xi, v) \in d_j a_j b_j, \quad j = 1, 2,
\end{array} \right. \quad (6.13)
$$

$$
g(\xi, v) = \left\{ \begin{array}{ll}
-2i\sigma'_j, & (\xi, v) \in b_j c_j d_j, \quad j = 1, 2, \\
-2\alpha_j, & (\xi, v) \in a_j b_j, \quad j = 1, 2, \\
2(\pi - \beta_j), & (\xi, v) \in d_j a_j, \quad j = 1, 2.
\end{array} \right. \quad (6.14)
$$
Figure 6.5: The contour \( \gamma \) and the canonical cross-sections \( a, b \) of the Riemann surface \( \mathcal{R} \).

The function \( \Phi(\zeta, u) \) satisfies the symmetry condition

\[
\Phi(\zeta, u) = \overline{\Phi(\bar{\zeta}, -u(\zeta))},
\]

has simple poles at the points \((c_j, u(c_j))\) \((j = 1, 2)\), is bounded at the points \((b_j, u(b_j)), (d_j, u(d_j))\) and the infinity points of the surface \( \mathcal{R} \), and has a zero at the point \((e, u(e))\).

### 6.1.4 Canonical Function \( X(\zeta, u) \)

As before we need to find the canonical function \( X(\zeta, u) \) which satisfies the symmetry condition (6.15) and the homogeneous boundary conditions (6.13), (6.14):

\[
X^+(\xi, v) = -X^-(\xi, v), \ (\xi, v) \in d_1a_1b_1 \cup d_2a_2b_2.
\]

The solution to this problem can be given in the form:

\[
X(\zeta, u) = \exp \left\{ \frac{1}{4} \sum_{j=1}^{2} \int_{d_ja_jb_j} \left( 1 + \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} - \frac{1}{2} \int_{\gamma} \left( 1 + \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} \right\},
\]

(6.16)

where \( \gamma \) is a curve with an arbitrarily fixed beginning point \((\eta_0, u(\eta_0))\) and an unknown end point \((\zeta_0, u(\zeta_0))\), \( a \) and \( b \) are the canonical cross-sections of the Riemann surface \( \mathcal{R} \) (fig. 6.5) and \( m \) is an integer. The point \((\zeta_0, u(\zeta_0))\) and the integer \( m \) can be found from the solution of the Jacobi inversion problem:

\[
\zeta_0 = k \operatorname{sn} \left\{ \int_{\eta_0}^{0} \frac{d\xi}{p^{1/2}(\xi)} - \frac{1}{4} \sum_{j=1}^{2} \int_{d_ja_jb_j} \frac{d\xi}{u(\xi)} \right\},
\]

(6.17)

and

\[
m = \frac{1}{2K'(k)} \Im \left\{ \int_{\eta_0}^{\zeta_0} \frac{d\xi}{p^{1/2}(\xi)} - \frac{1}{4} \sum_{j=1}^{2} \int_{d_ja_jb_j} \frac{d\xi}{u(\xi)} \right\}, \quad K'(k) = \frac{i}{2} \int_{a}^{b} \frac{d\xi}{u(\xi)}.
\]
if \((\zeta_0, u(\zeta_0))\) lies on the upper sheet of the Riemann surface \(\mathcal{R}\) and
\[
m = \frac{1}{2K'(k)} \Im \left\{ -\int_{\zeta_0}^0 \frac{d\xi}{p^{1/2}(\xi)} + \int_{\eta_0}^0 \frac{d\xi}{p^{1/2}(\xi)} - \frac{1}{4} \sum_{j=1}^{2} \int_{d_ja_jb_j} \frac{d\xi}{u(\xi)} \right\}
\]
if \((\zeta_0, u(\zeta_0))\) lies on the lower sheet of the Riemann surface \(\mathcal{R}\).

Observe, that the function \(X(\zeta, u)\) defined by the formula (6.16) has the following singularities on the Riemann surface \(\mathcal{R}\):
\[
X(\zeta, u) = \begin{cases} 
O((\zeta - b_j)^{1/2}) & \text{as } (\zeta, u) \to (b_j, u(b_j)), \ j = 1, 2, \\
O((\zeta - d_j)^{1/2}) & \text{as } (\zeta, u) \to (d_j, u(d_j)), \ j = 1, 2, \\
O(\zeta - \eta_0) & \text{as } (\zeta, u) \to (\eta_0, u(\eta_0)), \\
O((\zeta - \zeta_0)^{-1}) & \text{as } (\zeta, u) \to (\zeta_0, u(\zeta_0)).
\end{cases}
\]

6.1.5 Solution of the Inhomogeneous Riemann-Hilbert Problem and Additional Conditions

The canonical function \(X(\zeta, u)\) allows us to factorize the coefficient (6.13) of the inhomogeneous Riemann-Hilbert boundary value problem as
\[
G(\xi, v) = X^+(\xi, v) \cdot [X^-(\xi, v)]^{-1}, \ (\xi, v) \in l_1 \cup l_2.
\]

By substituting the last expression into the inhomogeneous condition (6.12) we obtain
\[
\Phi^+(\xi, v) = \Phi^-(\xi, v) + \frac{g(\xi, v)}{X^+(\xi, v)}, \ (\xi, v) \in l_1 \cup l_2.
\]

This problem can be solved explicitly:
\[
\Phi(\zeta, u) = X(\zeta, u)\{\Psi(\zeta, u) + \Omega(\zeta, u)\}, \quad (6.18)
\]
where
\[
\Psi(\zeta, u) = \frac{1}{2\pi i} \sum_{j=1}^{2} \left( -\alpha_j \int_{a_jb_j} + (\pi - \beta_j) \int_{d_ja_j} -i\sigma_j' \int_{b_jc_jd_j} \right) \frac{(1 + u/v)d\xi}{X^+(\xi, v)(\xi - \zeta)} \quad (6.19)
\]
and \(\Omega(\zeta, u)\) is a rational function chosen so that the solution (6.18) satisfies all other conditions of the Riemann-Hilbert problem. In particular, the function \(\Phi(\zeta, u)\) must have the following properties:

- has simple poles at the points \(\zeta = c_j, \ j = 1, 2,\)
- bounded at the infinity points,
- has a simple zero at the point \(\zeta = e,\)
- bounded at the points \(\zeta = d_j, \ j = 1, 2,\)
- bounded at the point \(\zeta = \zeta_0.\)
• does not have a zero at the point $\zeta = \eta_0$.

The function (6.18) additionally must satisfy the symmetry condition (6.15). Observe, that the functions $X(\zeta, u)$ and $\Psi(\zeta, u)$ are symmetric. Thus, to preserve the symmetry of the function $\Phi(\zeta, u)$ the function $\Omega(\zeta, u)$ must be symmetric.

We can satisfy some of the conditions mentioned above by making a smart choice of the function $\Omega(\zeta, u)$. After that to satisfy the rest of the conditions we will need to state several additional conditions for the parameters $a_j, b_j, c_j, d_j, e, k$ and $\delta_j$.

We can choose the function $\Omega(\zeta, u)$ as follows:

$$\Omega(\zeta, u) = M_0 + \frac{iM_1(u(\zeta) + u(c_1))}{\zeta - c_1} + \frac{iM_2(u(\zeta) + u(c_2))}{\zeta - c_2} + \frac{(M_3 + iM_4)(u(\zeta) + u(\eta_0))}{\zeta - \eta_0} - \frac{(M_3 - iM_4)(u(\zeta) - u(\eta_0))}{\zeta - \eta_0}. \quad (6.20)$$

If the function $\Omega(\zeta, u)$ is given by the formula (6.20) then the function $\Phi(\zeta, u)$ has simple poles at the points $\zeta = c_j$ and does not have zeros at the points $\zeta = \eta_0$. Additionally, the function $\Omega(\zeta, u)$ contains five unknowns $M_j, j = 0, 1, \ldots, 4$ which are necessary to satisfy other conditions imposed on the solution $\Phi(\zeta, u)$.

Firstly, it is necessary to eliminate the simple pole of the function $\Phi(\zeta, u)$ at the point $\zeta = \zeta_0$. This pole appears due to the Jacobi inversion problem and has no physical meaning. This means that the following complex condition must be satisfied:

$$\Psi(\zeta_0, u(\zeta_0)) + \Omega(\zeta_0, u(\zeta_0)) = 0. \quad (6.21)$$

Similarly, to eliminate the singularities at the points $\zeta = d_j, j = 1, 2$, we obtain two real conditions:

$$\Psi(d_j, u(d_j)) + \Omega(d_j, d_j) = 0, \quad j = 1, 2. \quad (6.22)$$

The function $\Phi(z, u)$ must have a zero at the point $\zeta = e$ due to the fact that this point is the preimage of the point $z = \infty$. At the point $z = \infty$ the velocity is equal to $v_\infty$ and hence the function $\omega(\zeta)$ must vanish. Thus, we obtain one additional complex condition

$$\Psi(e, u(e)) + \Omega(e, u(e)) = 0. \quad (6.23)$$

The point $\zeta = \infty$ is mapped into the regular point of the flow domain $\bar{D}$. Hence, the velocity must be bounded at this point. This means that the function $\Phi(\zeta, u)$ is bounded at the point $\zeta = \infty$. This leads to the additional real condition:

$$M_1 + M_2 + 2M_4 + \frac{1}{2\pi} \sum_{j=1}^{2} \left( -\alpha_j \int_{a_jb_j} + (\pi - \beta_j) \int_{d_ja_j} - i\sigma_j \int_{b_jc_jd_j} \right) \frac{d\zeta}{vX^+(\zeta, u)} = 0. \quad (6.24)$$

There are several additional conditions coming from the geometric considerations. Firstly, the lengths of the sides of the wedges are fixed, which gives us four real conditions:

$$\int_{d_ja_j} \left| \frac{df}{d\zeta} \right| |d\zeta| = \lambda_{j1}, \quad \int_{a_jb_j} \left| \frac{df}{d\zeta} \right| |d\zeta| = \lambda_{j2}, \quad j = 1, 2. \quad (6.25)$$
The vertical and the horizontal distances between the vertices of the wedges are given, which adds one more complex condition:

\[ \int_{a_1a_2} \frac{df}{d\zeta} d\zeta = \mu_1 + i\mu_2. \]  \hspace{1cm} (6.26)

It is also reasonable to assume that each cavity and the wedge can be enclosed by a closed contour, which is equivalent to the single-valuedness of the conformal mapping:

\[ \int_{l_j^*} \frac{df}{d\zeta} d\zeta = 0, \quad j = 1, 2, \]  \hspace{1cm} (6.27)

where \( l_j^* \) is a smooth contour enclosing the cut \( l_j \) and not crossing the other cut.

Finally, the condition (6.6) needs to be satisfied. Thus, in total we have eighteen real conditions for eighteen real unknown variables \( a_j, b_j, c_j, d_j, e, \delta_j, j = 1, 2, M_k, k = 0, 1, \ldots, 4 \) and \( N \). Observe, that the last eight variables are included in the system (6.6), (6.21) - (6.27) only linearly and can be eliminated from the system. Hence, to restore the conformal mapping we need to solve the system of ten transcendental equations with ten unknowns \( a_j, b_j, c_j, d_j, e, k \).

### 6.2 Two Supercavitating Wedges in a Presence of Free Surface

Consider a problem of a flow around two supercavitating wedges near a free surface of the liquid (fig. 6.6). The speed of the liquid on the free surface is equal to the speed at infinity \( v_\infty \) and the pressure on the free surface is the same as the pressure at infinity \( p_\infty \).

The conditions (6.1)-(6.3) need to be modified for this problem in the following way:

\[ \text{Im} \, w(z) = \begin{cases} 
\psi_j, & z \in A_jB_jC_jD_jA_j, \quad j = 1, 2, \\
\psi_0, & z \in E_1E_2,
\end{cases} \]  \hspace{1cm} (6.28)
where $\psi_0, \psi_1, \psi_2$ are real constants.

The flow domain $\mathcal{D}$ is triply-connected, thus we should choose a triply-connected auxiliary domain $\mathcal{D}$. Take the domain $\mathcal{D}$ to be the exterior of three cuts $l_0 = [k_0, \infty]$, $l_1 = [0, 1]$ and $l_2 = [-k_2, -k_1]$ (fig. 6.7).

Let $z = f(\zeta)$ be a conformal mapping from the auxiliary domain $\mathcal{D}$ onto the flow domain $\tilde{\mathcal{D}}$. Similarly to the previous section consider the function $p^{1/2}(\zeta)$:

$$p^{1/2}(\zeta) = i \sqrt{(\zeta - 1)(\zeta - k_0)(\zeta + k_1)(\zeta + k_2)}.$$ 

The chosen branch of the function $p^{1/2}(\zeta)$ is shown on the fig. 6.8.

As before, we have the following boundary condition for the function $dw/d\zeta$:

$$\text{Im} \frac{dw}{d\zeta} = 0, \quad \zeta \in l_0 \cup l_1 \cup l_2.$$ 

The function $dw/d\zeta$ has zeros at the points $\zeta = a_j$ and $\zeta = c_j$:

$$\frac{dw(a_j)}{d\zeta} = 0, \quad \frac{dw(c_j)}{d\zeta} = 0, \quad j = 1, 2.$$ 

The function $dw/d\zeta$ has a pole of the second order at the point $\zeta = e$:

$$\frac{dw}{d\zeta} = O((\zeta - e)^{-2}), \quad \zeta \to e.$$ 

Due to the asymptotics of the conformal mapping $z = f(\zeta)$ at infinity we obtain a condition

$$\frac{dw}{d\zeta} = O(\zeta^{-3/2}), \quad \zeta \to \infty.$$
As before we can continue the function $dw/dζ$ onto the whole Riemann surface $\mathcal{R}$ of the function $u^2 = p(ζ)$ and conclude that the function $dw/dζ$ is a rational function on the surface $\mathcal{R}$. Then the function $dw/dζ$ is in the following form:

$$
\frac{dw}{dζ} = N \frac{u(ζ) + u(e) + i(N_1ζ + N_2)(ζ - e)^2}{u(ζ)(ζ - e)^2},
$$

where $N$ is a real constant and

$$
N_1 = \frac{i}{a_2 - a_1} \left( \frac{u(a_2) + u(e)}{(a_2 - e)^2} - \frac{u(a_1) + u(e)}{(a_1 - e)^2} \right),
$$

$$
N_2 = \frac{i}{a_2 - a_1} \left( a_2 \frac{u(a_1) + u(e)}{(a_1 - e)^2} - a_1 \frac{u(a_2) + u(e)}{(a_2 - e)^2} \right).
$$

In order for the function (6.31) to have simple zeros at the points $ζ = c_j$ two additional real conditions must be imposed:

$$
u(c_j) + u(e) + i(N_1c_j + N_2)(c_j - e)^2 = 0, \quad j = 1, 2.
$$

Consider the logarithmic hodograph variable $ω(ζ)$. Extend the function $ω(ζ)$ to the whole Riemann surface $\mathcal{R}$ by the symmetry. Then for this new function $Φ(ζ, u)$ we obtain the following Riemann-Hilbert boundary value problem:

**Formulation.** Find all the functions $Φ(ζ, u)$ analytic in $\mathcal{R} \setminus (l_0 \cup l_1 \cup l_2)$, Hölder continuous up to the boundary $l_0 \cup l_1 \cup l_2$, with the boundary values satisfying the following boundary condition:

$$
Φ^+(ξ, v) = G(ξ, v)Φ^-(ξ, v) + g(ξ, v), \quad (ξ, v) ∈ l_0 \cup l_1 \cup l_2,
$$

where

$$
G(ξ, v) = \begin{cases} 
1, & (ξ, v) ∈ b_1c_1d_1 \cup b_2c_2d_2 \cup l_0, \\
-1, & (ξ, v) ∈ d_1a_1b_1, \quad j = 1, 2,
\end{cases}
$$

$$
g(ξ, v) = \begin{cases} 
-2iα_j, & (ξ, v) ∈ b_1c_jd_j, \quad j = 1, 2, \\
-2α_j, & (ξ, v) ∈ a_jb_j, \quad j = 1, 2, \\
2(π - β_j), & (ξ, v) ∈ d_ja_j, \quad j = 1, 2, \\
0, & (ξ, v) ∈ l_0.
\end{cases}
$$

The function $Φ(ζ, u)$ satisfies the symmetry condition

$$
Φ(ζ, u) = \overline{Φ(ζ, -u(ζ))},
$$

has simple poles at the points $(c_j, u(c_j))$, $j = 1, 2$, is bounded at the points $(b_j, u(b_j))$, $(d_j, u(d_j))$ and the infinity points of the surface $\mathcal{R}$, and has a zero at the point $(e, u(e))$.

Introduce a canonical function $X(ζ, u)$ which satisfies the homogeneous condition (6.33):

$$
X^+(ξ, v) = -X^-(ξ, v), \quad (ξ, v) ∈ d_ja_jb_j, \quad j = 1, 2.
$$
Figure 6.9: Canonical cross-sections of the Riemann surface $\mathcal{R}$ and the curves $\gamma_j$.

The solution to this problem is given by the function:

$$X(\zeta, u) = \exp \left\{ \frac{1}{4} \sum_{j=1}^{2} \int_{d_ja_jb_j} \left( 1 + \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} - \sum_{j=1}^{2} \left[ \frac{1}{2} \int_{\gamma_j} \left( 1 + \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} + \frac{1}{2} \int_{\gamma_j} \left( 1 - \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} + m_j \int_{a_j} \frac{u(\zeta)}{u(\xi)} \frac{d\xi}{\xi - \zeta} \right] \right\}, \quad (6.37)$$

where $\gamma_j$ are the contours on the Riemann surface $\mathcal{R}$ with the arbitrarily fixed beginning points $\eta_1 = (\eta_1, u(\eta_1))$ and $\eta_2 = (\eta_2, u(\eta_2))$ and the unknown end points $\zeta_1 = (\zeta_1, u(\zeta_1))$ and $\zeta_2 = (\zeta_2, u(\zeta_2))$ (fig. 6.9), and $m_1$ and $m_2$ are two integers. The points $\zeta_1$, $\zeta_2$ and the integers $m_1$, $m_2$ need to be found from the solution to the Jacobi inversion problem.

The solution to the Jacobi inversion problem in this case is given by the following formulas:

$$\zeta_j = \frac{\epsilon_1 - (-1)^j \sqrt{2\epsilon_2 - \epsilon_1^2}}{2}, \quad j = 1, 2, \quad (6.38)$$

$$m_\nu = \text{Re}(\hat{g}_\nu), \quad \hat{g}_\nu = g_\nu - \kappa_\nu - \sum_{j=1}^{2} \hat{\omega}_\nu(\zeta_j), \quad (6.39)$$

where

$$\hat{\omega} = T\omega, \quad T = A^{-1}, \quad A = A_{\nu j},$$

$$B_{\nu j} = \sum_{m=1}^{2} T_{\nu m} B_{m j},$$

$$\omega_\nu(\zeta) = \int_{(k_0,0)}^{(\zeta, u)} \frac{1}{2} \frac{\xi^{\nu-1}d\xi}{u(\xi)}, \quad \nu = 1, 2,$$

$$g_\nu = \sum_{j=1}^{2} T_{\nu j} \eta_j + \kappa_\nu, \quad \nu = 1, 2,$$

$$\eta_\nu = \sum_{j=1}^{2} \left( \frac{1}{4} \int_{d_ja_jb_j} \frac{\xi^{\nu-1}d\xi}{u(\xi)} - \int_{(\eta_j, u(\eta_j))}^{(k_0,0)} \frac{\xi^{\nu-1}d\xi}{p^{1/2}(\xi)} \right),$$

$$\epsilon_\nu = 2 \sum_{j=1}^{2} \sum_{m=1}^{2} T_{jm} \int_{a_j}^{p^{1/2}(\tau)} \frac{\tau^{m+\nu-1}d\tau}{p^{1/2}(\tau)} - \text{res}_{q=\infty} \frac{\zeta^{\nu} \mathcal{F}'(q)}{\mathcal{F}(q)},$$

$$\zeta_1, \zeta_2, \text{ and } m_1, m_2$$
The canonical function (6.37) has the following singularities:

\[
X(\zeta, u) = \begin{cases} 
O((\zeta - b_j)^{1/2}) & \text{as } (\zeta, u) \to (b_j, u(b_j)), \ j = 1, 2, \\
O((\zeta - d_j)^{-1/2}) & \text{as } (\zeta, u) \to (d_j, u(d_j)), \ j = 1, 2, \\
O(\zeta - \eta_j) & \text{as } (\zeta, u) \to (\eta_j, u(\eta_j)), \ j = 1, 2, \\
O((\zeta - \xi_j)^{-1}) & \text{as } (\zeta, u) \to (\xi_j, u(\xi_j)), \ j = 1, 2.
\end{cases}
\]

The solution to the inhomogeneous Riemann-Hilbert problem is given by the formulas (6.18), (6.19) with the canonical function \(X(\zeta, u)\) given by the formula (6.37). The rational function \(\Omega(\zeta, u)\) has the form:

\[
\Omega(\zeta, u) = M_0 + \sum_{j=1}^{2} \frac{(iM_j(u(\zeta) + u(c_j))}{\zeta - c_j} + \frac{(M_{j+2} + iM_{j+4})(u(\zeta) + u(\eta_j))}{\zeta - \eta_j} - \frac{(M_{j+2} - iM_{j+4})(u(\zeta) - \overline{u(\eta_j)})}{\zeta - \overline{\eta_j}}.
\]

where \(M_k, k = 0, 1, \ldots, 6, \) are real constants.

Thus, the solution to the problem of the supercavitating flow around two wedges under the free surface involves the twenty two real unknowns \(k_0, k_1, k_2, a_j, b_j, c_j, d_j, \delta_j (j = 1, 2), e, N, M_k, k = 0, 1, \ldots, 7.\) We have already stated two real conditions (6.32) for these unknowns. Below we state twenty additional real equations.

To eliminate the simple poles of the function (6.18) at the points \((\zeta_j, u(\zeta_j))\) we must have:

\[
\Psi(\zeta_j, u(\zeta_j)) + \Omega(\zeta_j, u(\zeta_j)) = 0, \ j = 1, 2,
\]

which gives us two complex equations. To eliminate the singularities of the order 1/2 at the points \(\zeta = d_j\) the following two real conditions must be satisfied:

\[
\Psi(d_j, u(d_j)) + \Omega(d_j, u(d_j)) = 0, \ j = 1, 2.
\]

The function \(\Phi(\zeta, u)\) must be bounded at infinity which leads to two real conditions:

\[
M_1 + M_2 + 2M_5 + 2M_6 + \Psi_0 = 0,
\]

\[
\sum_{j=1}^{2} (M_jc_j + 2 \Im \eta_j M_{j+2} + 2 \Re \eta_j M_{j+4}) + \Psi_1 = 0,
\]

where

\[
\Psi_k = \frac{1}{2\pi} \sum_{j=1}^{2} \left(-\alpha_j \int_{a_jb_j} + (\pi - \beta_j) \int_{d_ja_j} -i\sigma_j' \int_{b_jc_jd_j} \right) \frac{\xi^kd\xi}{vX^+(\xi, v)}, \ k = 0, 1,
\]

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are real numbers.

The function $\Phi(\zeta, u)$ must have a zero at the point $\zeta = e$:

$$\Psi(e, u(e)) + \Omega(e, u(e)) = 0.$$  \hspace{1cm} (6.45)

The equation (6.45) is a real condition.

To fix the distance from one of the wedges (say, the first wedge) to the free surface we fix the distance $h_1$ between the free streamline and the streamline which separates at the vertex of the wedge as $\text{Re} z \to -\infty$:

$$h_1 v_\infty = \text{Im} \int_{a_1}^{\xi_0} \frac{dw}{d\zeta} d\zeta,$$  \hspace{1cm} (6.46)

where $\zeta = e_0$ is some point on the cut $l_0$ not coinciding with the point $\zeta = e$, and the integration is taken over a path joining the points $\zeta = a_1$ and $\zeta = e_0$ and not intersecting the cuts $l_0$, $l_1$, $l_2$ (although, the path of integration may touch the cuts or partially coincide with some of them).

The lengths of the sides of the wedges and the horizontal and the vertical distances between the vertices of the wedges are fixed, which gives six additional real conditions (6.25), (6.26). The conditions (6.27) of the single-valuedness of the conformal mapping are also valid.

Thus, in total we obtain twenty two real equation (6.25) - (6.27), (6.32), (6.41) - (6.46) for the twenty two real unknowns. The unknowns $\delta_1, \delta_2, N, M_k, k = 0, 1, \ldots, 6$, are included in this system only linearly and, hence, can be eliminated from the system. This system must be solved numerically. After that the derivative $df/d\zeta$ of the conformal mapping can be obtained by the formula (6.4).

### 6.3 Two Wedges in a Jet. A Symmetric Case.

Consider a supercavitating flow around two wedges in a jet (fig. 6.10). Assume that the wedges $BAD$ and $B'A'D'$ and the boundaries of the jet are symmetric with respect to the central line $E_1^- E_2^-$. The speed on the boundaries of the jet is equal to the speed of the flow at infinity $v_\infty$, and the speeds on the boundaries of the cavities are the same and equal to $v_c$. The angles the sides of the upper wedge initially make with the horizontal direction are $\alpha_0$ and $\beta_0$ correspondingly, the angles which the sides of the lower wedge make with the horizontal direction are equal to $\pi - \beta_0$ and $\pi - \alpha_0$. The angles of the rotation $\delta$ and $\delta'$ of the wedges are initially unknown and need to be found as a part of the solution. Due to the symmetry of the problem we have $\delta' = -\delta$. The lengths of the sides of the wedge are $AB = A'D' = \lambda_1$ and $AD = A'B' = \lambda_2$.

Thus, the jet with the supercavitating wedges is symmetric with respect to the central line $E_1^- E_2^-$. Hence, the central line $E_1^- E_2^-$ is a streamline. Then we can consider only the upper half of the flow domain (denote it as $\tilde{D}$) and treat the streamline $E_1^- E_2^-$ as a rigid boundary.

Consider a complex potential $w(z) = \varphi(z) + i\psi(z)$ defined and analytic in the domain $\tilde{D}$. The function $w(z)$ must satisfy the following boundary conditions:

$$\text{Im} w(z) = \begin{cases} \psi_1, & z \in ABCDA, \\ \psi_0^\pm, & z \in E_1^\pm E_2^\pm, \end{cases}$$  \hspace{1cm} (6.47)
Figure 6.10: A symmetric flow around two wedges in a jet.

\[
\begin{align*}
|dw| &= \begin{cases} 
v_c, & z \in BCD, 
v_\infty, & z \in E_1^+ E_2^+, \end{cases} \quad (6.48) \\
\arg \frac{dw}{dz} &= \begin{cases} 
-\alpha, & z \in AB, 
\pi - \beta, & z \in DA, 
0, & z \in E_1^- E_2^-, \end{cases} \quad (6.49)
\end{align*}
\]

where \( \alpha = \alpha_0 + \delta, \beta = \beta_0 + \delta. \)

Choose the auxiliary domain \( D \) as in the Chapter 3 (fig. 4.6). Consider the conformal mapping \( z = f(\xi) \) from the auxiliary domain \( D \) onto the domain \( \tilde{D} \). As before, assume that the contour \( l_1 \) is mapped into the boundary of the cavity and the wedge, and the contour \( l_0 \) is mapped into the free boundary \( E_1^+ E_2^+ \) and the central streamline \( E_1^- E_2^- \).

Following the method of the Chapter 3 we can see that the derivative \( dw/d\zeta \) is determined by the formulas (4.36) and (4.37). The function \( \Phi(\zeta, u) \) can be found from the formulas:

\[ \Phi(\zeta, u) = X(\zeta, u)(\Psi(\zeta, u) + \Omega(\zeta, u)), \quad (6.50) \]

where the canonical function \( X(\zeta, u) \) is defined by

\[
X(\zeta, u) = \exp \left\{ \frac{1}{4} \int_{\mathcal{L}} \left( 1 + \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} - \frac{1}{2} \int_{\gamma} \left( 1 + \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} - \frac{1}{2} \int_{\gamma} \left( 1 - \frac{u(\zeta)}{u(\xi)} \right) \frac{d\xi}{\xi - \zeta} - m \int_{\eta} \frac{u(\zeta)}{u(\xi)} \frac{d\xi}{\xi - \zeta} \right\},
\]

where \( \mathcal{L} = dab \cup e_1 e_2, \gamma \) is a contour with the given beginning point \( \eta_0 = (\eta_0, u(\eta_0)) \) and the unknown end point \( \zeta_0 = (\zeta_0, u(\zeta_0)) \). The canonical cross-sections on the Riemann surface \( \mathcal{R} \) are taken in the same way as in the Chapter 3. The point \( \zeta_0 \) and the integer \( m \) are found from the solution of the Jacobi inversion problem:

\[
\zeta_0 = \text{sn}^2 \left( \frac{i\sqrt{m}}{2} \eta^* \right), \quad \eta^* = \int_0^{\eta_0} \frac{d\xi}{p^{1/2}(\xi)} + \frac{1}{4} \int_{\mathcal{L}} \frac{d\xi}{u(\xi)},
\]

\[
m = -\frac{\sqrt{m}}{4K} \text{Im} \left( \int_{\gamma} \frac{d\xi}{u(\xi)} - \frac{1}{4} \int_{\mathcal{L}} \frac{d\xi}{u(\xi)} \right), \quad K = \frac{i\sqrt{m}}{2} \int_m^{\infty} \frac{d\xi}{p^{1/2}(\xi)}.
\]

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The functions $\Psi(\zeta, u)$ and $\Omega(\zeta, u)$ are defined by the formulas:

$$\Psi(\zeta, u) = \frac{1}{2\pi i} \left( -\alpha \int_{ab} + (\pi - \beta) \int_{da} -i\sigma' \int_{bcd} \right) \frac{(1 + u/v)d\xi}{X^+(\xi, v)(\xi - \zeta)},$$

$$\Omega(\zeta, u) = M_0 + \frac{iM_1(u(\zeta) + u(e))}{\zeta - c} + \frac{(M_2 + iM_3)(u(\zeta) + u(\eta_0))}{\zeta - \eta_0} - \frac{(M_2 - iM_3)(u(\zeta) - u(\eta_0))}{\zeta - \eta_0}.$$  

The resulting formulas contain ten real unknown variables: $a, b, d, e_0, k, \delta, M_0, M_1, M_2, M_3$. To find these unknowns we have the following equations:

$$\Psi(\zeta_0, u(\zeta_0)) + \Omega(\zeta_0, u(\zeta_0)) = 0, \quad (6.51)$$

$$\Psi(d, u(d)) + \Omega(d, u(d)) = 0, \quad (6.52)$$

$$\Psi(e_0, u(e_0)) + \Omega(e_0, u(e_0)) = 0, \quad (6.53)$$

$$M_1 + 2M_3 + \frac{1}{2\pi} \left( \alpha \int_{ab} - (\pi - \beta) \int_{da} + i\sigma' \int_{bcd} \right) \frac{d\xi}{vX^+(\xi, v)} = 0, \quad (6.54)$$

$$\int_{ab} \left| \frac{df}{d\xi} \right| |d\zeta| = \lambda_1, \quad \int_{da} \left| \frac{df}{d\xi} \right| |d\zeta| = \lambda_2, \quad (6.55)$$

$$\int_{l_1^*} \frac{df}{d\zeta} d\zeta = 0, \quad (6.56)$$

where $l_1^*$ is a contour enclosing the cut $l_1$ but not crossing the cut $l_0$. Finally, we have a condition determining the position of the wedges with respect to the boundaries of the jet:

$$\text{Im} \int_a^{e_1} \frac{dw}{d\zeta} = h_1 v_\infty,$$

where the point $e_1 \in e_2 e_1$ (fig. 4.6) and $h_1$ is the limit of the distance between the free streamline $E_1^+ E_2^+$ of the jet and the central line $E_1^- E_2^-$ as $\text{Re} z \to -\infty$. Observe, that the unknowns $\delta, M_0, M_1, M_2, M_3$ are included in the system of equations (6.51)-(6.56) only linearly and can be easily eliminated from the system. Finally, we have a system of five nonlinear transcendental equations with respect to five unknowns $a, b, d, e_0, k$.

Consider a jet of the width $h = 20$ with two supercavitating wedges with the angles $\alpha_0 = \beta_0 = \pi/3$ and with the sides’ lengths $\lambda_1 = \lambda_2 = 1$. The computations have been made for four different sets of the parameters with the precision $\epsilon = 10^{-6}$. The point $e_1$ in all of the cases lies on the upper side of the cut $l_0$ and the point $e_2$ lies on the lower side. Some of the results of the computations are provided in the table 6.1.

The circulation and the lift and drag coefficients have been computed similarly to the previous chapters. The results for the upper wedge are given in the table 6.2.

The cavity and the jet profiles are shown on the fig. 6.11 for different values of the cavitation number $\sigma$. The dashed line corresponds to the cavitation number $\sigma = 0.6$, the solid line corresponds to the cavitation number $\sigma = 0.5$ and the dash-dot line corresponds to the cavitation number $\sigma = 0.45$. The slight differences in the position of the vertices...
Table 6.1: The values of the parameters $a, b, d, e_0, m$ and $\delta$ for different initial values of $\sigma$ and $h_1$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$h_1$</th>
<th>$a$</th>
<th>$b$</th>
<th>$d$</th>
<th>$e_0$</th>
<th>$m$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.45</td>
<td>10</td>
<td>0.968869</td>
<td>0.986837</td>
<td>0.928402</td>
<td>1.033128</td>
<td>1.000967</td>
<td>0.031505</td>
</tr>
<tr>
<td>0.5</td>
<td>10</td>
<td>0.981163</td>
<td>0.991951</td>
<td>0.956723</td>
<td>1.019562</td>
<td>1.000357</td>
<td>0.037218</td>
</tr>
<tr>
<td>0.6</td>
<td>10</td>
<td>0.940759</td>
<td>0.975584</td>
<td>0.863656</td>
<td>1.067038</td>
<td>1.003831</td>
<td>0.022250</td>
</tr>
<tr>
<td>0.5</td>
<td>8</td>
<td>0.940385</td>
<td>0.975504</td>
<td>0.866594</td>
<td>1.098127</td>
<td>1.001228</td>
<td>0.031567</td>
</tr>
</tbody>
</table>

Table 6.2: The values of the circulation $\Gamma$ and lift and drag coefficients $C_X$ and $C_Y$ for different initial values of $\sigma$ and $h_1$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$h_1$</th>
<th>$C_X$</th>
<th>$C_Y$</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.45</td>
<td>10</td>
<td>1.092097</td>
<td>0.04322</td>
<td>-1.159897</td>
</tr>
<tr>
<td>0.5</td>
<td>10</td>
<td>1.131685</td>
<td>0.03826</td>
<td>-0.993298</td>
</tr>
<tr>
<td>0.6</td>
<td>10</td>
<td>1.211326</td>
<td>0.02968</td>
<td>-0.569694</td>
</tr>
<tr>
<td>0.5</td>
<td>8</td>
<td>1.131226</td>
<td>0.03843</td>
<td>-0.925245</td>
</tr>
</tbody>
</table>

Figure 6.11: The cavity and the jet profiles for different cavitation numbers $\sigma$: (1) $\sigma = 0.6$, (2) $\sigma = 0.5$, (3) $\sigma = 0.45$. 

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Figure 6.12: The cavity and the jet profiles for different distances $h_1$: (1) $h_1 = 10$, (2) $h_1 = 8$.

$A$ and $A'$ of the wedges for different cavitation number appear because the position of the wedge is fixed not by the distance between the vertex of the wedge and the free surface but by the distance between the streamline which separates at the vertex of the wedge and the free surface as $Re z \to \infty$, i.e. the distance between the streamline and the free surface at infinity is fixed, but the actual distance between the tip of the wedge and free surface needs to be found as a part of the solution. This distance is different for different cavitation numbers $\sigma$ and also not necessary equal to $h_1 = 10$. Hence, the difference in the position of the wedges.

The cavity and the jet profiles are shown on the fig. 6.12 for the same cavitation number $\sigma = 0.5$ and two distances $h_1$. The solid line corresponds to $h_1 = 10$ and the dashed line corresponds to $h_1 = 8$. Observe the significant difference in the lengths of the cavities for these two cases. As before, we may conclude that the proximity of the free boundary decreases the size of the cavity. Another thing to note that even the wedges in the second case are closer to the free surfaces of the jet, the “wave” which they cause on the free surfaces is smaller than for the first case, when the wedges are farther away from the free surfaces. The explanation to this phenomenon may be in the smaller size of the cavities in the second case.
Bibliography


Vita

Anna Y. Zemlyanova was born in December 1982, in Cheboksary, Russia. She completed her undergraduate studies in mathematics with the highest honors at Chuvash State University, June 2003. She earned a Candidate of Sciences in mathematics and physics degree from Chuvash State University in June 2005 and a Master of Science degree in mathematics from Louisiana State University in May 2007. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2010.