SOME TRACKING PROBLEMS FOR
AEROSPACE MODELS WITH INPUT CONSTRAINTS

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Abstract

We study tracking controller design problems for key models of planar vertical takeoff and landing (PVTOL) aircraft and unmanned air vehicles (UAVs). The novelty of our PVTOL work is the global boundedness of our controllers in the decoupled coordinates, the positive uniform lower bound on the thrust controller, the applicability of our work to cases where the velocity measurements may not be available, the uniform global asymptotic stability and uniform local exponential stability of our closed loop tracking dynamics, the generality of our class of trackable reference trajectories, and the input-to-state stability of the controller performance under actuator errors of arbitrarily large amplitude. The significance of our UAV results is the generality of the trackable trajectories, the input-to-state stability properties of the tracking dynamics with respect to additive uncertainty on the controllers, and our ability to satisfy command amplitude and command rate constraints as well as state dependent command constraints and a state constraint on the velocity. Our work is based on a Matrosov approach for converting a nonstrict Lyapunov function for the UAV tracking dynamics into a strict one, in conjunction with asymptotic strict Lyapunov function methods and bounded backstepping.
Chapter 1
Background on Control Theory

In this chapter, we introduce the basic control concepts that will be used throughout this dissertation, and some motivation for our approach. This portion of the chapter largely follows Professor Michael Malisoff’s February 2012 colloquium at the University of Texas at Dallas [28]. Then we derive two benchmark aerospace models that we use in later chapters.

Control theory is an application–oriented branch of mathematics that deals with the basic principles underlying the analysis and design of control systems. A dynamical system, in general, is a system of equations that can describe the behavior of a real-world object, such as an airplane, a biomass growth, or an electric circuit. To control means to influence the behavior of an object by adjusting some parameters to achieve a desired goal. For example, we can often adjust an airplane’s thrust and roll angle to make the plane follow a desired trajectory.

A key feature in control applications is that the control parameter we adjust is a function of the time, the state, or both. This stands in contrast with the standard situation in the theory of differential equations, where the parameter we adjust is generally a vector of constants. While differential equations theory focuses on understanding the solution set of given dynamical systems, control theory usually begins with a desired behavior for the dynamics and then focuses on specifying the functional parameters in the dynamics to achieve the desired behavior. For example, we often wish to force all trajectories to track a prescribed reference trajectory, which is equivalent to achieving asymptotic stability properties for the tracking error dynamics.
There are often limitations caused by the nature of a physical model. For example, during a takeoff, the thrust of a plane cannot be negative. On the other hand, mathematics itself puts some limitations on what can be achieved, because the functions appearing in the model must have certain properties that guarantee the well definedness of the flow map. This ensures that there are unique maximal solutions from all initial configurations in the state space.

As mentioned in [49], there are two main branches of control theory. One focuses on optimization of the behavior of a controlled object, while the other deals with the constraints imposed by uncertainty about the model or its environment. Here we focus on the latter branch, so we use information about the current state to design feedback controllers that force all trajectories of the system into a desired behavior under uncertainty. We next make these concepts precise mathematically.

### 1.1 Control Systems

In this work, we understand a dynamical system to mean a finite dimensional system of deterministic, continuous-time, ordinary differential equations with two parameters, which can be written in compact form as

\[
\dot{X} = F(t, X, u(t, X), \delta(t)), \quad X \in \mathcal{X},
\]

with the conventions that we omit the dependence of the state variable $X$ on time and dots indicate derivatives with respect to time. Systems of this form are called control systems. Unlike the usual ordinary differential equations case where the parameters are constants, the two parameters $u(t, X)$ and $\delta(t)$ in (1.1) are functions. The controller function $u(t, X)$ is designed to achieve the control objective, which depends on the model under study. The controllers we construct will be uniformly bounded in $t$, meaning there is a nondecreasing function $\alpha$ such that
|u(t, X)| \leq \alpha(|X|) throughout its domain. The second parameter \( \delta \) takes its values in the set of all functions \( \delta : [0, +\infty) \to D \) valued in a given disturbance set \( D \).

The \( \delta \)'s represent uncertainty such as modeling errors, or disturbances acting on the system such as wind gusts, so we cannot choose them. In the control engineering literature, the disturbances \( \delta \) are taken as piecewise continuous functions. We always take our uncertainties \( \delta \) to be measurable essentially bounded functions.

We always assume that \( F \) is at least \( C^1 \). The state space \( \mathcal{X} \) is an open subset of a Euclidean space. Throughout this work, it will be the full Euclidean space, unless otherwise noted. The sets \( D \) and \( \mathcal{X} \) are determined by the specific application. For example, \( D \) can be a small enough ball around the origin that ensures that the controller \( u(t, X) \) maintains good performance of the system under the uncertainties \( \delta \). We will specify \( D \) as needed. We return to the issue of restrictions on the magnitudes of the disturbances as part of our analysis of the PVTOL and UAV models.

Specifying a controller \( u(t, X) \) in (1.1) leads to a system

\[
\dot{X} = G(t, X, \delta(t)), \quad X \in \mathcal{X}
\]  

(1.2)

with one parameter, where \( G(t, X, \delta) = F(t, X, u(t, X), \delta) \). Systems obtained from specifying a controller in a control system are called \emph{closed loop systems}. Since we take our disturbances to be measurable essentially bounded functions, and since our controllers \( u(t, X) \) and \( F(t, x, u, \delta) \) will be \( C^1 \) and uniformly bounded in \( t \), it follows that for all choices of the function \( \delta \) and all controls, the function \( H(t, X) = G(t, X, \delta(t)) = F(t, X, u(t, X), \delta(t)) \) satisfies the following two assumptions that ensure standard existence and uniqueness properties of the system (1.1):

1. For each \( t \in [0, +\infty) \), the function \( X \mapsto H(t, X) \) is \( C^1 \). For each \( X \in \mathcal{X} \), the function \( t \mapsto H(t, X) \) is locally integrable.
2. For every compact set $K \subset X$, there exists a locally integrable function $\alpha_K$ such that

$$\left| \frac{\partial H}{\partial X}(t, X) \right| \leq \alpha_K(t) \quad \text{for all } X \in K \text{ and } t \geq 0.$$ 

By local integrability, we mean that each point $X_0 \in X$ admits a locally integrable function $\mu$ such that $|\mathcal{H}(t, X_0)| \leq \mu(t)$ for almost all $t$. The preceding conditions ensure that for each initial condition $X(t_0) = X_0$ and each disturbance function $\delta$, there is a unique maximal solution $t \mapsto X(t, t_0, X_0, \delta)$ for the initial value problem given by $\dot{X} = g(t, X, \delta(t))$ and $X(t_0) = X_0$. We also assume that the system (1.1) is forward complete, which means that for all $t \geq t_0$, all initial times $t_0$, all $X_0 \in X$, and all $\delta$, the unique solution $t \mapsto X(t, t_0, X_0, \delta)$ of (1.1) satisfying $X(t_0) = X_0$ is defined for all $t \in [t_0, +\infty)$. This rules out finite time blow ups, as would be the case with $\dot{X} = 1 + X^2$. We always assume $\mathcal{F}(t, 0, 0, 0) = 0$ for all $t$, so the system has a zero equilibrium when the current state, the control, and the disturbance are all zero.

### 1.2 Stability Definitions

We next review two generalizations of the well known uniform global asymptotic stability (UGAS) property. These generalizations are due to Eduardo Sontag [48, 50]. For simplicity, we assume in what follows that $X = \mathbb{R}^n$ for any $n \in \mathbb{N}$. To facilitate our discussion, we use the following definitions:

**Definition 1.1.** We use the following classes of comparison functions:

1. $\mathcal{K}$ is the class of all continuous, strictly increasing functions $\gamma : [0, +\infty) \rightarrow [0, +\infty)$ such that $\gamma(0) = 0$;

2. $\mathcal{K}_\infty$ is the class of all functions $\gamma \in \mathcal{K}$ such that $\gamma(r) \rightarrow +\infty$ as $r \rightarrow +\infty$;
3. $\mathcal{KL}$ is the class of all continuous functions $\beta : [0, +\infty)^2 \rightarrow [0, +\infty)$ such that
   (a) for every $s \geq 0$, the function $\beta(\cdot, s)$ is of class $\mathcal{K}$ and (b) for every $r \geq 0$, the function $\beta(r, \cdot)$ is non-increasing with $\beta(r, s) \rightarrow 0$ as $s \rightarrow +\infty$; and

4. $\mathcal{KK}$ is the class of all continuous functions $\gamma : [0, +\infty)^2 \rightarrow [0, +\infty)$ such that $\gamma(s, \cdot) : [0, +\infty) \rightarrow [0, +\infty)$ and $\gamma(\cdot, t) : [0, +\infty) \rightarrow [0, +\infty)$ are in class $\mathcal{K}_\infty$ for all $s > 0$ and $t > 0$.

Notice that class $\mathcal{K}_\infty$ functions $\alpha$ are positive definite, i.e., $0$ at $0$ and positive at all positive values. They are also proper, in the sense that $\alpha(s) \rightarrow +\infty$ as $s \rightarrow +\infty$. More generally, we have the following definitions:

**Definition 1.2.** A function $V : [0, +\infty) \times \mathcal{X} \rightarrow [0, +\infty)$ is positive definite provided there are positive definite functions $\underline{\alpha} : [0, +\infty) \rightarrow [0, +\infty)$ and $\overline{\alpha} : [0, +\infty) \rightarrow [0, +\infty)$ such that $\underline{\alpha}(|X|) \leq V(t, X) \leq \overline{\alpha}(|X|)$ for all $t \geq 0$ and $X \in \mathcal{X}$. If $\underline{\alpha}$ is also proper, then we call $V$ proper.

We can now define the generalizations of UGAS. In what follows, $|\delta|_\mathcal{I}$ for any function $\delta$ and any subset $\mathcal{I}$ of its domain means the essential supremum of the restriction of $\delta$ to $\mathcal{I}$, $|\delta|_\infty$ is its essential supremum on $[0, \infty)$, and $\mathcal{M}_\mathcal{L}$ is the set of all measurable essentially bounded functions $\delta : [0, \infty) \rightarrow \mathcal{L}$ valued in any set $\mathcal{L}$. Also, $|| \cdot ||$ is the supremum norm, and for any constant $\mu > 0$, $\mu \mathcal{B}_n$ denotes the closed ball of radius $\mu$ centered at the origin in $\mathbb{R}^n$. When $\mu = 1$, we write this ball as $\mathcal{B}_n$, omitting the radius 1 in the notation. Consider a system of the form

$$\dot{X} = \mathcal{G}(t, X, \delta(t)), \quad X \in \mathcal{X}$$

(1.3)

with disturbances $\delta$ valued in some disturbance set $\mathcal{D}$ that satisfies the regularity assumptions we discussed above for all choices of the disturbance $\delta$, which implies that it has a well defined forward complete flow map. This includes the important
case of controlled closed loop systems with disturbances. Here is the corresponding
definition of input-to-state stability (ISS) [48] for (1.3):

**Definition 1.3.** We say that (1.3) satisfies ISS provided there exist functions
\( \beta \in K\mathcal{L} \) and \( \gamma \in \mathcal{K}_\infty \) such that for all initial conditions \( X(t_0) = X_0 \), all disturbance functions \( \delta \) valued in \( \mathcal{D} \), all initial times \( t_0 \geq 0 \), and all times \( t \geq t_0 \), we have

\[
|X(t, t_0, X_0, \delta)| \leq \beta(|X_0|, t - t_0) + \gamma(|\delta|_{[t_0, t]})
\]

(1.4)

for all \( t \geq t_0 \).

Therefore, if (1.3) satisfies ISS, then all trajectories for bounded disturbances are uniformly bounded by a term that decays to zero (produced by \( \beta \)) and an overshoot term, so bounded disturbances always yield bounded trajectories. A less restrictive condition on (1.3) is the following integral input-to-state stability (iISS) condition [50]:

**Definition 1.4.** We say that (1.3) satisfies iISS provided there exist functions \( \beta \in K\mathcal{L}, \gamma \in \mathcal{K}_\infty, \) and \( \bar{\gamma} \in \mathcal{K}_\infty \) such that for all initial conditions \( X(t_0) = X_0 \), all disturbance functions \( \delta \) valued in \( \mathcal{D} \), all initial times \( t_0 \geq 0 \), and all times \( t \geq t_0 \), we have

\[
\bar{\gamma}(|X(t, t_0, X_0, \delta)|) \leq \beta(|X_0|, t - t_0) + \int_{t_0}^{t} \bar{\gamma}(|\delta(m)|)dm
\]

(1.5)

for all \( t \geq t_0 \).

We also use ISS and iISS to mean input-to-state stable and integral input-to-state stable, respectively, and similarly for (uniform) global asymptotic stability. The iISS property allows bounded disturbances \( \delta \) to produce unbounded trajectories, but it means that finite energy disturbances give bounded trajectories, when the energy is represented in the integral term and finite energy means \( \int_{t_0}^{+\infty} \bar{\gamma}(|\delta(m)|)dm \) converges. We also use the notion of practical ISS, which is the requirement that
there exist $\beta \in KL$ and $\gamma \in KK$ such that $|X(t, t_0, X_0, \delta)| \leq \beta(|X_0|, t - t_0) + \gamma(|X_0|, |\delta|_{[t_0,t]})$ along all trajectories of (1.3). Therefore, practical ISS gives ISS for each compact set $\mathcal{C}$ of initial conditions, because we can maximize over $X_0$ in $\mathcal{C}$ in the term $\gamma(|X_0|, |\delta|_{[t_0,t]})$. We define practical iISS analogously, by replacing the integrand in (1.5) with $\overline{\gamma}(|X_0|, |\delta(m)|)$ for some function $\gamma^\sharp \in KK$. The following example illustrates how iISS is less restrictive than ISS:

**Example 1.5.** Consider the one dimensional system

$$\dot{X} = -\frac{X}{1 + X^2} + \delta$$  \hspace{1cm} (1.6)

with the state space $\mathcal{X} = \mathbb{R}$ and the disturbance set $\mathcal{D} = \mathbb{R}$. This system is not ISS, because putting the constant disturbance $\delta(t) = 2$ in (1.6) produces unbounded trajectories. However, it is iISS. The iISS property can be shown using an iISS Lyapunov function; see Example 1.8.

For the special case of systems

$$\dot{X} = G(t, X), \hspace{0.5cm} X \in \mathcal{X}$$  \hspace{1cm} (1.7)

without disturbances, ISS and iISS agree with the uniform global asymptotic stability (UGAS) property, which states that there is a function $\beta \in KL$ such that

$$|X(t, t_0, X_0)| \leq \beta(|X_0|, t - t_0)$$  \hspace{1cm} (1.8)

for all $t \geq t_0$, $t_0 \geq 0$, and $X_0 \in \mathcal{X}$, which is the conjunction of the standard local stability condition of the origin and the usual global attractivity condition. For time invariant systems, we always take the initial time $t_0 = 0$, and then we use global asymptotic stability (GAS) to mean UGAS. The system (1.7) is *uniformly locally exponentially stable (ULES)* provided there are positive constants $\Delta$, $c_1$, and $c_2$ such that for all initial times $t_0 \geq 0$ and all initial conditions $X(t_0) = X_0 \in \Delta B_n$, we have $|X(t, t_0, X_0)| \leq c_1 |X_0| e^{-c_2(t-t_0)}$ for all $t \geq t_0$.
One relationship between the comparison functions is the $\mathcal{KL}$ Lemma [50], whose statement is as follows: Each function $\beta \in \mathcal{KL}$ admits functions $\alpha_i \in \mathcal{K}_\infty$ such that $\beta(s,t) \leq \alpha_1(e^{-t}\alpha_2(s))$ for all $(s,t) \in [0, +\infty)^2$. Hence, we can always replace the upper bound in the UGAS condition with $\alpha_1(e^{t_0-t}\alpha_2(|X_0|))$ for certain functions $\alpha_i \in \mathcal{K}_\infty$ for $i = 1, 2$, and similarly for the $\mathcal{KL}$ functions in the ISS and iISS estimate. However, doing so can make the estimates less tight, e.g., $\alpha_1(e^{t_0-t}\alpha_2(|X_0|)) - \beta(|X_0|, t - t_0)$ can be large for some choices of $t$, $t_0$, and $X_0$.

While we will not study the tightness of our iISS, ISS, and UGAS estimates in this work, a key advantage of ISS and iISS is that they apply for all choices of $\delta : [0, +\infty) \to \mathcal{D}$. Hence, we can certify a guaranteed level of performance under worst case disturbances, which can be important in engineering applications.

### 1.3 Lyapunov Function Methods

Many important systems enjoy ISS and iISS, and this has led to a large robustness analysis literature; see [51] for a recent survey. However, since the flow map for a nonlinear system usually cannot be derived in closed form, it may be difficult to check the stability conditions in the ISS, iISS, and UGAS estimates. The Lyapunov function approach can often overcome this difficulty. Let us review how this can be done. This section largely follows [51]. Here is a key definition:

**Definition 1.6.** An iISS Lyapunov function for (1.3) is any proper positive definite $C^1$ function $V$ that admits functions $\gamma \in \mathcal{K}_\infty$ and positive definite function $\alpha$ such that

$$V_i(t, X) + V_X(t, X)G(t, X, \delta) \leq -\alpha(X) + \gamma(|\delta|)$$

holds for all $t \geq 0$ and $X \in \mathcal{X}$. If, in addition, $\alpha$ is proper, then we call $V$ an ISS Lyapunov function.
We often denote the left side of (1.9) simply by ˙\(V\). Definition 1.6 should be compared with the corresponding definitions for systems without disturbances [29]:

**Definition 1.7.** A (nonstrict) Lyapunov function for a system of the form \(\dot{X} = G(t,X)\) is any proper positive definite \(C^1\) function \(V\) that admits a nonnegative valued function \(W\) such that \(V_t(t,X) + V_X(t,X)G(t,X) \leq -W(X)\) for all \(t \geq 0\) and \(X \in \mathcal{X}\). If, in addition, \(W\) is positive definite, then we call \(V\) a strict Lyapunov function for the system.

Therefore, ISS and iISS Lyapunov functions are special kinds of strict Lyapunov functions. The existence of an iISS (resp., ISS) Lyapunov function for a perturbed system (1.3) ensures that the system is iISS (resp., ISS) [49]. This includes the classical result that states that the existence of a strict Lyapunov function ensures that the undisturbed system is UGAS. Strict Lyapunov functions can also lead to explicit constructions of the comparison functions in the ISS, iISS, and UGAS estimates [29]. We can sometimes also use strict Lyapunov functions to prove ISS with respect to additive uncertainty on the right hand side of the dynamics, at least when the disturbance set \(D\) is a small enough ball around the origin. However, the following example illustrates how this cannot be done in general:

**Example 1.8.** The function \(V(X) = \ln(1 + X^2)\) is a strict Lyapunov function for the one dimensional system \(\dot{X} = -\frac{X}{1+X^2}\) with state space \(\mathcal{X} = \mathbb{R}\), because \(\dot{V} \leq -\frac{X^2}{(1+X^2)^2}\) along all of its trajectories, which gives global asymptotic stability. Moreover, \(V\) is an iISS Lyapunov function for

\[
\dot{X} = -\frac{X}{1+X^2} + \delta
\]

for the disturbance set \(D = \mathbb{R}\). However, for each constant \(\delta > 0\), we can find an initial state \(X_0\) such that the trajectory for (1.10) starting at \(X_0\) is unbounded,
so (1.10) cannot be ISS, regardless of how small a neighborhood $\mathcal{D}$ of 0 we pick for the disturbance set. For example, for any constant disturbance $\bar{\delta} \in (0, 0.5)$ the trajectory for (1.10) starting at $X_0 = 4/\bar{\delta}$ is unbounded.

By contrast, here is a general situation where strict Lyapunov functions can give robustness results. Many engineering models are control-affine, meaning the right side is an affine function of the input. Assume that we have a controller $u$ such that the control affine system

$$\dot{X} = \mathcal{G}(t, X) \overset{\text{def}}{=} f(t, X) + g(t, X)u(t, X)$$

(1.11)
in closed loop with $u$ evolving on the state space $\mathbb{R}^n$ is UGAS to the origin. We assume that $\mathcal{G}$ satisfies the assumptions we gave above. Then standard converse Lyapunov function theory provides a strict Lyapunov function $V$ for (1.11) on $\mathbb{R}^n$ [29]. We can then conclude that

$$\dot{X} = f(t, X) + g(t, X)[u(t, X) - D_X V(t, X) \cdot g(t, X) + \delta]$$

(1.12)
is iISS with respect to actuator errors $\delta$ for any choice of the disturbance set $\mathcal{D}$. This is the case because we can use the triangle inequality

$$|D_X V(t, X) \cdot g(t, X)||\delta| \leq \frac{1}{2}|D_X V(t, X) \cdot g(t, X)| + \frac{1}{2}|\delta|^2$$

(1.13)
to prove that $V$ is an iISS Lyapunov function for (1.12). In fact, if we set $W(X) = \inf_t \{-[V_t(t, X) + V_X(t, X)\mathcal{G}(t, X)]\}$, then along all trajectories of (1.12), we have

$$\dot{V} \leq -W(X) - |D_X V(t, X) \cdot g(t, X)|^2 + |D_X V(t, X) \cdot g(t, X)||\delta|
\leq -W(X) - \frac{1}{2}|D_X V(t, X) \cdot g(t, X)|^2 + \frac{1}{2}|\delta|^2$$

(1.14)
$$\leq -W(X) + \frac{1}{2}|\delta|^2,$$

which is the iISS decay condition. Moreover, we can select the strict Lyapunov function $V$ for (1.11) such that the function $W(X) = \inf_t \{-[V_t(t, X) + V_X(t, X)\mathcal{G}(t, X)]\}$
is proper, in which case (1.14) gives ISS of (1.12); see [39]. The added controller component $-D_X V(t, X) \cdot g(t, X)$ in (1.12) is called an LgV controller (since it is the negative of the Lie derivative of $V$ along the vector field $g$). This procedure of adding a term to the controller to get robustness with respect to additive uncertainties is called robustification. The LgV controller is only explicitly given when we know the gradient $\nabla V$, and this has been one motivation for a large literature on the ‘strictification’ process of converting nonstrict Lyapunov functions into strict Lyapunov functions [29]. The value of the strictification process is that it is often much easier to construct nonstrict Lyapunov functions than it is to construct strict ones. The Matrosov method is one approach to strictification that combines a nonstrict Lyapunov function with a collection of auxiliary functions which roughly speaking decay at points where the nonstrict Lyapunov function may not necessarily decay [29, 34]. We use the Matrosov approach to solve a class of tracking problems for UAVs in Chapter 5.

1.4 Two Important Aerospace Models

We next describe two control systems from aerospace engineering. The first models an aircraft moving in a vertical plane, and the second is for a constant altitude UAV. These will be benchmark models for our tracking analysis in later chapters.

1.4.1 Planar Vertical Takeoff and Landing Aircraft

The complete dynamics of the vertical takeoff and landing (VTOL) aircraft are very complex. To consider these models in their most general form, one needs to take many factors into account, including flexibility of the wings and fuselage, aeroelastic effects, and the dynamic of the engine. In this work, we consider the
plane to be a rigid body, so that we do not have to consider the preceding features, but we still obtain a model that is detailed enough to apply the results of its analysis to design control laws for more complex aircraft models. We use the planar vertical takeoff and landing (PVTOL) aircraft model [22], where the aircraft moves in a vertical plane. The following derivation of the PVTOL model largely follows [42], but see [22] or [27, Chapter 1] for a more detailed derivation.

The model equations of the motion are

\[
\begin{align*}
    m \ddot{x} &= -\sin \theta \cos \alpha (F'_1 + F'_2) + \cos \theta \sin \alpha (F'_1 - F'_2) \\
    m \ddot{y} &= \cos \theta \cos \alpha (F'_1 + F'_2) + \sin \theta \sin \alpha (F'_1 - F'_2) - mg \\
    J \ddot{\theta} &= (\ell \cos \alpha + \Delta \sin \alpha) (F'_1 - F'_2),
\end{align*}
\]

where \( m \) is the mass of the aircraft, \((x, y)\) is the position, \( \theta \) is the angular velocity, \( \alpha \) is the angle made by the engines, \( F'_1 \) and \( F'_2 \) are forces acting on the aircraft, \( g \) is the gravitational constant, \( J > 0 \) is the moment mass inertia, \( \ell \) is half of the distance between the two engines, and \( \Delta \) is the distance from the center of the engine to the bottom of the wing. See Figure 1.1.

![FIGURE 1.1. PVTOL Aircraft in its Vertical Plane](image)
We divide the first two equations by $m$ and the third equation by $J$ to obtain
\[
\begin{aligned}
\ddot{x} &= -\sin \theta \frac{\cos \alpha (F_1' + F_2')}{m} + \cos \theta \frac{\sin \alpha (F_1' - F_2')}{m} \\
\ddot{y} &= \cos \theta \frac{\cos \alpha (F_1' + F_2')}{m} + \sin \theta \frac{\sin \alpha (F_1' - F_2')}{m} - g \\
\ddot{\theta} &= \frac{(\ell \cos \alpha + \Delta \sin \alpha)(F_1' - F_2')}{J}
\end{aligned}
\]  
(1.16)

To simplify notation, we put
\[
\varepsilon = \frac{J \sin \alpha}{m(\ell \cos \alpha + \Delta \sin \alpha)},
\]
and we choose the controls
\[
u_1 = \frac{\cos \alpha (F_1' + F_2')}{m} \quad \text{and} \quad
u_2 = \frac{(\ell \cos \alpha + \Delta \sin \alpha)(F_1' - F_2')}{J}.
\]  
(1.18)

This gives the PVTOL system [22]
\[
\begin{aligned}
\dot{x} &= -u_1 \sin(\theta) + \varepsilon u_2 \cos(\theta) \\
\dot{y} &= u_1 \cos(\theta) + \varepsilon u_2 \sin(\theta) - g \\
\dot{\theta} &= u_2.
\end{aligned}
\]  
(1.19)

where $u_1$ is the thrust out of the bottom, $u_2$ is called the rolling moment controller, and $\varepsilon$ is called the coupled parameter. The controller $u_1$ is required to be nonnegative valued. This can be written as a first order system, by introducing the new variables $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ in the standard way to get a six dimensional dynamics. We will design tracking controllers for this six dimensional dynamics that force all of its trajectories to track a broad class of desired reference trajectory $\mathcal{E}_r(t) = (x_r(t), \dot{x}_r(t), y_r(t), \dot{y}_r(t), \theta_r(t), \dot{\theta}_r(t))$, while satisfying certain input constraints and ISS properties. This means that we must render the dynamics of the tracking error
\[
\mathcal{E} = \begin{pmatrix} x - x_r(t), \dot{x} - \dot{x}_r(t), y - y_r(t), \dot{y} - \dot{y}_r(t), \theta - \theta_r(t), \dot{\theta} - \dot{\theta}_r(t) \end{pmatrix}
\]  
(1.20)

ISS with respect to additive uncertainties on the controllers. We return to the PVTOL model in Chapter 3.
1.4.2 Four State Unmanned Air Vehicle

We next consider a model for the kinematics of an unmanned air vehicle (UAV) operating at a constant altitude. While simple, it contains the key features that are needed to design controllers for high level formation flight of real UAVs [40, 43]. Its full derivation is as follows. We introduce two coordinate systems, the inertial frame \((x_0, y_0)\) corresponding the usual \((x, y)\) axes and a moving frame \((x_c, y_c)\) corresponding to the direction of the UAV, as illustrated in Figure 1.2.

The heading velocity \(v_h\) of the vehicle is aligned with the \(x_c\) direction. Therefore, the heading velocity in the \((x_c, y_c)\) frame has the form

\[
v^c_h = \begin{bmatrix} v \\ 0 \end{bmatrix},
\]

so the heading velocity in the \((x_0, y_0)\) frame is

\[
v^0_h = \begin{bmatrix} (v^0_h)_x \\ (v^0_h)_y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix}.
\]
It follows that the dynamics of the center of mass \((x, y)\) of the UAV are

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
v \cos \theta \\
v \sin \theta
\end{bmatrix}.
\] (1.23)

If the heading angle and heading velocity are controlled, then we also have equations of the form \(\dot{\theta} = \omega_\theta\) and \(\dot{v} = \omega_v\). Using the change of input from [43] and allowing additive uncertainties on the controls gives the control formulas

\[
\omega_\theta = \alpha_\theta (\theta_c - \theta + \Delta) \quad \text{and} \quad \omega_v = \alpha_v (v_c - v + \delta).
\] (1.24)

The changes of inputs are used to represent autopilots, so the positive constants \(\alpha_\theta\) and \(\alpha_v\) are called autopilot constants. This gives the model

\[
\begin{align*}
\dot{x} & = v \cos(\theta) \\
\dot{y} & = v \sin(\theta) \\
\dot{\theta} & = \alpha_\theta (\theta_c - \theta + \Delta) \\
\dot{v} & = \alpha_v (v_c - v + \delta)
\end{align*}
\] (1.25)

for the UAV. As in the PVTOL model, our control objectives will be to track certain reference trajectories while satisfying certain input constraints and ISS properties. In fact, we can design our UAV controllers to satisfy both command amplitude and command rate constraints, meaning \(\theta_c, v_c\), and their time derivatives along the closed loop trajectories are guaranteed to remain in suitable prescribed intervals.

We discuss tracking problems for the UAV model and the precise conditions on the reference trajectories and input restrictions in Chapters 4-5, after we present our analysis for the PVTOL tracking problems.
Chapter 2
Bounded Backstepping Theorem

Our tracking results for the PVTOL model are based on a repeated application of our key bounded backstepping theorem. We state and prove this theorem in this chapter. The basic idea of backstepping is to build controllers for nonlinear systems that can be transformed into an appropriate triangular form. Then one studies the controller design problems for lower dimensional subsystems, and then augments the dynamics component by component to get feedback stabilizers for the full systems. Backstepping has been applied in many areas, but it has serious limitations. For example, many engineering problems have input constraints, but the controllers obtained from the usual backstepping methods are not necessarily even bounded, and therefore may not be of practical use. Even when backstepping leads to bounded controllers, it can be difficult to prove that the closed loop system satisfies ISS with respect to additive uncertainties $\delta$ on the controls, which are common in engineering applications. See [24, 29, 45] for more on backstepping. Our bounded backstepping theorem overcomes some of these limitations, by producing a user friendly general formula that lends itself to the design of thrust and rolling moment tracking controllers for the PVTOL model.

2.1 Useful Classes of Functions

We use the key functions $\sigma_\ell, \varphi_\ell : \mathbb{R} \to \mathbb{R}$ defined by

$$\sigma_\ell(x) = \frac{2\ell}{\pi} \arctan \left( \frac{\pi x}{2\ell} \right)$$

(2.1)

and

$$\varphi_\ell(x) = 1 - \frac{1}{B_\ell} \int_{4\ell}^{\max\{4\ell, \min\{|x|, 6\ell\}\}} (q - 4\ell)^4(q - 6\ell)^3dq,$$

(2.2)
where
\[
B_\ell = \int_{4\ell}^{6\ell} (q - 4\ell)^4(q - 6\ell)^4 dq
\]  
(2.3)
for each constant \( \ell > 0 \). The constant \( B_\ell \) is chosen such that \( \varphi_\ell \) is a compactly supported smoothed indicator function for the interval \([-6\ell, 6\ell]\); see Figure 2.1 for examples of \( \varphi_\ell \) for different choices of \( \ell \).

In fact, we have the following key properties of \( \sigma_\ell \) and \( \varphi_\ell \):

**Lemma 2.1.** For each constant \( \ell > 0 \), we have (a) \( \sigma_\ell'(x) \in [0, 1] \) for all \( x \in \mathbb{R} \), (b) \( \sigma_\ell(x) \geq 0.75x \) for all \( x \in [0, \ell/4] \), (c) \(|\sigma_\ell(x)| \leq \ell \) for all \( x \in \mathbb{R} \), (d) \( \varphi_\ell : \mathbb{R} \to [0, 1] \) is \( C^4 \) and even, (e) \( \varphi_\ell(x) = 1 \) on \([-4\ell, 4\ell]\), (f) \( \varphi_\ell(x) = 0 \) when \(|x| \geq 6\ell\), and (g) \( \ell \sup_{x \in \mathbb{R}} |\varphi_\ell'(x)| = \frac{315}{256} \).

Property (g) holds because \( B_\ell = \frac{256\ell^9}{315} \), so
\[
\ell \sup_{x \in \mathbb{R}} |\varphi_\ell'(x)| = \frac{\ell}{B_\ell} \max_{q \in [4\ell, 6\ell]} (q - 4\ell)^4(q - 6\ell)^4 = \frac{315}{256}.
\]  
(2.4)
The rest of Lemma 2.1 follows from simple calculations, and by matching the left and right derivatives of \( \varphi_\ell \) at \( 4\ell, -4\ell, 6\ell, \) and \(-6\ell \).

**Remark 2.2.** The results in this chapter remain true if our functions \( \sigma_\ell \) and \( \varphi_\ell \) from (2.1)-(2.2) are replaced by any \( C^4 \) functions \( \sigma_\ell : \mathbb{R} \to \mathbb{R} \) and \( \varphi_\ell : \mathbb{R} \to \mathbb{R} \).
that satisfy the requirements from Lemma 2.1. Our strategy of using integrals to smooth corners to approximate indicator functions has been used in other contexts. See for example [30] where this is done to approximate the opening and closing of a microelectromechanical relay.

Lemma 2.1 implies that the functions \( \varphi_\ell(x)x \) and \( \varphi^{(i)}_\ell(x)x \) are all bounded for each derivative \( i = 1, 2, 3 \). Also, for each constant \( \ell > 0 \), we can use properties (c) and (g) of Lemma 2.1 to define the function \( U_\ell : \mathbb{R}^2 \rightarrow \mathbb{R} \) by

\[
U_\ell(Z) = \frac{-\sigma_\ell(2Z_2 + \sigma_\ell(\ell Z_1)\varphi_\ell(Z_2)) - \ell\sigma'_\ell(\ell Z_1)\varphi_\ell(Z_2)Z_2}{2 + \sigma_\ell(\ell Z_1)\varphi'_\ell(Z_2)}. \tag{2.5}
\]

Using the compact support of \( \varphi_\ell \) from Lemma 2.1, we can easily prove:

**Lemma 2.3.** For each constant \( \ell > 0 \), (I) the functions

\[
\frac{\partial U_\ell}{\partial Z}(Z), \quad Z_2 \frac{\partial U_\ell}{\partial Z_1}(Z), \quad Z_2^2 \frac{\partial^2 U_\ell}{\partial Z_1^2}(Z), \quad \frac{\partial^2 U_\ell}{\partial Z_2^2}(Z), \quad \text{and} \quad Z_2 \frac{\partial^2 U_\ell}{\partial Z_1 \partial Z_2}(Z) \tag{2.6}
\]

are all bounded and (II) \( \sup_{Z \in \mathbb{R}^2} |U_\ell(Z)| \leq 2(6 \ell^2 + \ell) \).

### 2.2 Statement of Theorem

Here is our key bounded backstepping theorem:

**Theorem 2.4.** Let \( \Theta : [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R} \) be any \( C^1 \) function that admits a constant \( \ell > 0 \) such that

\[
\sup_{t \geq 0} |\Theta(t, X)| \leq \frac{\ell}{16} \min\{1, |X_1|\} \quad \text{for all} \quad X = (X_1, X_2) \in \mathbb{R}^2. \tag{2.7}
\]

Let \( \dot{S} = E(t, S) \) be any system on some Euclidean state space \( \mathbb{R}^p \) that is UGAS and ULES, with \( E \in C^1 \). Assume that \( \partial E/\partial S \) is bounded. Let \( L : [0, +\infty) \times \mathbb{R}^2 \times \mathbb{R}^p \rightarrow \mathbb{R} \) be any \( C^1 \) function that admits a constant \( \bar{L} \) such that \( |L(t, X, S)| \leq \bar{L}|S| \) for
all \( t \geq 0, \ X \in \mathbb{R}^2, \) and \( S \in \mathbb{R}^p. \) Let \( \bar{\eta} \geq 0 \) be any constant. Then

\[
\begin{cases}
\dot{X}_1 = X_2 + \Theta(t, X) \\
\dot{X}_2 = \beta_{\ell, \bar{\eta}}(t, X) + L(t, X, S) + \eta \\
\dot{S} = E(t, S)
\end{cases}
\]  

(2.8)

in closed loop with the bounded \( C^1 \) feedback

\[
\beta_{\ell, \bar{\eta}}(t, X) = \frac{-\left[1 + \frac{172\bar{\eta}}{\ell}\right]\sigma_{\ell}(2X_2 + \sigma_{\ell}(\ell X_1)\varphi_{\ell}(X_2)) - \ell\sigma'_{\ell}(\ell X_1)\varphi_{\ell}(X_2)X_2 + \Theta(t, X)}{2 + \sigma_{\ell}(\ell X_1)\varphi'_{\ell}(X_2)}
\]  

(2.9)

admits a function \( \alpha \in \mathcal{K}_\infty \cap C^1 \) and a constant \( c > 0 \) such that

\[
|Y(t)| \leq \alpha(|Y(t_0)|)e^{-c(t-t_0)/16} + \frac{12}{c}\sqrt{\ell^2 + 1} \left(1 + \sqrt{\frac{2}{c\ell}}\right)|\eta|_{\infty}
\]  

(2.10)

along all trajectories \( Y = (X, S) \) of (2.8) for all \( t_0 \geq 0, \ t \geq t_0, \) and measurable functions \( \eta : [0, +\infty) \to \bar{\eta}\mathcal{B}_1, \) and therefore is UGAS and ULES when \( \eta \equiv 0, \) and ISS with respect to disturbances \( \eta : [0, +\infty) \to \mathcal{B}_{\bar{\eta}}. \)

The preceding result implies that for any constant \( \bar{\eta} > 0, \) we can choose the feedback such that (2.8) is ISS with respect to disturbances that are bounded by \( \bar{\eta}. \) Moreover, \( \bar{\eta} \) can be taken as large as we desire, so we get ISS with respect to disturbances of arbitrarily large amplitude through an appropriate choice of \( \bar{\eta} \) in the feedback. Also, the estimate

\[
|\beta_{\ell, \bar{\eta}}(t, X)| \leq 2\ell \left(1 + \frac{172\bar{\eta}}{\ell}\right)(1 + 7\ell)
\]  

(2.11)

holds throughout its domain, and \((\partial/\partial X)\beta_{\ell, \bar{\eta}}(t, X)\) is bounded if \((\partial/\partial X)\Theta(t, X)\) is bounded.

2.3 Proof of Theorem

The strategy of the proof of Theorem 2.4 is to use asymptotic quadratic strict Lyapunov functions, which are defined in the same way as standard strict Lyapunov
functions except the Lyapunov decay estimate along trajectories is only required for large enough times. We use this consequence of Gronwall’s Inequality:

**Lemma 2.5.** Assume that the $C^1$ system

$$\dot{Y} = f(t,Y), \ Y \in \mathbb{R}^n$$

(2.12)

is ULES and UGAS, and that $\partial f/\partial Y$ is bounded. Then there is a constant $c > 0$ and a function $\alpha \in K_\infty \cap C^1$ such that for each initial condition $Y(t_0) = Y_o \in \mathbb{R}^n$, the corresponding solution $Y(t,t_0,Y_o)$ of (2.12) satisfies $|Y(t,t_0,Y_o)| \leq \alpha(|Y_o|)e^{-c(t-t_0)}$ for all $t \geq t_0$.

**Proof.** Assume that (2.12) is ULES on some closed ball $\Delta B_n$ of some radius $\Delta > 0$ centered at the origin. The UGAS assumption provides a constant $\Delta_1 \in (0,\Delta]$ such that all trajectories of (2.12) with initial states $Y(t_0) \in \Delta_1 B_n$ remain in $\Delta B_n$ for all $t \geq t_0$, as well as a $C^1$ increasing function $\gamma_\Delta : [0,\infty) \to [0,\infty)$ satisfying the following condition: If $Y_o \in \mathbb{R}^n$ and if $t_0 \geq 0$ is any constant, then $Y(t,t_0,Y_o) \in \Delta_1 B_n$ for all $t \geq t_0 + \gamma_\Delta(|Y_o|)$. Pick any constants $c > 0$ and $\bar{K} > 1$ such that

$$|Y(s,s_0,p)| \leq \bar{K}|p|e^{-c(s-s_0)}$$

(2.13)

for all $p \in \Delta B_n$, $s_0 \geq 0$, and $s \geq s_0$. Since $\partial f/\partial Y$ is bounded by some constant $\bar{f} > 0$ and $f(t,0) = 0$ for all $t \geq 0$, we have

$$|Y(t,t_0,Y_o)| \leq |Y_o| + \int_{t_0}^t |f(s,Y(s,t_0,Y_o))| ds$$

(2.14)

$$\leq |Y_o| + \int_{t_0}^t \bar{f}|Y(s,t_0,Y_o)| ds$$

for all $t \geq t_0 \geq 0$ and $Y_o \in \mathbb{R}^n$. This follows from the Mean Value Theorem applied to $G_s(Y) = f(s,Y)$ for each $s \geq t_0$, which gives $|f(s,Y(s,t_0,Y_o))| = |f(s,Y(s,t_0,Y_o)) - f(s,0)| \leq \bar{f}|Y(s,t_0,Y_o)|$. Hence, Gronwall’s Inequality [24, Appendix A] gives $|Y(t,t_0,Y_o)| \leq \alpha_o(|Y_o|)$ for all $Y_o \in \mathbb{R}^n$, $t_0 \geq 0$, and $t \in$
[t_o, t_o + \gamma_\Delta(|Y_o|)] if we take \( \alpha_o(r) = re^{f_\gamma(r)} \). If \( t \geq t_o + \gamma_\Delta(|Y_o|) \), and if we set \( t_* = t_o + \gamma_\Delta(|Y_o|) \), then the weak semigroup property \( Y(t, s, p) = Y(t, r, Y(r, s, p)) \) for all \( p \in \mathbb{R}^n \) and \( t \geq r \geq s \geq 0 \) gives

\[
|Y(t, t_o, Y_o)| = |Y(t, t_*, Y(t_*, t_o, Y_o))|
\leq K|Y(t_*, t_o, Y_o)|e^{-c(t-t_*)}
\leq K\alpha_o(|Y_o|)e^{-c(t-t_o-\gamma_\Delta(|Y_o|))}.
\]

Also, \( |Y(t, t_o, Y_o)| \leq \alpha_o(|Y_o|)e^{-c(t-t_o)}e^{c\gamma_\Delta(|Y_o|)} \) for all \( t \in [t_o, t_o + \gamma_\Delta(|Y_o|)] \). Therefore, we can satisfy the requirements with \( \alpha(r) = K\alpha_o(r)e^{c\gamma_\Delta(r)} \).

We now return to the proof of Theorem 2.4. Using our bound (2.11) on \( \beta_{\ell,\eta}(t, X) \), it follows from our growth conditions on \( \Theta \) and \( L \) that (2.8) is forward complete. We can use Lemma 2.5 to fix a constant

\[
c \in \left(0, \min\{0.75, 0.5\ell\}\right)
\]

(2.15)

and a function \( \alpha_S \in C^1 \cap \mathcal{K}_\infty \) such that

\[
|S(t)| \leq \alpha_S(|S(t_o)|)e^{-c(t-t_o)}
\]

(2.16)

along all trajectories of \( \dot{S} = E(t, S) \) in the rest of the proof. We construct a function \( \alpha \in \mathcal{K}_\infty \cap C^1 \) such that (2.10) holds along all trajectories \( Y = (X, S) \) of (2.8) for all \( t_o \geq 0 \) and \( t \geq t_o \) when \( |\eta|_\infty \leq \bar{\eta} \), which will give the UGAS, ULES, and ISS properties.

The variable \( Z_1 = 2X_2 + \sigma_\ell(\ell X_1)\varphi_\ell(X_2) \) satisfies

\[
\dot{Z}_1 = \left[2 + \sigma_\ell(\ell X_1)\varphi'_\ell(X_2)\right]\beta_{\ell,\eta}(t, X) + L(t, X, S) + \eta
\]

(2.17)

\[
+ \ell\sigma'_\ell(\ell X_1)\varphi_\ell(\eta)X_2 + \Theta(t, X)
\]

along all trajectories of (2.8). Our choice (2.9) of the feedback \( \beta_{\ell,\eta} \) therefore gives

\[
\dot{Z}_1 = -\left[1 + \frac{172\eta}{\ell}\right]\sigma_\ell(Z_1) + \left[2 + \sigma_\ell(\ell X_1)\varphi'_\ell(X_2)\right]L(t, X, S) + \eta.
\]

(2.18)
We now consider trajectories \((X_1(t), Z_1(t), S(t))\) of the dynamics of the variable \((X_1, Z_1, S)\) for any fixed measurable function \(\eta : [0, +\infty) \to \tilde{\eta}\mathcal{B}_1\). In all of what follows, we use \(t_o\) to denote the initial time of our trajectories. Also, all inequalities should be understood to hold for all \(t \geq t_o\), unless otherwise indicated.

**Step 1:** We first construct a \(C^1\) function \(\alpha_Z \in \mathcal{K}_\infty\) such that
\[
|Z_1(t)| \leq \alpha_Z((Z_1(t_o), S(t_o))) e^{-0.5c(t-t_o)} + \frac{6}{c} |\eta|_\infty \quad (2.19)
\]
for all \(t \geq t_o\) and all trajectories \(Z_1\). To this end, we first build a \(C^1\) function \(\mathcal{T}_o \in \mathcal{K}_\infty\) such that for all \(t_o \geq 0\), all trajectories \(Z_1 : [t_o, +\infty) \to \mathbb{R}\), and all \(t \geq t_o + \mathcal{T}_o((Z_1(t_o), S(t_o)))\), we have \(|Z_1(t)| \leq \ell/32\). If \(t \geq t_o\) is such that \(|Z_1(t)| \geq \ell/32\), then parts (b), (c), and (g) of Lemma 2.1 give \(|\sigma_\ell(Z_1(t))| \geq |\sigma_\ell(\ell/32)| \geq \frac{3\ell}{128}\) and \(|\sigma_\ell(\ell X_1(t)) \varphi'_\ell(X_2(t))| \leq 2\), and (2.16) and (2.18) give
\[
\text{sign}\{Z_1(t)\} \dot{Z}_1(t) \leq -\frac{3\ell}{128} \left[1 + 172 \frac{\ell}{2}\right] + 4 (\bar{L}|S(t)| + |\eta(t)|)
\leq -\frac{\ell}{128} + 4L \alpha_S(|S(t_o)|) e^{-c(t-t_o)}.
\]
Therefore, \(\frac{d}{dt} |Z_1(t)| < 0\) at all times \(t \geq t_o + \mathcal{T}_o((Z_1(t_o), S(t_o)))\) for which \(|Z_1(t)| \geq \ell/32\), where
\[
\mathcal{T}_o(r) = \frac{2}{c} \ln \left(\frac{512 \ell}{c} \alpha_S(r) + 1\right),
\]
because \(-\ell/128 + 4L \alpha_S(|S(t_o)|) e^{-c(t-t_o)} < 0\) for such values of \(t\). If \(|Z_1(t)| > \ell/32\) at some time \(\check{t} \geq t_o\), and if \(\check{t}\) is the smallest time on \([t_o, \check{t}]\) such that \(|Z_1(r)| \geq \ell/32\) for all \(r \in [\check{t}, \check{t}]\), then either \(|Z_1(t)| = \ell/32\) or else \(\check{t} = t_o\). (The first possibility occurs if there is a time \(r \in [t_o, \check{t}]\) at which \(|Z_1(r)| < \ell/32\), and the second occurs if there is no such time.) Hence, \(|Z_1(t)| \leq \frac{\ell}{32} + |Z_1(t_o)|\), so integrating (2.20) over \([\check{t}, \check{t}]\) gives
\[
\frac{\ell}{32} \leq |Z_1(t)|
\leq |Z_1(t)| - \frac{\ell}{128} (\check{t} - t) + \frac{4L}{c} \alpha_S(|S(t_o)|)
\leq \frac{\ell}{32} + |Z_1(t_o)| - \frac{\ell}{128} (\check{t} - t) + \frac{4L}{c} \alpha_S(|S(t_o)|)\quad (2.21).
\]
Then canceling $\ell/32$ from both sides of (2.21) gives
\[ t - \bar{t} \leq \left( |Z_1(t_o)| + \frac{4\ell}{c} \alpha_S(|S(t_o)|) \right) \frac{128}{\ell}. \] (2.22)

Hence, we can take
\[ T_o(r) = 2T_s(r) + \left[ r + \frac{4\ell}{c} \alpha_S(r) \right] \frac{128}{\ell}. \]

In fact, if $(Z_1(t_o), S(t_o)) \neq 0$, and if there were a time $\bar{t} \geq t_o + T_o((Z_1(t_o), S(t_o)))$ such that $|Z_1(\bar{t})| > \ell/32$, and if $t$ is the smallest time on $[t_o, \bar{t}]$ such that $|Z_1(r)| \geq \ell/32$ for all $r \in [t, \bar{t}]$, then $|Z_1(t)|$ decreases on $[t, \bar{t}]$, since (2.22) gives $t - t_o = \bar{t} - t_0 - (\bar{t} - t) > T_o((Z_1(t_o), S(t_o))) > 0$, but then $|Z_1(t)| = \ell/32$ and $|Z_1(t)|$ is strictly decreasing in a neighborhood of $t$, which is a contradiction. If $(Z_1(t_o), S(t_o)) = 0$, then (2.20) gives $\frac{d}{dt}|Z_1(t)| < 0$ for all times $t > t_o$ for which $|Z_1(t)| \geq \frac{\ell}{32}$, so $|Z_1(t)|$ never goes above $\ell/32$. We conclude that $|Z_1(t)| \leq \frac{\ell}{32}$ if $t - t_o \geq T_o((Z_1(t_o), S(t_o)))$.

Setting $t_\ast = t_o + T_o((Z_1(t_o), S(t_o)))$, it follows that since $|Z_1(t)| \leq \ell/32$ for all $t \geq t_\ast$, part (b) of Lemma 2.1 and our choice of $c \in (0, 0.75)$ give $Z_1(t)\sigma_t(Z_1(t)) \geq cZ_1^2(t)$ for all $t \geq t_\ast$. Then the choice $W(Z_1) = \frac{1}{2}Z_1^2$ and (2.18) give
\[ \dot{W}(Z_1) \leq -Z_1\sigma_t(Z_1) + 4|Z_1|(|\bar{L}|S(t) + |\eta(t)|) \]
\[ \leq -cW(Z_1) + \frac{16}{c}[|L|^2|S(t)|^2 + \eta^2(t)] \] (2.23)
for all $t \geq t_\ast$, where the second inequality used Hölder’s Inequality to get
\[ 4|Z_1|(|\bar{L}|S(t) + |\eta(t)|) \leq 0.5cZ_1^2 + \frac{8}{c}(\bar{L}|S(t)| + |\eta(t)|)^2 \]
\[ \leq 0.5cZ_1^2 + \frac{16}{c}[|L|^2|S(t)|^2 + \eta^2(t)]. \] (2.24)

Also, (2.18) and the Fundamental Theorem of Calculus give
\[ |Z_1(t_\ast)| \leq |Z_1(t_o)| + (t_\ast - t_o)\left\{ \ell + 172\bar{\eta} + 4[\bar{L}\alpha_S(|S(t_o)|) + \bar{\eta}] \right\}. \] (2.25)

Hence, multiplying both sides of the last inequality from (2.23) by $e^{c(t-t_\ast)}$ and integrating the result on $[t_\ast, t]$ for any $t \geq t_\ast$, taking square roots, and then using
(2.16) and the general relation \( \sqrt{p+q} \leq \sqrt{p} + \sqrt{q} \) for all \( p \geq 0 \) and \( q \geq 0 \) gives

\[
|Z(t)| \leq \left[ |Z(t_*)| + \frac{6L}{c} \alpha_S(|S(t_0)|) \right] e^{-0.5c(t-t_*)} + \frac{6}{c} |\eta|_{\infty}
\]

\[
\leq \left[ \alpha_o(|(Z(t_0), S(t_0))|) + \frac{6L}{c} \alpha_S(|S(t_0)|) \right] \times e^{-0.5c(t-t_*)} e^{cT_o(|(Z(t_0), S(t_0))|)} e^{-0.5c(t_* - t_0)} + \frac{6}{c} |\eta|_{\infty},
\]

where \( \alpha_o(r) = r + T_o(r)[\ell + 176\bar{\eta} + 4L\alpha_S(r)] \). Hence, (2.19) holds with \( \alpha_Z(r) = [\alpha_o(r) + \frac{6}{c} \bar{\alpha}_S(r)] e^{cT_o(r)} \).

**Step 2:** We next construct a function \( \alpha_X \in C^1 \cap K_{\infty} \) such that

\[
|X(t)| \leq \alpha_X(|(X(t_0), Z(t_0), S(t_0))|) e^{-c(t-t_0)/16}
\]

\[
+ \sqrt{\frac{2}{\ell^2}} \frac{12}{c^2} |\eta|_{\infty} \quad \forall t \geq t_0
\]

along all trajectories \( X_1 \) from (2.8). First note that (2.8) and our choice \( Z_1 = 2X_2 + \sigma_\ell(\ell X_1)\phi_\ell(X_2) \) give

\[
\dot{X}_1 = -0.5\sigma_\ell(\ell X_1) + \Theta(t, X) + d(t),
\]

where \( d(t) = 0.5Z_1(t) - 0.5\sigma_\ell(\ell X_1(t))[\phi_\ell(X_2(t)) - 1] \). Since \( |\sigma_\ell(\ell X_1(t))\phi_\ell(X_2(t))| \leq \ell \) for all \( t \), and since

\[
|Z_1(t)| = |2X_2 + \sigma_\ell(\ell X_1(t))\phi_\ell(X_2(t))| \leq \frac{\ell}{32}
\]

when \( t-t_0 \geq T_o(|(Z_1(t_0), S(t_0))|) \), we also have \( |X_2(t)| \leq 4\ell \) and so also \( \phi_\ell(X_2(t)) = 1 \) when \( t-t_0 \geq T_o(|(Z_1(t_0), S(t_0))|) \), by part (e) of Lemma 2.1. It follows that \( |d(t)| = 0.5|Z_1(t)| \leq \ell/64 \) when \( t-t_0 \geq T_o(|(Z_1(t_0), S(t_0))|) \). Moreover, the time derivative of \( V(X_1) = \frac{1}{2}X_1^2 \) along all trajectories of (2.8) is

\[
\dot{V} = -0.5X_1\sigma_\ell(\ell X_1) + X_1\Theta(t, X) + X_1d(t).
\]

We consider these two possible cases:
1) \(|X_1| \geq 1/4\). Then \(X_1 \sigma_t(\ell X_1) \geq |X_1| \sigma_t(\ell/4) \geq \frac{32}{16} |X_1|\), by part (b) of Lemma 2.1. Therefore, our bound \(\ell/16\) on \(\Theta\) from (2.7) and (2.30) give

\[
\dot{V} \leq -\frac{32}{32} |X_1| + X_1 \Theta(t, X) + X_1 d(t) \\
\leq -\frac{32}{32} |X_1| + \frac{\ell}{16} |X_1| + |X_1||d(t)| \\
\leq \left[-\frac{\ell}{32} + |d(t)|\right] |X_1|.
\] (2.31)

2) \(|X_1| \leq 1/4\). Then \(X_1 \sigma_t(\ell X_1) \geq 0.75 \ell X_1^2\), by part (b) of Lemma 2.1. By (2.7), we have \(|\Theta(t, X)| \leq (\ell/16)|X_1|\) everywhere, so Hölder’s Inequality \(X_1 d(t) \leq \frac{\ell}{4} X_1^2 + \frac{1}{16} d^2(t)\) applied to (2.30) gives

\[
\dot{V} \leq -\frac{32}{8} X_1^2 + X_1 \Theta(t, X) + X_1 d(t) \\
\leq -\frac{32}{8} X_1^2 + \frac{\ell}{16} X_1^2 + X_1 d(t) \\
\leq -\frac{\ell}{16} X_1^2 + \frac{1}{16} d^2(t).
\] (2.32)

Arguing as we did to construct \(\mathcal{T}_o\), we can build a function \(\mathcal{T}_1 \in \mathcal{K}_\infty \cap C^1\) such that \(|X_1(t)| \leq 1/4\) for all \(t \geq t_o + \mathcal{T}_1((X_1(t_o), Z_1(t_o), S(t_o)))\). In fact, \(\dot{V} \leq -\ell |X_1(t)|/64 < 0\), when \(|X_1(t)| \geq 1/4\) and \(t - t_o \geq \mathcal{T}_o((X_1(t_o), Z_1(t_o), S(t_o)))\) both hold, by Case 1) and the fact that \(|d(t)| = 0.5|Z_1(t)| \leq \ell/64\) for such \(t\). (In particular, \(|X_1(t)|\) never goes above 1/4 if \((X_1(t_o), Z_1(t_o), S(t_o)) = 0\).) On the other hand, if \(|X_1(t)| \geq 1/4\) for all \(t\) on some interval \([t^*, \bar{t}]\) with \(t^* \geq t_o\), and if \(\bar{t}\) is the smallest time \(r \geq t^*\) such that \(|X_1(t)| \geq 1/4\) for all \(t \in [r, \bar{t}]\), then either \(\bar{t} = t^*\), or else \(|X_1(\bar{t})| = 1/4\), so Case 1), (2.19), and (2.28) combine to give

\[
\frac{1}{32} \leq V(X_1(\bar{t})) \\
\leq V(X_1(\bar{t})) - \frac{\ell}{64} \left(\frac{1}{4}(\bar{t} - t^*)\right) \\
\leq V(X_1(\bar{t})) + \frac{1}{32} - \frac{\ell}{256} (\bar{t} - t^*) \\
\leq 0.5 \left[|X_1(\bar{t})| + \mathcal{T}_o((Z_1(t_o), S(t_o)))\right]\left\{2\ell + \alpha Z((Z_1(t_o), S(t_o))) + \frac{64}{e}\right\}^2 \\
+ \frac{1}{32} - \frac{\ell}{256} (\bar{t} - t^*),
\]
since \(|d(t)| \leq \ell/64\) for all \(t \geq t_*\). Hence, canceling 1/32 from both sides gives
\[
\bar{t} - t \leq (128/\ell)[|X_1(t_0)| + T_o((\mathcal{Z}_1(t_0), S(t_0)))]\{2\ell + \alpha_Z((\mathcal{Z}_1(t_0), S(t_0)))+ 6\bar{n}/c\}^2.
\]
Therefore, we can take \(T_1(r) = 2T_o(r) + (128/\ell)[r + T_o(r)]\{2\ell + \alpha_Z(r) + 6\bar{n}/c\}^2\).

To show why this choice of \(T_1\) works, notice that if there existed a time \(\bar{t}\) such that \(\bar{t} - t_0 \geq T_1((X_1(t_0), \mathcal{Z}_1(t_0), S(t_0)))\neq 0\) and \(|X_1(\bar{t})| \geq 1/4\), then our formula for \(T_1\) implies that \(\bar{t} \geq t_*\). Hence, choosing \(t\) as in the previous paragraph and setting \(W(t_0) = (X_1(t_0), \mathcal{Z}_1(t_0), S(t_0))\) for brevity gives
\[
\frac{128}{\ell} [\max(|W_1(t_0)| + T_o(|W(t_0)|)]\{2\ell + \alpha_Z(|W(t_0)|) + 6\bar{n}/c\}^2 + \bar{t} - t_0 \geq \bar{t} - t + \bar{t} - t_0 \geq T_1(|W_1(t_0)|).
\]
This gives \(\bar{t} - t_0 \geq 2T_0((X_1(t_0), \mathcal{Z}_1(t_0), S(t_0)))\) and therefore also \(\ell \geq t_*\), so Case 1) gives \(\dot{V} < 0\) on \([t, \bar{t}]\) which is a contradiction because \(|X_1(t)| = 1/4\). On the other hand, if \(\bar{t} - t_0 \geq T_1((X_1(t_0), \mathcal{Z}_1(t_0), S(t_0))) = 0\) and \(|X_1(\bar{t})| \geq 1/4\), then \(|\mathcal{Z}_1(t)|\) never goes above \(\ell/32\), so Case 1) gives \(\dot{V} < 0\) when \(X_1(t) = 1/4\) so \(|X_1|\) cannot go above 1/4.

Hence, \(|X_1(t)| \leq \frac{1}{4}\) on \([t_q, +\infty)\), where \(t_q = t_o + T_1((X_1(t_0), \mathcal{Z}_1(t_0), S(t_0)))\). On this interval, Case 2) gives
\[
\dot{V} \leq -cV + \frac{d^2(t)}{\ell} = -cV + \frac{1}{4\ell}Z_1^2(t),
\]
since we chose \(c \in (0, \ell)\). Multiplying (2.34) through by \(e^{(t-t_q)/8}\), using (2.19) and (2.28), integrating over \([t_q, t]\), and taking square roots gives
\[
|X_1(t)| \leq \left[|X_1(t_q)| + \frac{3}{\sqrt{6\ell}}\alpha_Z((X_1(t_0), \mathcal{Z}_1(t_0), S(t_0)))\right] e^{-c(t-t_q)/16} + \sqrt{\frac{2}{\ell}}\frac{12}{c\sqrt{2}}|\eta|_\infty
\leq e^{-c(t-t_q)/16}\alpha_3((X_1(t_o), \mathcal{Z}_1(t_0), S(t_0))) + \sqrt{\frac{2}{\ell}}\frac{12}{c\sqrt{2}}|\eta|_\infty,
\]
where
\[
\alpha_3(r) = r + T_1(r)\left(2\ell + \alpha_Z(r) + \frac{6\bar{n}}{c}\right) + \frac{3\alpha_Z(r)}{\sqrt{\ell}}.
\]
This gives (2.27) with $\alpha_X(r) = \alpha_3(r)e^{cT_1(r)/16}$.

**Step 3:** Combining the estimates (2.16), (2.19), and (2.27) gives a function $\alpha_{X,Z} \in C^1 \cap K_\infty$ such that

$$|(Z_1(t), X_1(t), S(t))| \leq \alpha_{X,Z} \left(|(Z_1(t_o), X_1(t_o), S(t_o))|e^{-c(t-t_o)/16} + \frac{12}{\epsilon} \left(1 + \sqrt{\frac{2}{\epsilon c}} \right) \eta_\infty \right)$$

along all trajectories $(Z_1, X_1, S)$ for all initial times $t_o \geq 0$ and all $t \geq t_o$. The construction of $\alpha$ to satisfy (2.10) follows, because Lemma 2.1 gives

$$|(X, S)| \leq \sqrt{\ell^2 + 1} |(Z_1, X_1, S)| \leq 3(\ell^2 + 1)|X, S|$$

everywhere. This proves Theorem 2.4.

**Remark 2.6.** The following observations will be useful in Section 3.7, in our analysis of tracking problems for PVTOL aircraft models where velocity observations may not be available. The preceding proof shows that when $\eta \equiv 0$, the function

$$G_\ell(X) = \frac{1}{2}X_1^2 + \frac{1}{\ell \epsilon} \left\{ 2X_2 + \sigma_\ell(\ell X_1) \varphi_\ell(X_2) \right\}^2$$

satisfies

$$\frac{d}{dt}G_\ell(X(t)) \leq -d_\ell |X(t)|^2 + \frac{32\ell^2}{c^2} |S(t)|^2$$

and $G_\ell(X(t)) \geq d_\ell |X(t)|^2$ along all trajectories of (2.8) for all times $t \geq t_o + T_1(|(X_1(t_o), S(t_o))|)$ and all initial times $t_o \geq 0$, where

$$d_\ell = \min \left\{ \frac{c_\ell}{10}, \frac{1}{2\ell} \right\} \min \left\{ \ell^2, 0.5, \frac{1}{2\ell} \right\}$$

and $c_\ell = \min\{0.75, 0.5\ell\}$. To see why, notice that if $|\ell X_1| \leq |X_2|$, then $X_1^2 + Z_1^2 = X_1^2 + [2X_2 + \sigma_\ell(\ell X_1) \varphi_\ell(X_2)]^2 \geq |X|^2$; while if $|\ell X_1| \geq |X_2|$, then we instead have

$$X_1^2 + Z_1^2 \geq (2\ell^2 X_1^2 + Z_1^2) \min \left\{ \frac{1}{2\ell^2}, 1 \right\} \geq (\ell^2 X_1^2 + X_2^2) \min \left\{ \frac{1}{2\ell^2}, 1 \right\} \geq \min \left\{ 0.5, \ell^2, \frac{1}{2\ell^2} \right\} |X|^2.$$
Also, \( G_\ell(X) = \frac{1}{2}X_1^2 + \frac{1}{c_\ell}Z_1^2 = V(X_1) + \frac{2}{\ell c_\ell}W(Z_1) \) everywhere. The estimates now follow from (2.23) and (2.34), because

\[
\frac{d}{dt}G_\ell(X(t)) \leq -\min \left\{ \frac{c_\ell}{16}, \frac{1}{2\ell} \right\} (X_1^2 + Z_1^2) + \frac{32L^2}{c_\ell^2} |S(t)|^2
\]

along all trajectories of (2.8) for all times \( t \geq t_o + T_1((X_1(t_o), Z_1(t_o), S(t_o))) \) and all initial times \( t_o \geq 0 \).
Chapter 3
Planar Vertical Takeoff and Landing Model

In this chapter, we use our bounded backstepping theorem from page 18 of Chapter 2 to design controllers that ensure tracking for a broad class of PVTOL trajectories. First we consider the case where the full state is available for measurement. Then, we combine our results with the observer approach from [12] to generate tracking controllers that do not require velocity measurements.

3.1 Discussion on Model

Since its introduction in [22], the planar vertical takeoff and landing (PVTOL) aircraft model has become a benchmark dynamical system in aerospace engineering, and it is of continuing ongoing research interest [3, 10, 15]. Recall from Section 1.4.1 above that the PVTOL model is

\[
\begin{align*}
\dddot{x} &= -\bar{u}_1 \sin(\theta) + \epsilon u_2 \cos(\theta) \\
\dddot{y} &= \bar{u}_1 \cos(\theta) + \epsilon u_2 \sin(\theta) - g \\
\ddot{\theta} &= u_2,
\end{align*}
\]  

(3.1)

where \((x, y)\) gives the lateral and vertical coordinates of the center of mass of the aircraft, \(\theta\) is the roll angle relative to the horizon, the control \(\bar{u}_1\) is the thrust directed out of the bottom, \(g\) is the gravitational constant, the control \(u_2\) is the rolling moment, and the constant \(\epsilon\) gives the coupling between the roll moment and the lateral force [3].

It is a simplified model with the minimal number of states and inputs that has the main features needed to design controllers for real aircraft [22]; see Figure 3.1.
FIGURE 3.1. PVTOL Airplane

We are using a bar on the thrust controller because it is convenient to use a change of feedback to decouple the coordinates. In fact, the coordinates
\[ z_1 = x - \varepsilon \sin(\theta), \]
\[ z_2 = \dot{x} - \varepsilon \dot{\theta} \cos(\theta), \]
\[ w_1 = y + \varepsilon (\cos(\theta) - 1), \]
\[ w_2 = \dot{y} - \varepsilon \dot{\theta} \sin(\theta), \]
\[ \xi_1 = \theta, \]
\[ \xi_2 = \dot{\theta}, \]
and new input
\[ u_1 = \bar{u}_1 - \varepsilon \xi_2^2 \]
from [38] transform (3.1) into [38]

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -u_1 \sin(\xi_1) \\
\dot{w}_1 &= w_2 \\
\dot{w}_2 &= u_1 \cos(\xi_1) - g \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= u_2.
\end{align*}
\] (3.2)

The main literature on (3.2) is divided into set point stabilization (e.g., [38, 44, 54]), and tracking or path following (e.g., [10, 11, 12, 26, 31, 32]). The challenges in designing PVTOL stabilizers are that \( u_1 \) must be nonnegative and that the system is underactuated. Much of the PVTOL literature uses output feedbacks that only depend on \((z_1, w_1, \xi_1)\). One can design globally exponentially stable observers for the velocities; see [12] and Section 3.7 below, and [3, 55] for recent work on state feedback tracking controllers.
Given a reference trajectory for (3.2), it is natural to ask whether we can design feedback controllers $u_1$ and $u_2$ that force all trajectories of (3.2) to track the reference trajectory, for all initial configurations. This is the problem of rendering the tracking error dynamics for (3.2) uniformly globally asymptotically stable. Recall from p.13 that the tracking error encodes the difference between the current state and the reference trajectory vector at each time $t$. Several significant papers gave sufficient conditions guaranteeing that such controllers can be constructed [3, 12]. However, one would hope to establish uniform global asymptotic stability of the tracking dynamics by globally bounded controllers. Also, it is important for the controllers to perform well under uncertainty, so it is also important to have controllers that give ISS with respect to actuator errors, which are additive uncertainties on the controllers. In this chapter, we use our bounded backstepping theorem from Chapter 2 to achieve these additional boundedness and key robustness objectives.

### 3.2 Literature Review

The fundamental importance of the PVTOL model has led to a vast PVTOL literature involving a variety of techniques. In their original work [22], Hauser et al. used approximate input-output linearization to get bounded tracking and asymptotic stability for (3.2). Later work [52] by Teel developed small gain theory for systems in feedforward form that gives stabilization results for the PVTOL model as a special case, including robustness to uncertainty in the coupling parameter $\varepsilon$. In [32], Martin et al. extended [22] by giving output tracking results for a class of slightly or strongly non-minimum phase systems that includes the PVTOL. The main idea in [32] was to use the output at the Huygens center of oscillation, which is a fixed point with respect to the aircraft body, and then the controller was defined on a suitable subset of the state space. Also, [46, Section 6.1] designed PVTOL
aircraft state feedbacks under the assumption that the coupling parameter $\varepsilon$ is zero and then selected the controller parameters to mitigate the effects of nonzero values of $\varepsilon$. Then [26] gave optimal control methods that led to nonlinear state feedback controllers that give hovering control that is robust to uncertainty in the coupling parameter $\varepsilon$. See [6, Section VI.C] for stabilization of equilibrium points under linear dynamic stabilizers.

Subsequent work [38] by Olfati-Saber from 2002 used a change of coordinates from [37] to design a state controller that stabilizes a zero velocity configuration and allows larger values of the parameter $\varepsilon$. Also in 2002, Marconi et al. [31] used an internal-based model approach and nested saturations to design an autopilot for the autonomous landing of a PVTOL aircraft on a ship whose deck oscillates under high seas. See [4] for output tracking along a circle. Later work [13] by Francisco et al. used forwarding results from [36] to design distributed delay nested saturation feedbacks that give global asymptotic stability.

The PVTOL literature on path following can be summarized as follows. Tracking leads to controllers that have an a priori parametrization of the curve to be followed, while path following does not involve such a parametrization. See [10] for path following of Jordan curves using continuous feedback based on finite time stabilization for initial states near the desired configuration. An advantage of path following is that it can mitigate the effects of moving along a path too quickly [10]. However, the PVTOL tracking error dynamics are amenable to global Lyapunov function methods. Lyapunov methods have the advantage that they can lead to ISS proofs, which is important for certifying good performance under worst case disturbances. Therefore, tracking and path following are both important.

One natural approach to the PVTOL dynamics involves backstepping [12]. See [54], whose feedback law leads to a cascade structure that minimizes the norm
of the interconnection term between subsystems. When designing PVTOL controllers, it is important to take the maximum amplitude of the feedbacks into account. On the other hand, standard backstepping techniques do not in general lead to bounded feedback stabilizers. There have been several generalizations of backstepping that give bounded feedbacks [14, 29, 33]. See [29, Chapter 7] where bounded backstepping was used to track certain sinusoidal PVTOL trajectories.

The work [3] gave globally stabilizing tracking controllers for a specific class of reference trajectories when $u_1$ is bounded, and semiglobal stability when both $u_1$ and $u_2$ are bounded. The PVTOL output feedback tracking controllers in [12] were based on several changes of coordinates, Lyapunov’s direct method, Barbalat’s Lemma, and backstepping. However, the controllers in [12] are not bounded. Moreover, the thrust control $\bar{u}_1$ in [12] is not guaranteed to be bounded below by a positive constant. Since the existing work on global tracking for (3.2) is based on Barbalat’s Lemma, it does not lend itself to ISS. Our controllers for (3.2) are necessarily more complex than those of [3, 12]. However, to the best of our knowledge, the results to follow are original and significant because of (a) the global boundedness of our controllers $u_1$ and $u_2$ and the uniform positive lower bound on $\bar{u}_1$, (b) the applicability of our work to cases where the velocity measurements may not be available, (c) the uniform global asymptotic stability and uniform local exponential stability of our closed loop tracking dynamics, (d) our allowing a rather general class of reference trajectories, and (e) our use of ISS to quantify the performance under actuator errors of arbitrarily large amplitude.

### 3.3 Tracking Objective

We begin by choosing any reference trajectory $\mathcal{E}_r = (z_{1r}, z_{2r}, w_{1r}, w_{2r}, \xi_{1r}, \xi_{2r}) : [0, +\infty) \to \mathbb{R}^6$. This means that there exists a reference input $u_r = (u_{1r}, u_{2r})$ such
that for all $t \geq 0$, we have

\[
\begin{cases}
\dot{z}_1(t) = z_2(t) \\
\dot{z}_2(t) = -u_1(t) \sin(\xi_1(t)) \\
\dot{w}_1(t) = w_2(t) \\
\dot{w}_2(t) = u_1(t) \cos(\xi_1(t)) - g \\
\dot{\xi}_1(t) = \xi_2(t) \\
\dot{\xi}_2(t) = u_2(t) \end{cases}
\] (3.3)

We wish to design bounded tracking controllers that ensure tracking for reference trajectories that satisfy the following:

**Assumption 1.** (1) The functions $E_r$ and $u_r$ are $C^2$. (2) There is a constant $c_1 \in (0, \pi/2)$ such that $\xi_{1r}(t) \in [-\pi/2 + c_1, \pi/2 - c_1]$ for all $t \geq 0$. (3) The functions $\dot{\xi}_{1r}, \ddot{\xi}_{1r}, u_r, \dot{u}_r$, and $\ddot{u}_r$ are all bounded. (4) There is a constant $c_2 > 0$ such that $\inf_{t \geq 0} u_{1r}(t) \geq c_2$.

Equivalently, we must design bounded $C^1$ feedbacks $u_i$ to drive the error variables $\tilde{z}_i(t) = z_i(t) - z_{ir}(t)$, $\tilde{w}_i(t) = w_i(t) - w_{ir}(t)$, and $\tilde{\xi}_i(t) = \xi_i(t) - \xi_{ir}(t)$ to 0 for $i = 1, 2$. This means that the $u_i$'s must render the tracking dynamics

\[
\begin{cases}
\dot{\tilde{z}}_1 = \tilde{z}_2 \\
\dot{\tilde{z}}_2 = -u_1 \sin(\xi_1) + u_{1r}(t) \sin(\xi_{1r}(t)) \\
\dot{\tilde{w}}_1 = \tilde{w}_2 \\
\dot{\tilde{w}}_2 = u_1 \cos(\xi_1) - u_{1r}(t) \cos(\xi_{1r}(t)) \\
\dot{\tilde{\xi}}_1 = \tilde{\xi}_2 \\
\dot{\tilde{\xi}}_2 = u_2 - u_{2r}(t) \end{cases}
\] (3.4)

UGAS.

**Remark 3.1.** The physical and technical constraints of the system provide input restrictions. Hence, there are positive constants $\bar{u}_i$ such that $\bar{u}_0 \leq u_{1r}(t) \leq \bar{u}_1$
and \( |u_2(t)| \leq \bar{u}_2 \) must hold for all \( t \geq 0 \). The positive lower bound \( \bar{u}_0 \) is used to avoid the 0 thrust. We can use these actuator constraints to give sufficient conditions for a trajectory to be trackable. In fact, take any \( C^4 \) function \( R_r = (z_{1r}, z_{2r}, w_{1r}, w_{2r}, \xi_{1r}, \xi_{2r}) : [0, +\infty) \to \mathbb{R}^6 \) whose first four derivatives are bounded, and assume that \( \inf_{t \geq 0} [\ddot{w}_{1r}(t) + g] > 0 \). Then the following conditions are easily shown to be equivalent:

\[ C_1 \] \( R_r \) satisfies (3.3) for all \( t \geq 0 \) for some \( C^2 \) input \( u_r = (u_{1r}, u_{2r}) : [0, +\infty) \to \mathbb{R}^2 \) for which \( u_{1r}(t) \) is nonnegative for all \( t \geq 0 \).

\[ C_2 \] \( \xi_{1r}(t) = \arcsin(-\ddot{z}_{1r}(t)/\{(\ddot{z}_{1r}(t))^2 + (\ddot{w}_{1r}(t) + g)^2\}^{1/2}) \), \( \dot{z}_{1r}(t) = z_{2r}(t) \), \( \ddot{w}_{1r}(t) \), and \( \dot{\xi}_{1r}(t) = \xi_{2r}(t) \) hold for all \( t \geq 0 \).

The implication \([C_1] \Rightarrow [C_2]\) follows by using the second and fourth equations in (3.3) to solve for \( u_{1r} \). In this case, (3.3) holds for all \( t \geq 0 \) with \( u_{1r}(t) = \{(\ddot{z}_{1r}(t))^2 + (\ddot{w}_{1r}(t) + g)^2\}^{1/2} \) and \( u_{2r}(t) = \ddot{\xi}_{1r}(t) \), and there are constants \( c_1 \in (0, \pi/2) \) and \( c_2 > 0 \) such that \( \xi_{1r}(t) \in [-\pi/2 + c_1, \pi/2 - c_1] \) for all \( t \geq 0 \) and \( \inf_{t \geq 0} u_{1r}(t) \geq c_2 \). Hence, the trajectory satisfies all of our assumptions. Moreover, we satisfy the input restrictions if we also have \( \bar{u}_0 \leq \{(\ddot{z}_{1r}(t))^2 + (\ddot{w}_{1r}(t) + g)^2\}^{1/2} \leq \bar{u}_1 \) and \( |\dddot{\xi}_{1r}(t)| \leq \bar{u}_2 \) for all \( t \geq 0 \). See Sections 3.9-3.10 for more details and an application to a specific tracking problem.

### 3.4 Thrust Control Out of the Bottom

Recall the functions \( \varphi_\ell, \sigma_\ell, \) and \( U_\ell \) we defined in Section 2.1. Using part (II) of Lemma 2.3 on page 18 and the constants \( c_i > 0 \) defined above, we have \( \inf_{t \geq 0} \{u_{1r}(t) \cos(\xi_{1r}(t))\} > 0 \), and we can fix a small enough constant \( \lambda > 0 \) such that

\[ v(t, \bar{z}) = \arctan \left( \tan(\xi_{1r}(t)) - \frac{U_\lambda(\bar{z})}{u_{1r}(t) \cos(\xi_{1r}(t))} \right) \]  

(3.5)
admits a constant $c_3 \in (0, c_1)$ such that $v(t, \tilde{z}) \in [-\pi/2 + c_3, \pi/2 - c_3]$ for all $t \geq 0$ and $\tilde{z} \in \mathbb{R}^2$. We choose the control component

$$u_1(t, \tilde{z}, \tilde{w}) = \frac{1}{\cos(v)} \left[ u_{1r}(t) \cos(\xi_{1r}(t)) + U_\lambda(\tilde{w}) \right]$$

(3.6)

for the thrust controller. By reducing $\lambda > 0$ without relabeling and again using part (II) of Lemma 2.3, we can assume that $u_1$ is everywhere nonnegative. We then set

$$K(t, \tilde{z}) = \frac{1}{1 + \tan^2(v)} \frac{1}{u_{1r}(t) \cos(\xi_{1r}(t))} ,$$

(3.7)

and we define the functions $S_\lambda$ and $T_\lambda$ by

$$S_\lambda(t, \tilde{z}, \tilde{w}) = -\dot{\xi}_{1r}(t) + \frac{\{1 + \tan^2(\xi_{1r}(t))\} \dot{\xi}_{1r}(t)}{1 + \tan^2(v)}$$

$$+ K(t, \tilde{z}) \left[ \frac{U_\lambda(\tilde{z})}{u_{1r}(t) \cos(\xi_{1r}(t))} \{ u_{1r}(t) \cos(\xi_{1r}(t)) \} - \frac{\partial U_\lambda(\tilde{z})}{\partial \tilde{z}} \right]$$

$$- K(t, \tilde{z}) \frac{\partial U_\lambda(\tilde{z})}{\partial \tilde{z}} \{ \sin(\xi_{1r}(t)) u_{1r}(t) - \sin(v) u_1 \}$$

(3.8)

and

$$T_\lambda(t, \varpi_1, \tilde{z}, \tilde{w}) = u_1 K(t, \tilde{z}) \frac{\partial U_\lambda(\tilde{z})}{\partial \tilde{z}} \{ \sin(\varpi_1 + v) - \sin(v) \} ,$$

(3.9)

where $\varpi_1 = \xi_1 - v$, and where $v$ and $u_1$ are from (3.5)-(3.6). Since $\dot{\tilde{z}}_2$ and $\dot{\tilde{w}}_2$ are bounded, Lemma 2.3 and our Assumption 1 on $E_r$ implies that the time derivative $\dot{S}_\lambda$ along all trajectories of (3.4) is bounded. Fix a constant $a > 0$ such that

$$\max \left\{ \left| \frac{\partial T_\lambda}{\partial \varpi_1}(t, \varpi_1, \tilde{z}, \tilde{w}) \right|, \left| T_\lambda(t, \varpi_1, \tilde{z}, \tilde{w}) \right| \right\} \leq \frac{a}{16}$$

(3.10)
everywhere. Notice that our tracking dynamics (3.4) with the choices (3.6) and $\varpi_2 = \xi_2 - S_\lambda$ can be rewritten as

$$
\begin{align*}
\dot{\hat{z}}_1 &= \hat{z}_2 \\
\dot{\hat{z}}_2 &= -\frac{\sin(\varpi_1 + v)}{\cos(v)} \left[u_{1r}(t) \cos(\xi_{1r}(t)) + U_\lambda(\tilde{w})\right] + u_{1r}(t) \sin(\xi_{1r}(t)) \\
\dot{\tilde{w}}_1 &= \tilde{w}_2 \\
\dot{\tilde{w}}_2 &= \left[\cos(\varpi_1 + v) - 1\right] u_{1r}(t) \cos(\xi_{1r}(t)) + U_\lambda(\tilde{w}) \frac{\cos(\varpi_1 + v)}{\cos(v)} \\
\dot{\varpi}_1 &= \varpi_2 - T_\lambda(t, \varpi_1, \tilde{z}, \tilde{w}) \\
\dot{\varpi}_2 &= u_3
\end{align*}
$$

where $u_3 = u_2 - u_{2r}(t) - \dot{S}_\lambda$ and $v$ depends on $(t, \tilde{z})$. Then the UGAS and ULES properties for (3.4) are equivalent to those of (3.11), so we have reduced the stabilization problems for (3.4) to those for (3.11).

### 3.5 Main Tracking Theorem

Since $\dot{S}_\lambda$, $u_{2r}$, and (3.6) are $C^1$ and bounded, we will have our bounded feedbacks for the PVTOL tracking dynamics, once we design a $C^1$ bounded feedback $u_3$ that renders (3.11) UGAS and ULES. Our construction of $u_3$ is:

**Theorem 3.2.** Let the constants $a > 0$ and $\lambda > 0$ satisfy the requirements from Section 3.4. Then

$$
u_3(t, \tilde{z}, \tilde{w}, \varpi) = \frac{-\sigma_a(2\varpi_2 + \sigma_a(a\varpi_1)\varphi_a(\varpi_2)) - a\sigma'_a(a\varpi_1)\varphi_a(\varpi_2)[\varpi_2 - T_\lambda(t, \varpi_1, \tilde{z}, \tilde{w})]}{2 + \sigma_a(a\varpi_1)\varphi'_a(\varpi_2)} \quad (3.12)$$

is bounded and $C^1$ and renders (3.11) UGAS and ULES. Hence, the controller $u_1$ from (3.6) and the rolling moment controller $u_2 = u_3 + u_{2r}(t) + \dot{S}_\lambda$ render (3.4) UGAS and ULES. $\square$
Proof. The dynamics (3.11) in closed loop with (3.12) is forward complete because its right side grows linearly in the state uniformly in \( t \). Therefore, the fact that there exist a function \( \alpha_1 \in C^1 \cap K_\infty \) and a constant \( \tilde{c}_1 > 0 \) such that the UGAS and ULES estimate

\[
|\varpi(t)| \leq \alpha_1(|\varpi(t_0)|) e^{-\tilde{c}_1(t-t_0)}
\]  

(3.13)

holds along all trajectories of (3.11) follows from our bounded backstepping theorem (Theorem 2.4) with the choice \( S(t) \equiv 0 \), (3.10), and the fact that (3.12) agrees with the controller (2.9) when we take \( L \equiv 0 \), \( \ell = a \), \( X = \varpi \), \( \bar{\eta} = 0 \), and \( \Theta(t, \varpi) = -\mathcal{T}_\lambda(t, \varpi, \tilde{z}(t), \tilde{w}(t)) \).

Next note that the \( \tilde{w} \) dynamics in (3.11) can be written as

\[
\begin{align*}
\dot{\tilde{w}}_1 &= \tilde{w}_2 \\
\dot{\tilde{w}}_2 &= U_\lambda(\tilde{w}) + L(t, \tilde{w}, \varpi)
\end{align*}
\]  

(3.14)

for an appropriate function \( L \) that admits a constant \( \bar{L} > 0 \) so that \(|L(t, \tilde{w}, \varpi)| \leq \bar{L}|\varpi_1| \) for all \( t \geq 0 \). The time variable in \( L \) includes the effects of \( \tilde{z} \), which enter through the function \( v(t, \tilde{z}) \). Hence, the fact that there exist a function \( \alpha_2 \in C^1 \cap K_\infty \) and a constant \( \tilde{c}_2 > 0 \) such that the UGAS and ULES estimate

\[
|((\tilde{w}(t), \varpi(t)))| \leq \alpha_2\left(||(\tilde{w}(t_0), \varpi(t_0))||\right) e^{-\tilde{c}_2(t-t_0)}
\]  

(3.15)

holds all trajectories of the \((\tilde{w}, \varpi)\) subsystem of (3.11) also follows from Theorem 2.4, this time applied with \( X = \tilde{w} \), \( \Theta \equiv 0 \), \( \bar{\eta} = 0 \), and \( S = \varpi \).

Finally, notice that \( U_\lambda(\tilde{z}) = [\tan(\xi_1(t)) - \tan(v)]u_1(t) \cos(\xi_1(t)) \) everywhere. Hence, the \( \tilde{z} \) subdynamics in (3.11) can be rewritten as

\[
\begin{align*}
\dot{\tilde{z}}_1 &= \tilde{z}_2 \\
\dot{\tilde{z}}_2 &= U_\lambda(\tilde{z}) + \mathcal{L}(t, \tilde{z}, \tilde{w}, \varpi),
\end{align*}
\]  

(3.16)

where

\[
\mathcal{L}(t, \tilde{z}, \tilde{w}, \varpi) = \frac{\sin(v) - \sin(\xi_1)}{\cos(v)} \left[u_1(t) \cos(\xi_1(t)) + U_\lambda(\tilde{w})\right] - \tan(v)U_\lambda(\tilde{w}).
\]
Using the properties of $v$ and $U_\lambda$, we can find a constant $\bar{L} > 0$ such that $|\mathcal{L}(t, \tilde{z}, \tilde{w}, \varpi)| \leq \bar{L}|(\tilde{w}, \varpi)|$ everywhere. Then the assumptions of Theorem 2.4 are satisfied with $X = \tilde{z}$, $S = (\tilde{w}, \varpi)$, $L = \mathcal{L}$, $\bar{L} = \bar{\mathcal{L}}$, $\Theta \equiv 0$, and $\bar{\eta} = 0$, so Theorem 2.4 gives a function $\alpha_3 \in C^1 \cap \mathcal{K}_\infty$ and a constant $\tilde{c}_3 > 0$ such that

$$\left| (\tilde{z}(t), \tilde{w}(t), \varpi(t)) \right| \leq \alpha_3 \left( \left| (\tilde{z}(t_0), \tilde{w}(t_0), \varpi(t_0)) \right| \right) e^{-\tilde{c}_3(t-t_0)}$$

along all trajectories of (3.11), which gives the desired conclusions. \[\square\]

**Remark 3.3.** See Section 3.8 for our extension to cases where there are actuator errors. We cannot eliminate $T_\lambda$ the way we eliminated $S_\lambda$, because the unboundedness of $\dot{\varpi}_1$ implies that $\dot{T}_\lambda$ is unbounded.

### 3.6 Input Constraints and Controller Bounds

**Remark 3.4.** We can derive explicit global bounds on our controllers $u_i$. To get the bound on the controller $u_1$ from (3.6), first pick any constant $b \in (0, 1)$. By reducing the constant $\lambda > 0$ from Section 3.4, we can assume that

$$0 < \lambda < \min \left\{ 1, \frac{b \alpha_4}{14} \cos(\pi/2 - c_1) \right\}, \quad (3.17)$$

where the $c_i$’s satisfy Assumption 1. Then part (II) of Lemma 2.3 implies that

$$\frac{|U_\lambda(Z)|}{u_{1r}(t) \cos(\xi_{1r}(t))} \leq \frac{2\lambda(6\lambda + 1)}{c_2 \cos(\pi/2 - c_1)} \leq b \quad (3.18)$$

for all $Z \in \mathbb{R}^2$ and all $t \geq 0$. Since $\cos(\arctan(q)) = 1/(1+q^2)^{1/2}$ holds for all $q \in \mathbb{R}$, we conclude from our formula (3.5) for $v(t, \tilde{z})$ that $1/\cos(v) = (1 + \tan^2(v))^{1/2} = (1 + [U_\lambda(\tilde{z})/\{u_{1r}(t) \cos(\xi_{1r}(t))\} - \tan(\xi_{1r}(t)))]^{1/2} \in [1, 1 + [\tan(\pi/2 - c_1) + b]^2]$ and $u_{1r}(t) \cos(\xi_{1r}(t)) + U_\lambda(\tilde{w}) \in [(1 - b)c_2 \cos(\pi/2 - c_1), (1 + b)\sup\left\{u_{1r}(p) : p \geq 0\right\}]$ hold for all $t \geq 0$, $\tilde{z} \in \mathbb{R}^2$, and $\tilde{w} \in \mathbb{R}^2$. Combining the preceding estimates gives
the controller bounds

\[(1 - b)c_2 \cos(\pi/2 - c_1)\]

\[\leq u_1(t, \tilde{z}, \tilde{w})\]

\[\leq \{1 + [\tan(\pi/2 - c_1) + b]^2\}(1 + b) \sup\{u_{1r}(p) : p \geq 0\}\]

for all \(t \geq 0\) and \((\tilde{z}, \tilde{w}) \in \mathbb{R}^4\). Hence, for all constants \(\tilde{u}_0 \in (0, (1 - b)c_2 \cos(\pi/2 - c_1))\) and \(\tilde{u}_1 \geq \{1 + [\tan(\pi/2 - c_1) + b]^2\}(1 + b) \sup\{u_{1r}(p) : p \geq 0\}\), we satisfy \(\tilde{u}_0 \leq u_1(t, \tilde{z}, \tilde{w}) \leq \tilde{u}_1\) for all \(t \geq 0, \tilde{z} \in \mathbb{R}^2,\) and \(\tilde{w} \in \mathbb{R}^2\). Taking the constant \(c_1\) from Assumption 1 close enough to \(\pi/2\) (which can be done by restricting to reference trajectories such that \(\inf_{t \geq 0}[\tilde{w}_{1r}(t) + g]\) is large enough and arguing as in Remark 3.1) and \(b\) close enough to 0, we can then satisfy the actuator constraints on \(u_1\) if \(\tilde{u}_0 < c_2\) and \(\tilde{u}_1 > \sup\{u_{1r}(p) : p \geq 0\}\). Also, \(u_1(t, \tilde{z}, \tilde{w})\) has a uniform positive lower bound, which is important for avoiding the zero thrust. This differs from [12], where the controller \(u_1\) is not necessarily bounded away from zero. See Remark 3.6 for analogous bounds for \(u_2\).

**Remark 3.5.** Recall from our decoupling change of coordinates from p.29 that the thrust out of the bottom is \(\bar{u}_1 = u_1 + \varepsilon \xi_2^2\). This will not be globally bounded, because \(\xi_2\) is unbounded. However, simple calculations allow us to combine our UGAS estimate for (3.11), the triangle inequality, and the coordinate changes that transformed (3.4) into (3.11) to construct a function \(\alpha^* \in K_{\infty}\) such that

\[|\xi_2(t)| \leq |\xi_{2r}(t)| + |\tilde{\xi}_2(t)| \leq |\xi_{2r}(t)| + \alpha^*\left(|(\tilde{z}, \tilde{w}, \tilde{\xi})(t_0)|\right)\]

(3.20)

along all trajectories of (3.4). Hence,

\[u_1(t, \tilde{z}(t), \tilde{w}(t)) \leq \bar{u}_1(t, \tilde{z}(t), \tilde{w}(t), \xi_2(t))\]

\[\leq u_1(t, \tilde{z}(t), \tilde{w}(t)) + 2\varepsilon\left\{\xi_{2r}^2(t) + \left[\alpha^*\left(|(\tilde{z}, \tilde{w}, \tilde{\xi})(t_0)|\right)\right]^2\right\}\]

(3.21)

holds along all trajectories of (3.4). Since \(\xi_{2r}\) is bounded, this gives finite positive upper and lower bounds on \(\bar{u}_1\) in terms of our bounds on \(u_1\) from Remark 3.4 and
the norm of the initial state of (3.4). We can use \( \alpha^* \in K_\infty \) to ensure that the overflow term \( 2\varepsilon \{ \xi_{2r}^2(t) + [\alpha^* |(\tilde{z}, \tilde{w}, \tilde{\xi})(t_0)|]^2 \} \) is small enough, either by further restricting the reference trajectory such that \( \sup\{|\xi_{2r}(t)| : t \geq 0\} \) is small enough, or by restricting to state trajectories that start close enough to the reference trajectory to make \( |(\tilde{z}, \tilde{w}, \tilde{\xi})(t_0)| \) small enough. In fact, if \( \tilde{u}_1 \) is the maximum allowable thrust out of the bottom, and if we further restrict the reference trajectories as in Remark 3.4 such that \( \sup\{|u_1(t, \tilde{z}, \tilde{w}) : t \geq 0, (\tilde{z}, \tilde{w}) \in \mathbb{R}^4 \} < \tilde{u}_1 \), then \( \tilde{u}_1 \) is also bounded above \( \tilde{u}_1 \) if

\[
2\varepsilon \{ \xi_{2r}^2(t) + [\alpha^* |(\tilde{z}, \tilde{w}, \tilde{\xi})(t_0)|]^2 \} \leq \tilde{u}_1 - \sup \{ u_1(p, \tilde{z}, \tilde{w}) : p \geq 0, (\tilde{z}, \tilde{w}) \in \mathbb{R}^4 \}.
\] (3.22)

Combined with our positive lower bound on \( u_1 \) from Remark 3.4, we conclude that our actuator envelope is satisfied.

**Remark 3.6.** We can combine the ideas from Remarks 3.4-3.5 to derive global bounds on our controller \( u_2 \) from Theorem 3.2. In fact, Lemma 2.3 implies that the time derivative \( \dot{S}_\lambda \) of \( S_\lambda \) along all trajectories of (3.4) has some finite global bound \( \bar{S}_\lambda \). Hence, Lemmas 2.1 and 2.3 and the bound on \( T_\lambda \) from (3.10) give

\[
\sup \{ |u_2(t, \tilde{z}, \tilde{w}, \varpi)| : t \geq 0, (\tilde{z}, \tilde{w}, \varpi) \in \mathbb{R}^6 \}
\leq 2a(6a + 1) + \frac{a^2}{8} + \sup\{|u_{2r}(p)| : p \geq 0\} + \bar{S}_\lambda ,
\] (3.23)

where the \( \frac{a^2}{8} \) term comes from the estimate

\[
\left| \frac{a\sigma'_a(a\varpi_1)\varphi_a(\varpi_2)T_\lambda(t, \varpi_1, \tilde{z}, \tilde{w})}{2 + \sigma_a(a\varpi_1)\varphi'_a(\varpi_2)} \right| \leq \frac{a^2}{8} .
\] (3.24)

While finite, the upper bound in (3.23) could exceed the physical constraints of the system, but we can get a tighter bound by using the UGAS estimate on (3.11) as in Remark 3.5. This gives an upper bound depending on a \( K_\infty \) function of the norm of the initial state of the tracking dynamics (3.11). The details are as follows. Our
controller $u_2$ is such that $u_2(t, 0, 0, 0) = u_{2r}(t)$ for all $t \geq 0$. Therefore, we can use our bounds on the reference trajectories and reference inputs and their derivatives to find a function $\tilde{\alpha} \in \mathcal{K}_\infty$ such that

$$|u_2(t, \tilde{z}, \tilde{w}, \varpi)| \leq \sup\{|u_{2r}(p)| : p \geq 0\} + \tilde{\alpha}((\tilde{z}, \tilde{w}, \varpi))$$  \hspace{1cm} (3.25)

for all $t \geq 0$ and all $(\tilde{z}, \tilde{w}, \varpi) \in \mathbb{R}^6$. Combining (3.25) with the UGAS estimate on (3.11) therefore gives a function $\alpha^{**} \in \mathcal{K}_\infty$ such that

$$|u_2(t, \tilde{z}(t), \tilde{w}(t), \varpi(t))| \leq \sup\{|u_{2r}(p)| : p \geq 0\}$$

$$+ \alpha^{**}((\tilde{z}(t_0), \tilde{w}(t_0), \varpi(t_0)))$$  \hspace{1cm} (3.26)

along all trajectories of (3.11). Hence, given any bound $\tilde{u}_2 > 0$ on the rolling moment, and given a reference trajectory for which the corresponding control component $u_{2r} = \dot{\xi}_{2r}$ satisfies $\sup\{|u_{2r}(p)| : p \geq 0\} < \tilde{u}_2$, we can find a region $\mathcal{R}_b$ in the state space containing the reference trajectory such that $|u_2(t, \tilde{z}(t), \tilde{w}(t), \varpi(t))| \leq \tilde{u}_2$ along all trajectories of (3.11) and all values attained by the rolling moment controller satisfy the actuator envelope, when the system starts in $\mathcal{R}_b$. This is done by simply choosing $\mathcal{R}_b$ such that $\alpha^{**}((\tilde{z}(t_0), \tilde{w}(t_0), \varpi(t_0))) \leq \tilde{u}_2 - \sup\{|u_{2r}(p)| : p \geq 0\}$ for all trajectories starting in $\mathcal{R}_b$. We leave the construction of $\alpha^{**}$ to the reader, but we demonstrate in our simulations in Section 3.10 how the control inputs satisfy the input restrictions.

### 3.7 Tracking Without Velocity Measurements

If only the variables $z_1$, $w_1$, and $\xi_1$ are measured, then we can achieve our tracking objective using the observer approach from [12]. We apply the approach as follows. First, the proofs of our bounded backstepping theorem (Theorem 2.4 on page 18) with $\tilde{\eta} = 0$ and Theorem 3.2 provide a positive definite proper function $\mathcal{V}_o(\tilde{z}, \tilde{w}, \varpi)$
and a $C^1$ function $\Gamma_o : [0, +\infty) \to (0, +\infty)$ such that $\dot{V}_o$ is negative definite along all trajectories of the tracking dynamics (3.11) in closed loop with (3.12) for all times $t \geq t_o + \Gamma_o(||(\tilde{z}, \tilde{w}, \varpi)(t_o)||)$, and such that $V_1 = \ln(1 + V_o)$ has a bounded gradient. To construct $V_o$, notice that the proof of Theorem 2.4 implies that the positive definite function $G_\ell(X) = 0.5X_1^2 + \frac{1}{\ell c_\ell}\{2X_2 + \sigma_\ell(\ell X_1)\varphi_\ell(X_2)\}^2$ (3.27) has a positive definite quadratic lower bound and admits a function $\Gamma \in K_{\infty} \cap C^1$ and a constant $d_\ell > 0$ such that

$$\frac{d}{dt}G_\ell(X(t)) \leq -d_\ell |X(t)|^2 + \frac{32\bar{L}^2}{c_\ell^2}|S(t)|^2$$ (3.28)

along all trajectories of (2.8) when $\eta = 0$ and $t - t_o \geq \Gamma(||(X(t_o), S(t_o))||)$, where $c_\ell = \min\{0.75, 0.5\ell\}$; see Remark 2.6. Applying the preceding construction successively with $(X, S) = (\varpi, 0)$, then $(X, S) = (\tilde{w}, \varpi)$, and finally with $(X, S) = (\tilde{z}, (\tilde{w}, \varpi))$ as in the proof of Theorem 3.2 provides positive constants $A_i$ such that $V_o(\tilde{z}, \tilde{w}, \varpi) = G_\lambda(\tilde{z}) + A_2[A_1G_a(\varpi) + G_\lambda(\tilde{w})]$ satisfies the requirements, where the constant $A_1$ is chosen to cancel the term $\frac{32\bar{L}^2|\varpi(t)|^2}{c_\ell^2a}$ in the decay estimate on $G_\lambda(\tilde{w})$, and then the constant $A_2$ is chosen to cancel the term $\frac{32\bar{L}^2|\tilde{w}(\varpi)(t)||^2}{c_\ell^2a}$ in the decay estimate for $G_\lambda(\tilde{z})$.

Using the coordinate changes that we used to transform (3.4) into (3.11), one easily checks that the feedback $u_1,s(t, \tilde{z}, \tilde{w})$ defined in (3.6) and

$$u_2,s(t, \tilde{z}, \tilde{w}, \tilde{\xi}) = u_3(t, \tilde{z}, \tilde{w}, \tilde{\xi}_1 + \xi_1r(t) - v(t, \tilde{z}), \tilde{\xi}_2 - S_\lambda(t, \tilde{z}, \tilde{w})) + u_2r(t) + \dot{S}_\lambda(t, \tilde{z}, \tilde{w})$$

are globally Lipschitz in the state $(\tilde{z}, \tilde{w}, \tilde{\xi})$ uniformly in $t$ and admit a proper positive function $V_2(t, \tilde{z}, \tilde{w}, \tilde{\xi})$ and a $C^1$ function $\Gamma_2 : [0, +\infty) \to (0, +\infty)$ such that $\dot{V}_2$ is negative definite along all of the closed loop trajectories of (3.4) with $u_1 = u_{1,s}$ and
\[ u_2 = u_{2,s} \text{ for all times } t \geq t_0 + \Gamma_2, \text{ and such that } |(\partial V_2/\partial \tilde{z})(t, \tilde{z}, \tilde{w}, \tilde{\xi})|, \]
\[ |(\partial V_2/\partial \tilde{w})(t, \tilde{z}, \tilde{w}, \tilde{\xi})|, \text{ and } |(\partial V_2/\partial \tilde{\xi})(t, \tilde{z}, \tilde{w}, \tilde{\xi})| \text{ are all bounded. In fact, we can take} \]
\[ \mathcal{V}_2(t, \tilde{z}, \tilde{w}, \tilde{\xi}) = \mathcal{V}_1(\tilde{z}, \tilde{w}, \tilde{\xi}_1, t) - v(t, \tilde{z}), \tilde{\xi}_2 - S_\lambda(t, \tilde{z}, \tilde{w}). \]  
(3.29)

Next consider the augmented dynamics
\[
\begin{align*}
\dot{\tilde{z}}_1 &= \tilde{z}_2 \\
\dot{\tilde{z}}_2 &= -u_{1,s}(t, \tilde{z}, \tilde{w}) \sin(\xi_1) + u_{1r}(t) \sin(\xi_{1r}(t)) \\
\dot{\tilde{w}}_1 &= \tilde{w}_2 \\
\dot{\tilde{w}}_2 &= u_{1,s}(t, \tilde{z}, \tilde{w}) \cos(\xi_1) - u_{1r}(t) \cos(\xi_{1r}(t)) \\
\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 \\
\dot{\tilde{\xi}}_2 &= u_{2,s}(t, \tilde{z}, \tilde{w}, \tilde{\xi}) - u_{2r}(t) \\
\dot{\hat{z}}_1 &= \hat{z}_2 + k_1(\tilde{z}_1 - \hat{z}_1) \\
\dot{\hat{z}}_2 &= -u_{1,s}(t, \hat{z}, \hat{w}) \sin(\xi_1) + u_{1r}(t) \sin(\xi_{1r}(t)) + k_2(\tilde{z}_1 - \hat{z}_1) \\
\dot{\hat{w}}_1 &= \hat{w}_2 + k_3(\tilde{w}_1 - \hat{w}_1) \\
\dot{\hat{w}}_2 &= u_{1,s}(t, \hat{z}, \hat{w}) \cos(\xi_1) - u_{1r}(t) \cos(\xi_{1r}(t)) + k_4(\tilde{w}_1 - \hat{w}_1) \\
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + k_5(\tilde{\xi}_1 - \hat{\xi}_1) \\
\dot{\hat{\xi}}_2 &= u_{2,s}(t, \hat{z}, \hat{w}, \hat{\xi}) - u_{2r}(t) + k_6(\tilde{\xi}_1 - \hat{\xi}_1) 
\end{align*}
\]  
(3.30)

where the \( k_i \) are any positive constants and the hats indicate estimates (so \( \hat{z}_1 \) represents an estimate of \( \tilde{z}_1 \) and likewise for the other components). We prove:

**Theorem 3.7.** The dynamics (3.30) are UGAS and ULES to the origin.

**Proof.** The system (3.30) is forward complete, because its right side grows linearly in the state uniformly in \( t \). Also, the linear time invariant dynamics for the error
\[ Y_e = (\tilde{z} - \hat{z}, \tilde{w} - \hat{w}, \tilde{\xi} - \hat{\xi}) \]  
(3.31)

is uniformly globally exponentially stable to zero. This and the boundedness of the gradient of \( \mathcal{V}_2 \) in the state imply that the \((\tilde{z}, \tilde{w}, \tilde{\xi})\) dynamics satisfies the necessary
UGAS and ULES estimates, using an integral ISS argument. To get the integral ISS estimate, notice that the \((\tilde{z}, \tilde{w}, \tilde{\xi})\) subdynamics in (3.30) can be written as
\[
\left(\dot{\tilde{z}}, \dot{\tilde{w}}, \dot{\tilde{\xi}}\right) = F(t, \tilde{z}, \tilde{w}, \tilde{\xi}) + \left(0, \Delta u_{1,s} \sin(\xi_1), 0, -\Delta u_{1,s} \cos(\xi_1), 0, -\Delta u_{2,s}\right),
\]
where \(\Delta u_{i,s} = u_{i,s}(t, \tilde{z}, \tilde{w}, \tilde{\xi}) - u_{i,s}(t, \dot{\tilde{z}}, \dot{\tilde{w}}, \dot{\tilde{\xi}})\) for \(i = 1, 2\) and \(F(t, \tilde{z}, \tilde{w}, \tilde{\xi})\) is the right side of (3.4) in closed loop with the feedbacks \(u_{1,s}\) and \(u_{2,s}\) defined above.

We can find a positive definite function \(\alpha_\ast\) and a constant \(\bar{U} > 0\) such that \(\dot{\mathcal{V}}_2 \leq -\alpha_\ast(|(\tilde{z}, \tilde{w}, \tilde{\xi})|)\) along all trajectories of (3.4) for all \(t \geq t_o + \Gamma_2((\tilde{z}, \tilde{w}, \tilde{\xi})(t_o))\), and such that \(|\Delta u_{i,s}| \leq \bar{U}|\mathcal{V}_e|\) everywhere for \(i = 1, 2\), where \(\mathcal{V}_2\) and \(\Gamma_2\) are from the previous paragraph. Since \(\mathcal{V}_2\) has a uniformly bounded gradient in the state, we can then find a constant \(\bar{B} > 0\) such that
\[
\dot{\mathcal{V}}_2 \leq -\alpha_\ast(|(\tilde{z}, \tilde{w}, \tilde{\xi})|) + \bar{B}|\mathcal{V}_e(t)|
\]
along all trajectories of (3.30) for all times such that \(t \geq t_o + \Gamma_2((\tilde{z}, \tilde{w}, \tilde{\xi})(t_o))\). In fact, we can take \(\bar{B} = 3\bar{V}\bar{U}\) where \(\bar{V}\) is the uniform bound on the state gradient for \(\mathcal{V}_2\). Condition (3.33) is the standard integral ISS Lyapunov function decay condition except it is only required for large times. Since \(\mathcal{V}_e\) converges exponentially to zero, and since similar reasoning applies to the \((\tilde{z}, \tilde{w}, \tilde{\xi})\) dynamics, the result follows from standard arguments, which we summarize next.

Standard integral ISS arguments [5] construct functions \(\beta \in \mathcal{KL}\) and \(\gamma_1 \in \mathcal{K}_\infty\) such that \(\gamma_1 \left(|(\tilde{z}, \tilde{w}, \tilde{\xi})(t)|\right) \leq \beta \left(|(\tilde{z}, \tilde{w}, \tilde{\xi})(\xi)|, t - \xi \right) + 2\bar{B} \int_{t}^{\xi} |\mathcal{V}_e(r)| dr\) when \(t \geq \xi \geq t_o + \Gamma_2((\tilde{z}, \tilde{w}, \tilde{\xi})(t_o))\). Through a suitable choice of the constants \(k_i\) in (3.30), we can assume that \(|\mathcal{V}_e(t)| \leq |\mathcal{V}_e(t_o)| \exp(-(t - t_o))\) everywhere. Let \(\bar{S} > 0\) be any constant, and choose \(\Gamma_3 \in C^1 \cap \mathcal{K}_\infty\) depending on \(\bar{S}\) such that
\[
2\bar{B} \int_{t_o}^{t} |\mathcal{V}_e(r)| dr \leq 2\bar{B} \int_{t_o}^{\infty} |\mathcal{V}_e(r)| dr \leq 2\bar{B} \left[|\tilde{z}(t_o)| + |\tilde{w}(t_o)|\right] e^{-\bar{S} - t_o}
\]
(3.34)
if we fix $t = t_o + \Gamma_3(||(\tilde{z}, \tilde{w}, \tilde{\xi}, \hat{z}, \hat{w}, \hat{\xi})(t_o)||)$. By enlarging $\Gamma_3$ as needed without relabeling, we can assume that $\Gamma_3(||(\tilde{z}, \tilde{w}, \tilde{\xi}, \hat{z}, \hat{w}, \hat{\xi})(t_o)||) \geq \Gamma_2(||(\tilde{z}, \tilde{w}, \tilde{\xi})(t_o)||)$ everywhere. We can use Gronwall’s Inequality to find a function $H \in C^1 \cap K_\infty$ so that $\left|\left|(\tilde{z}, \tilde{w}, \tilde{\xi})(t)\right|\right| \leq H\left(\left|\left|(\tilde{z}, \tilde{w}, \tilde{\xi}, \hat{z}, \hat{w}, \hat{\xi})(t_o)\right|\right|\right)$, this provides a function $\Gamma_4 \in C^1 \cap K_\infty$ depending on $\bar{S}$ such that

\[ \beta\left(\left|\left|(\tilde{z}, \tilde{w}, \tilde{\xi})(t)\right|\right|, t - t_o \right) \leq \beta\left(H\left(\left|\left|(\tilde{z}, \tilde{w}, \tilde{\xi}, \hat{z}, \hat{w}, \hat{\xi})(t_o)\right|\right|\right), t - t_o \right) \leq \frac{\bar{S}}{2} \]  

(3.35)

when $t - t_o \geq \Gamma_4(||(\tilde{z}, \tilde{w}, \tilde{\xi}, \hat{z}, \hat{w}, \hat{\xi})(t_o)||)$. Combining (3.34) and (3.35) shows that $\gamma_1(||(\tilde{z}, \tilde{w}, \tilde{\xi})(t)||) \leq \bar{S}$ if $t - t_o = (t - t) + (t - t_o) \geq \Gamma_4(||(\tilde{z}, \tilde{w}, \tilde{\xi}, \hat{z}, \hat{w}, \hat{\xi})(t_o)||) + \Gamma_3(||(\tilde{z}, \tilde{w}, \tilde{\xi})(t_o)||)$, so the $(\tilde{z}, \tilde{w}, \tilde{\xi})$ subsystem satisfies the UGAS estimate. Similar arguments show that the $(\hat{z}, \hat{w}, \hat{\xi})$ subsystem of (3.30) satisfies the UGAS estimate. Finally, analyzing the local properties of (3.30) and recalling the ULES property from Theorem 3.2 shows that (3.30) is ULES. This uses the fact that ULES systems admit quadratic Lyapunov functions in a neighborhood of the origin [24].

\[ \square \]

### 3.8 Input-to-State Stability of Tracking Dynamics

We can also use our bounded backstepping theorem to show that the perturbed PVTOL error dynamics

\[
\begin{align*}
\dot{\hat{z}}_1 &= \hat{z}_2 \\
\dot{\hat{z}}_2 &= -[u_1 + \delta_1] \sin(\xi_1) + u_{1r}(t) \sin(\xi_{1r}(t)) \\
\dot{\hat{w}}_1 &= \hat{w}_2 \\
\dot{\hat{w}}_2 &= [u_1 + \delta_1] \cos(\xi_1) - u_{1r}(t) \cos(\xi_{1r}(t)) \\
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 \\
\dot{\hat{\xi}}_2 &= u_2 - u_{2r}(t) + \delta_2
\end{align*}
\]

(3.36)
with actuator errors $\delta_i$, in closed loop with the feedbacks we designed above, is ISS with respect to measurable essentially bounded actuator errors $\delta : [0, +\infty) \to \eta B_2$, where the feedback formulas must now depend on the bound $\eta$ on the disturbance. The argument is similar to the proof of Theorem 3.2 except with actuator errors added in the control channels. We can allow any bound $\eta$, through a proper choice of the feedbacks. We illustrate this robustness property in Section 3.10.

### 3.9 Trackable Reference Trajectories

Our Assumption 1 from p.33 holds for a broad class of reference trajectories and corresponding reference inputs, and so is not too restrictive. For example, assume that $(z_{1r}, w_{1r}) : [0, +\infty) \to \mathbb{R}^2$ is any bounded $C^4$ (but not necessarily periodic) function such that $\inf_{t \geq 0} [\ddot{w}_{1r}(t) + g] > 0$ and whose first four derivatives are globally bounded. Then Remark 3.1 from p.34 shows that the PVTOL reference dynamics (3.3) are satisfied with the reference inputs

$$u_{1r} = \sqrt{(\ddot{z}_{1r})^2 + (\ddot{w}_{1r} + g)^2} \quad \text{and} \quad u_{2r} = \dot{\xi}_{1r},$$

(3.37)

and with $\xi_{2r} = \dot{\xi}_{1r}$, $z_{2r} = \dot{z}_{1r}$, $w_{2r} = \dot{w}_{1r}$, and

$$\xi_{1r} = \arcsin \left( \frac{-\ddot{z}_{1r}}{\sqrt{(\ddot{z}_{1r})^2 + (\ddot{w}_{1r} + g)^2}} \right).$$

(3.38)

Also, $u_r \in C^2$ because $(z_{1r}, w_{1r}) \in C^4$. Therefore, our assumptions are satisfied by the corresponding reference trajectory $(z_{1r}, z_{2r}, w_{1r}, w_{2r}, \xi_{1r}, \xi_{2r}) : [0, +\infty) \to \mathbb{R}^6$. Positivity of $\ddot{w}_{1r}(t) + g$ holds for circular trajectories $(z_{1r}(t), w_{1r}(t)) = g_o(\bar{K} + \cos(t), \bar{K} + \sin(t))$ for any constants $\bar{K} \geq 1$ and $g_o \in (0, g)$, so we can track trajectories along these circles. In the next section, we illustrate this tracking in simulations.
3.10 Simulating the Tracking Dynamics

To illustrate our method, we ran several Mathematica simulations. We took the reference profile

\[
(z_{1r}(t), w_{1r}(t)) = 5(1.5 + \cos(t), 1.5 + \sin(t)) ,
\]

the coupling parameter \( \varepsilon = 1 \), and the actuator envelopes \( 4 \leq \bar{u}_1 \leq 16 \) and \(-10 \leq u_2 \leq 2 \). As we saw in the preceding section, the corresponding reference trajectory is obtained by taking \( z_{2r} = \dot{z}_{1r} \), \( w_{2r} = \dot{w}_{1r} \), \( \xi_{1r} \) as defined in (3.38), and \( \xi_{2r} = \dot{\xi}_{1r} \). The reference inputs are

\[
u_{1r} = \sqrt{\dot{z}_{1r}^2 + (\dot{w}_{1r} + 9.81)^2} \quad \text{and} \quad u_{2r} = \dot{\xi}_{1r} .
\]

They satisfy

\[
4.81 \leq \nu_{1r}(t) \leq 14.81 \quad \text{and} \quad |u_{2r}(t)| \leq 1.42781
\]

for all \( t \geq 0 \). Simple calculations show that the requirements from Section 3.4 are satisfied with \( \lambda = .266 \) and \( a = 10.14 \), so Theorem 3.2 gives UGAS and ULES of the corresponding PVTOL tracking error dynamics (3.11) in closed loop with the feedback (3.12).

Using the preceding data, we performed two simulations. First, we simulated (3.11) with the initial state

\[
(\tilde{z}_1(0), \tilde{z}_2(0), \tilde{w}_1(0), \tilde{w}_2(0), \varpi_1(0), \varpi_2(0)) = (0.31, 0.31, 0.21, 0.41, 0.41)
\]

at the initial time \( t_0 = 0 \), the disturbance \( \delta \equiv 0 \in \mathbb{R}^2 \), and the controller

\[
u_3 = \frac{-[1+172\bar{\eta}/a]\sigma_a \left( 2\varpi_2 + \sigma_a(\alpha \varpi_1) \varphi_a(\varpi_2) \right) - \sigma_a'(\alpha \varpi_1) \varphi_a'(\varpi_2) \left[ \varpi_2 - T_\lambda(t, \varpi_1, \tilde{z}, \tilde{w}) \right]}{\varpi_2 + \sigma_a(\alpha \varpi_1) \varphi_a'(\varpi_2)}
\]

corresponding to the disturbance bound \( \bar{\eta} = 0.5 \), in accordance with Section 3.8. In the following figures, we report our numerical results. In Figure 3.2, the reference
trajectory \((z_{1r}(t), w_{1r}(t))\) from (3.39) is blue and dashed, the simulated trajectory \((z_1(t), w_1(t))\) is red and solid, and the plot covers the tracking times \(t = 20\) to \(t = 50\); Figure 3.3 shows the trajectory for the roll angle \(\theta = \varpi_1 + \nu\) and Figure 3.4 shows the closed loop thrust input \(\bar{u}_1\) and the closed loop rolling moment control \(u_2\).

![Figure 3.2. PVTOL Center of Mass Tracking Without Disturbances](image)

**FIGURE 3.2.** PVTOL Center of Mass Tracking Without Disturbances
Blue and Dashed: Reference States \((z_{1r}(t), w_{1r}(t))\). Red and Solid: Closed Loop States \((z_1(t), w_1(t))\). Plot Covers Times \(t = 20\) to \(t = 50\).

![Figure 3.3. PVTOL Rolling Angle Tracking Without Disturbances](image)

**FIGURE 3.3.** PVTOL Rolling Angle Tracking Without Disturbances

![Figure 3.4. Closed Loop PVTOL Controls Without Disturbances](image)

**FIGURE 3.4.** Closed Loop PVTOL Controls Without Disturbances
Left: Thrust Control \(\bar{u}_1\). Right: Rolling Moment Control \(u_2\).
In Figure 3.5, we plot the convergence of the states $\tilde{z}_1, \tilde{z}_2, \tilde{w}_1, \tilde{w}_2, \varpi_1,$ and $\varpi_2$ from (3.11) to 0 without disturbances.

Our second simulation was done in the same way as our first, except we added the sinusoidal actuator error

$$\delta_2(t) = 0.25 \sin(t) \quad (3.44)$$
in the $u_3$ channel in (3.11) such that instead of $\dot{\omega}_2 = u_3$, we now have $\dot{\omega}_2 = u_3 + 0.25 \sin(t)$. The next figures show our numerical results under the actuator error. In Figure 3.6, we plot the tracking of $(z_{1r}(t), w_{1r}(t))$ over times $t = 20$ to $t = 50$, and Figure 3.7 shows the closed loop control values for $\tilde{u}_1$ and $u_2$.

![Figure 3.6. PVTOL Center of Mass Tracking With Disturbances. Blue and Dashed: Reference States $(z_{1r}(t), w_{1r}(t))$. Red and Solid: Closed Loop States $(z_1(t), w_1(t))$. Plot Covers Times $t = 20$ to $t = 50$.](image)

![Figure 3.7. PVTOL Controls With Disturbances Left: Thrust Control $\tilde{u}_1$. Right: Rolling Moment Control $u_2$.](image)

The plot for the rolling angle $\theta$ was similar to the one from our first simulation, and therefore is not shown. However, the corresponding trajectory components for (3.11) exhibited a sinusoidal motion that is similar to the disturbance (3.44). Figure 3.8 shows the corresponding trajectory components of (3.11).
Comparing the simulations for the undisturbed and disturbed cases illustrates how introducing the sinusoidal disturbance keeps the tracking errors from converging to zero, although our theory guarantees ISS properties with respect to $\delta_2$. Moreover, our controllers respect the prescribed actuator envelopes and therefore are physically viable.

**Remark 3.8.** Our simulations show how the thrust controller $\bar{u}_1 = u_1 + \varepsilon \xi_2^2$ is bounded away from zero, since $\bar{u}_1$ remains above 4. In fact, our assumptions from Section 3.3 are satisfied with $c_1 = 1.0359$ and $c_2 = 4.81$, so our choice $\lambda = 0.266$.
for the constant in the feedback formula (3.6) for \( u_1 \) and the choice \( b = 0.9 \) give 
\[ 0 < \lambda < \min\{1, bc_2 \cos(\pi/2-c_1)/14\}. \]
It follows from Remark 3.4 that the controller \( u_1 \) satisfies 
\[ u_1(t, \tilde{z}, \tilde{w}) \geq (1 - b)c_2 \cos(\pi/2 - c_1) = 0.480979 \]
on \([0, +\infty) \times \mathbb{R}^4\), so we have a guaranteed positive lower bound on the thrust \( \bar{u}_1 \) out of the bottom. By 
reducing \( \lambda \) further, we can satisfy 
\[ 0 < \lambda < \min\{1, bc_2 \cos(\pi/2-c_1)/14\} \]
for smaller values of \( b \) and thereby get much larger lower bounds for \( u_1 \). For example, with 
\( b = 0.05 \), the uniform lower bound on \( u_1 \) is 
\[ (1 - b)c_2 \cos(\pi/2 - c_1) = 4.569301. \]
This differs from [12], where there is no guaranteed positive lower bound on \( u_1 \).

**Remark 3.9.** We can also track along Cassini’s Oval [10]
\[
(z_{1r}(t), w_{1r}(t)) = R(t)(\cos(t), \sin(t)),
\]
where
\[
R(t) = \sqrt{a^2 \cos(2t) + b^4 - (a^2 \sin(2t))^2},
\]
for certain choices of the constants \( a_*>0 \) and \( b_* > a_* \) when we take the gravitational constant \( g = 9.81 \). For example, with the choices \( a_* = 2.65 \) and \( b_* = 2.9 \), Mathematica gives 
\[ \dot{w}_{1r}(t) + g \geq 0.552321 \] for all \( t \geq 0 \). It follows from our discussion from Section 3.9 that we can track reference trajectories with the center of mass profile (3.45) using the parameter values \( a_* = 2.65 \) and \( b_* = 2.9 \). See Figure 3.9 for a Mathematica plot of Cassini’s Oval for these values of the parameters.

**FIGURE 3.9.** Cassini’s Oval
Chapter 4
Lemmas on ISS and Trackability

We next provide key lemmas that we need in our analysis of our UAV tracking problems in the next chapter. The first gives general conditions under which an iISS Lyapunov function can be used to prove ISS under a suitably small bound on the admissible disturbances. Then we give several criteria that ensure that certain trajectories are trackable in our four state UAV model.

4.1 Using iISS Lyapunov Functions to Prove ISS

We again consider nonlinear systems

\[
\dot{X} = G(t, X, \delta(t)), \quad X \in \mathcal{X}
\]

under our assumptions from Chapter 1. As noted in Chapter 1, standard arguments [51] show that (4.1) is ISS (resp., integral ISS) when it admits an ISS (resp., integral ISS) Lyapunov function. Also, ISS implies integral ISS but not conversely. For example,

\[
\dot{X} = -\frac{X}{1 + X^2} + \delta
\]

with the state and disturbance set \( \mathcal{X} = \mathcal{D} = \mathbb{R} \) admits the integral ISS Lyapunov function \( V(X) = \ln(1 + X^2) \). However, it is not ISS, even if we restrict the disturbance set \( \mathcal{D} \), because for any constant disturbance \( \bar{\delta} \in (0, 0.5) \) the trajectory for the system starting at \( X_0 = 4/\bar{\delta} \) is unbounded. It is therefore natural to search for nondegeneracy conditions on an iISS Lyapunov function for a system of the form
(4.1) that ensure that (4.1) is also ISS with respect to disturbances of sufficiently small magnitude. The following lemma provides such conditions:

**Lemma 4.1.** Assume that (4.1) is integral ISS for $X = \mathbb{R}^n$ and $D = \mathbb{R}^m$, and that there exist an integral ISS Lyapunov function $V$, a positive definite function $\alpha$, a function $\gamma \in K_\infty$, and constants $\rho_0 > 0$ and $\rho_* > 0$ such that:

(a) $\dot{V} \leq -\alpha(X) + \gamma(|\delta|)$ along all trajectories of (4.1) and

(b) $\alpha(X) > \rho_*$ for all $X \in \mathbb{R}^n \setminus \rho_0 B_n$.

Then for each constant $\lambda \in (0, 1)$, the system (4.1) is ISS with respect to disturbances valued in $\gamma^{-1}(\lambda \rho_*) B_m$. \hfill \Box

**Proof.** Fix any constant $\lambda \in (0, 1)$, and set $\delta_M = \gamma^{-1}(\lambda \rho_*)$. Pick $\underline{\alpha}, \bar{\alpha} \in K_\infty$ such that $\underline{\alpha}(|X|) \leq V(t, X) \leq \bar{\alpha}(|X|)$ for all $t \geq 0$ and $X \in \mathbb{R}^n$. Our assumptions provide constants $\rho_i > 0$ and a function $\alpha_0 \in K_\infty$ such that if $|\delta|_\infty \leq \delta_M$, then:

(i) $\dot{V} \leq -\rho_2$ whenever $V(t, X) \geq \rho_1$ and (ii) $\dot{V} \leq -\alpha_0(|X|) + \gamma(|\delta|)$ whenever $V(t, X) \leq \rho_1$. For example, we can satisfy the requirements by choosing $\rho_1 = \bar{\alpha}(\rho_0)$, $\rho_2 = (1 - \lambda)\rho_*$, and

$$
\alpha_0(r) = \frac{r}{1 + \frac{r}{\bar{\alpha}(\rho_0)}} \min \left\{ \alpha(p) : \min \{r, \bar{\alpha}^{-1}(\rho_1)\} \leq |p| \leq \bar{\alpha}^{-1}(\rho_1) \right\},
$$

(4.3)

because if $V(t, X) \geq \rho_1$, then $\bar{\alpha}(|X|) \geq \bar{\alpha}(\rho_0)$ and then condition (b) applies. Set $T(r) = \underline{\alpha}(r)/\rho_2$, and take any trajectory $X(t)$ of (4.1) with any $\delta \in M_{[-\delta_M, \delta_M]}$ for any initial time $t_0 \geq 0$. If $V(t_0, X(t_0)) \geq \rho_1$, then $V(t, X(t)) \leq \rho_1$ for all $t \geq t_0 + T(|X(t_0)|)$, because $V(t, X(t)) \leq V(t_0, X(t_0)) - \rho_2(t - t_0)$ as long as $t \geq t_0$ is such that $V(t, X(t)) \geq \rho_1$ and because $V$ is nonnegative valued. (We used the fact that if $V(t_s, X(t_s)) \leq \rho_1$ for some $t_s \geq 0$, then condition (i) gives $V(t, X(t)) \leq \rho_1$ for all $t \geq t_s$.)
Condition (ii) and standard ISS arguments [51] now allow us to construct functions \( \beta_s \in \mathcal{K}\mathcal{L} \) and \( \alpha_5 \in \mathcal{K}_\infty \) such that \(|X(t)| \leq \beta_s(|X(t)|, t-t) + \alpha_5(|\delta|_{[t,t]})\) for all \( t \geq t \geq 0 \), all \( \delta \in \mathcal{M}_{[-\delta_M,\delta_M]} \), and all \( t \geq 0 \) such that \( V(t, X(t)) \leq \rho_1 \). Hence, if \( V(t_0, X(t_0)) \leq \rho_1 \), then \(|X(t)| \leq \beta_s(|X(t_0)|, t-t_0) + \alpha_5(|\delta|_{[t_0,t]})\) for all \( t \geq t_0 \).

If \( V(t_0, X(t_0)) \geq \rho_1 \) and \( t \geq t_0 + T(|X(t_0)|) \), then take \( t = t_0 + T(|X(t_0)|) \) to get
\[
|X(t)| \leq \beta_s(\alpha^{-1}(\bar{\alpha}(|X(t_0)|)), t-t_0 - T(|X(t_0)|)) + \alpha_5(|\delta|_{[t_0,t]})\), because \( V(t, X(t)) \leq V(t_0, X(t_0)) \) for all \( t \geq t_0 \). If \( V(t_0, X(t_0)) \geq \rho_1 \) and \( t-t_0 \in [0, T(|X(t_0)|)] \), then we have
\[
|X(t)| \leq \alpha^{-1}(\bar{\alpha}(|X(t_0)|))\exp(T(|X(t_0)|) - t + t_0)). 
\tag{4.4}
\]

Combining all three cases gives
\[
|X(t)| \leq \beta_\sharp(|X(t_0)|, t-t_0) + \alpha_5(|\delta|_{[t_0,t]}) \tag{4.5}
\]
along all trajectories of (4.1) with \( \delta \in \mathcal{M}_{[-\delta_M,\delta_M]} \), where
\[
\beta_\sharp(s,t) = \beta_s(s,t) + \beta_s(\alpha^{-1}(\bar{\alpha}(s)), \max\{0, t-T(s)\}) + \alpha^{-1} \left( \bar{\alpha}(s)\exp(T(s) - t) \right)
\]
is \( \mathcal{K}\mathcal{L} \). Therefore, (4.5) is the desired ISS estimate.

\[\Box\]

4.2 Sufficient Conditions for Trackability

Recall from Section 1.4.2 that the four state UAV model is
\[
\begin{cases}
\dot{x} = v \cos(\theta) \\
\dot{y} = v \sin(\theta) \\
\dot{\theta} = \alpha_\theta(\theta_c - \theta + \Delta) \\
\dot{v} = \alpha_\nu(v_c - v + \delta)
\end{cases} \tag{4.6}
\]
where \( \delta \) and \( \Delta \) are uncertainties and \( \theta_c \) and \( v_c \) are the controllers we are to design.

The states are the center of mass \((x, y)\), the heading angle \( \theta \), and the velocity \( v \). The
positive constants $\alpha_v$ and $\alpha_\theta$ are associated with the autopilots. One of our aims is to design controllers such that the tracking dynamics for (4.6) for suitable reference trajectories enjoys ISS properties with respect to $(\delta, \Delta)$. In this dissertation, we are interested in tracking UAV trajectories that satisfy the following conditions:

**Assumption 2.** The $C^2$ function $R_\ast = (x_\ast, y_\ast, \theta_\ast, v_\ast) : \mathbb{R} \to \mathbb{R}^3 \times (0, +\infty)$ is such that (A) $x_\ast, y_\ast, \dot{\theta}_\ast, \ddot{\theta}_\ast, v_\ast, \dot{v}_\ast$ are bounded, (B) $\dot{x}_\ast(t) = v_\ast(t) \cos(\theta_\ast(t))$ and $\dot{y}_\ast(t) = v_\ast(t) \sin(\theta_\ast(t))$ hold for all $t \in \mathbb{R}$, and (C) there is a constant $c > 0$ such that

$$\min \{\inf \{v_\ast(t) : t \in \mathbb{R}\}, \inf \{v_\ast(t) + \dot{v}_\ast(t)/\alpha_v : t \in \mathbb{R}\} \} \geq c. \quad \square$$

Condition (C) combines the no-stall condition that $v_\ast$ has a uniform positive lower bound with a nondegeneracy condition on $v_\ast(t) + \dot{v}_\ast(t)/\alpha_v$ which will be needed to design velocity controllers with uniform positive lower bounds. We postpone the design of the controllers until the next chapter. Instead, we use the rest of this chapter to explore the set of all reference trajectories $R_\ast = (x_\ast, y_\ast, \theta_\ast, v_\ast)$ that satisfy Assumption 2. We will see show how Assumption 2 holds for many standard and more complex figures that prevail in real UAV applications.

To this end, we first give a useful preliminary result. Given any reference trajectory $R_\ast = (x_\ast, y_\ast, \theta_\ast, v_\ast)$ satisfying Assumption 2, we can easily express $\theta_\ast(t)$ and $v_\ast(t)$ in terms of $x_\ast(t)$ and $y_\ast(t)$, using the relations

$$\dot{x}_\ast(t) = v_\ast(t) \cos(\theta_\ast(t)), \quad \dot{y}_\ast(t) = v_\ast(t) \sin(\theta_\ast(t)) \quad (4.7)$$

from Assumption 2(B). In fact, we can square both equations in (4.7) and sum the results and take square roots to get $v_\ast$. Also, if we differentiate both sides of the equations in (4.7), then multiply the new first equation by $-\sin(\theta_\ast(t))$ and the new second equation by $\cos(\theta_\ast(t))$, then add the results, and then substitute in $\cos(\theta_\ast(t)) = \dot{x}_\ast(t)/v_\ast(t)$ and $\sin(\theta_\ast(t)) = \dot{y}_\ast(t)/v_\ast(t)$, then we can solve for $\dot{\theta}_\ast(t)$. 

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Doing so gives \( \dot{\theta}_* = \frac{\hat{x}_* \ddot{y}_* - \hat{y}_* \ddot{x}_*}{v_*^2} \) and therefore also

\[
v_*(t) = \sqrt{[\dot{x}_*(t)]^2 + [\dot{y}_*(t)]^2} \quad \text{and} \quad (4.8a)
\]

\[
\theta_*(t) = \pm \arccos \left[ \frac{\dot{x}_*(0)}{v_*(0)} \right] + \int_0^t \frac{1}{v_*^2(s)} [\dot{x}_*(s) \ddot{y}_*(s) - \dot{y}_*(s) \ddot{x}_*(s)] ds \quad (4.8b)
\]

where the “+” (respectively, “−”) is used when \( \dot{y}_*(0) \) is nonnegative (respectively, negative), although we can add any integer multiple of \( 2\pi \) to \( \theta_*(t) \) to get other solutions \( \theta_*(t) \). Conversely, given any \( C^3 \) function \( (x_*, y_*) : \mathbb{R} \to \mathbb{R}^2 \) for which \( v_*(t) = \sqrt{[\dot{x}_*(t)]^2 + [\dot{y}_*(t)]^2} \) satisfies part (C) of Assumption 2, we can show that the formulas from (4.8a)-(4.8b) satisfy (4.7) for all \( t \in \mathbb{R} \). This gives the following sufficient conditions for reference paths \( (x_*, y_*) : \mathbb{R} \to \mathbb{R}^2 \) to be the first two components of trackable UAV reference trajectories:

**Proposition 4.2.** Let \( (x_*, y_*) : \mathbb{R} \to \mathbb{R}^2 \) be any bounded \( C^3 \) function whose first three derivatives are bounded, and define \( v_* \) as in (4.8a). If part (C) of Assumption 2 holds, then \( (x_*, y_*, \theta_*, v_*) \) with the choices (4.8a)-(4.8b) satisfies Assumption 2.

\[ \square \]

**Proof.** It suffices to verify (4.7) for all \( t \in \mathbb{R} \). For all real values \( \bar{t} \) and \( c_* \) for which \((\dot{x}_*(\bar{t}), \dot{y}_*(\bar{t})) = v_*(\bar{t})(\cos(c_*), \sin(c_*))\), the Implicit Function Theorem, applied to the function \( \mathcal{G}(t, \lambda) = (\dot{x}_*(t) - v_*(t) \cos(\lambda), \dot{y}_*(t) - v_*(t) \sin(\lambda)) \), gives an open interval \( \mathcal{I}_{\bar{t}} \) and a \( C^1 \) function \( \lambda_* : \mathcal{I}_{\bar{t}} \to \mathbb{R} \) such that \( \lambda_*(\bar{t}) = c_* \) and

\[
\dot{x}_*(t) = v_*(t) \cos(\lambda_*(t)) \quad \text{and} \quad \dot{y}_*(t) = v_*(t) \sin(\lambda_*(t)) \quad (4.9)
\]

hold for all \( t \in \mathcal{I}_{\bar{t}} \). Solving for \( \lambda_* \) as above (except with \( \dot{\theta}_* \) replaced by \( \lambda_* \)) gives

\[
\dot{\lambda}_*(t) = [\dot{x}_*(t) \ddot{y}_*(t) - \dot{y}_*(t) \ddot{x}_*(t)] / v_*^2(t) \quad \text{for each} \quad t \in \mathcal{I}_{\bar{t}}.
\]

Hence, if we have a \( C^1 \) solution \( \lambda_* : [0, t_{max}) \to \mathbb{R} \) for (4.9) such that \( \lambda_*(0) = \pm \arccos[\dot{x}_*(0)/v_*(0)] \), defined up to some maximal time \( t_{max} > 0 \), then it agrees with the formula for \( \theta_* \) from (4.8b) on
If $t_{\text{max}} < +\infty$, then the Implicit Function Theorem gives a constant $\varepsilon > 0$ and a $C^1$ solution $\Gamma_* : (t_{\text{max}} - \varepsilon, t_{\text{max}} + \varepsilon) \to \mathbb{R}$ of the system $\dot{x}_*(t) = v_*(t) \cos(\Gamma_*(t))$ and $\dot{y}_*(t) = v_*(t) \sin(\Gamma_*(t))$ that satisfies $\lambda_*(t_{\text{max}}) = \Gamma_*(t_{\text{max}})$. Solving for $\dot{\Gamma}_*$ on $(t_{\text{max}} - \varepsilon, t_{\text{max}} + \varepsilon)$ as we did for $\dot{\lambda}_*$ gives

$$
\dot{\lambda}_*(t) = \frac{\dot{x}_*(t)\dot{y}_*(t) - \dot{y}_*(t)\dot{x}_*(t)}{v_*^2(t)} = \dot{\Gamma}_*(t)
$$

(4.10)
on $(t_{\text{max}} - \varepsilon, t_{\text{max}})$. Hence, $\lambda_*(t) = \Gamma_*(t)$ on $(t_{\text{max}} - \varepsilon, t_{\text{max}})$, so we can extend the solution $\lambda_*$ of (4.9) to $[0, t_{\text{max}} + \varepsilon)$, contradicting maximality of $t_{\text{max}}$. Similar arguments apply for negative times. Hence, we have a solution of (4.9) on $\mathbb{R}$ that agrees with $\theta_*$, so (4.8a)-(4.8b) satisfy (4.7) for all $t \in \mathbb{R}$.

4.3 Tracking Circles and Figure 8’s

Let $\alpha_v > 0$ be the autopilot constant from our UAV dynamics (4.6). The following result on trackability of ellipses is an easy consequence of Proposition 4.2:

**Proposition 4.3.** Let $a > 0$ and $b > 0$ be any constants such that

$$\alpha_v > \frac{|b^2 - a^2|}{\min\{a^2, b^2\}}.$$  

(4.11)

Let $c_x \in \mathbb{R}$ and $c_y \in \mathbb{R}$ be any constants and choose the elliptical trajectory

$$(x_*, y_*)(t) = (c_x, c_y) + (a \cos(t), -b \sin(t)).$$

(4.12)

Then $(x_*, y_*, \theta_*, v_*)$, with $v_*$ and $\theta_*$ given by (4.8a)-(4.8b), satisfies Assumption 2.

\[Q.E.D.]

**Proof.** The inequalities $v_*(t) = \sqrt{[\dot{x}_*(t)]^2 + [\dot{y}_*(t)]^2} \geq \min\{a, b\}$ and

$$v_*(t) + \frac{\dot{v}_*(t)}{\alpha_v} \geq \frac{1}{v_*^2(t)}[a^2 \sin^2(t) + b^2 \cos^2(t) + \frac{1}{\alpha_v}(a^2 - b^2) \sin(t) \cos(t)]$$

$$\geq \frac{1}{\max\{a, b\}}[\min\{a^2, b^2\} - \frac{1}{\alpha_v}|b^2 - a^2|]$$

(4.13)

$$> 0$$

hold for all $t \in \mathbb{R}$, by (4.11). Hence, the result follows from Proposition 4.2. \[Q.E.D.]
Proposition 4.4. Let $d > 1/4$ be any constant. Assume that
\[ \alpha_v > \frac{16d(4d + 1)}{8d - 1}. \] (4.14)

Choose
\[ (x_*, y_*)(t) = (\sqrt{d}\cos(t), d\cos(t)\sin(t)). \] (4.15)

Then $(x_*, y_*, \theta_*, v_*)$, with $v_*$ and $\theta_*$ given by (4.8a)-(4.8b), satisfies Assumption 2.
\[ \square \]

Proof. On the interval $[0, 1]$, the polynomial $Q(p) = 4d^2p^2 + (d - 4d^2)p + d^2$ has the unique minimum
\[ Q \left( \frac{4d - 1}{8d} \right) = \frac{8d - 1}{16}. \] (4.16)

Hence,
\[ v_*^2(t) = 4d^2\sin^4(t) + (d - 4d^2)\sin^2(t) + d^2 \geq \frac{8d - 1}{16} \] (4.17)
for all $t \in \mathbb{R}$. Also,
\[ v_*(t) + \frac{\dot{v}_*(t)}{\alpha_v} \geq \frac{1}{\alpha_v v_*(t)} \left\{ \alpha_v [4d^2\sin^4(t) + (d - 4d^2)\sin^2(t) + d^2] \\ + 8d^2\sin^3(t)\cos(t) + (d - 4d^2)\sin(t)\cos(t) \right\} \\ \geq \frac{1}{\alpha_v v_*(t)} \left\{ \frac{\alpha_v(8d-1)}{16} - |4d^2(2\sin^2(t) - 1) + d| \right\} \\ \geq \frac{1}{\alpha_v v_*(t)} \left\{ \frac{\alpha_v(8d-1)}{16} - d(4d + 1) \right\}, \] (4.18)
which has a uniform positive lower bound over $\mathbb{R}$, by our lower bound assumption (4.14) on $\alpha_v$. The result now follows from Proposition 4.2 as in Proposition 4.3. \[ \square \]

Proposition 4.3 covers all circles of any radius $r > 0$ for all choices of $\alpha_v$, by taking $a = b = r$. By enlarging $a$, $b$, and $d$ in Propositions 4.3-4.4, we get arbitrarily large elliptical and figure eight paths that lie along the relations
\[ \left[ \frac{x - c_x}{a} \right]^2 + \left[ \frac{y - c_y}{b} \right]^2 = 1 \quad \text{and} \quad y^2 = x^2(d - x^2). \] (4.19)
Then Propositions 4.3-4.4 give lower bounds on the values of the autopilot constant $\alpha_v$ for which the ellipses and figure eights are trackable. See Figure 4.1 for trackable figure 8’s for different values of the parameter $d$.

**FIGURE 4.1.** Trackable Figure 8’s
Plots of $(x_*, y_*)(t) = (\sqrt{d}\cos(t), d\cos(t)\sin(t))$ for $d = 2$ (Red and Solid), $d = 6$ (Blue and Dashed) and $d = 10$ (Black and Dotted)

### 4.4 Tracking Bounded Trajectories and Swirls

Our work applies to much more complex trackable reference trajectories as well. For example, one can also find bounded trajectories satisfying Assumption 2, as follows. For simplicity, we first take $v_*(t) \equiv 10$. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be any odd $C^2$ function that admits a constant $c_\ast \in (\pi/2, \pi)$ such that $\lim_{s \to +\infty} \sigma(s) = c_\ast$, fix any constant $R > 0$, and consider the functions $F_k(t) = \sigma(k \sin(Rt))$ and $I(k) = \int_0^{\pi/R} \cos(F_k(m))dm$ parameterized by constants $k \geq 0$. Then $I(0) = \pi/R$, and the Lebesgue Dominated Convergence Theorem gives $\lim_{k \to +\infty} I(k) = \pi \cos(c_\ast)/R < 0$. Since $k \mapsto I(k)$ is continuous, the Intermediate Value Theorem gives a constant $\bar{g} > 0$ such that $I(\bar{g}) = 0$. Take $\theta_\ast = F_{\bar{g}}$. Since $\theta_\ast$ is odd, we get

$$\int_{-\pi/R}^{\pi/R} v_*(t)\sin(\theta_*(t))dt = \int_{-\pi/R}^{\pi/R} v_*(t)\cos(\theta_*(t))dt = 0.$$ (4.20)
Hence, \( x_*(t) = \int_0^t v_*(s) \cos(\theta_*(s)) \, ds \) and \( y_*(t) = \int_0^t v_*(s) \sin(\theta_*(s)) \, ds \) are bounded, by the \( 2\pi/R \) periodicity of their integrands, so we get a bounded pair \((x_*, y_*)\) satisfying requirement (B) from Assumption 2. If, in addition, \( \sigma' \) and \( \sigma'' \) are bounded, then the corresponding bounded reference trajectory \( \mathcal{R}_* = (x_*, y_*, \theta_*, 10) \) satisfies all requirements from Assumption 2. We can use numerical methods such as bisection to solve for \( g \). For example, if we take \( R = 0.24 \) and \( \sigma(s) = 1.5 \arctan(s) \), then \( \int_{-\pi/R}^{\pi/R} \sin(\theta_*(t)) \, dt = \int_{-\pi/R}^{\pi/R} \cos(\theta_*(t)) \, dt = 0 \) when \( \bar{g} = 3.38321412225 \), and all of our requirements are met.

Here is a different construction of a bounded reference trajectory \( \mathcal{R}_* \) that satisfies Assumption 2. Take

\[
C(t) = \frac{1 - kt^6(t - \pi)^6}{1 + kt^6(t - \pi)^6},
\]

(4.21)

where \( k = 0.040905 \) is chosen such that \( \int_0^\pi C(t) \, dt = 0 \), and define \( \theta_* : [0, \pi] \rightarrow \mathbb{R} \) by \( \theta_*(t) = \arccos(C(t)) \). We extend \( \theta_* \) to \( \mathbb{R} \) by requiring it to be odd and have period \( 2\pi \). This extension, which we also call \( \theta_* \), is easily shown to be \( C^2 \), by checking that its one-sided first and second derivatives are 0 at all integer multiples of \( \pi \). Moreover, it satisfies (4.20) when we pick \( R = 1 \) and \( v_*(t) \equiv 10 \), so the arguments above show that the corresponding trajectory \( \mathcal{R}_* \) is bounded. Also, condition (C) from Assumption 2 holds, so all of our requirements are met. The corresponding trajectory \((x_*(t), y_*(t)) = (\int_0^t v_*(s) \cos(\theta_*(s)) \, ds, \int_0^t v_*(s) \sin(\theta_*(s)) \, ds)\) does a figure eight.

Finally, taking nonconstant velocities \( v_* \) gives more complex reference trajectories where the path for \((x_*, y_*)\) is neither an ellipse nor a figure eight. For example, take \( v_*(t) = 20 - 10 \cos^2(0.2t), \theta_*(t) = t \), and the autopilot constant \( \alpha_v = 0.192 \) from [43]. Then Assumption 2 holds with \( x_*(t) = -4.16667 \sin(0.6t) + 15 \sin(t) - 1.78571 \sin(1.4t) \) and \( y_*(t) = 9.04762 + 4.16667 \cos(0.6t) - 15 \cos(t) + \)

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1.78571 \cos(1.4t). Figure 4.2 gives a sketch of this “swirl” reference position trajectory, using Mathematica.

\[ x_*(t) = -4.16667 \sin(0.6t) + 15 \sin(t) - 1.78571 \sin(1.4t) \]
\[ y_*(t) = 9.04762 + 4.16667 \cos(0.6t) - 15 \cos(t) + 1.78571 \cos(1.4t) \]

In the next chapter, we use some of these observations to illustrate our tracking results.
Chapter 5
Tracking for the UAV Model

In this chapter, we design tracking controllers for the UAV model from Section 1.4.2. We show how to track trajectories satisfying our Assumption 2 from Section 4.2 while satisfying several important constraints. These constraints are (a) admissible ranges on the controller values, (b) admissible ranges for the command rates, which are the time derivatives of the controllers along the closed loop trajectories, and (c) bounds on the heading angle rate that are relevant for UAVs operating under coordinated turning conditions. As we saw in the preceding chapter, Assumption 2 holds for many trajectories. However, it is far from clear how to design the tracking controllers to ensure ISS of the UAV tracking dynamics with respect to additive uncertainty on the controllers, under our constraints. We will overcome this challenge using a strictification of a nonstrict Lyapunov function. We begin with some background on UAV models and UAV control problems.

5.1 Literature Review

The constrained nonlinear tracking control problem for fixed wing small UAV is a challenging topic that is of continuing ongoing research interest [2, 21, 25, 43]. The constraints stem from the positive lower and upper bounds on the velocity (which are related to the airspeed) and saturation constraints on the heading rate (which come from restrictions on the pitch rate and roll angle). While the UAV dynamics are related to those of nonholonomic mobile robots, standard mobile robot tracking designs, such as those of [23], do not apply because the UAV velocity must remain positive [43].
As in [40, 43], we assume that the UAVs have standard autopilots, so the models are first order for heading and Mach hold and second order for altitude hold. As we saw in Section 1.4.2, this gives the important benchmark model [40]

\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \alpha_\theta (\theta_c - \theta) \\
\dot{v} &= \alpha_v (v_c - v + \delta)
\end{align*}
\] (5.1)

where we omit the altitude subdynamics \( \ddot{h} = -\alpha_h \dot{h} + \alpha_h (h^c - h) \) since altitude controllers \( h^c \) are available [7]. As before, \((x, y)\) is the position of the UAV with respect to an inertial coordinate system, \( \theta \) is the heading (course) angle, the ground speed \( v \) is the inertial velocity, \( \alpha_\theta \) and \( \alpha_v \) are positive constants associated with the autopilot, the controllers \( \theta_c \) and \( v_c \) are to be determined, and the unknown perturbation \( \delta \) can be expected under model uncertainty [43] or actuator errors. For simplicity, we only added uncertainty to the velocity controller, but see Section 5.6 for extensions to cases where there is additive uncertainty on both controls.

The paper [40] was one of the first works on close formation flight control, and more complex UAV models now exist. However, the underactuated kino-dynamic representation (5.1) is justifiable for high-level formation flight control of UAVs and therefore is of considerable importance [43].

When \( \alpha_v \) in (5.1) is large relative to \( \alpha_\theta \) and \( \delta \) is negligible, \( v \) converges to \( v_c \) quickly relative to the total response time, and then one can consider the three dimensional reduced dynamics for \((x, y, \theta)\) obtained by setting \( v \equiv v_c \) in (5.1) and dropping the velocity dynamics [2, 43]. There are bounded tracking controllers available for this reduced model. For example, [43] proves the key input-to-state stability (ISS) property with respect to additive uncertainty on the controls, lead-
ing to velocity controllers for the reduced model with positive upper and lower bounds. Also, [43] experimentally validated controllers for the three state model, by simulating a more complex UAV model for transitioning through targets under multiple dynamic threats. See also [2] for bounds on both controls for the reduced model and exponential stability for teams of UAVs [9, 16]. Many other methods have been proposed for UAVs, e.g., cooperative games, differential flatness, and linearization; see [19, 41, 47, 53], which include pursuit of targets and collision avoidance. However, the theoretical analysis in [2, 43] is very specific to the three state model, and to the best of our knowledge, there are no known tracking controllers that respect given amplitude and command rate or state dependent constraints and achieve ISS or integral ISS with respect to uncertainties for general classes of reference trajectories in the important model (5.1). This necessitates our ISS analysis for (5.1).

In this chapter, we build controllers $\theta_c$ and $v_c$ for (5.1) that apply for all values of $\alpha_{\theta}$ and $\alpha_v$ and all reference trajectories satisfying Assumption 2. Unlike the existing results for UAV models, we use ‘strictification’ [29, 34]. This is a Matrosov approach for transforming a nonstrict Lyapunov function for the tracking dynamics into a strict Lyapunov function, which then gives ISS under suitable restrictions on the magnitude of the disturbances. Our work is primarily focused on a methodological and mathematical development, rather than being focused on a specific real-world UAV application or experiments. However, three important features of our controllers are that they (a) fulfill amplitude and rate constraints, including positive lower bounds on $v_c$ which arise from the physical constraints of the aircraft, (b) give integral ISS or ISS with respect to additive uncertainties on the controls under appropriate restrictions on their sup norms, and (c) can track a wide class of reference trajectories for which a suitable weighted sum of the ref-
erence velocities and accelerations satisfies a mild nondegeneracy condition. The command amplitude and rate constraint sets for $v_c$ are intervals $[\underline{v}_a, \overline{v}_a]$ and $[\underline{v}_r, \overline{v}_r]$ respectively with constant endpoints and $\underline{v}_a > 0$, and similarly for $\theta_c$ except $\theta_c$ will not be bounded unless the reference angle $\theta_*(t)$ is a bounded function and $\theta_c$ has no sign constraint; see Section 5.5 for more details on our command amplitude and rate constraints, as well as results for state dependent constraints. Moreover, our simulations will illustrate good controller performance. Therefore, our work has significant theoretical novelty.

5.2 Tracking Dynamics

In this section, we obtain the tracking dynamics corresponding to (5.1) and reference trajectories satisfying Assumption 2. For the convenience of the reader, we repeat the assumption here:

\textbf{Assumption 2.} The $C^2$ function $R_*(x_*, y_*, \theta_*, v_*) : \mathbb{R} \to \mathbb{R}^3 \times (0, +\infty)$ is such that (A) $x_*$, $y_*$, $\dot{\theta}_*$, $\ddot{\theta}_*$, $v_*$, and $\dot{v}_*$ are bounded, (B) $\dot{x}_*(t) = v_*(t) \cos(\theta_*(t))$ and $\dot{y}_*(t) = v_*(t) \sin(\theta_*(t))$ hold for all $t \in \mathbb{R}$, and (C) there is a constant $c > 0$ such that $\min\{\inf\{v_*(t) : t \in \mathbb{R}\}, \inf\{v_*(t) + \dot{v}_*(t)/\alpha_v : t \in \mathbb{R}\}\} \geq c$. \hfill \Box

It is convenient to use the new coordinates $\psi = -\sin(\theta)x + \cos(\theta)y$ and $\xi = \cos(\theta)x + \sin(\theta)y$ to transform (5.1) into

\[
\begin{align*}
\dot{\psi} &= -\alpha_\theta \xi (\theta_c - \theta) \\
\dot{\xi} &= \alpha_\theta \psi (\theta_c - \theta) + v \\
\dot{\theta} &= \alpha_\theta (\theta_c - \theta) \\
\dot{v} &= \alpha_v (v_c - v + \delta)
\end{align*}
\] (5.2)
We also take \( \psi_* = -\sin(\theta_*)x_* + \cos(\theta_*)y_* \), \( \xi_* = \cos(\theta_*)x_* + \sin(\theta_*)y_* \), and the changes of feedbacks

\[
\begin{align*}
v_c(t, S) &= v_N(S - (\psi_*, \xi_*, \theta_*, v_*)) + v_*(t) + \frac{\dot{v}_*(t)}{\alpha_v} \quad \text{and} \\
\theta_c(t, S) &= \theta_N(t, S - (\psi_*, \xi_*, \theta_*, v_*)) + \theta_*(t) + \frac{\dot{\theta}_*(t)}{\alpha_\theta},
\end{align*}
\]

where \( S = (\psi, \xi, \theta, v) \), and where the new controls \( v_N \) and \( \theta_N \) will be constructed such that \( v_N(0) = \theta_N(t, 0) = 0 \) for all \( t \in \mathbb{R} \); see (5.9). Recalling part (B) of Assumption 2 gives \( \dot{\psi}_*(t) = -\dot{\theta}_*(t)\xi_*(t) \) and \( \dot{\xi}_*(t) = \dot{\theta}_*(t)\psi_*(t) + v_*(t) \) for all \( t \in \mathbb{R} \). Taking the tracking variables \( \tilde{\psi} = \psi - \psi_*(t) \), \( \tilde{\xi} = \xi - \xi_*(t) \), \( \tilde{\theta} = \theta - \theta_*(t) \), and \( \tilde{v} = v - v_*(t) \), it follows that the dynamics of the tracking error \( E = (\tilde{\psi}, \tilde{\xi}, \tilde{\theta}, \tilde{v}) \) are

\[
\begin{align*}
\dot{\tilde{\psi}} &= -\dot{\theta}_*(t)\tilde{\xi} + \alpha_\theta[\tilde{\xi} + \xi_*(t)][\tilde{\theta} - \theta_N], \\
\dot{\tilde{\xi}} &= \dot{\theta}_*(t)\tilde{\psi} + \tilde{v} - \alpha_\theta[\tilde{\psi} + \psi_*(t)][\tilde{\theta} - \theta_N] \\
\dot{\tilde{\theta}} &= \alpha_\theta(-\tilde{\theta} + \theta_N) \\
\dot{\tilde{v}} &= \alpha_v(-\tilde{v} + v_N + \delta).
\end{align*}
\]

Hence, we can achieve all of our tracking objectives by designing the new controllers \( \theta_N \) and \( v_N \) for (5.4).

### 5.3 Persistency of Excitation

The following consequence of Assumption 2 will be key to our strictification procedure:

**Lemma 5.1.** If \( \mathcal{R}_* = (x_*, y_*, \theta_*, v_*) : \mathbb{R} \to \mathbb{R}^3 \times (0, +\infty) \) satisfies Assumption 2, then there exist constants \( c_0 > 0 \) and \( T > 0 \) such that

\[
\int_t^{t+T} [\dot{\theta}_*(s)]^2 ds \geq c_0
\]

for all \( t \in \mathbb{R} \).

**Proof.** We prove the lemma by contradiction. Suppose that there were no constants \( c_0 > 0 \) and \( T > 0 \) such that \( \int_t^{t+T} [\dot{\theta}_*(s)]^2 ds \geq c_0 \) for all \( t \in \mathbb{R} \). Then for each \( p \in \mathbb{N} \),
we could find a \( t_p \in \mathbb{R} \) such that \( \int_{t_p}^{t_p+p} \left[ \dot{\theta}_*(s) \right]^2 \, ds \leq \frac{1}{p^4} \). Hence, for all \( p \in \mathbb{N} \) and \( s_p \in [t_p, t_p+p] \), Jensen’s Inequality gives

\[
\left( \theta_*(s_p) - \theta_*(t_p) \right)^2 = \left[ \int_{t_p}^{s_p} \dot{\theta}_*(s) \, ds \right]^2 \leq \left[ \int_{t_p}^{t_p+p} |\dot{\theta}_*(s)| \, ds \right]^2 \leq p \int_{t_p}^{t_p+p} |\dot{\theta}_*(s)|^2 \, ds \leq \frac{1}{p^4}. \tag{5.6}
\]

Since \( v_* \) is bounded, condition (B) of Assumption 2 and (5.6) then give

\[
x_*(t_p+p) - x_*(t_p) = \int_{t_p}^{t_p+p} v_*(s) \cos(\theta_*(t_p)) \, ds + J(p)
\]

for some function \( J \) for which \( |J(p)| \leq \sup \{v_*(s) : s \in \mathbb{R} \} / p \to 0 \) as \( p \to +\infty \). If there were a subsequence \( \mathcal{L}_{p_j} \) of the sequence \( \mathcal{L}_p = \cos(\theta_*(t_p)) \) converging to some nonzero limit \( \mathcal{L}_* \), then \( \int_{t_{p_j}}^{t_{p_j}+p_j} v_*(s) \cos(\theta_*(t_{p_j})) \, ds \to \pm \infty \) as \( j \to +\infty \) (by our positive lower bound on \( v_* \)). Combining these limits with (5.7) contradicts the boundedness of \( x_* \), so \( \lim_{p \to +\infty} \cos(\theta_*(t_p)) = 0 \). Using \( \dot{y}_*(t) = v_*(t) \sin(\theta_*(t)) \) and similar reasoning shows that \( \lim_{p \to +\infty} \sin(\theta_*(t_p)) = 0 \), which is a contradiction because \( |(\cos(\theta_*(t_p)), \sin(\theta_*(t_p)))| = 1 \) for all \( p \).

We refer to the conclusion of Lemma 5.1 as the **persistency of excitation (PE)** condition. By reducing \( c_0 \) from Lemma 5.1 or the constant \( c > 0 \) from Assumption 2 without relabeling, we will assume that \( c = c_0 \).

### 5.4 Main UAV Theorem

It remains to design the control components \( \theta_N \) and \( v_N \) in (5.3). Fix any tuning design constant \( k > 0 \). We introduce the functions

\[
Q_1 = 0.5[\tilde{\psi}_*^2 + \tilde{\xi}_*^2], \quad Q_2 = 0.5\tilde{v}^2, \quad Q_3 = 0.5\tilde{\theta}^2,
\]

\[
Q_4 = Q_3(\tilde{\theta}) + Q_2(\tilde{v}) + k\sqrt{Q_1 + M} - k,
\]

\[
M = \tilde{\xi} \tilde{v}, \quad N = -\dot{\theta}_*(t) \tilde{\psi} \tilde{\xi}, \quad \text{and} \quad P = N + \left( \frac{1}{T} \int_T^t \dot{\theta}_*^2(\ell) \, d\ell \right) \tilde{\psi}^2.
\]

\[
(5.8)
\]
where $T$ is from Lemma 5.1 and the reference trajectory $(x^*, y^*, \theta^*, v^*)$ satisfies Assumption 2. We prove:

**Theorem 5.2.** The dynamics (5.4) for the tracking error $\mathcal{E} = (\tilde{\psi}, \tilde{\xi}, \tilde{\theta}, \tilde{v})$, in closed loop with

$$v_N(\mathcal{E}) = -k\frac{\tilde{\xi}}{2\alpha_v\sqrt{Q_1 + 1}} \quad \text{and} \quad \theta_N(t, \mathcal{E}) = k\frac{\tilde{\psi}_s(t) - \tilde{\xi}_s(t)}{2\sqrt{Q_1 + 1}}, \quad \text{(5.9)}$$

are integral ISS with respect to $\delta \in \mathcal{M}_R$. Also, we can find a constant $\delta_M > 0$ such that the closed loop dynamics are ISS with respect to $\delta \in \mathcal{M}_{[-\delta_M, \delta_M]}$, and a constant $\tilde{c} > 0$ and a polynomial $G$ such that

$$U^*(t, \mathcal{E}) = \left[U(t, \mathcal{E}) + 1\right]^{1/3} - 1, \quad \text{where}$$

$$U(t, \mathcal{E}) = P(t, \mathcal{E}) + \tilde{c}[Q_4(\mathcal{E}) + k]M(\mathcal{E}) + G(Q_4(\mathcal{E}))$$

is an integral ISS Lyapunov function for the closed loop dynamics with $\delta \in \mathcal{M}_R$. In particular, the controllers (5.3) uniformly globally asymptotically stabilize all trajectories of (5.1) to $\mathcal{R}_*$ when $\delta \equiv 0$. \hfill \Box

**Proof. Step 1: Nonstrict Lyapunov Decay.** We will refer to (5.4) in closed loop with (5.9) as the closed loop tracking dynamics. Along all of its trajectories, our functions from (5.8) satisfy

$$\dot{Q}_1 = \tilde{\xi}\tilde{v} + \alpha_\theta[\tilde{\psi}_s(t) - \tilde{\xi}_s(t)][\tilde{\theta} - \theta_N], \quad \dot{Q}_2 = \alpha_v(-\tilde{v}^2 + \tilde{v}v_N + \tilde{v}\delta),$$

and

$$\dot{Q}_3 = \alpha_\theta[-\tilde{\theta} + \theta_N]\tilde{\theta}. \quad \text{Hence, our choices (5.9) of $\theta_N$ and $v_N$ give}$$

$$\dot{Q}_4 = -\alpha_v\tilde{v}^2 + \left[\alpha_v v_N + k\frac{\tilde{\xi}}{2\sqrt{Q_1 + 1}}\right] \tilde{v}$$

$$+ \alpha_\theta[\theta_N - \tilde{\theta}] \left[\tilde{\theta} + k\frac{\tilde{\psi}_s(t) - \tilde{\xi}_s(t)}{2\sqrt{Q_1 + 1}}\right] + \alpha_v \tilde{v}\delta \quad \text{(5.11)}$$

along the closed loop tracking dynamics, where

$$W = \alpha_\theta[\tilde{\theta} - \theta_N]^2 + \alpha_v\tilde{v}^2, \quad \text{(5.12)}$$

so $Q_4$ is a weak Lyapunov function, because $\dot{Q}_4 \leq 0$ when $\delta = 0$. We will transform $Q_4$ into the desired strict Lyapunov function.
\[ \dot{P} = -\frac{1}{T} \int_{t-T}^{t} \dot{\theta}_s^2(\ell) d\ell \tilde{\psi}^2 - \tilde{\theta}_s(t) \tilde{\psi} \tilde{\xi} \]

Recalling the PE condition (5.5) from Lemma 5.1 (with \( c = c_0 \) without loss of generality) and setting \( a_1 = ||\tilde{\theta}_s|| + T||\dot{\theta}_s||, a_2 = \sqrt{\alpha_\theta} (||\dot{\theta}_s|| + T||\dot{\theta}_s^2||)(1 + ||\xi_s|| + ||\varphi_s||) + \frac{||\psi_s||}{\sqrt{\alpha_v}}, a_3 = ||\dot{\theta}_s^2|| + \frac{T\alpha}{c} \), \( a_4 = a_3 + \frac{8T\alpha^2}{c} \), \( a_5 = \frac{32T\alpha^2}{c} \) and \( a_6 = 1 + \frac{8T\alpha^2}{c} \), condition (5.13) and our choice of \( W \) give

\[ \dot{P} \leq -\frac{cT^2}{3} + a_1 |\tilde{\psi} \tilde{\xi}| + ||\dot{\theta}_s^2|| \tilde{\xi}^2 + a_2 [||\tilde{\xi}|| + \tilde{\xi}^2 + |\tilde{\psi}| + \tilde{\psi}^2 + |\tilde{\psi} \tilde{\varphi}|] \sqrt{W} \]

\[ \leq -\frac{cT^2}{4} + a_3 \tilde{\xi}^2 + a_2 [||\tilde{\xi}|| + \tilde{\xi}^2 + |\tilde{\psi}| + \tilde{\psi}^2 + |\tilde{\psi} \tilde{\varphi}|] \sqrt{W} \]

\[ \leq -\frac{cT^2}{2} + a_4 \tilde{\xi}^2 + a_5 [\tilde{\xi}^2 + \tilde{\psi}^2] W + a_6 W \]

where the first inequality used

\[ \max\{\sqrt{\alpha_\theta} |\tilde{\theta} - \theta_N|, \sqrt{\alpha_v} |\tilde{\psi}|\} \leq \sqrt{W} \] and

\[ \int_{t-T}^{t} \int_{s}^{t} \dot{\theta}_s^2(\ell) d\ell ds \leq \frac{T^2 ||\dot{\theta}_s||}{2}, \]

and the next two used Hölder’s Inequality to get

\[ a_1 |\tilde{\psi} \tilde{\xi}| \leq \frac{c}{\sqrt{T}} \tilde{\psi}^2 + \frac{T\alpha^2}{c} \tilde{\xi}^2, |\tilde{\psi} \tilde{\varphi}| \leq \tilde{\xi}^2 + \tilde{\psi}^2, a_2 (||\tilde{\xi}|| + |\tilde{\psi}|) \sqrt{W} \leq a_2 \tilde{\xi}^2 + \frac{8T\alpha^2}{c} \tilde{\psi}^2 + W \left(1 + \frac{8T\alpha^2}{c} \right), \]

and \( 2a_2 (\tilde{\xi}^2 + \tilde{\psi}^2) \sqrt{W} \leq 2a_2 (\tilde{\xi}^2 + \tilde{\psi}^2) \left\{ \frac{c}{16T\alpha^2} + \frac{16T\alpha^2}{c} W \right\} \).

By (5.9),

\[ \dot{M} = -\frac{0.5k\tilde{\xi}^2}{\sqrt{Q_{1+T}}} + \dot{\theta}_s(t) \tilde{\psi} \tilde{\varphi} \]

\[ + \tilde{\psi}^2 - \alpha_\theta \{ \tilde{\psi} + \psi_s(t) \} (\tilde{\theta} - \theta_N) \tilde{\psi} - \alpha_v \tilde{\varphi} + \tilde{\xi} \tilde{\alpha}_v \delta \]

(5.16)
\[
\dot{M} \leq -\frac{0.5k\xi^2}{\sqrt{Q_1+1}} + ||\dot{\theta}_*|| |\ddot{\psi}\bar{v}| + a_7||\ddot{\psi}| + 1]W + \alpha_v|\ddot{\xi}\bar{v}| + \alpha_v\ddot{\xi}\delta \\
\leq -\frac{0.25k\xi^2}{\sqrt{Q_1+1}} + ||\dot{\theta}_*|| |\ddot{\psi}\bar{v}| + a_7||\ddot{\psi}| + 1]W \\
+ \alpha_v^2/(Q_1+1)\ddot{v}^2/k + \alpha_v\ddot{\xi}\delta \\
\leq -\frac{0.25k\xi^2}{\sqrt{Q_1+1}} + \frac{1}{\sqrt{\alpha_v}}||\dot{\theta}_*|| |\ddot{\psi}|\sqrt{W} + a_8\sqrt{Q_1+1}W + \alpha_v\ddot{\xi}\delta 
\tag{5.17}
\]

along all trajectories of the closed loop tracking dynamics, using \(\max\{\sqrt{\alpha_v}|\dot{\theta} - \theta_N|, \sqrt{\alpha_v}|\ddot{\psi}|\} \leq \sqrt{W}, |\ddot{\psi}| + 1 \leq 3(\ddot{\psi}^2/2 + 1)^{1/2} \leq 3\sqrt{Q_1+1},\) and the special case \(\alpha_v|\ddot{\xi}| \leq \frac{k\ddot{\xi}^2}{4\sqrt{Q_1+1}} + \frac{\alpha_v^2}{k} \sqrt{Q_1+1}\ddot{v}^2\) of Hölder’s Inequality, and where \(a_7 = \sqrt{\alpha_v}(1 + ||\ddot{\psi}||) + \frac{1}{\alpha_v}\) and \(a_8 = 3a_7 + \alpha_v/k.\)

**Step 3: Constructing the Polynomial \(G\) in (5.10).** Set \(a_9 = ||\dot{\theta}_*||/\sqrt{\alpha_v}.\) Since \(Q_4 + k \geq k\sqrt{Q_1+1},\) (5.17) gives

\[
\frac{4a_9+1}{k^2}(Q_4 + k)\dot{M} \leq -[a_4 + 1]|\ddot{\xi}|^2 + [a_4 + 1]\frac{4a_9}{k^2}(Q_4 + k)|\ddot{\psi}|\sqrt{W} \\
+ [a_4 + 1]\frac{4a_9}{k^2}(Q_4 + k)\sqrt{Q_1+1}W \\
+ \alpha_v(a_4 + 1)\frac{4}{k^2}(Q_4 + k)|\ddot{\xi}|\delta \tag{5.18}
\]

Taking \(\alpha_1(s) = \frac{4}{k^2}[a_4 + 1](s+k),\) we deduce from adding (5.14) and (5.18) and then using the special case \([a_4 + 1]\frac{4a_9}{k^2}(Q_4 + k)|\ddot{\psi}|\sqrt{W} \leq \frac{c}{kT}\ddot{\psi}^2 + \frac{16\beta}{c}[a_4 + 1]\frac{4a_9}{k^2}(Q_4 + k)^2W\) of Hölder’s Inequality that

\[
\dot{P} + \alpha_1(Q_4)\dot{M} \leq -\frac{c}{kT}\ddot{\psi}^2 - |\ddot{\xi}|^2 + [a_6 + a_5(\ddot{\xi}^2 + \ddot{\psi}^2)]W \\
+ [a_4 + 1]\frac{4a_9}{k^2}(Q_4 + k)|\ddot{\psi}|\sqrt{W} \\
+ [a_4 + 1]\frac{4a_9}{k^2}(Q_4 + k)\sqrt{Q_1+1}W \\
+ \alpha_v(a_4 + 1)\frac{4}{k^2}(Q_4 + k)|\ddot{\xi}|\delta \tag{5.19}
\]

where \(\mathcal{U}(\mathcal{E}) = a_5(\ddot{\xi}^2 + \ddot{\psi}^2) + [a_6 + \frac{16\beta}{c}[a_4 + 1]\frac{4a_9}{k^2}(Q_4 + k)^2W + [a_4 + 1]4a_8(Q_4 + k)\sqrt{Q_1+1}/k^2.\)
Using the inequality $k\sqrt{Q_1 + 1} \leq k + Q_4$, we also have $\tilde{\xi}^2 + \tilde{\psi}^2 = 2Q_1 \leq 2\{(k + Q_4)^2/k^2 - 1\}$, so

$$\dot{P} + \tilde{c}(Q_4 + k)\dot{M} = \dot{P} + \alpha_1(Q_4)\dot{M} \leq -\frac{c}{4T}\tilde{\psi}^2 - \tilde{\xi}^2 + \mathcal{U}_N(Q_4)\dot{W} + \tilde{c}\alpha_v(Q_4 + k)|\tilde{\xi}||\delta|,$$  

(5.20)

where $\tilde{c} = \frac{4}{k^2}(a_4 + 1)$ and $\mathcal{U}_N(m) = a_6 + \left[\frac{2a_5}{k^2} + \frac{16T}{c}[a_4 + 1]^2\frac{a_5^2}{k^2} + \frac{c_8}{k}\right](k + m)^2$. Choose the third degree polynomial

$$\mathcal{G}(l) = 2\alpha_2(l) + 2l + 2\int_0^l \left[\mathcal{U}_N(m) + \tilde{c}\left\{m + (m + k)^2/k^2\right\}\right]dm,$$  

(5.21)

where $\alpha_2(l) = (\alpha_1(l) + ||\dot{\theta}_\ast||\{(k + l)^2/k^2 - 1\} + l\alpha_1(l)$ and $\alpha_1(l) = \tilde{c}(l + k)$ as before. Since $|M(\tilde{\nu}, \tilde{\xi})| = |\tilde{\nu}\tilde{\xi}| \leq \frac{1}{2}\tilde{\nu}^2 + \frac{1}{2}\tilde{\xi}^2 \leq Q_4 + \frac{1}{k^2}(k + Q_4)^2$, it follows from (5.11) and (5.20) that the function $U$ from (5.10) satisfies

$$\dot{U} \leq -\frac{c}{4T}\tilde{\psi}^2 - \tilde{\xi}^2 + \left[-\mathcal{G}'(Q_4) + \mathcal{U}_N(Q_4) + \tilde{c}\left\{Q_4 + \frac{(Q_4 + k)^2}{k^2}\right\}\right]W \nonumber$$

$$+ [\tilde{c}|M| + \mathcal{G}'(Q_4)]\alpha_v|\tilde{\nu}\delta| + \tilde{c}\alpha_v(Q_4 + k)|\tilde{\xi}||\delta|. \quad (5.22)$$

Step 4: Stability Properties Using $U^3$ from (5.10). Since

$$|N| \leq ||\dot{\theta}_\ast|||\tilde{\nu}\tilde{\xi}| \leq ||\dot{\theta}_\ast||\left\{\frac{(k + Q_4)^2}{k^2} - 1\right\}$$  

(5.23)

and

$$|M| \leq Q_4 + \left\{\frac{(k + Q_4)^2}{k^2} - 1\right\}, \quad (5.24)$$

we have $P + \alpha_1(Q_4)\dot{M} \geq N + \alpha_1(Q_4)\dot{M} \geq -\alpha_2(Q_4)$. Hence, the above choice (5.21) of $\mathcal{G}$ gives constants $c_0 \in (0, 1)$ and $d_0 > 0$ such that $d_0(Q_4^3 + 1) \geq U \geq c_0Q_4^3$ everywhere. Also, (5.22) gives $\dot{U} \leq -\frac{c}{4T}\tilde{\psi}^2 - \tilde{\xi}^2 + \alpha_v\tilde{c}(Q_4 + k)|\tilde{\xi}||\delta| - \frac{1}{2}\mathcal{G}'(Q_4)W + \alpha_v[\tilde{c}\{Q_4 + \frac{1}{k^2}(k + Q_4)^2\} + \mathcal{G}'(Q_4)]|\tilde{\nu}\delta|$ along the closed loop tracking dynamics. Since $|\tilde{\xi}| \leq 2(1 + Q_4/k)$ holds everywhere, we can then use our choice $W = \alpha_\theta[\tilde{\theta} -$
\[
\theta_N^2 + \alpha_v \tilde{v}^2 \text{ to find constants } c_1 > 0 \text{ and } c_2 > 0 \text{ such that } G'(l) \leq c_2 [1 + l^2] \text{ and }
\]
\[
\dot{U} \leq - \frac{c}{4T} \tilde{\psi}^2 - \tilde{\xi}^2 + c_1 G'(Q_4) \left[ |\tilde{\nu}| + |\delta| \right] - \frac{1}{2} G'(Q_4) \left[ \alpha_{\theta} \tilde{\theta} - \theta_N \right]^2 + \alpha_v \tilde{v}^2 
\leq - \frac{c}{4T} \tilde{\psi}^2 - \tilde{\xi}^2 + c_1 c_2 [1 + Q_4^2] \left[ \frac{c_1}{\alpha_v} \delta^2 + |\delta| \right] 
\leq - \frac{1}{4} G'(Q_4) \left[ \alpha_{\theta} \tilde{\theta} - \theta_N \right]^2 + \alpha_v \tilde{v}^2 
\]
(5.25)
hold everywhere, where we used the inequality \(|\tilde{\nu}| \leq \frac{\alpha_v}{4c_1} \tilde{v}^2 + \frac{c_1}{\alpha_v} \delta^2\).

Therefore, \(U^2 = (1 + U)^{1/3} - 1\) satisfies \(\dot{U}^2 \leq - \alpha_3(E) + \alpha_4(|\delta|)\) along all trajectories of the closed loop tracking dynamics, where
\[
\alpha_3(E) = \inf \left\{ \frac{\mathcal{H}(t, E)}{3[1 + U(t, E)]^{2/3}} : t \geq 0 \right\}, 
\]
(5.26)
and with the choices
\[
\alpha_4(l) = c_1 c_2 \left( \frac{c_1}{\alpha_v} + 1 \right) \left\{ l + l^2 \right\} 
\]
(5.27)
and
\[
\mathcal{H}(t, E) = \frac{c}{4T} \tilde{\psi}^2 + \tilde{\xi}^2 + \frac{1}{4} G'(Q_4(E)) \left[ \alpha_{\theta} \tilde{\theta} - \theta_N \right]^2 + \alpha_v \tilde{v}^2 \right]. 
\]
(5.28)
The formula for \(\alpha_4\) follows by recalling that \(U \geq c_0 Q_4^3\) with \(c_0 \in (0, 1)\), and then separately considering points \(E\) where \(Q_4 \leq 1\) and \(Q_4 \geq 1\) (to cancel the \([1 + Q_4^2]\) in (5.25) with \(1/[3(1 + U)^{2/3}]\)). By our choice of \(\theta_N\) from (5.9), the function \(\alpha_3\) is positive definite, and \(U^2\) is proper and positive definite. Hence, \(U^2\) is an integral ISS Lyapunov function for the closed loop tracking dynamics, which are therefore integral ISS.

We can also find constants \(\rho_0 > 0\) and \(\rho_* > 0\) such that \(\alpha_3(E) > \rho_*\) for all \(E \in \mathbb{R}^4 \setminus \rho_0 B_4\). This is because for each constant \(\rho_0 > 0\), we have
\[
\inf \left\{ \frac{G'(Q_4(E))}{[1 + U(t, E)]^{2/3}} : t \in \mathbb{R}, |E| \geq \rho_0 \right\} > 0, 
\]
(5.29)
using the bounds
\[
d_0(Q_4^3 + 1) \geq U \geq c_0 Q_4^3, 
\]
(5.30)
and by separately considering the cases where $|\tilde{\theta}, \tilde{v}| \to +\infty$, or $|\tilde{\theta}, \tilde{v}|$ stays bounded and $|(\tilde{\psi}, \tilde{\xi})| \to +\infty$. Hence, ISS follows from Lemma 4.1 from the preceding chapter. This proves the theorem.

\[\square\]

5.5 Satisfying Input and State Constraints

The expressions for our commands are obtained by substituting the control components $v_N$ and $\theta_N$ from (5.9) into (5.3), where $\alpha_{\theta}$ and $\alpha_v$ are the autopilot constants from (5.1). They only depend on $t$, $x$, $y$, and $\theta$. In real UAV applications, there are restrictions on the admissible control values, e.g., amplitude constraints for the controls requiring all values of the controller to lie in suitable intervals, rate constraints that restrict the admissible values of the time derivative of the controls along the closed loop trajectories, and state constraints that could require positive lower bounds on the velocity state or other conditions. In this section, we show how to use the tuning parameter $k$ from our controls to satisfy all of these constraints.

5.5.1 Command Amplitude Constraints

To see how we can satisfy control amplitude constraints, first note that the relation

$$\max\{\|\tilde{\xi}\|, \|\tilde{\psi}\|\} \leq \sqrt{2}\sqrt{Q_1} + 1 \quad (5.31)$$

gives $|v_N| \leq \bar{v}_k$ and $|\theta_N| \leq \bar{\theta}_k$ on the entire state space, where $\bar{v}_k = \frac{k}{\sqrt{2}\alpha_v}$ and $\bar{\theta}_k = \sqrt{2}k \max\{||\xi||, ||\psi||\}$ tend to 0 as $k \to 0^+$. Hence, if $[v_a, \bar{v}_a]$ is a desired velocity actuator amplitude envelope (with constant positive endpoints) and if we choose the trackable trajectory $R_*$ such that it admits a constant $\varepsilon > 0$ such that

$$v_a + \varepsilon < v_*(t) + \frac{\dot{v}_a(t)}{\alpha_v} < \bar{v}_a - \varepsilon \quad (5.32)$$
for all \( t \in \mathbb{R} \), then we can choose \( k \) small enough such that \( v_a < v_c < \bar{v}_a \) on the full state space to satisfy the amplitude constraints. Analogous considerations apply to \( \theta_c \), except \( \theta_c \) is only bounded when \( \theta_* \) is bounded; see (5.3).

### 5.5.2 Command Rate Constraints

To satisfy command rate constraints, take any bound \( \delta_M \) on the disturbance \( \delta \) (e.g., the ISS bound from Theorem 5.2). By reducing \( k \), we assume that

\[
\max \left\{ ||v_N||, ||\theta_N|| \right\} \leq 0.5. \tag{5.33}
\]

Then \( |\bar{v}(t)| \leq 1 + |\bar{v}(t_0)| + \delta_M \) and \( |\bar{\theta}(t)| \leq 1 + |\bar{\theta}(t_0)| \) along all trajectories of the UAV tracking dynamics (5.4). Hence, Assumption 2 and simple calculations based on the estimate \( \max \{ |\bar{\xi}|, |\bar{\psi}| \} \leq \sqrt{2} \sqrt{Q_1} + 1 \) prove that

\[
\max \{ |\dot{v}_N|, |\dot{\theta}_N| \} \leq k \bar{H} (1 + |(\bar{\theta}, \bar{v})(t_0)| + \delta_M) \tag{5.34}
\]

along all trajectories of (5.4) with disturbances bounded by \( \delta_M \), where

\[
\bar{H} = \max \left\{ 3, \frac{2}{\alpha_v} \right\} \left( 1 + ||\dot{\theta}_*|| + ||\xi_*|| + ||\dot{\psi}_*|| + ||\psi_*|| \right) \left( 9\alpha_\theta + 4 \right). \tag{5.35}
\]

Assume that \([\bar{\theta}_r, \bar{\theta}_r]\) and \([v_r, \bar{v}_r]\) are the desired command rate envelopes (with constant endpoints) and that there is a constant \( \varepsilon > 0 \) such that

\[
\bar{\theta}_r + \varepsilon < \dot{\theta}_*(t) + \ddot{\theta}_*(t)/\alpha_\theta < \bar{\theta}_r - \varepsilon \quad \text{and} \quad \bar{v}_r + \varepsilon < \dot{v}_*(t) + \ddot{v}_*(t)/\alpha_v < \bar{v}_r - \varepsilon \tag{5.36}
\]

hold for all \( t \in \mathbb{R} \). Then for each constant \( B > 0 \), we can find a constant \( \bar{K}(B) \) such that: For all \( k \in (0, \bar{K}(B)) \), we have

\[
\bar{\theta}_r < \dot{\theta}_c < \bar{\theta}_r \quad \text{and} \quad v_r < \dot{v}_c < \bar{v}_r \tag{5.37}
\]

along all trajectories of (5.4) for which \( |(\bar{\theta}, \bar{v})(t_0)| \leq B \) and \( |\delta|_\infty \leq \delta_M \). This is done by picking \( k \) such that \( \max \{ ||\dot{v}_N||, ||\dot{\theta}_N|| \} \) is small enough. This semiglobal bound
on the command rates is useful because \(|(\hat{\theta}, \hat{v})(t_0)|\) is known.

5.5.3 State Dependent Constraints and State Constraints

We can also ensure that the closed loop system satisfies state dependent constraints, i.e., for a given constant \(c_* > 0\), we can design the controller such that \(|\hat{\theta}| \leq \frac{c_*}{v}\) holds along all admissible trajectories. This is important for UAVs operating under coordinated turning conditions, where \(\theta\) cannot change too quickly when \(v\) is large.

To see how this can be done, note that \(|\hat{\theta}(t)| \leq ||\theta|_N| + |\hat{\theta}(t_0)|, |\hat{v}(t)| \leq ||v|_N| + |\hat{v}(t_0)| + |\delta|_\infty\), and so also

\[
|\hat{\theta}| = \alpha_\theta|\theta_c - \theta| \\
= \alpha_\theta|\theta_N + \theta_* + \hat{\theta}_*/\alpha_\theta - \theta| \\
\leq \alpha_\theta|\theta_N - \hat{\theta}| + ||\hat{\theta}_*|| \\
\leq 2\alpha_\theta||\theta_N|| + ||\hat{\theta}_*|| + \alpha_\theta|\hat{\theta}(t_0)|
\]

and \(v \leq ||v|_N| + |\hat{v}(t_0)| + |\delta|_\infty + ||v_*||\) hold along all of the closed loop trajectories.

Hence, the constraint \(|\hat{\theta}| \leq \frac{c_*}{v}\) holds if

\[
||\hat{\theta}_*|| + \alpha_\theta|\hat{\theta}(t_0)| \leq \frac{c_*}{2(||v_*|| + |\hat{v}(t_0)|)} \quad \text{and} \quad |\delta|_\infty \leq \frac{||v_*|| + |\hat{v}(t_0)|}{6}, \quad \text{(5.38)}
\]

and \(k\) is small enough. In fact, if \(k\) is chosen small enough such that \(v_N\) and \(\theta_N\) satisfy the state dependent input constraint

\[
\max\{||\theta_N||, ||v_N||\} \leq \min\left\{\frac{c_*}{8\alpha_\theta(||v_*|| + |\hat{v}(t_0)|)}, \frac{||v_*|| + |\hat{v}(t_0)|}{6}\right\}, \quad \text{(5.39)}
\]

then our bounds on \(|\delta|_\infty\) and \(v_N\) give \(v \leq \frac{4}{3}(||v_*|| + |\hat{v}(t_0)|)\), hence

\[
|\hat{\theta}| \leq 2\alpha_\theta||\theta_N|| + \frac{c_*}{2(||v_*|| + |\hat{v}(t_0)|)} \leq \frac{3c_*}{3(||v_*|| + |\hat{v}(t_0)|)} \leq c_*/v. \quad \text{(5.40)}
\]

For the important special case where \((\hat{\theta}(t_0), \hat{v}(t_0)) = 0\), it follows that the state constraint \(|\hat{\theta}| \leq c_*/v\) holds if \(R_*\) satisfies \(||\hat{\theta}_*|| \leq \frac{c_*}{2||v_*||}\) and \(|\delta|_\infty\) and \(k\) are small
enough. In that case, we also have \( v_\ast - |\delta|_\infty - |v_N| \leq v \leq |v_\ast| + |\delta|_\infty + v_\ast \), so keeping \( k \) and \( |\delta|_\infty \) small also ensures the state constraint that \( v \) stays close to \( v_\ast \).

5.6 Additive Uncertainty on Both Controls

We can also prove practical iISS and practical ISS properties for the tracking dynamics for the more complex UAV model

\[
\begin{aligned}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= \alpha_\theta (\theta_c - \theta + \Delta) \\
\dot{v} &= \alpha_v (v_c - v + \delta)
\end{aligned}
\]  

(5.41)

with additive uncertainty on both controls, provided we assume that \( \Delta \) is bounded by some constant \( \bar{\Delta} > 0 \) satisfying the smallness condition

\[
\alpha_\theta ||\dot{\theta}_\ast|| \bar{\Delta} < \frac{c}{2T},
\]  

(5.42)

where \( c \) and \( T \) are from the PE condition. This is done by noting that \( |(\tilde{\theta}, \tilde{v})(t)| \leq 2(1 + |(\tilde{\theta}, \tilde{v})(t_0)| + \delta_M + \bar{\Delta}) \) along the corresponding tracking dynamics when the tuning parameter \( k > 0 \) is small enough, and allowing the constants in the ISS estimate to depend on \( |(\tilde{\theta}, \tilde{v})(t_0)| \), which is a semiglobal ISS because the overshoot terms in the estimates depend on \( |(\tilde{\theta}, \tilde{v})(t_0)| \).

Here is a sketch of how to prove the extension. We indicate the changes needed in the proof of Theorem 5.2. The disturbance \( \Delta \) adds \( \alpha_\theta (\tilde{\theta} - \theta_N) \Delta \) to \( \dot{Q}_4 \) in (5.11), and \( \{\theta_\ast \alpha_\theta [(\tilde{\xi} + \xi_\ast)\tilde{\xi} - (\tilde{\psi} + \psi_\ast)\tilde{\psi}] - (2\alpha_\theta/T) \int_{t-T}^t \int_s^t \dot{\theta}_\ast^2 (\ell) d\ell \} \Delta \) to \( \dot{P} \) in (5.13). Hölder’s Inequality gives a constant \( b_0 > 0 \) such that the terms added to \( \dot{P} \) are bounded by \( 2||\dot{\theta}_\ast||\alpha_\theta \Delta \tilde{\psi}^2 + b_0[\tilde{\xi}^2 + |\tilde{\psi}\tilde{\xi}| + \Delta^2] \), so analogous reasoning to the argument that gave (5.14) and our smallness condition (5.42) on \( \bar{\Delta} \) provides
constants $b_i > 0$ such that $\dot{P} \leq -b_1 \tilde{v}^2 + b_2 (\tilde{\xi}^2 + [\tilde{\xi}^2 + \tilde{\psi}^2] W + W + \Delta^2)$. Similarly, the term $\alpha_\theta (\tilde{\psi} + \psi_\ast) \tilde{v} \Delta$ we must add to $\dot{M}$ is bounded above by $\bar{\Delta} \alpha_\theta |\tilde{v}| + \alpha_\theta ||\psi_\ast|| \tilde{v}^2 + \alpha_\theta ||\psi_\ast|| \Delta^2$, so (5.17) must be replaced by

$$\dot{M} \leq -\frac{k \tilde{\xi}^2}{4\sqrt{Q_1 + 1}} + b_3 (|\tilde{\psi}| \sqrt{W} + \sqrt{Q_1 + 1} W + \alpha_v |\tilde{\xi}| \delta + \Delta^2)$$

(5.43)

for a suitable constant $b_3 > 0$. Combining the decay estimates on $P$ and $M$ as in the proof of Theorem 5.2 gives $\dot{P} + \alpha_1 (Q_4) \dot{M} \leq -b_4 \tilde{v}^2 - \tilde{\xi}^2 + \bar{U}_N^2 (Q_4) W + b_5 (Q_4 + k) [|\tilde{\xi}| |\delta| + \Delta^2]$ for suitable constants $b_i > 0$ and a suitable second degree polynomial $\bar{U}_N^2$. Then we can find a third degree polynomial $G_1$ and constants $\tilde{c}_1 > 0$ and $b_i > 0$ such that the time derivative of $U_1 = P + \tilde{c}_1 (Q_4 + k) M + G_1 (Q_4)$ satisfies $\ddot{U}_1 \leq -b_6 \tilde{v}^2 - \tilde{\xi}^2 + b_7 (1 + Q_4^2) (\delta^2 + |\delta| + \Delta^2 + |\Delta|) - b_8 G_1^2 (Q_4) W$ along the new closed loop tracking dynamics, where $b_7$ depends on $|(\tilde{\theta}, \tilde{v})(t_0)|$. Then $U_1^2 = (U_1 + 1)^{1/3} - 1$ is the desired integral ISS Lyapunov function. This also gives ISS, under a smaller bound $\bar{\Delta}$ on $\Delta$, by reasoning as in the last part of the proof of Theorem 5.2. The argument is very similar to the proof of Theorem 5.2, so we leave the details to the reader.

5.7 Simulating the UAV Tracking Dynamics

We first took the reference trajectory $(x_\ast(t), y_\ast(t), \theta_\ast(t), v_\ast(t)) = (10 \sin(t), 10 - 10 \cos(t), t, 10)$ where the command is for the UAV to orbit a point at the constant speed of 10m/s. We simulated the UAV tracking dynamics (5.4) in closed loop with our controllers $v_N$ and $\theta_N$ from (5.9), with the initial error $\mathcal{E}(0) = (1, 1, 1, 1)$ and $\delta = 0.15 \sin(0.05t)$. Following [43, Section 5], we took $\alpha_v = 0.192$ and $\alpha_\theta = 0.55$, the actuator envelope [7, 13] for $v_c$, and the tuning constant $k = 1$ for our controls.

The controller $\theta_c$ is unbounded because $\theta_\ast$ is unbounded; see our controller formulas.
In Figure 5.1, we plot \((x(t), y(t))\), the tracking errors, and the closed loop controller values. As in [43], the units are radians, seconds, and meters.

\[
\theta_*(t) = 1.5 \arctan(3.38321412225 \sin(0.24t))
\] (5.44)

from Section 4.4 and \(v_*(t) \equiv 10\) with initial state \((x_*(0), y_*(0)) = (1, 1)\). We again simulated (5.4) with the feedbacks (5.9) and \(E(0) = (1, 1, 1, 1)\), with the same choices of \(\alpha_v, \alpha_\theta, k, \delta\), and the \(v_c\) actuator envelope as in our first simulation, but this time we also added the wind disturbance \(\Delta = 0.1\) to the angle controller \(\theta_c\).
and the actuator envelope $[-2, 3.1]$ on $\theta_c$. Since $\theta_*$ and $\dot{\theta}_*$ are bounded, we can satisfy an actuator envelope on $\theta_c$; see our control formulas (5.3). In Figure 5.2, we plot the simulated trajectory $(x(t), y(t))$ on $[250, 350]$, and the corresponding error components $(x(t) - x_*(t), y(t) - y_*(t), \theta(t) - \theta_*(t), v(t) - v_*(t))$ and the closed loop controller values.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.2.png}
\caption{UAV Tracking Figure 8 with Uncertainties in Both Controls}
\end{figure}

Top Panels: Center of Mass Path $(x(t), y(t))$ (Left) and Tracking Errors $x - x_*$ (Right). Second Row from Top: $y - y_*$ (Left) and $\theta - \theta_*$ (Right). Third Row from Top: $v - v_*$ (Left) and Closed Loop Control $\theta_c$ (Right). Bottom Panel: Closed Loop Control $v_c$. Units are Seconds, Radians, and Meters.

The units are the same as in Figure 5.1. Due to the disturbances, the tracking errors no longer converge to zero. This illustrates the effect of the disturbances. Nevertheless, our simulations show the good tracking performance and robustness under the actuator disturbances. Moreover, the controlled velocities are bounded away from zero and so respect the no-stall constraint.
Chapter 6
Future Research Topics

We proved tracking results for important benchmark aerospace models using novel asymptotic Lyapunov function, bounded backstepping, and strictification methods under realistic input constraints. In this chapter, we present some remarks about possible extensions and possible techniques for proving the extensions.

6.1 Tracking Under Input Delays

Our PVTOL aircraft and UAV controllers are functions of the current value of the state and the current time, possibly after augmenting the system with an observer. However, current state values might not always be available for use in the controllers. This motivates the search for controllers that only depend on the current time and time delayed values of the state, which produce closed loop systems of the form

\[ \dot{X}(t) = f(t, X(t), u(t, X(t - \tau)), \delta(t)), \quad X(t) \in \mathbb{R}^n \]  

(6.1)

with disturbances \( \delta \) and a constant delay \( \tau > 0 \). The Lyapunov function machinery we used in the preceding chapters does not apply to time delayed systems such as (6.1). However, we have the following definition from [35], in which \( X_t \) is defined by \( X_t(\theta) = X(t + \theta) \) for all \( \theta \in [-r, 0] \), and \( \mathcal{C}_n(I) \) is the set of all continuous functions \( h : I \to \mathbb{R}^n \) on any interval \( I \):

**Definition 6.1.** A continuous functional \( U : [0, \infty) \times \mathcal{C}_n(\mathbb{R}) \to [0, \infty) \) is called an ISS Lyapunov-Krasovski functional (ISS-LKF) for (6.1) provided that for all trajectories \( X(t) \equiv X(t, t_0, X_0, \delta) \) of (6.1) corresponding to all possible initial conditions
\(X(t_0) = X_0\) and all measurable essentially bounded disturbances \(\delta\), the function \(t \mapsto U(t, X_t)\) is locally absolutely continuous and there exist functions \(\alpha_i \in \mathcal{K}_\infty\) for \(i = 1, 2, 3, 4\) such that for all \(\phi \in \mathcal{C}_n([\tau, 0])\), all trajectories \(X(t)\) of (6.1), and all \(t \geq t_0 + \tau\), we have (a) \(\alpha_1(|\phi(0)|) \leq U(t, \phi) \leq \alpha_2(|\phi|_{[\tau, 0]})\) and (b) the time derivative \(D_t U(t, X_t)\) of \(U(t, X_t)\) satisfies \(D_t U(t, X_t) \leq -\alpha_3(U(t, X_t)) + \alpha_4(|\delta|_{[t_0, t]})\) almost everywhere.

A key difference between an ISS-LKF \(U(t, \phi)\) and a standard ISS Lyapunov function is that \(U(t, \phi)\) is evaluated at continuous \(\mathbb{R}^n\)-valued functions \(\phi \in \mathcal{C}_n(\mathbb{R})\) defined on the real line and times \(t \geq 0\), which is why we use the term functional instead of function. Under standard conditions on the dynamics [35], the existence of an ISS-LKF implies ISS in the following sense:

**Definition 6.2.** We call (6.1) input-to-state stable (ISS) provided there exist functions \(\beta \in \mathcal{K}\mathcal{L}\) and \(\gamma \in \mathcal{K}_\infty\) such that

\[
|X(t, t_o, X_o, \delta)| \leq \beta(|X_o|_{[t_o-\tau, t_o]}, t - t_o) + \gamma(|\delta|_{[t_o, t]})
\]  

(6.2)

for all \(t_o \geq 0, X_o \in \mathcal{C}_n([t_o - \tau, t_o]), t \geq t_o,\) and measurable essentially bounded disturbances \(\delta\).

Unlike the undelayed case, we have initial functions in the ISS estimate (6.2), instead of initial states. However, just as in the undelayed case, knowing an ISS-LKF makes it possible to construct the comparison functions in the ISS estimate. Therefore, one possible extension of our work would be to use strictification methods to build ISS-LKF’s and to prove ISS properties of the UAV and PVTOL aircraft tracking dynamics with respect to additive uncertainty on the controllers, under delayed feedback controls.
6.2 Other Extensions

In many applications, teams of UAVs are used for cooperative surveillance. This gives rise to more complex tracking problems whose possible control objectives could include (a) forcing all trajectories of all of the UAVs to track prescribed reference trajectories or (b) forcing the UAVs to maintain a formation while achieving some other control objective. Then the problem of collision avoidance becomes important. This motivates the search for tracking controllers for teams of UAVs that maintain state constraints under time delays in the controls, or under additive uncertainty on the controllers. Moreover, the autopilot constants in UAV models or other model parameters might not be known. This suggests the problem of using adaptive control methods for parameter identification in UAV models, in conjunction with an ISS analysis under time delays in, and additive uncertainty on, the controls. We leave these extensions for future work.
References


Vita

Aleksandra Gruszka was born in 1981 in Poland. She finished her Magister in Mathematics with a specialty in Applied Probability and Mathematical Statistics at the Universytet Wroclawski in 2006, under the direction of Professor Andrzej Hulanicki. She earned her Master of Science degree in mathematics from Louisiana State University in 2009. She won a plaque for being one of five finalists for the 2011 Best Student Paper Award at the 2011 American Control Conference in San Francisco, where she also won a best presentation award in her session. She was also one of the 12 US graduate students who were selected to present their work at the Association for Women in Mathematics sessions at the 2012 Joint Mathematics Meetings in Boston. She is currently a candidate for the degree of Doctor of Philosophy in mathematics which will be awarded in August 2012.