C° INTERIOR PENALTY METHODS FOR CAHN-HILLIARD EQUATIONS

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Abstract

In this work we study $C^0$ interior penalty methods for Cahn-Hilliard equations. In Chapter 1 we introduce Cahn-Hilliard equations and the time discretization that leads to linear fourth order boundary value problems. In Chapter 2 we review related fundamentals of finite element methods and multigrid methods. In Chapter 3 we formulate the discrete problems for linear fourth order boundary value problems with the boundary conditions of the Cahn-Hilliard type, which are called $C^0$ interior penalty methods, and we carry out the convergence analysis. In Chapter 4 we consider multigrid methods for the $C^0$ interior penalty methods. We present two smoothing schemes and compare their performance. In Chapter 5 we apply the $C^0$ interior penalty methods and the time discretization scheme to nonlinear time-dependent Cahn-Hilliard equations. Numerical examples for phase separation and image processing are presented.
Chapter 1

Introduction

The goal of this chapter is to introduce Cahn-Hilliard equations, which are non-linear fourth order time-dependent problems. We will discretize the equations in time using the time discretization scheme in [15]. The time discretization leads to linear fourth order boundary value problems which are the model problems of Chapter 3 and Chapter 4. At the end of this chapter, we will provide an outline of this dissertation.

1.1 Cahn-Hilliard Equations

The Cahn-Hilliard equation describes the dynamics of phase separation phenomena [36], by which the two mixed components in binary fluid separate and form regions pure in each component. Let \( \Omega \) be the spatial domain and \( u(x, t) \) \((0 \leq u \leq 1)\) denote the concentration of one component at the location \( x \) at time \( t \). Then the Cahn-Hilliard equation is

\[
\frac{\partial u}{\partial t} = -\nabla \cdot J \tag{1.1a}
\]

\[
J = -M \nabla (-\epsilon^2 \Delta u + W'(u)) \tag{1.1b}
\]
where \( W(u) = \frac{1}{4}u^2(u - 1)^2 \), \( M \) is the diffusional mobility and \( \epsilon \) represents the thickness of the transition regions between the two components. We consider the following boundary conditions:

\[
\mathbf{n} \cdot \mathbf{J} = \mathbf{n} \cdot \nabla u = 0 \tag{1.2}
\]

where \( \mathbf{n} \) denotes the outer normal of \( \partial \Omega \).

The boundary condition (1.2) implies mass conservation in \( \Omega \), i.e.,

\[
\int_{\Omega} u(x,t) dx = \text{Constant}
\]

Under some assumptions, there exists a solution to the Cahn-Hilliard equation (1.1) with the boundary conditions (1.2) and a given initial condition [48].

In this thesis, we assume \( \Omega \subset \mathbb{R}^2 \) and \( M = 1 \). After a rescaling in time, the Cahn-Hilliard equation (1.1) (1.2) has the following form (\( T > 0 \))

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\epsilon \Delta^2 u + \frac{1}{\epsilon} \Delta W'(u) \quad \text{in } \Omega \times (0,T), \tag{1.3a} \\
\frac{\partial u}{\partial n} &= \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on } \partial \Omega \times (0,T), \tag{1.3b} \\
u(x,0) &= u_0 \quad \text{in } \Omega. \tag{1.3c}
\end{align*}
\]

where \( u_0 \) is the given initial condition.

Bertozzi et al. proposed a modified Cahn-Hilliard equation for image inpainting [14, 15]. Let \( g : \Omega \to \mathbb{R} \) be the input image and \( D \subset \Omega \) be the inpainting domain where the image information is missing. The goal of image inpainting is to fill in the missing part \( D \). The modified Cahn-Hilliard equation replaces (1.3a) by

\[
\frac{\partial u}{\partial t} = -\epsilon \Delta^2 u + \frac{1}{\epsilon} \Delta W'(u) + \lambda(x)(g - u) \quad \text{in } \Omega \times (0,T) \tag{1.4}
\]
where
\[
\lambda(x) = \begin{cases} 
0 & \text{if } x \in D, \\
\lambda_0 & \text{if } x \in \Omega \setminus D.
\end{cases}
\]

and \(\lambda_0\) is a predefined positive constant. The added term, with sufficiently large \(\lambda_0\), is to keep the recovered image close to the input image in the region where the image is known (cf. Proposition 5.3 in [14]). The global existence and uniqueness can be established [14]. The model takes an initial guess \(u_0\) and fills in the missing part as \(u\) evolves.

Next we discretize the Cahn-Hilliard equation (1.3) in time. We will follow [15] and use a semi-implicit scheme obtained by convex splitting. The convex splitting provides unconditional stable schemes as stated by the following theorem [55].

**Theorem 1.1.** Let \(V\) be a Hilbert space with inner product \((\cdot, \cdot)_V\) and norm \(\|\cdot\|_V\), \(V'\) be the dual space of \(V\) and \(E \in V'\) satisfying the following conditions:

(a) \(E\) can be written as
\[
E = E_1 - E_2.
\]

where \(E_1\) and \(E_2\) are strictly convex. Moreover, \(E, E_1, E_2\) are continuously differentiable up to second order.

(b) \(\nabla^2 E(w)\) has a uniform lower bound of its eigenvalues, i.e.,
\[
\langle \nabla^2 E(w)v, v \rangle \geq \lambda \|v\|^2_{V'} \quad \forall v, w \in V,
\]

for some \(\lambda \in \mathbb{R}\) where \(\langle \cdot, \cdot \rangle\) denotes the canonical bilinear form between a vector space and its dual and \(\nabla^2\) denotes the second order Fréchet derivative which is the generalization of the Hessian matrix to Banach spaces [43].
(c) Let $\nabla$ denote the first order Fréchet derivative,

$$\hat{\lambda} = \sup \{ c \in \mathbb{R} : \langle \nabla E_2(v) - \nabla E_2(w), v - w \rangle \geq c \| v - w \|^2 \quad \forall v, w \in V \}.$$ 

and $\hat{\lambda} \geq -\lambda/2$.

Suppose $U^{n+1}, U^n \in V$ satisfy

$$\langle U^{n+1}, w \rangle_V = \langle U^n, w \rangle_V - k \langle \nabla E_1(U^{n+1}) - \nabla E_2(U^n), w \rangle \quad \forall w \in V.$$ 

for some $k > 0$. Then $E(U^{n+1}) \leq E(U^n)$.

Proof. The proof is based on [55].

By by condition (b), we have

$$E(v) - E(w) \leq \langle \nabla E(v), v - w \rangle - \frac{\lambda}{2} \| v - w \|^2 \quad \forall v, w \in V. \quad (1.6)$$

$$E(U^{n+1}) - E(U^n) \leq \langle \nabla E(U^{n+1}), U^{n+1} - U^n \rangle - \frac{\lambda}{2} \| U^{n+1} - U^n \|^2_V$$

$$= \langle \nabla E_1(U^{n+1}) - \nabla E_2(U^n), U^{n+1} - U^n \rangle - \frac{\lambda}{2} \| U^{n+1} - U^n \|^2_V$$

$$- \left( \frac{1}{k} [U^{n+1} - U^n, U^{n+1} - U^n] \right)_V$$

$$+ \langle \nabla E_1(U^{n+1}) - \nabla E_2(U^n), U^{n+1} - U^n \rangle$$

$$= -\langle \nabla E_2(U^{n+1}) - \nabla E_2(U^n), U^{n+1} - U^n \rangle$$

$$- \left( \frac{\lambda}{2} + \frac{1}{k} \right) \| U^{n+1} - U^n \|^2_V$$

$$\leq -\hat{\lambda} \| U^{n+1} - U^n \|^2 \quad - \left( \frac{\lambda}{2} + \frac{1}{k} \right) \| U^{n+1} - U^n \|^2_V$$

$$= (\hat{\lambda} - \frac{\lambda}{2} - \frac{1}{k}) \| U^{n+1} - U^n \|^2$$

$$\leq 0$$
The Cahn-Hilliard equation (1.3) can be viewed as a gradient flow using an $H^{-1}$ norm for the energy

$$E_{ch} = \int_{\Omega} \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) dx.$$ 

We split $E$ as

$$E_{ch} = E_{ch}^1 - E_{ch}^2$$

(1.7)

where

$$E_{ch}^1 = \int_{\Omega} \frac{\epsilon}{2} |\nabla u|^2 + \frac{C_1}{2} |u|^2 dx$$

(1.8)

and

$$E_{ch}^2 = \int_{\Omega} \frac{1}{\epsilon} W(u) + \frac{C_1}{2} |u|^2 dx.$$ 

(1.9)

So if $C_1$ is sufficiently large (which is of order $O(\frac{1}{\epsilon})$), then $E_{ch}^1$ and $E_{ch}^2$ satisfy the conditions (a), (b), (c) in Theorem 1.1. Therefore, the following time-stepping scheme will not increase the energy functional $E_{ch}$.

$$\frac{u^{n+1} - u^n}{\Delta t} = -\nabla H^{-1}(E_{ch}^1(u^{n+1}) - E_{ch}^2(u^n))$$

$$= -\epsilon \Delta^2 u^{n+1} + C_1 \Delta u^{n+1} + \frac{1}{\epsilon} \Delta W'(u^n) - C_1 \Delta u^n$$

(1.10)

where $u^n$ is an approximation of $u(x, \Delta t)$ ($n \geq 0$). Similarly, we can apply the convex splitting to the modified Cahn-Hilliard equation (1.4) [14, 15] and obtain the following equation:

$$\frac{u^{n+1} - u^n}{\Delta t} = -\epsilon \Delta^2 u^{n+1} + C_1 \Delta u^{n+1} - C_2 u^{n+1}$$

$$+ \frac{1}{\epsilon} \Delta W'(u^n) + \lambda(x)(i - u^n) - C_1 \Delta u^n + C_2 u^n$$

(1.11)
where the zero order terms come from the $L_2$ gradient flow for the energy
\[ \lambda_0 \int_{\Omega} (f - u)^2 dx \]
and its convex splitting [15], the constant $C_2$ is of order $O(\lambda_0)$.

In both case, we need to solve a fourth order boundary value problem of the
following form at each time step:

\[
\Delta^2 u - \beta \Delta u + \gamma u = f \quad \text{in } \Omega \tag{1.12a}
\]
\[
\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on } \partial \Omega \tag{1.12b}
\]

where $\beta, \gamma > 0$. In this dissertation, we will focus on the finite element methods
for the fourth order boundary value problem (1.12). We only consider 2D problems
for simplicity. The extension to 3D problems is straightforward.

The finite element methods for (1.12) start with the following weak form.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $V = \{v \in H^2(\Omega) : \partial v/\partial n = 0 \text{ on } \partial \Omega\}$
and $f \in L^2(\Omega)$. We consider the following problem:

Find $u \in V$ such that

\[
a(u, v) = \int_{\Omega} f v dx \quad \forall v \in V, \tag{1.13}
\]

where

\[
a(w, v) = \int_{\Omega} D^2 w : D^2 v dx + \int_{\Omega} \beta \nabla w \cdot \nabla v dx + \int_{\Omega} \gamma w v dx \quad \forall v, w \in H^2(\Omega),
\]

for some positive constant $\beta$ and $\gamma$, and

\[
D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}.
\]

**Remark 1.2.** The boundary condition $\frac{\partial \Delta u}{\partial n} = 0$ in (1.12b) is a natural bound-
ary condition and can be recovered automatically from the weak form (1.13) if \( u \in H^4(\Omega) \).

### 1.2 Thesis Outline

In Chapter 2 we provide a brief introduction of finite element methods and multigrid methods. The goal of this chapter is to provide the general framework of finite element methods. The multigrid algorithms presented in this chapter are also used with modification in Chapter 4. In Chapter 3 we introduce \( C^0 \) interior penalty methods for the linear fourth order boundary value problem (1.13) resulting from the time discretization of Cahn-Hilliard equations. \( C^0 \) interior penalty methods, which use \( C^0 \) Lagrange finite elements, are discontinuous Galerkin finite element methods for fourth order problems. However, due to the weak regularity of the solution to the continuous problem, the traditional convergence analysis for nonconforming finite element methods becomes problematic. We prove the convergence of the methods using the medius analysis proposed in [60] that combines the techniques from \textit{a priori} and \textit{a posteriori} analyses. To our best knowledge, this is the first time a rigorous analysis has ever been carried out for discontinuous Galerkin methods for linear fourth order problems with the boundary conditions of the Cahn-Hilliard type. We also derive a reliable and efficient error estimator that can be used for adaptive mesh refinement. Numerical results are presented. Even though we only consider 2D problems for simplicity, the formulation of the discrete problem and the convergence analysis developed in this chapter can be carried over to 3D problems. In Chapter 4 we extend the multigrid methods for the \( C^0 \) interior penalty methods in [34] to the boundary conditions of the Cahn-Hilliard type. Two smoothing schemes are presented. One is the unpreconditioned Richardson relaxation and the other one uses multigrid solves for second order
problems as preconditioners. The performance of these two smoothing schemes is carefully compared. We also investigate the performance of the multigrid algorithms on Graphics Processing Units (GPU). In Chapter 5 we apply the $C^0$ interior penalty methods to phase separation and image processing. The numerical experiments presented in this chapter indicate that $C^0$ interior penalty methods, coupling with the time stepping scheme described in Chapter 1, can solve the Cahn-Hilliard equations.
Chapter 2

Finite Element Methods

2.1 Sobolev Spaces

Let $\Omega \in \mathbb{R}^2$ be a bounded domain and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$, define

$$D^{\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

where the differentiation is understood in the sense of weak derivatives [54]. Sobolev semi-norms/norms when $k$ is a positive integer are defined as follows:

$$|v|_{H^k(\Omega)} = \left( \sum_{|\alpha| = k} \|D^{\alpha}v\|_{L^2(\Omega)}^2 \right)^{1/2}$$  \hspace{1cm} (2.1)

$$\|v\|_{H^k(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^{\alpha}v\|_{L^2(\Omega)}^2 \right)^{1/2}$$  \hspace{1cm} (2.2)

Define the Sobolev spaces $H^k(\Omega)$ as follows:

$$H^k(\Omega) = \{ v \in L^2(\Omega) : \|v\|_{H^k(\Omega)} < \infty \},$$  \hspace{1cm} (2.3)
Sobolev spaces $H^s(\Omega)$ can also be defined for non-integer $s$ [1, 78]. Let $s = k + \theta$ for some integer $k \geq 0$ and $0 < \theta < 1$, define

$$|v|_{H^s(\Omega)} = \left( \sum_{|\alpha| = k} \iint_{\Omega \times \Omega} \frac{|D^\alpha v(x) - D^\alpha v(y)|^2}{|x - y|^{2+2\theta}} \, dx \, dy \right)^{1/2},$$  \hspace{1cm} \text{(2.4)}$$

$$\|v\|_{H^s(\Omega)} = \left( \|v\|_{H^k(\Omega)}^2 + |v|_{H^s(\Omega)}^2 \right)^{1/2},$$  \hspace{1cm} \text{(2.5)}$$

$$H^s(\Omega) = \{ v \in L^2(\Omega) : \|v\|_{H^s(\Omega)} < \infty \}.$$  \hspace{1cm} \text{(2.6)}$$

Similarly, we can define $H^s(\partial \Omega)$,

$$|v|_{H^s(\partial \Omega)} = \left( \sum_{|\alpha| = k} \iint_{\partial \Omega \times \partial \Omega} \frac{|D^\alpha v(x) - D^\alpha v(y)|^2}{|x - y|^{2+2\theta}} \, dS_x \, dS_y \right)^{1/2},$$  \hspace{1cm} \text{(2.7)}$$

$$\|v\|_{H^s(\partial \Omega)} = \left( \|v\|_{H^k(\partial \Omega)}^2 + |v|_{H^s(\partial \Omega)}^2 \right)^{1/2},$$  \hspace{1cm} \text{(2.8)}$$

$$H^s(\partial \Omega) = \{ v \in L^2(\partial \Omega) : \|v\|_{H^s(\partial \Omega)} < \infty \}.$$  \hspace{1cm} \text{(2.9)}$$

\textbf{Theorem 2.1.} The Sobolev space $H^k(\Omega)$ is a Hilbert space [1, 32, 54].

The following theorem allows us to define “boundary value” for the Sobolev functions with enough regularity.

\textbf{Theorem 2.2 (Trace Theorem [59]).} Let $\Omega \subset \mathbb{R}^2$ be a sufficiently smooth domain and $s > l + 1/2$. Then there exists a bounded linear operator

$$Tr : H^s(\Omega) \to \Pi_{j=0}^l H^{s-j-1/2}(\partial \Omega)$$

such that

$$Tr v = \left( v|_{\partial \Omega}, \ldots, \frac{\partial^j v}{\partial n^j}|_{\partial \Omega} \right) \quad \forall v \in C^\infty(\bar{\Omega}).$$

\textbf{Remark 2.3.} The trace theorem can be extended to polygonal domains [59].
Hereafter, when referring to the boundary value of Sobolev functions, we mean $Tr v$ and drop the trace operator $Tr$ for simplicity.

Define

$$H^k_0(\Omega) = \{ v \in H^k(\Omega) : D^\alpha v = 0 \text{ on } \partial\Omega \text{ for } |\alpha| < k \}. \quad (2.10)$$

The following theorem states that $|\cdot|_{H^k(\Omega)}$ is a norm on $H^k_0(\Omega)$.

**Theorem 2.4** (Poincaré Inequality [54]). Let $\Omega$ be a bounded polygonal domain. Then there exists a constant $C$ such that

$$\|v\|_{H^k(\Omega)} \leq C|v|_{H^k(\Omega)} \quad \forall v \in H^k_0(\Omega). \quad (2.11)$$

### 2.2 Finite Element Methods

In this section, we will provide a brief introduction of finite element methods. Let $\Omega$ be a bounded polygonal domain and $f \in L_2(\Omega)$.

Consider Poisson’s equation

$$-\Delta u = f \quad \text{ in } \Omega, \quad (2.12a)$$

$$u = 0 \quad \text{ on } \partial\Omega. \quad (2.12b)$$

Finite element methods start with the weak form of (2.12).

Find $u \in H^1_0(\Omega)$ such that

$$a(u, v) = F(v) \quad \forall v \in H^1_0(\Omega) \quad (2.13)$$

where the bilinear form $a(\cdot, \cdot)$ is defined as

$$a(w, v) = \int_{\Omega} \nabla w \cdot \nabla v \, dx \quad \forall w, v \in H^1_0(\Omega)$$
and the linear functional $F(\cdot)$ is defined as

$$F(v) = \int_\Omega f v dx \quad \forall v \in H^1_0(\Omega).$$

The existence and uniqueness of the solution for the problem (2.13) follow from the Poincaré Inequality and the Riesz Representation Theorem [32]. Moreover, it can be shown that the solution for the problem (2.13) has higher regularity than just $H^1(\Omega)$ [59]. The following theorem states the regularity of the solution $u$ which is crucial for the convergence of finite element methods.

**Theorem 2.5.** The problem (2.13) has a unique solution. Moreover,

$$\|u\|_{H^{1+\alpha}(\Omega)} \leq C\|f\|_{L^2(\Omega)} \quad (2.14)$$

where $1/2 < \alpha \leq 1$. The constant $\alpha$ depends on the interior angles at the corners of $\Omega$.

**Remark 2.6.** On smooth domains, the Shift Theorem [54] states that if $f \in H^m(\Omega)$, then $u \in H^{m+2}(\Omega)$. However, on polygonal domains, the Shift Theorem fails in general, that is, the solution $u$ may not be in $H^2(\Omega)$ even if the given data $f \in C^\infty(\ol{\Omega})$.

The constant $\alpha$ will be referred to as the index of elliptic regularity. We will see later that the constant $\alpha$ affects the convergence of finite element methods.

Next, we will discretize the weak formulation (2.13). To do that, we first introduce the concept of finite element [40, 32]. The following definition is taken verbatim from [32].

**Definition 2.7.** Let
(i) $K \subset \mathbb{R}^n$ be a bounded closed set with nonempty interior and piecewise smooth boundary (the element domain),

(ii) $\mathcal{P}$ be a finite-dimensional space of functions on $K$ (the space of shape functions) and

(iii) $\mathcal{N} = \{N_1, N_2, \ldots, N_k\}$ be a basis for the dual space of $\mathcal{P}$ (the set of nodal variables).

Then $(K, \mathcal{P}, \mathcal{N})$ is called a finite element.

Note that the values of $N_1(v), N_2(v), \ldots, N_k(v)$ uniquely identify a function $v$ in $\mathcal{P}$.

A class of finite elements which are commonly used for second order problems is the family of Lagrange finite elements. In particular the linear Lagrange element is defined as follows: $K$ is a triangle, $\mathcal{P}$ is the set of linear functions and $\mathcal{N} = \{N_1, N_2, N_3\}$ with $N_i(v) = v(p_i) (1 \leq i \leq 3)$ where $p_i (1 \leq i \leq 3)$ are the three vertices of $K$ (Figure 2.1a). Higher order Lagrange elements that take $\mathcal{P}$ to be higher order polynomials can also be defined [32] (Figure 2.1b).

Next we discretize the domain.

**Definition 2.8.** A triangulation of a domain $\Omega$ is a finite collection of $\{T_i\}$ such
Figure 2.2: Two divisions of a square domain: (a) is a triangulation and (b) is not.

that

- \( T_i \) are triangles and

- \( \{\text{interior of } T_i\} \cap \{\text{interior of } T_j\} = \emptyset \) for \( i \neq j \) and

- \( \cup T_i = \overline{\Omega} \)

- no vertex of any triangle lies in the interior of an edge of another triangle

Note that this definition of triangulation does not allow “hanging nodes” (Figure 2.2).

Once we have the triangulation \( \mathcal{T}_h \) of \( \Omega \), we can associate each triangle in the triangulation with a \( \mathcal{P}_d \) Lagrange finite element and obtain a finite dimension function space defined on \( \Omega \):

\[
V_h = \{ v \in C(\overline{\Omega}) : v|_T \in \mathcal{P}_d \ \forall T \in \mathcal{T}_h \text{ and } v = 0 \text{ on } \partial \Omega \}, \tag{2.15}
\]

where \( \mathcal{P}_d \) denotes the polynomials of degree less than or equal to \( d \). Note that we also incorporate the boundary condition in \( V_h \) so that the functions in \( V_h \) are zero at \( \partial \Omega \).

A function in \( V_h \) can be uniquely identified by the nodal variable evaluations in the interior of \( \Omega \). For example, in Figure 2.3, the domain \( \Omega \) is a square and the
Figure 2.3: $P_1$ Lagrange finite element space on a square. The domain is a square triangulated by eight triangles. $V_h$ defined in (2.15) ($d = 1$) has only one degree of freedom that is the function value at the center.

triangulation consists of eight triangles. If we take $d = 1$ in (2.15), then a function in $V_h$ can be uniquely identified by its value at the center. Hence, $\dim V_h = 1$.

The discrete problem for (2.13) is defined as follows:

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

(2.16)

where $a(\cdot, \cdot)$ and $F(\cdot)$ are the same as in (2.13).

**Remark 2.9.** The space $V_h$ is a finite dimensional subspace of $H^1_0(\Omega)$ (cf. [40]). The finite element method (2.16) is hence called a conforming finite element method.

**Remark 2.10.** Because of the fact that $V_h \subset H^1_0(\Omega)$, the Poincaré inequality holds for $V_h$. The existence and the uniqueness of the solution for the discrete problem (2.16) immediately follows.

From (2.13), (2.16) and Remark 2.10, we have the following Galerkin Orthogonality

$$a(u - u_h, v_h) = F(v_h) \quad \forall v_h \in V_h.$$  

(2.17)

An abstract error estimate follows from the Galerkin Orthogonality.
Define the energy norm
\[
\|v\|_E = (a(v, v))^{1/2} \quad \forall v \in H^1_0(\Omega).
\] (2.18)

**Theorem 2.11** (Céa). Let \( u \) and \( u_h \) solve (2.13) and (2.16), respectively. Then
\[
\|u - u_h\|_E = \min_{v_h \in V_h} \|u - v_h\|_E. 
\] (2.19)

**Proof.** For all \( v_h \in V_h \),
\[
\|u - u_h\|_E^2 = a(u - u_h, u - u_h) \\
= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\
= a(u - u_h, u - v_h) \\
\leq \|u - u_h\|_E \|u - v_h\|_E \\
\]
Therefore,
\[
\|u - u_h\|_E \leq \min_{v_h \in V_h} \|u - v_h\|_E. 
\]
\[\square\]

**Remark 2.12.** Céa’s Theorem shows that the solution of the discrete problem (2.16) provides the best approximation to the solution of the continuous problem (2.13) in the subspace \( V_h \) under the energy norm \( \|\cdot\|_E \).

To derive a concrete error estimate, we need to find a function in \( V_h \) that can provide a good approximation of \( u \). This is achieved by interpolation.

Let \((T, P_d, \mathcal{N})\) be the \( P_d \) Lagrange finite element, the set \( \{\varphi_i; 1 \leq i \leq k\} \subset P_d \) be the basis dual to \( \mathcal{N} \). The local nodal interpolation operator \( \Pi^T_h \) for the \( P_d \)
Lagrange finite element is defined as follows [32, 40]:

\[ \Pi_T^h v = \sum_{i=1}^{k} N_i(v) \varphi_i \quad \forall v \in C(T). \]

**Theorem 2.13.** Let \( \Pi_T^h \) be the local nodal interpolation operator for \( P_d \) Lagrange element on the triangle \( T \), \( h_T \) be the diameter of \( T \) and \( 0 \leq s \leq t \), \( t > 1 \), \( d \geq t - 1 \). we have the following estimate for the interpolation error [40].

\[ |v - \Pi_T^h v|_{H^s(T)} \leq C_T h_T^{t-s} |v|_{H^t(T)} \quad \forall v \in H^t(T) \quad (2.20) \]

where the positive constant \( C_T \) depends only on the minimal angle of \( T \).

Let \( \Pi_h \) be the global nodal interpolation operator defined on the triangulation \( \mathcal{T}_h \) as follows. For \( v \in C(\Omega) \),

\[ (\Pi_h v)|_T = \Pi_T^h v \quad \forall T \in \mathcal{T}_h. \]

We have the following a priori error estimate.

**Theorem 2.14.** Let \( u \) and \( u_h \) solve (2.13), (2.16), respectively. Then

\[ \|u - u_h\|_E \leq C h^{t-1} |u|_{H^t(\Omega)} \quad (2.21) \]

where \( h = \max_{T \in \mathcal{T}_h} h_T \), \( t > 1 \), \( d \geq t - 1 \) and the constant \( C \) depends on the minimal angle of all the triangles in \( \mathcal{T}_h \).
Proof. By (2.18), (2.14), (2.19) and (2.20)

\[ \| u - u_h \|_E = \min_{v_h \in V_h} \| u - v_h \|_E \]
\[ \leq \| u - \Pi_h u \|_E \]
\[ = \left( \sum_{T \in \mathcal{T}_h} |u - \Pi_h u|_{H^1(T)}^2 \right)^{1/2} \]
\[ \leq \left( \sum_{T \in \mathcal{T}_h} C_T^2 h_T^{2(t-1)} |u|_{H^t(T)}^2 \right)^{1/2} \]
\[ \leq Ch^{t-1} |u|_{H^t(\Omega)} \]

where \( C = \max_{T \in \mathcal{T}_h} C_T \) and \( h = \max_{T \in \mathcal{T}_h} h_T \). \( \square \)

Remark 2.15.

- From (2.21), we can see that how well the solution of the discrete problem (2.16) approximates the solution of the continuous problem (2.13) depends on the regularity of the solution. For smooth solutions, we can use higher order polynomials to get better approximations.

- The second last equality in the proof motivates the use of meshes consisting of different sizes of triangles. In the region where the function has high regularity, we use large triangles; in the region where the function has low regularity, we use small triangles. Adaptive mesh techniques explore this idea and will be presented in Chapter 3.

By (2.14), we obtain the convergence of the solution of the discrete problem (2.16) to the solution of the continuous problem (2.13) as the mesh size \( h \to 0 \).

**Theorem 2.16.** Let \( u \) and \( u_h \) solve (2.13) and (2.16), respectively. Then

\[ \| u - u_h \|_E \leq Ch^\alpha \| f \|_{L^2(\Omega)} \]  \hfill (2.22)
where \( h = \max_{T \in T_h} h_T \) and the constant \( C \) depends on the minimal angle of all triangles in \( T_h \).

### 2.3 Multigrid Methods

It can be shown that the condition number of the discrete system (2.13) grows at the order of \( O(h^{-2}) \) [32]. On one hand, we need to choose a small mesh size in order to obtain a small discretization error. On the other hand, a small mesh size leads to an ill-conditioned linear system. In this section, we will introduce multigrid methods for the model problem (2.12). More detailed introductions for multigrid methods can be found in [19, 32, 62, 67].

Let \( T_0, T_1, \ldots \) be a sequence of triangulation of \( \Omega \). In this section, we assume that \( T_k (k > 0) \) is a regular refinement of \( T_{k-1} \), that is, the triangles in \( T_k \) is obtained by connecting the midpoints of all edges in \( T_{k-1} \) and dividing each triangle in \( T_{k-1} \) into four similar triangles. Let \( h_k = \max_{T \in T_k} h_T \) \((k \geq 0)\). Then

\[
h_k = \frac{1}{2} h_{k-1}. \tag{2.23}
\]

Let \( V_k \) be the \( P_d \) Lagrange finite element space associated with \( T_k \). Then

\[
V_0 \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k \ldots \tag{2.24}
\]

Let \( N_k \) be the set of all the nodes for the \( P_d \) Lagrange finite element space which are interior to \( \Omega \). Define a mesh-dependent inner product \((\cdot, \cdot)_k\) on \( V_k \) as follows:

\[
(v_k, w_k)_k = \sum_{p \in N_k} v_k(p)w_k(p). \quad \forall v_k, w_k \in V_k. \tag{2.25}
\]

Let \((\cdot, \cdot)\) be the canonical bilinear form between a vector space and its dual.
Define the operator $A_k : V_k \rightarrow V_k'$ by

$$\langle A_k v_k, w_k \rangle = a(v_k, w_k) \quad \forall v_k, w_k \in V_k.$$  \hfill (2.26)

Let the operator $I^k_{k-1} : V_{k-1} \rightarrow V_k$ be the natural injection, i.e.,

$$I^k_{k-1} v = v \quad \forall v \in V_{k-1},$$  \hfill (2.27)

the operator $I^{k-1}_k : V'_k \rightarrow V'_{k-1}$ be the transpose of $I^k_{k-1}$, i.e.,

$$\langle I^{k-1}_k v, w \rangle = \langle v, I^k_{k-1} w \rangle \quad \forall v \in V'_k, w \in V_{k-1},$$  \hfill (2.28)

and the operator $H_k : V'_k \rightarrow V_k$ be the operator defined by

$$(H_k \zeta_k, w_k)_k = \langle \zeta_k, w_k \rangle \quad \forall \zeta_k \in V'_k, w_k \in V_k.$$  

**Remark 2.17.** If we choose the canonical nodal basis and its dual for $V_k$ and $V'_k$, $H_k$ is represented by the identity matrix.

Next we define $k^{th}$ level iteration.

Let $MG_V(k, m, z_0, \phi_k)$ (resp. $MG_W(k, m, z_0, \phi_k)$, $MG_F(k, m, z_0, \phi_k)$) be an approximation to the solution of

$$A_k z = \phi_k$$

on the $k^{th}$ level with initial guess $z_0 \in V_k$ and $m$ presmoothing and $m$ postsmoothing steps obtained by the following algorithm:

**Algorithm 2.18.** ($V$-cycle Multigrid algorithm $MG_V(k, m, z_0, \phi_k)$)

**Input:** integer $k$, $z_0 \in V_k$, and $\phi_k \in V'_k$

**Output:** $z_{out} \in V_k$
if $k = 0$, then take $z_{out}$ to be the exact solution $A_0^{-1}\phi_0$
otherwise proceed as follows:

*Presmoother: For $j = 1, 2, \ldots, m$,

$$z_j = z_{j-1} + \lambda_k H_k(\phi_k - A_k z_{j-1})$$

where $\lambda_k$ is chosen such that $\rho(\lambda_k H_k A_k) < 2$ where $\rho(\cdot)$ denotes the spectral radius.
Note that $\lambda_k$ is of order of a constant independent of $k$.

*Coarse grid correction:

$$z_{m+1} = z_m + I_{k-1}^k MG_V(k-1, m, 0, r_{k-1}),$$

where $r_{k-1} = I_{k}^{k-1}(\phi_k - A_k z_m)$.

*Postsomoother: For $j = m+2, m+3, \ldots, 2m+1$,

$$z_j = z_{j-1} + \lambda_k H_k(\phi_k - A_k z_{j-1}).$$

Finally,

$$z_{out} = z_{2m+1}.$$

**Algorithm 2.19.** *(W-cycle multigrid algorithm $MG_W(k, m, z_0, \phi_k)$)* The same as Algorithm 2.18 except that the coarse grid correction step is replaced by

*Coarse grid correction:

$$z_{m+\frac{1}{2}} = MG_W(k-1, m, 0, r_{k-1}),$$

$$z_{m+1} = z_m + I_{k-1}^k MG_W(k-1, m, z_{m+\frac{1}{2}}, r_{k-1}).$$

**Algorithm 2.20.** *(F-cycle multigrid algorithm $MG_F(k, m, z_0, \phi_k)$)* The same as
Algorithm 2.18 except that the coarse grid correction step is replaced by coarse grid correction:

\[
\begin{align*}
    z_{m+\frac{1}{2}} &= MG_F(k-1, m, 0, r_{k-1}), \\
    z_{m+1} &= z_m + I_{k-1}^k MG_V(k-1, m, z_{m+\frac{1}{2}}, r_{k-1}).
\end{align*}
\]

**Remark 2.21.** The presmoothing and postsmoothing steps are just Richardson relaxation (Remark 2.17). Richardson relaxation is effective for decreasing the high frequency errors but not low frequency errors which is corrected by the coarse grid correction step. For the interpretation of the multigrid methods in the notion of “frequency”, refer to [81].

We have the following convergence property for the \(k^{th}\) level iteration defined in Algorithm 2.18-2.20 [25, 26].

**Theorem 2.22** (Convergence of the \(k^{th}\) Level Iteration). Let \(MG(k, m, z_0, \phi_k)\) be \(k^{th}\) level V-cycle, W-cycle or F-cycle iteration. There exists mesh-independent constants \(C\) such that

\[
\|z - MG(k, m, z_0, \phi_k)\|_E \leq \frac{C}{C + m^\alpha} \|z - z_0\|_E \quad \forall z_0 \in V_k, \ m \geq 1. \quad (2.29)
\]

Hereafter, we define the **contraction number** by

\[
\gamma_{k,m} = \frac{\|z - MG(k, m, z_0, \phi_k)\|_E}{\|z - z_0\|_E}. \quad (2.30)
\]

**Remark 2.23.** It follows immediately from Theorem 2.22 that

\[
\gamma_{k,m} \leq \delta \quad \forall m \geq 1,
\]

where \(\delta \in (0, 1)\) is independent of \(k\). This result can also be found in [22, 23, 58, 

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On $k^{th}$ level, the discrete problem (2.16) can be written as

$$A_k u_k = f_k$$  \hfill (2.31)

where

$$\langle f_k, v_k \rangle = \int_{\Omega} f v_k \, dx \quad \forall v_k \in V_k.$$  

Let $MG$ be any of the $MG_V, MG_W, MG_F$ defined in Algorithm 2.18-Algorithm 2.20. If we use the output of the $k - 1$ level iteration as the initial guess for the $k^{th}$ iteration and apply the $k^{th}$ level iteration multiple times, we can obtain an approximation to the equation (2.31).

**Algorithm 2.24. (Full Multigrid Algorithm)**

For $k = 0$, $\bar{u}_0 = A_0^{-1} f_0$.

For $k \geq 1$, $\bar{u}_k$ is computed recursively by

\[
\begin{align*}
    u_{k,0} &= I_{k-1}^k \bar{u}_{k-1}, \\
    u_{k,i} &= MG(k, m, u_{k,i-1}, f_k), \quad 1 \leq i \leq \ell, \\
    \bar{u}_k &= u_{k,\ell}.
\end{align*}
\]

The following theorem shows the error of the approximation obtained from the full multigrid algorithm defined in Algorithm 2.24 is comparable to the discretization error of the finite element methods. Moreover, the computational cost is optimal up to a factor.

**Theorem 2.25 (Convergence of Full Multigrid [32]).** Let $u_k$ be the exact solution of the discrete problem (2.31), $\bar{u}_k$ be the output of full multigrid algorithm defined in Algorithm 2.24 and $u$ be the solution of the continuous problem (2.13). If the
$k^{th}$ level iteration is a contraction with a contraction number independent of $k$, the integer $\ell$ in Algorithm 2.24 is sufficiently large, and $d \geq t - 1$, then there exists a constant $C$ such that

$$
\|u_k - \tilde{u}_k\|_E \leq C h_k^{t-1} |u|_{H^t(\Omega)}
$$

(2.32)

Moreover, the computational cost (the number of multiplication and addition operations) is $O(n_k)$ where $n_k$ is the dimension of $V_k$. 
Chapter 3

$C^0$ Interior Penalty Methods

In this chapter, we will formulate and analyze $C^0$ interior penalty methods. We will also obtain a reliable and efficient error estimator. The estimator is then used for adaptive mesh refinement. The presentation here is largely taken verbatim from [29]. We only consider 2D problems for simplicity, but the analysis can be carried over to 3D problems.

3.1 Formulation

We will consider a more general form of problem (1.12) to allow the second boundary condition to be nonhomogeneous. Let $\Omega \in \mathbb{R}^2$ be a bounded polygonal domain, $\beta, \gamma \geq 0$, $f \in L^2(\Omega)$, and $q = \partial (\Delta \phi) / \partial n$ on $\partial \Omega$, where $\phi \in H^4(\Omega)$ and $\partial \phi / \partial n = 0$ on $\partial \Omega$. Consider the following fourth order boundary value problem:

\begin{align}
\Delta^2 u - \beta \Delta u + \gamma u &= f \quad \text{in } \Omega \quad (3.1a) \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega \quad (3.1b) \\
\frac{\partial \Delta u}{\partial n} &= q \quad \text{on } \partial \Omega. \quad (3.1c)
\end{align}
Define
\[ V = \{ v \in H^2(\Omega) : \partial v / \partial n = 0 \text{ on } \partial \Omega \}, \quad (3.2) \]
where \( \partial / \partial n \) denotes the outward normal derivative.

The weak formulation of the problem (3.1) is defined as follows:
Find \( u \in V \) such that
\[ a(u, v) = (f, v) - \langle q, v \rangle \quad \forall v \in V, \quad (3.3) \]
where \((\cdot, \cdot)\) (resp. \(\langle \cdot, \cdot \rangle\)) denotes the \(L_2\) inner product on \( \Omega \) (resp. on \( \partial \Omega \)),
\[ a(w, v) = \int_\Omega D^2 w : D^2 v dx + \int_\Omega \beta \nabla w \cdot \nabla v dx + \int_\Omega \gamma w v dx \quad \forall v, w \in H^2(\Omega), \]
and
\[ D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}. \]

In the case where \( \gamma > 0 \), the problem (3.3) is uniquely solvable; in the case where \( \gamma = 0 \), we assume the compatibility condition
\[ \int_\Omega f dx = \int_{\partial \Omega} q ds \quad (3.4) \]
so that the problem (3.3) is also solvable and the solutions differ by an additive constant.

Define \( V^* \) by
\[ V^* = \begin{cases} V & \text{if } \gamma > 0 \\ \{ v \in V : v(p_*) = 0 \} & \text{if } \gamma = 0 \end{cases}, \]
where \( p_* \) is a corner in \( \Omega \). Then the unique solution of the problem (3.3) in \( V^* \)
satisfies the following elliptic regularity estimate (cf. Appendix A in [29]):

\[ |u|_{H^{2+\alpha}(\Omega)} \leq C_\Omega \left[ \|f\|_{L^2(\Omega)} + (1 + \gamma^{1/2}) \|\varphi\|_{H^4(\Omega)} \right], \tag{3.5} \]

where \(0 < \alpha \leq 1\). The constant \(\alpha\), which depends only on the interior angles of \(\Omega\), will be referred to as the index of elliptic regularity.

In this chapter, we will develop a \(C^0\) interior penalty method that uses only \(C^0\) Lagrange element for solving the problem (3.1).

Let \(T_h\) be a triangulation of \(\Omega\) and \(V_h\) be the \(P_2\) Lagrange finite element space associated with \(T_h\), i.e.,

\[ V_h = \{v \in C(\bar{\Omega}) : v_T = v|_T \in P_2(T) \quad \forall T \in T_h\}. \]

Define

\[ V_h^* = \begin{cases} V_h & \text{if } \gamma > 0 \\ \{v \in V_h : v(p_\ast) = 0\} & \text{if } \gamma = 0 \end{cases}. \]

We will use the following notations:

- \(h_T=\)diameter of the triangle \(T\) (\(h = \max_{T \in T_h} h_T\))
- \(v_T=\)restriction of the function \(v\) to the triangle \(T\)
- \(|T|=\)area of the triangle \(T\)
- \(\mathcal{E}_h=\)the set of edges of the triangles in \(T_h\)
- \(\mathcal{E}_h^i=\)the subset of \(\mathcal{E}_h\) consisting of the edges interior to \(\Omega\)
- \(\mathcal{E}_h^b=\)the subset of \(\mathcal{E}_h\) consisting of the edges along \(\partial\Omega\)
- \(\mathcal{V}_h=\)the set of vertices of the triangles in \(T_h\)
• $\mathcal{V}_h^i$=the subset of $\mathcal{V}_h$ consisting of the vertices interior to $\Omega$
• $\mathcal{V}_h^\partial$=the subset of $\mathcal{E}_h$ consisting of the vertices along $\partial \Omega$
• $|e|$=the length of the edge $e$
• $m_e$=the midpoint of the edge $e$
• $\mathcal{T}_p$=the set of the triangles in $\mathcal{T}_h$ that share the common vertex $p$
• $\mathcal{T}_e$=the set of the triangles in $\mathcal{T}_h$ that share the common edge $e$
• $|\mathcal{T}_p|$ (resp. $|\mathcal{T}_e|$)=the number of the triangles in $\mathcal{T}_p$ (resp. $|\mathcal{T}_e|$)
• $\mathcal{E}_p$=the set of the edges in $\mathcal{E}$ that share the common vertex $p$
• $\mathcal{V}_T$=the set of the three vertices of $T$
• $\mathcal{E}_T$=the set of the three edges of $T$
• $\mathcal{T}_e$=the set of the triangles in $\mathcal{T}_h$ such that $e \in \mathcal{E}_T$ ($e \in \mathcal{E}_h^\partial$)

Next we define the jump and the mean on an edge. Let

$$H^k(\Omega, \mathcal{T}_h) = \{ v \in H^1(\Omega) : v_T \in H^k(T) \quad \forall T \in \mathcal{T}_h \}.$$
Let \( e \in \mathcal{E}_h^i \) be shared by the two triangles \( T_\pm \in \mathcal{T}_h \) (cf. Figure 3.1). We take \( n_e \) to be the unit normal of \( e \) pointing from \( T_- \) to \( T_+ \) and define

\[
\begin{align*}
\left[ \frac{\partial v_h}{\partial n_e} \right] &= \frac{\partial v_{T_+}}{\partial n_e} \bigg|_e - \frac{\partial v_{T_-}}{\partial n_e} \bigg|_e \quad \forall v \in H^2(\Omega, \mathcal{T}_h), \\
\left[ \frac{\partial^2 v_h}{\partial n^2_e} \right] &= \frac{\partial^2 v_{T_+}}{\partial n^2_e} \bigg|_e - \frac{\partial^2 v_{T_-}}{\partial n^2_e} \bigg|_e \quad \forall v \in H^3(\Omega, \mathcal{T}_h), \\
\left\{ \frac{\partial v_h}{\partial n_e} \right\} &= \frac{1}{2} \left( \frac{\partial v_{T_+}}{\partial n_e} \bigg|_e + \frac{\partial v_{T_-}}{\partial n_e} \bigg|_e \right) \quad \forall v \in H^2(\Omega, \mathcal{T}_h), \\
\left\{ \frac{\partial^2 v_h}{\partial n^2_e} \right\} &= \frac{1}{2} \left( \frac{\partial^2 v_{T_+}}{\partial n^2_e} \bigg|_e + \frac{\partial^2 v_{T_-}}{\partial n^2_e} \bigg|_e \right) \quad \forall v \in H^3(\Omega, \mathcal{T}_h).
\end{align*}
\]

Let \( e \in \mathcal{E}_h^b \) be an edge of \( T \in \mathcal{T}_h \). We take \( n_e \) to be the unit normal of \( e \) pointing outside \( \Omega \) and define

\[
\begin{align*}
\left[ \frac{\partial v_h}{\partial n_e} \right] &= -\frac{\partial v_T}{\partial n_e} \bigg|_e \quad \forall v \in H^2(\Omega, \mathcal{T}_h), \\
\left\{ \frac{\partial^2 v_h}{\partial n^2_e} \right\} &= \frac{\partial^2 v_T}{\partial n^2_e} \bigg|_e \quad \forall v \in H^3(\Omega, \mathcal{T}_h).
\end{align*}
\]

We first derive an integration by parts formula for the biharmonic operator \( \Delta^2 \).

For \( w, v \in H^4(T) \) we have

\[
\int_T (\Delta^2 w) v \, dx = \int_{\partial T} \frac{\partial \Delta w}{\partial n} v \, ds - \int_T (\Delta \Delta w) \cdot \nabla v \, dx
\]

\[(3.6)\]

\[
= \int_{\partial T} \frac{\partial \Delta w}{\partial n} v \, ds - \sum_{i=1}^2 \int_T (\nabla \cdot \nabla w_{x_i}) v_{x_i} \, dx
\]

\[
= \int_{\partial T} \frac{\partial \Delta w}{\partial n} v \, ds - \int_{\partial T} \left( \frac{\partial}{\partial n} \Delta w \right) \cdot \nabla v \, ds + \int_T D^2 w : D^2 v \, dx
\]

\[
= \int_{\partial T} \frac{\partial \Delta w}{\partial n} v \, ds - \int_{\partial T} \frac{\partial^2 w}{\partial n \partial t} \frac{\partial v}{\partial t} \, ds - \int_{\partial T} \frac{\partial^2 w}{\partial n^2} \frac{\partial v}{\partial n} \, ds + \int_T D^2 w : D^2 v \, dx.
\]
Therefore, if the solution \( u \) has enough regularity to allow integration by parts on \( T \in T_h \), we have, for \( v \in H^4(T) \)

\[
\int_T (\Delta^2 u) v \, dx = \int_{\partial T} \frac{\partial \Delta u}{\partial n} v \, ds - \int_{\partial T} \frac{\partial^2 u}{\partial n \partial t} \frac{\partial v}{\partial t} \, ds - \int_{\partial T} \frac{\partial^2 u}{\partial n^2} \frac{\partial v}{\partial n} \, ds + \int_T D^2 u : D^2 v \, dx.
\]

Summing up all \( T \in T_h \), we have for \( \forall v \in H^4(\Omega, T_h) \cap C^0(\bar{\Omega}) \),

\[
\int_\Omega (\Delta^2 u) v \, dx = \sum_{e \in E_h} \int_e \left( \frac{\partial \Delta u}{\partial n} \right) v \, ds - \sum_{e \in E_h} \int_e \left( \frac{\partial^2 u}{\partial n \partial t} \right) \frac{\partial v}{\partial t} \, ds + \sum_{e \in E_h} \int_e \left( \frac{\partial^2 u}{\partial n^2} \right) \frac{\partial v}{\partial n} \, ds + \sum_{T \in T_h} \int_T D^2 u : D^2 v \, dx.
\]  

(3.7)

Note that we are using the boundary conditions \( \frac{\partial \Delta u}{\partial n} = q \) on \( \partial \Omega \) and \( \frac{\partial u}{\partial n} = 0 \) on \( \partial \Omega \) (and hence \( \frac{\partial^2 u}{\partial n \partial t} = 0 \) on \( \partial \Omega \)). We can formulate our discrete problem: Find \( u_h \in V_h^* \) such that

\[
A_h(u_h, v_h) = (f, v_h) - \langle q, v_h \rangle \quad \forall v_h \in V_h^*,
\]  

(3.8)

where for \( w_h, v_h \in V_h \)

\[
A_h(w_h, v_h) = \sum_{T \in T_h} \int_T D^2 w_h : D^2 v_h \, dx
\]

\[
+ \sum_{e \in E_h} \int_e \left( \frac{\partial^2 w_h}{\partial n^2} \right) \left[ \frac{\partial v_h}{\partial n} \right] ds + \sum_{e \in E_h} \int_e \left( \frac{\partial^2 v_h}{\partial n^2} \right) \left[ \frac{\partial w_h}{\partial n} \right] ds
\]

\[
+ \sum_{e \in E_h} \sigma_e \int_e \left[ \frac{\partial w_h}{\partial n} \right] \left[ \frac{\partial v_h}{\partial n} \right] ds
\]

\[
+ \int_\Omega \beta \nabla w_h \cdot \nabla v_h \, dx + \int_\Omega \gamma w_h v_h \, dx.
\]  

(3.9)
and the constant $\sigma \geq 1$ is a penalty parameter.

**Remark 3.1.** The discrete bilinear form $A_h(\cdot, \cdot)$ is independent of the choice of $n_e$ (or equivalently, the choice of $T_{\pm}$).

Let $v$ be an arbitrary quadratic polynomial on a triangle $T$, $w \in H^2(T)$ such that $w$ vanishes at the three vertices of the triangle $T$, $e_i (1 \leq i \leq 3)$ be the three edges of $T$, and $n_i, t_i (1 \leq i \leq 3)$ be the outer normal and tangential vector on edge $e_i (1 \leq i \leq 3)$, respectively. By (3.6) we have

$$\int_T D^2 v : D^2 w \, dx = \int_{\partial T} \frac{\partial^2 v}{\partial n \partial t} \frac{\partial w}{\partial t} \, ds + \int_{\partial T} \frac{\partial^2 v}{\partial n^2} \frac{\partial w}{\partial n} \, ds = \sum_{i=1}^3 \frac{\partial^2 v}{\partial n_i \partial t_i} \int_{e_i} \frac{\partial w}{\partial t} \, ds + \int_{\partial T} \frac{\partial^2 v}{\partial n^2} \frac{\partial w}{\partial n} \, ds = \int_{\partial T} \frac{\partial^2 v}{\partial n^2} \frac{\partial w}{\partial n} \, ds.$$

For all $v \in V_h$ and $w \in H^2(\Omega, T_h)$ such that $w$ vanishes at all the vertices of $T_h$, we can sum up all $T \in T_h$, use the elementary equality $ab + cd = \frac{1}{2}(a + c)(b + d) + \frac{1}{2}(a - c)(b - d)$ and obtain the following equality,

$$\sum_{T \in T_h} \int_T D^2 v : D^2 w \, dx = \sum_{e \in E_h} \int_e \left\{ \frac{\partial^2 v}{\partial n_e^2} \right\} \left[ \frac{\partial w}{\partial n_e} \right] \, ds - \sum_{e \in E_h} \int_e \left[ \frac{\partial^2 v}{\partial n_e^2} \right] \left\{ \frac{\partial w}{\partial n_e} \right\} \, ds. \quad (3.10)$$

The formulation of $C^0$ interior penalty methods was first proposed by G. Engel et al. in [53]. In the case of Dirichlet boundary conditions where boundary data $u$ and $\partial u / \partial n$ are given, the integration by parts involving $u$ can be justified using the singular function representation of $u$ [33, 59]. In the convergence analysis which we will present later, we will use an approach which does not require the integration
by parts involving $u$.

Let $\| \cdot \|_h$ be defined by

$$
\| v \|_h = \left( a_h(v, v) + \sum_{e \in E_h} \frac{\sigma}{|e|} \left\| \left[ \frac{\partial v}{\partial n_e} \right] \right\|^2_{L^2(e)} \right)^{1/2} \quad \forall v \in H^2(\Omega, T_h),
$$

(3.11)

where the bilinear form $a_h(\cdot, \cdot)$ defined by

$$
a_h(w, v) = \sum_{T \in T_h} \int_T D^2 w : D^2 v dx + \int_\Omega (\beta \nabla w \cdot \nabla v + \gamma vw) \, dx
$$

(3.12)

for all $v, w \in H^2(\Omega, T_h)$, is the piecewise version of $a(\cdot, \cdot)$. Note that $\| \cdot \|_h$ is a norm on the space

$$
H^2_*(\Omega, T_h) = \begin{cases}
H^2(\Omega, T_h) & \text{if } \gamma > 0 \\
\{ v \in H^2(\Omega, T_h) : v(p_*) = 0 \} & \text{if } \gamma = 0
\end{cases}
$$

The following lemmas [33] imply that the discrete problem is a symmetric positive definite problem and hence uniquely solvable when the penalty parameter $\sigma$ is sufficiently large.

**Lemma 3.2.** There exists a constant $C_1$ which depends only on the shape regularity of $T_h$ such that

$$
|A_h(w, v)| \leq C_1 \| w \|_h \| v \|_h \quad \forall v, w \in V^*_h.
$$

(3.13)

*Proof.* By the trace theorem with scaling and standard inverse estimates, we have

$$
\sum_{e \in E_h} |e| \left\| \left[ \frac{\partial^2 v}{\partial n^2} \right] \right\|^2_{L^2(e)} \leq C^*_1 \sum_{T \in T_h} |v|_{H^2(T)}^2 \quad \forall v \in V^*_h
$$

(3.14)

where the constant $C^*_1$ depends only on the shape regularity of $T_h$. 

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By the Cauchy-Schwarz inequality and (3.14) we have

\[
\sum_{e \in \mathcal{E}_h} \int_{e} \left\{ \frac{\partial^2 w}{\partial n^2_e} \right\} \left[ \frac{\partial v}{\partial n_e} \right] ds \\
\leq \left( \sum_{e \in \mathcal{E}_h} |e| \left\| \left\{ \frac{\partial^2 w}{\partial n^2_e} \right\} \right\|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h} |e|^{-1} \left\| \left[ \frac{\partial v}{\partial n_e} \right] \right\|_{L^2(e)}^2 \right)^{1/2} \quad (3.15)
\]

\[
\leq \left( C_1^* \sum_{T \in \mathcal{T}_h} |w|^2_{H^2(T)} \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h} |e|^{-1} \left\| \left[ \frac{\partial v}{\partial n_e} \right] \right\|_{L^2(e)}^2 \right)^{1/2}
\]

\[
\leq (C_1^*)^{1/2} \|w\|_h \|v\|_h
\]

and

\[
\sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|} \int_{e} \left[ \left[ \frac{\partial w}{\partial n_e} \right] \left[ \frac{\partial v}{\partial n_e} \right] \right] ds \\
\leq \left( \sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|} \left\| \left[ \frac{\partial w}{\partial n_e} \right] \right\|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|} \left\| \left[ \frac{\partial v}{\partial n_e} \right] \right\|_{L^2(e)}^2 \right)^{1/2} \quad (3.16)
\]

\[
\leq \|w\|_h \|v\|_h
\]

The inequality (3.13) follows from (3.11), (3.15) and (3.16) \qed

**Lemma 3.3.** There exists a constant \( \sigma^* \) which depends only on the shape regularity of \( \mathcal{T}_h \) such that if \( \sigma \geq \sigma^* \), the discrete bilinear form is coercive with respect to \( \| \cdot \|_h \) on \( V_h^* \), i.e.,

\[
\mathcal{A}_h(v, v) \geq \frac{1}{2} \|v\|_h^2 \quad \forall v \in V_h^*.
\quad (3.17)
\]
Proof. By the Cauchy-Schwarz inequality and (3.14), we have

\[
\left| 2 \sum_{e \in E_h} \int_{e} \left\{ \left( \frac{\partial^2 v}{\partial n_e^2} \right) \left[ \frac{\partial v}{\partial n_e} \right] \right\} ds \right|
\leq \sum_{e \in E_h} \left( \rho |e| \left\| \left\{ \frac{\partial^2 v}{\partial n_e^2} \right\} \right\|_{L^2(e)}^2 + (\rho |e|)^{-1} \left\| \left[ \frac{\partial v}{\partial n_e} \right] \right\|_{L^2(e)}^2 \right)
\leq C^*_1 \rho \sum_{T \in T_h} |v|_{H^2(T)}^2 + \rho^{-1} \sum_{e \in E_h} |e|^{-1} \left\| \left[ \frac{\partial v}{\partial n_e} \right] \right\|_{L^2(e)}^2
\]

for all \( v \in V_h^* \) where \( \rho \) is arbitrary. If we take \( \rho = \frac{1}{2C^*_1} \) and let \( K = \sigma \rho \), by (3.18) we have

\[
\sum_{T \in T_h} |v|_{H^2(T)}^2 + \sum_{e \in E_h} \int_{e} \left\{ \left( \frac{\partial^2 v}{\partial n_e^2} \right) \left[ \frac{\partial v}{\partial n_e} \right] \right\} ds + \sum_{e \in E_h} \frac{\sigma}{|e|} \left\| \left[ \frac{\partial v}{\partial n_e} \right] \right\|_{L^2(e)}^2 \geq \frac{1}{2} \sum_{T \in T_h} |v|_{H^2(T)}^2 + \frac{K - 1}{K} \sum_{e \in E_h} \frac{\sigma}{|e|} \left\| \left[ \frac{\partial v}{\partial n_e} \right] \right\|_{L^2(e)}^2
\]

It follows from (3.11) and (3.19) that when \( K \geq 2 \), i.e., \( \sigma > 2 \rho^{-1} \), we have (3.17).

\( \square \)

To avoid the proliferation of constants, henceforth we will use the notation \( A \lesssim B \) (resp. \( B \gesim A \)) to represent the inequality \( A \leq \text{Constant} \times B \) (resp. \( B \geq \text{Constant} \times A \)), where the constant, unless otherwise specified, depends only on the shape regularity (or the minimum angle) of \( T_h \). The statement \( A \approx B \) is equivalent to \( A \lesssim B \) and \( B \gesim A \). In particular (3.13)-(3.17) imply that

\[
A_h(v, v) \approx \| v \|_{h}^2 \quad \forall v \in V_h^*
\]

for a sufficiently large \( \sigma \).

\( C^0 \) interior penalty methods have certain advantages over classical finite element methods for fourth order problems. First of all, they are simpler than \( C^1 \)
conforming finite element methods. In fact, the lowest order $C^0$ interior penalty methods are as simple as the classical nonconforming finite element methods [40]. Unlike classical nonconforming finite element methods that only use low order polynomials, $C^0$ interior penalty methods can use high order polynomials to capture smooth solutions efficiently. Compared to mixed finite element methods that lead to saddle point problems, $C^0$ interior penalty methods preserve the symmetric positive definiteness of the continuous problem. Furthermore, since the underlying finite element spaces are standard spaces for second order problems, multigrid solves for the second order problems can be used as natural preconditioners for the fourth order problems.

We note that other discontinuous Galerkin methods for Cahn-Hilliard equation have been investigated in [45, 56, 63]. However the analyses in these papers are carried out under regularity assumptions which are not valid even for convex polygonal domains. In this chapter, we will develop a rigorous analysis of discontinuous Galerkin methods for linear fourth order problems with Cahn-Hilliard boundary conditions on polygonal domains. We will use the medius analyses approach [60] that combines the techniques from a priori and a posteriori analyses. The main results of this chapter are obtained in [29].

3.2 Enriching Operators

In this section, we will construct a linear operator $E_h$ that connects the $C^0$ finite element space $V_h$ to the $C^1$ Hsieh-Clough-Tocher macro finite element space $\tilde{V}_h (\subset H^2(\Omega))$ (cf. Figure 3.2). Note that the operator $E_h$ and the Hsieh-Clough-Tocher macro finite element are only used in the convergence analysis but not in the computation of the $C^0$ interior penalty methods.

We will define $E_h : V_h \to \tilde{V}_h \cap V$ by specifying the degrees of freedom in the
Figure 3.2: Hsieh-Clough-Tocher macro finite element [32, 40]. The element domain is a triangle which is partitioned into three triangles by connecting the centroid to the three vertices. The shape functions are $C^1$ piecewise cubic functions with respect to the partition. The nodal variables consist of the evaluations of the function values at three vertices, the first order derivatives at the three vertices, and the normal derivatives at the midpoints of the three edges.

Hsieh-Clough-Tocher finite element. Given any $v \in V_h$, $E_h v$ is defined by averaging as follows.

Let $p \in V_h$ be any vertex, we define

$$E_h v(p) = v(p).$$  \hfill (3.20)

For a vertex $p \in V^i_h$, we define

$$\nabla E_h v(p) = \frac{1}{\lvert T_p \rvert} \sum_{T \in T_p} \nabla v_T(p).$$  \hfill (3.21)

Let $p$ be any vertex of $T_h$ along $\partial \Omega$ which is not a corner of $\Omega$. Then $p$ is interior to an edge of $\Omega$ with a unit outward normal $n$ and a unit tangent vector $t$. We define

$$\frac{\partial E_h v}{\partial n}(p) = 0,$$  \hfill (3.22)

$$\frac{\partial E_h v}{\partial t}(p) = \frac{1}{\lvert T_p \rvert} \sum_{T \in T_p} \frac{\partial v_T}{\partial t}(p).$$  \hfill (3.23)
At a corner $p$ of $\Omega$, we define

$$\nabla E_h v(p) = 0. \quad (3.24)$$

Let $m_e$ be the midpoint of an edge $e \in E^i_h$ with a unit normal vector $n$. We define

$$\frac{\partial E_h v}{\partial n}(m_e) = \frac{1}{2} \sum_{T \in T_e} \frac{\partial v_T}{\partial n}(m_e). \quad (3.25)$$

At the midpoint $m_e$ of an edge $e \in E^b_h$, we define

$$\frac{\partial E_h v}{\partial n}(m_e) = 0. \quad (3.26)$$

**Lemma 3.4.** There exists a positive constant $C$ depending only on the shape regularity of $T_h$ such that

$$\sum_{T \in T_h} h^{-4}_T \|v - E_h v\|^2_{L_2(T)} \lesssim \sum_{e \in E_h} |e|^{-1} \left\| \left[ \left[ \frac{\partial v}{\partial n_e} \right] \right] \right\|^2_{L_2(e)} \quad v \in V_h. \quad (3.27)$$

**Proof.** Let $v \in V_h$ and $T \in T_h$ be arbitrary. From scaling and (3.20), we have

$$\|v - E_h v\|^2_{L_2(T)} \approx \sum_{p \in V_T} h^4_T |\nabla (v_T - E_h v)(p)|^2 \sum_{m_e \in \partial T_e} \frac{\partial (v_T - E_h v)(m_e)}{\partial n}(m_e) \quad (3.28)$$

The terms involving $p \in V_T$ on the right-hand side of (3.28) can be estimated according to whether $p$ is interior to $\Omega$, on $\partial \Omega$ but not a corner of $\Omega$, or a corner of $\Omega$, and the terms involving $e$ can be estimated according whether $e$ is interior to $\Omega$ or on $\partial \Omega$.

Let $p \in V^i_h \cap V_T$. We can connect any triangles $T' \in T_p$ to $T$ through a chain of triangles $T_0, \ldots, T_{j_T}$ in $T_p$ such that $T_0 = T'$, $T_{j_T} = T$ and any two consecutive triangles $T_k$ and $T_{k-1}$ in this chain share a common edge $e_k$ that belongs to $E_p$. It
follows from (3.21), scaling and the continuity of \( v \) that

\[
|\nabla(v_T - E_hv)(p)|^2 = \left| \nabla v_T - \frac{1}{|T_p|} \sum_{T' \in T_p} \nabla v_{T'} \right|^2 \\
\lesssim \frac{1}{|T_p|} \sum_{T' \in T_p} |\nabla v_T(p) - \nabla v_{T'}(p)|^2 \\
\leq \sum_{T' \in T_p} \sum_{k=1}^{j_{T'}} |e_k|^{-1} \left\| \left[ \frac{\partial v}{\partial n_{e_k}} \right] \right\|_{L^2(e_k)}^2 \\
\lesssim \sum_{e \in \mathcal{E}_p} |e|^{-1} \left\| \left[ \frac{\partial v}{\partial n_{e}} \right] \right\|_{L^2(e)}^2.
\] (3.29)

Similarly, for \( e \in \mathcal{E}_h \cap \mathcal{E}_T \), it follows from (3.25) that

\[
\left| \frac{\partial (v_T - E_hv)}{\partial n}(m_e) \right|^2 \lesssim |e|^{-1} \left\| \left[ \frac{\partial v}{\partial n_{e}} \right] \right\|_{L^2(e)}^2.
\] (3.30)

If \( e \in \mathcal{E}_h \cap \mathcal{E}_T \), then (3.26) implies that

\[
\left| \frac{\partial (v_T - E_hv)}{\partial n}(m_e) \right|^2 \lesssim |e|^{-1} \left\| \left[ \frac{\partial v}{\partial n_{e}} \right] \right\|_{L^2(e)}^2.
\] (3.31)

If \( p \in \mathcal{V}_h \cap \mathcal{V}_T \) is not a corner of \( \Omega \), then \( p \) is the endpoint of an edge \( e_p \in \mathcal{E}_h \cap \mathcal{E}_T \) and it follows from (3.22) and (3.23) that

\[
|\nabla(v_T - E_hv)(p)|^2 = \left| \frac{\partial (v_T - E_hv)}{\partial n_{e_p}}(p) \right|^2 + \left| \frac{\partial (v_T - E_hv)}{\partial t_{e_p}}(p) \right|^2 \\
\lesssim |e_p|^{-1} \left\| \left[ \frac{\partial v}{\partial n_{e}} \right] \right\|_{L^2(e_p)}^2 + \sum_{e \in \mathcal{E}_p \cap \mathcal{E}_h} |e|^{-1} \left\| \left[ \frac{\partial v}{\partial n_{e}} \right] \right\|_{L^2(e)}^2
\] (3.32)

where we have used the fact that we can connect \( T \) by a chain of triangles to the triangle that has \( e_p \) as an edge.
Finally, if \( p \in \mathcal{V}_h \cap \mathcal{V}_T \) is a corner of \( \Omega \) (cf. Figure 3.3), then \( p \) is the endpoint of the edges \( e_\pm \in \mathcal{E}_h^b \) with outward unit normals \( n_\pm \) and it follows from (3.24) that

\[
\left| \nabla (v_T - E_h v)(p) \right|^2 \leq \left| \frac{\partial v_T}{\partial n_+} (p) \right|^2 + \left| \frac{\partial v_T}{\partial n_-} (p) \right|^2 \leq |e_+|^{-1} \| \frac{\partial v}{\partial n_+} \|_{L_2(e_+)}^2 + |e_-|^{-1} \| \frac{\partial v}{\partial n_-} \|_{L_2(e_-)}^2 \\
+ \sum_{e \in \mathcal{E}_+ \cap \mathcal{E}_h} |e|^{-1} \left\| \left[ \frac{\partial v}{\partial n_e} \right] \right\|_{L_2(e)}^2,
\]

(3.33)

where we have used the fact that \( T \) can be connected by chains of triangles to the two triangles that have either \( e_+ \) or \( e_- \) as an edge.

![Figure 3.3: A corner p in a triangulation](image)

Summing up (3.28) over \( T \in \mathcal{T}_h \), we obtain the estimate (3.27) from (3.28)-(3.33).

The corollary below follows immediately from scaling and standard inverse estimates [32, 40].

**Corollary 3.5.** There exists a positive constant \( C \) depending only on the shape...
regularity of $\mathcal{T}_h$ such that

$$
\sum_{e \in \mathcal{E}_h} \left( |e| \left\| \left\{ \frac{\partial^2 (v - E_h v)}{\partial n_e^2} \right\} \right\|_{L^2(e)}^2 + |e|^{-1} \left\| \left\{ \frac{\partial (v - E_h v)}{\partial n_e} \right\} \right\|_{L^2(e)}^2 \right)
+ |e|^{-3} \left\| v - E_h v \right\|_{L^2(e)}^2 + \sum_{T \in \mathcal{T}_h} \left| v - E_h v \right|_{H^2(T)}^2 \leq C \sum_{e \in \mathcal{E}_h} |e|^{-1} \left\| \left[ \frac{\partial v}{\partial n_e} \right] \right\|_{L^2(e)}^2 \quad \forall v \in V_h^*.
$$

(3.34)

Finally, it follows from (3.11), (3.27) and (3.34) that

$$
\left\| v - E_h v \right\|_h + \left\| E_h v \right\|_a \leq C \left( 1 + \beta^{1/2} h + \gamma^{1/2} h^2 \right) \left\| v \right\|_h \quad \forall v \in V_h^*.
$$

(3.35)

where

$$
\left\| v \right\|_a = a(v, v)^{1/2} = \left( |v|_{H^2(\Omega)}^2 + \beta |v|_{H^1(\Omega)}^2 + \gamma \| v \|_{L^2(\Omega)}^2 \right)^{1/2} \quad \forall v \in H^2(\Omega).
$$

(3.36)

### 3.3 Medius Analysis

The main goal of this section is to derive an abstract a priori error estimate using only the continuous problem (3.3) and the discrete problem (3.8) as in the case of conforming methods. We borrow some techniques from a posteriori analysis and avoid integration by parts involving the continuous solution $u$.

#### 3.3.1 Local Efficiency Estimates

Let $\bar{f}$ (resp. $\bar{q}$) be the piecewise constant function on $\Omega$ (resp. $\partial \Omega$) defined by $\bar{f}|_T = \frac{1}{|T|} \int_T f \, dx \quad \forall T \in \mathcal{T}_h$ (resp. $\bar{q}|_e = \frac{1}{|e|} \int_e q \, ds \quad \forall e \in \mathcal{E}_h^b$). Define

$$
\| v \|_{H^2(T)} = |v|_{H^2(T)} + \beta h_T |v|_{H^1(T)} + \gamma h_T^2 \left\| v \right\|_{L^2(T)} \quad \forall v \in H^2(\Omega, \mathcal{T}_h), T \in \mathcal{T}_h.
$$
Lemma 3.6. We have
\[ h_T^2 \| f + \beta \Delta v - \gamma v \|_{L^2(T)} \lesssim \| u - v \|_{H^2(T)} + h_T^2 \| f - \bar{f} \|_{L^2(T)} \] (3.37)
for all \( T \in T_h, \ v \in V^*_h \),
\[ |e|^{1/2} \left\| \left[ \frac{\partial^2 v}{\partial n_T^2} \right] \right\|_{L^2(e)} \leq \sum_{T \in T_e} \left( \| u - v \|_{H^2(T)} + h_T^2 \| f - \bar{f} \|_{L^2(T)} \right) \] (3.38)
for all \( e \in E^i_h, \ v \in V^*_h \),
\[ |e|^{3/2} \| q \|_{L^2(e)} \lesssim \| u - v \|_{H^2(T_e)} + |e|^{3/2} \| q - \bar{q} \|_{L^2(e)} + h_{T_e}^2 \| f - \bar{f} \|_{L^2(T_e)} \] (3.39)
for all \( e \in E^b_h, \ v \in V^*_h \).

Proof. The proof of the lemma is based on bubble function techniques in a posteriori error analysis [2, 79].

Let \( b_T \in P_6(T) \) be a bubble function vanishing to the second order on \( \partial T \), i.e. \( b_T \) and \( \nabla b_T \) vanish on \( \partial T \), and \( \phi_T = (\bar{f} + \beta \Delta v - \gamma v)b_T \) then
\[ \| \phi_T \|_{L^2(T)} \approx \| \bar{f} + \beta \Delta v - \gamma v \|_{L^2(T)}. \] (3.40)

We have
\[ \| \bar{f} + \beta \Delta v - \gamma v \|_{L^2(T)}^2 \approx \int_T (\bar{f} + \beta \Delta v - \gamma v) \phi_T dx \]
\[ = \int_T (\bar{f} - f) \phi_T dx + \int_T (f + \beta \Delta v - \gamma v) \phi_T dx. \]

Let \( \bar{\phi}_T \) be the trivial extension of \( \phi_T \) to \( \Omega \). Then \( \bar{\phi}_T \in V \). Note that
\[ \int_T D^2 v : D^2 \phi_T \, dx = 0. \] We have

\[
\int_T (f + \beta \Delta v - \gamma v) \phi_T \, dx = \int_T f \phi_T \, dx - (\int_T D^2 v : D^2 \phi_T \, dx + \int_T \beta \nabla v \cdot \nabla \phi_T \, dx + \int_T \gamma v \phi_T \, dx)
\]

\[ = \int_\Omega D^2 u : D^2 \tilde{\phi}_T \, dx + \int_\Omega \beta \nabla u \cdot \nabla \tilde{\phi}_T \, dx + \int_\Omega \gamma u \tilde{\phi}_T \, dx - \left( \int_T D^2 v : D^2 \phi_T \, dx + \int_T \beta \nabla v \cdot \nabla \phi_T \, dx + \int_T \gamma v \phi_T \, dx \right) \]  

(3.42)

\[ \leq |u - v|_{H^2(T)} |\phi_T|_{H^2(T)} + \beta |u - v|_{H^1(T)} |\phi_T|_{H^1(T)} + \gamma \|u - v\|_{L^2(T)} \|\phi_T\|_{L^2(T)} \]

\[ \lesssim (h^{-2}_T |u - v|_{H^2(T)} + h^{-1}_T |u - v|_{H^1(T)} + \gamma \|u - v\|_{L^2(T)}) \|\phi_T\|_{L^2(T)}. \]

The estimate (3.37) follows from (3.40)-(3.42).

Next we show (3.38). For \( e \in \mathcal{E}^i_h \), let \( T_e = T_+ \cup T_- \). Define a bubble function \( \xi_1 \in \mathcal{P}_1(T_e) \) as follows.

\[ \xi_1 = 0 \text{ on } e \text{ and } \partial \xi_1 / \partial n_e = \left[ \frac{\partial^2 v}{\partial n_e^2} \right] \text{ in } T_e. \]

Scaling yields the following estimate

\[ |\xi_1|_{H^1(T_e)} \approx |e|^{1/2} \left\| \left[ \frac{\partial^2 v}{\partial n_e^2} \right] \right\|_{L^2(e)} \approx \|\xi_1\|_{L^\infty(T_e)}. \]  

(3.43)

Let \( \xi_2 \in \mathcal{P}_8(T_e) \) satisfy

1. \( \xi_2 \) vanishes to the second order on \( (\partial T_+ \cup \partial T_-) \setminus e \), i.e., \( \xi_2 \) and \( \nabla \xi_2 \) vanish on \( (\partial T_+ \cup \partial T_-) \setminus e \)

2. \( \int_{T_e} \xi_2 \, dx = |T_+| + |T_-| \)

3. \( \xi_2 > 0 \) on \( e \).
Scaling yields

\[ \| \xi_2 \|_{L^\infty(T_e)} \approx 1 \approx \| \xi_2 \|_{H^1(T_e)}, \quad (3.44) \]

\[ \int E \xi_2 ds \approx |e|. \quad (3.45) \]

Let \( \phi = \xi_1 \xi_2 \) in \( T_e \). Note that \( \phi \) vanishes on \( \partial T_+ \cup \partial T_- \) and \( \partial \phi / \partial n \) vanishes on \( (\partial T_+ \cup \partial T_-) \setminus e \). Then by the Poincaré inequality, and (3.43)-(3.45), we have

\[ \| \phi \|_{L^2(T_e)} \lesssim |e| \| \phi \|_{H^1(T_e)} \]

\[ \leq |e| (|\xi_1|_{H^1(T_e)} \| \xi_2 \|_{L^\infty(T_e)} + |\xi_2|_{H^1(T_e)} \| \xi_1 \|_{L^\infty(T_e)}) \]

\[ \lesssim |e|^{3/2} \left\| \left[ \frac{\partial^2 v}{\partial n_e^2} \right] \right\|_{L^2(e)}. \quad (3.46) \]

By (3.3), (3.10) and (3.45) we have

\[ \left\| \left[ \frac{\partial^2 v}{\partial n_e^2} \right] \right\|_{L^2(e)}^2 \]

\[ = \left[ \frac{\partial^2 v}{\partial n_e^2} \right] \left[ \frac{\partial^2 v}{\partial n_e^2} \right] - \int_E \left[ \frac{\partial^2 v}{\partial n_e^2} \right]^2 \xi_2 ds \]

\[ = \int_E \left[ \frac{\partial^2 v}{\partial n_e^2} \right] \frac{\partial \phi}{\partial n_e} ds \]

\[ = \sum_{T \in T_e} - \int_T D^2 v : D^2 \phi dx \]

\[ = \sum_{T \in T_e} \int_T (\beta \Delta v + \gamma v - f) \phi dx + \sum_{T \in T_e} \int_T D^2(u - v) : D^2 \phi dx \]

\[ + \sum_{T \in T_e} \int_T \beta \nabla (u - v) \cdot \nabla \phi dx + \sum_{T \in T_e} \int_T \gamma (u - v) \phi dx. \]

The estimate (3.38) follows from (3.37), (3.46), (3.47) and inverse estimates.

Next we show (3.39).

Let \( e \in \mathcal{E}^b_h \) be arbitrary and \( \psi_{T_e} \in C^1(\bar{T}_e) \cap H^2(T_e) \) be a macro bubble function
(cf. Appendix B in [29]) such that $\psi_{T_e}$ vanishes to the second order (i.e., $\psi_{T_e}$ and $\nabla \psi_{T_e}$ vanish) on the two edges in $\mathcal{E}_{T_e} \setminus \{e\}$, $\partial \psi_{T_e} / \partial n = 0$ on $e$, $\|\psi_{T_e}\|_{L^2(e)}^2 \approx |e|$, $|\psi_{T_e}|_{H^2(T_e)} \lesssim h_{T_e}^{-3/2} \|\psi_{T_e}\|_{L^2(e)}$, $|\psi_{T_e}|_{H^1(T_e)} \lesssim h_{T_e}^{-1/2} \|\psi_{T_e}\|_{L^2(e)}$ and $\|\psi_{T_e}\|_{L^2(T_e)} \lesssim h_{T_e}^{1/2} \|\psi_{T_e}\|_{L^2(e)}$.

Let $v \in V^*_h$. It follows from (3.6) and the properties of $\psi_{T_e}$ on $\partial T_e$ that

$$\int_{T_e} D^2 v : D^2 \psi_{T_e} \, dx = \int_{\partial T_e} \left[ \left( \frac{\partial^2 v}{\partial n^2} \right) \left( \frac{\partial \psi_{T_e}}{\partial n} \right) + \left( \frac{\partial^2 v}{\partial n \partial t} \right) \left( \frac{\partial \psi_{T_e}}{\partial t} \right) \right] \, ds = 0$$

(3.48)

because $\psi_{T_e}$ vanishes at the two endpoints of $e$. From (3.3), (3.47) and inverse estimates, we find

$$\|\bar{q}\|_{L^2(e)}^2 \approx \int_e \bar{q}(\bar{q} \psi_{T_e}) \, ds$$

$$= \int_e \bar{q}(\bar{q} \psi_{T_e}) \, ds + \int_e (\bar{q} - q)(\bar{q} \psi_{T_e}) \, ds$$

$$= - \int_{T_e} \left[ D^2 u : D^2 (\bar{q} \psi_{T_e}) + \beta \nabla u \cdot \nabla (\bar{q} \psi_{T_e}) + \gamma u (\bar{q} \psi_{T_e}) \right] \, dx$$

$$+ \int_{T_e} f(\bar{q} \psi_{T_e}) \, dx + \int_e (\bar{q} - q)(\bar{q} \psi_{T_e}) \, ds$$

$$= - \int_{T_e} \left[ D^2 (u - v) : D^2 (\bar{q} \psi_{T_e}) + \beta \nabla (u - v) \cdot \nabla (\bar{q} \psi_{T_e}) + \gamma (u - v) (\bar{q} \psi_{T_e}) \right] \, dx$$

$$+ \int_{T_e} (f + \beta \Delta v - \gamma v)(\bar{q} \psi_{T_e}) \, dx + \int_e (\bar{q} - q)(\bar{q} \psi_{T_e}) \, ds$$

$$\leq |u - v|_{H^2(T_e)} |\bar{q} \psi_{T_e}|_{H^2(T_e)} + |u - v|_{H^1(T_e)} |\bar{q} \psi_{T_e}|_{H^1(T_e)}$$

$$+ \gamma \|u - v\|_{L^2(T_e)} \|\bar{q} \psi_{T_e}\|_{L^2(T_e)} + \|f + \beta \Delta v - \gamma v\|_{L^2(T_e)} \|\bar{q} \psi_{T_e}\|_{L^2(T_e)}$$

$$+ \|q - \bar{q}\|_{L^2(e)} \|\bar{q} \psi_{T_e}\|_{L^2(e)}$$

$$\lesssim \left( |e|^{-3/2} \|u - v\|_{H^2(T_e)} + |e|^{1/2} \|f + \beta \Delta v - \gamma v\|_{L^2(T_e)} + \|q - \bar{q}\|_{L^2(e)} \right) \|\bar{q}\|_{L^2(e)}.$$
which implies
\[ |e|^{3/2} \|q\|_{L^2(T_e)} \lesssim \|u-v\|_{H^2(T_e)} + |e|^2 \|f + \beta \Delta v - \gamma v\|_{L^2(T_e)} + |e|^{3/2} \|q - \bar{q}\|_{L^2(T_e)}. \tag{3.49} \]

The estimate (3.39) follows from (3.37), (3.49) and a triangle inequality. \hfill \Box

**Remark 3.7.** The integration by parts carried out in the derivation of (3.37)- (3.39) involves only piecewise polynomial functions. Thus all the estimates obtained in this section are valid under the assumption that \( u \in H^2(\Omega) \).

Define
\[ \|v\|_{H^2(\Omega, T_h)} = \left( \sum_{T \in T_h} \left[ |v|^2_{H^2(T)} + \beta h_T^2 |v|_{H^1(T)} + \gamma h_T^4 \|v\|_{L^2(T)}^2 \right] \right)^{1/2} \tag{3.50} \]
for all \( v \in H^2(\Omega, T_h) \).

**Remark 3.8.** We have
\[ \|v\|_{H^2(\Omega, T_h)} \leq \max(1, (\text{diam} \Omega)^2) \|v\|_h \quad \forall v \in H^2(\Omega, T_h). \tag{3.51} \]

Define
\[ \text{Osc}(f) = \left( \sum_{T \in T_h} h_T^4 \|f - \bar{f}\|_{L^2(T)}^2 \right)^{1/2}, \quad \text{Osc}(q) = \left( \sum_{e \in E_h^i} |e|^3 \|q - \bar{q}\|_{L^2(T)}^2 \right)^{1/2}. \tag{3.52} \]

**Remark 3.9.** \( \text{Osc}(f) \) and \( \text{Osc}(q) \) indicate the oscillations of the data. Under our assumptions on the given data \( f \) and \( q \), \( \text{Osc}(f) \) and \( \text{Osc}(q) \) are of order \( O(h^2) \). More precisely, we have \( \text{Osc}(f) \lesssim h^2 \|f\|_{L^2(\Omega)} \) and \( \text{Osc}(q) \lesssim h^4 \|\varphi\|_{H^4(\Omega)} \).

Summing up the squares of (3.37)-(3.39) over all \( T \in T_h, e \in E_h^i \) and \( e \in E_h^b \) respectively, we have the following estimates.
Corollary 3.10. We have the following estimates:

\[
\sum_{T \in T_h} h_T^4 \| f + \beta \Delta v - \gamma v \|_{L^2(T)}^2 \lesssim \| u - v \|_{H^2(\Omega, \mathcal{T}_h)}^2 + \text{Osc}(f)^2 \quad \forall v \in V_h^*, \tag{3.53}
\]

\[
\sum_{e \in \mathcal{E}_h^i} |e| \left\| \left[ \frac{\partial^2 v}{\partial n_e^2} \right] \right\|_{L^2(e)}^2 \lesssim \| u - v \|_{H^2(\Omega, \mathcal{T}_h)}^2 + \text{Osc}(f)^2 \quad \forall v \in V_h^*, \tag{3.54}
\]

\[
\sum_{e \in \mathcal{E}_h^b} |e|^3 \| q \|_{L^2(e)}^2 \lesssim \| u - v \|_{H^2(\Omega, \mathcal{T}_h)}^2 + \text{Osc}(f)^2 + \text{Osc}(q)^2 \quad \forall v \in V_h^*. \tag{3.55}
\]

3.3.2 A Priori Error Analysis

Theorem 3.11. Let \( u \in V^* \) and \( u_h \in V_h^* \) be the solution of (3.3) and (3.8) respectively. There exists a positive constant \( C \) depending only on the shape regularity of \( \mathcal{T}_h \) such that

\[
\| u - u_h \|_h \leq C \left[ 1 + (\beta + \gamma^{1/2})h^2 \right] \inf_{v \in V_h^*} \| u - v \|_h + \text{Osc}(f) + \text{Osc}(q). \tag{3.56}
\]

Proof. Let \( v \in V_h^* \) be arbitrary. We have

\[
\| u - u_h \|_h \lesssim \| u - v \|_h + \max_{w \in V_h^* \setminus \{0\}} \frac{\mathcal{A}_h(v - u_h, w)}{\| w \|_h}. \tag{3.57}
\]

Use (3.8) we can write the numerator on the right hand side of (3.57) as

\[
\mathcal{A}_h(v - u_h, w) = \mathcal{A}_h(v, w - E_h w) + \mathcal{A}_h(v, E_h w) - \langle f, w \rangle + \langle q, w \rangle. \tag{3.58}
\]
Since $u \in V$ and $E_h w \in V^*$ for $w \in V_h^*$, we have

$$A_h(v, E_h w) = a_h(v, E_h w) + \sum_{e \in \mathcal{E}_h} \int_e \left\{ \left( \frac{\partial^2 E_h w}{\partial n_e^2} \right) \left( \frac{\partial v}{\partial n_e} \right) \right\} ds$$

(3.59)

$$= a_h(v - u, E_h w) + \sum_{e \in \mathcal{E}_h} \int_e \left\{ \left( \frac{\partial^2 E_h w}{\partial n_e^2} \right) \left( \frac{\partial (v - u)}{\partial n_e} \right) \right\} ds$$

$$+ (f, E_h w) - \langle q, E_h w \rangle.$$

It follows from (3.10) that

$$A_h(v, w - E_h w) = \int_\Omega \left[ \beta \nabla v \cdot \nabla (w - E_h w) + \gamma v (w - E_h w) \right] dx$$

$$- \sum_{e \in \mathcal{E}_h} \int_e \left( \frac{\partial^2 v}{\partial n_e^2} \right) \left( \frac{\partial (w - E_h w)}{\partial n_e} \right) ds$$

$$+ \sum_{e \in \mathcal{E}_h} \int_e \left( \frac{\partial v}{\partial n_e} \right) \left( \frac{\partial^2 (w - E_h w)}{\partial n_e^2} \right) ds$$

$$+ \sum_{e \in \mathcal{E}_h} |e| \int_e \left( \frac{\partial v}{\partial n_e} \right) \left( \frac{\partial (w - E_h w)}{\partial n_e} \right) ds.$$  

(3.60)

Putting the formulas (3.57)-(3.60) together, we find

$$A_h(v - u_h, w) = a_h(v - u, E_h w) - \sum_{T \in \mathcal{T}_h} \int_T (f + \beta \Delta v - \gamma v)(w - E_h w) dx$$

$$- \beta \sum_{e \in \mathcal{E}_h} \int_e \left( \frac{\partial v}{\partial n_e} \right) (w - E_h w) ds$$

$$+ \sum_{e \in \mathcal{E}_h} \int_e \left( \frac{\partial^2 (E_h w)}{\partial n_e^2} \right) \left( \frac{\partial (v - u)}{\partial n_e} \right) ds + \langle q, w - E_h w \rangle$$

$$- \sum_{e \in \mathcal{E}_h} \int_e \left( \frac{\partial^2 v}{\partial n_e^2} \right) \left( \frac{\partial (w - E_h w)}{\partial n_e} \right) ds$$

$$+ \sum_{e \in \mathcal{E}_h} \int_e \left( \frac{\partial v}{\partial n_e} \right) \left( \frac{\partial^2 (w - E_h w)}{\partial n_e^2} \right) ds$$

$$+ \sum_{e \in \mathcal{E}_h} |e| \int_e \left( \frac{\partial v}{\partial n_e} \right) \left( \frac{\partial (w - E_h w)}{\partial n_e} \right) ds.$$  

(3.61)
From (3.11), (3.27), (3.34), (3.35) and scaling we have

\[
\begin{align*}
|a_h(v - u, E_h w)| & \leq \| u - v \|_h \| E_h w \|_a \\
& \lesssim \| u - v \|_h (1 + \beta^{1/2}h + \gamma^{1/2}h^2) \| w \|_h ,
\end{align*}
\]

(3.62)

and

\[
\left| \sum_{T \in T_h} \int_T (f + \beta \Delta v - \gamma v)(w - E_h w) \, dx \right| \\
\lesssim \left( \sum_{T \in T_h} h_T^4 \| f + \beta \Delta v - \gamma v \|_{L_2(T)}^2 \right)^{1/2} \| w \|_h ,
\]

(3.63)

By the Cauchy-Schwarz inequality, (3.11) and (3.34) we have

\[
\left| \beta \sum_{e \in E_h} \int_e \left[ \frac{\partial v}{\partial n_e} \right] (w - E_h w) ds \right| \\
\leq \beta \left( \sum_{e \in E_h} |e|^{-1} \left\| \left[ \frac{\partial (v - u)}{\partial n_e} \right] \right\|_{L_2(e)}^2 \right)^{1/2} \left( \sum_{e \in E_h} |e| \| w - E_h w \|_{L_2(e)}^2 \right)^{1/2}
\]

(3.64)

\[
\lesssim \beta h^2 \| u - v \|_h \| w \|_h ,
\]

and

\[
\left| \sum_{e \in E_h} \int_e \left\{ \frac{\partial^2 (E_h w)}{\partial n_e^2} \right\} \left[ \frac{\partial (v - u)}{\partial n_e} \right] ds \right| \\
\leq \left( \sum_{e \in E_h} |e| \left\{ \frac{\partial^2 (E_h w)}{\partial n_e^2} \right\} \right)^{1/2} \left( \sum_{e \in E_h} |e|^{-1} \left\| \left[ \frac{\partial (v - u)}{\partial n_e} \right] \right\|_{L_2(e)}^2 \right)^{1/2}
\]

(3.65)

\[
\lesssim \left( \sum_{e \in E_h} \left\| E_h w \|_{H^2(T)} \right\|^2 \right)^{1/2} \| u - v \|_h
\]

\[
\lesssim \| u - v \|_h \| w \|_h .
\]
It also follows from the Cauchy-Schwarz inequality, (3.11) and (3.34) that

\[ |\langle q, w - E_h w \rangle| \lesssim \left( \sum_{e \in \mathcal{E}_h^b} |e|^3 \|q\|_{L^2(e)}^2 \right)^{1/2} \|w\|_h, \]  

(3.66)

and

\[
\left| \sum_{e \in \mathcal{E}_h^b} \int_e \left[ \left[ \frac{\partial^2 v}{\partial n_e^2} \right] \left[ \frac{\partial (w - E_h w)}{\partial n_e} \right] \right] \right|^{1/2} \lesssim \left( \sum_{e \in \mathcal{E}_h^b} |e| \left\| \left[ \left[ \frac{\partial^2 v}{\partial n_e^2} \right] \right] \right\|_{L^2(e)} \right)^{1/2} \|w\|_h. \]

(3.67)

Similarly, we have

\[
\left| \sum_{e \in \mathcal{E}_h^b} \int_e \left[ \left[ \frac{\partial v}{\partial n_e} \right] \left[ \frac{\partial^2 (w - E_h w)}{\partial n_e^2} \right] \right] \right| \lesssim \|u - v\|_h \|w\|_h, \]

(3.68)

and

\[
\left| \sum_{e \in \mathcal{E}_h^b} \int_e \left[ \left[ \frac{\partial (v - w)}{\partial n_e} \right] \left[ \frac{\partial^2 (w - E_h w)}{\partial n_e^2} \right] \right] \right| \lesssim \|u - v\|_h \left( \sum_{e \in \mathcal{E}_h^b} |e|^{-1} \left\| \left[ \left[ \frac{\partial (w - E_h w)}{\partial n_e} \right] \right] \right\|_{L^2(e)}^2 \right)^{1/2} \|w\|_h. \]

(3.69)
Combining (3.57), (3.61)-(3.69), we have

\[ \| u - u_h \|_h \lesssim \left[ 1 + (\beta + \gamma^{1/2})h^2 \right] \| u - v \|_h + \left( \sum_{T \in \mathcal{T}_h} h_T^4 \| f + \beta \Delta v - \gamma v \|^2_{L^2(T)} \right)^{1/2} \]

\[ + \left( \sum_{e \in \mathcal{E}_h} |e|^3 \| q \|^2_{L^2(e)} \right)^{1/2} + \left( \sum_{e \in \mathcal{E}_h} |e| \left\| \left[ \frac{\partial^2 v}{\partial n_e} \right] \right\|_{L^2(e)}^2 \right)^{1/2}, \]

which together with (3.51) and (3.53)- (3.55) implies

\[ \| u - u_h \|_h \lesssim \left[ 1 + (\beta + \gamma^{1/2})h^2 \right] \| u - v \|_h + \text{Osc}(f) + \text{Osc}(q) \quad \forall v \in V_h^* \quad (3.70) \]

and hence the estimate (3.56).

At this point we can invoke the elliptic regularity of \( u \) to obtain a concrete error estimate for the \( C^0 \) interior penalty method. Let \( \Pi_h \) be the nodal interpolation operator for the quadratic Lagrange element that maps \( V^* \) into \( V_h^* \). Using a standard interpolation error estimate [32, 40] and the trace theorem with scaling, we find

\[ \| u - \Pi_h u \|^2_h = \sum_{T \in \mathcal{T}_h} \| u - \Pi_h u \|^2_{H^2(T)} + \beta \| u - \Pi_h u \|^2_{H^2(\Omega)} + \gamma \| u - \Pi_h u \|^2_{L^2(\Omega)} \]

\[ + \sum_{e \in \mathcal{E}_h} \sigma \left\| \left[ \frac{\partial^2(u - \Pi_h u)}{\partial n_e} \right] \right\|_{L^2(e)}^2 \]

\[ \lesssim \sigma \sum_{T \in \mathcal{T}_h} \| u - \Pi_h u \|^2_{H^2(T)} + \beta \| u - \Pi_h u \|^2_{H^2(\Omega)} + \gamma \| u - \Pi_h u \|^2_{L^2(\Omega)} \]

\[ \lesssim h^{2\alpha} \left( \sigma + \beta h^2 + \gamma h^4 \right) \| u \|^2_{H^{2+\alpha}(\Omega)}. \]

The following concrete error estimate is an immediate consequence of (3.5), Remark 3.9, Theorem 3.11 and (3.71).

**Theorem 3.12.** There is a positive constant \( C \) depending only on \( \Omega \) and the shape
regularity of $T_h$ such that

$$
\|u - u_h\|_h \leq C h^\alpha \sigma^{1/2} \left[ 1 + (\beta^2 + \gamma) h^4 \right] \left[ \|f\|_{L^2(\Omega)} + (1 + \gamma^{1/2}) \|\varphi\|_{H^4(\Omega)} \right]. \tag{3.72}
$$

**Remark 3.13.** The explicit dependence of the error estimate (3.72) on $\beta$ and $\gamma$ is useful for the analysis for Cahn-Hilliard equations where constant $\beta$ and $\gamma$ depend on various parameters.

### 3.4 A Posteriori Error Analysis

In this section, we introduce a residual-based error estimator for the quadratic $C^0$ interior penalty method and show that it is both reliable and efficient up to terms that decay at a higher order as $h \downarrow 0$. Let the residuals $\eta_T$ and $\eta_{e,i} (1 \leq i \leq 3)$ be given by

\[\begin{align*}
\eta_T &= h^2_T \|f + \beta \Delta u_h - \gamma u_h\|_{L^2(T)} \quad \forall T \in T_h, \\
\eta_{e,1} &= (\sigma |e|^{-1/2}) \left\| \left[ \frac{\partial u_h}{\partial n_e} \right] \right\|_{L^2(e)} \quad \forall e \in \mathcal{E}_h, \\
\eta_{e,2} &= |e|^{1/2} \left\| \left[ \frac{\partial^2 u_h}{\partial n_e^2} \right] \right\|_{L^2(e)} \quad \forall e \in \mathcal{E}_h^i, \\
\eta_{e,3} &= |e|^{3/2} \|q\|_{L^2(e)} \quad \forall e \in \mathcal{E}_h^b.
\end{align*}\]

The error estimator $\eta_h$ for the quadratic $C^0$ interior penalty method is defined by

$$
\eta_h = \left[ \sum_{T \in T_h} \eta_T^2 + \left[ 1 + (\beta^2 + \gamma) h^4 \right] \sum_{e \in \mathcal{E}_h} \eta_{e,1}^2 + \sum_{e \in \mathcal{E}_h^i} \eta_{e,2}^2 + \sum_{e \in \mathcal{E}_h^b} \eta_{e,3}^2 \right]^{1/2}. \tag{3.73}
$$

The following theorem indicates that the error estimator $\eta_h$ is reliable, i.e., it provides an upper bound of the true error.
**Theorem 3.14.** There exists a positive constant $C$ depending only on the shape regularity of $T_h$ such that

$$
\|u - u_h\|_h \leq C\eta_h.
$$

(3.74)

**Proof.** It follows from (3.11) that

$$
\|u - u_h\|_h^2 = a_h(u - u_h, u - u_h) + \sum_{e \in E_h} (\sigma|e|^{-1}) \left\| \left[ \frac{\partial u_h}{\partial n_e} \right] \right\|_{L^2(e)}^2.
$$

(3.75)

Since the second term on the right-hand side of (3.75) is precisely $\sigma^{-1} \sum_{e \in E_h} \eta_{e,1}^2$, we have only have to estimate the first term.

Let $\chi = E_h u_h \in V^*$. We have, by the Cauchy-Schwarz inequality,

$$
a_h(u - u_h, u - u_h) \leq 2a(u - \chi, u - \chi) + 2a_h(u_h - \chi, u_h - \chi),
$$

(3.76)

and by (3.12), (3.27) and the inverse estimate we find,

$$
a_h(u_h - \chi, u_h - \chi) \lesssim (1 + \beta h^2 + \gamma h^4) \sum_{e \in E_h} \eta_{e,1}^2.
$$

(3.77)

Note that $\|\cdot\|_a$ is a norm on $V^*$. This is obvious if $\gamma > 0$, and in the case where $\gamma = 0$ it follows from a Poincaré-Friedrichs inequality [73]. Therefore it follows from duality that

$$
\|u - \chi\|_a = \sup_{\phi \in V^* \setminus \{0\}} \frac{a(u - \chi, \phi)}{\|\phi\|_a},
$$

(3.78)

and

$$
a(u - \chi, \phi) = a(u, \phi) - A_h(u_h, \phi_h) + a_h(u_h - \chi, \phi) + A_h(u_h, \phi_h) - a_h(u_h, \phi) + (f, \phi - \phi_h) - (q, \phi - \phi_h) + a_h(u_h - \chi, \phi) + A_h(u_h, \phi_h) - a_h(u_h, \phi),
$$

(3.79)

where $\phi_h = \Pi_h \phi \in V_h^*$ is the nodal interpolant of $\phi \in V^*$.
It follows from (3.10) (since $\phi - \phi_h$ vanishes at the vertices of $T_h$) that

$$A_h(u_h, \phi_h) - a_h(u_h, \phi) = - \int_\Omega [\beta \nabla u_h \cdot \nabla (\phi - \phi_h)] \, dx$$

$$- \sum_{e \in E_h} \int_e \left[ \frac{\partial^2 u_h}{\partial n_e^2} \right] \left\{ \frac{\partial (\phi - \phi_h)}{\partial n_e} \right\} \, ds$$

$$+ \sum_{e \in E_h} \int_e \left[ \frac{\partial u_h}{\partial n_e} \right] \left\{ \frac{\partial^2 \phi_h}{\partial n_e^2} \right\} \, ds$$

$$+ \sum_{e \in E_h} \frac{\sigma}{|e|} \int_e \left[ \frac{\partial u_h}{\partial n_e} \right] \left[ \frac{\partial \phi_h}{\partial n_e} \right] \, ds$$

Substituting (3.80) into (3.79), we arrive at

$$a(u - \chi, \phi) = \sum_{T \in \mathcal{T}} \int_T (f + \beta \Delta u_h - \gamma u_h)(\phi - \phi_h) \, dx$$

$$- \langle q, \phi - \phi_h \rangle + a_h(u_h - \chi, \phi)$$

$$+ \sum_{e \in E_h} \int_e \left[ \frac{\partial u_h}{\partial n_e} \right] \left\{ \frac{\partial^2 \phi_h}{\partial n_e^2} \right\} \, ds$$

$$+ \sum_{e \in E_h} \frac{\sigma}{|e|} \int_e \left[ \frac{\partial u_h}{\partial n_e} \right] \left[ \frac{\partial \phi_h}{\partial n_e} \right] \, ds$$

$$- \sum_{e \in E_h} \int_e \left[ \frac{\partial^2 u_h}{\partial n_e^2} \right] \left[ \frac{\partial (\phi_h - \phi)}{\partial n_e} \right] \, ds$$

$$+ \beta \sum_{e \in E_h} \int_e \left[ \frac{\partial u_h}{\partial n_e} \right] (\phi - \phi_h) \, ds.$$

Next we will estimate the terms on the right-hand side of (3.81).

By (3.36), the Cauchy-Schwarz inequality and the standard interpolation error
estimates, we have
\[
\left| \sum_{T \in T} \int_T (f + \beta \Delta u_h - \gamma u_h)(\phi - \phi_h) \, dx \right|
\lesssim \left( \sum_{T \in T} \eta_T^2 \right)^{1/2} |\phi|_{H^2(\Omega)}
\]

Using the Cauchy-Schwarz inequality, the trace theorem with scaling and the standard interpolation error estimates and (3.36), we find
\[
|\langle q, \phi - \phi_h \rangle| \leq \left( \sum_{e \in E} |e|^{-1} \|q\|_{L_2(e)}^2 \right)^{1/2} \left( \sum_{e \in E} |e|^{-3} \|\phi - \phi_h\|_{L_2(e)}^2 \right)^{1/2}
\lesssim \left( \sum_{e \in E} \eta_{e,3}^2 \right)^{1/2} |\phi|_{H^2(\Omega)} \leq \left( \sum_{e \in E} \eta_{e,3}^2 \right)^{1/2} \|\phi\|_a.
\]

In view of (3.77), we have
\[
|a_h(u_h - \chi, \phi)| \lesssim (1 + \beta^{1/2} h + \gamma^{1/2} h^2) \left( \sum_{e \in E_h} \eta_{e,1}^2 \right)^{1/2} \|\phi\|_a.
\]

It follows from the trace theorem with scaling and (3.36) that
\[
\left| \sum_{e \in E_h} \int_e \left[ \left[ \frac{\partial u_h}{\partial n_e} \right] \left\{ \frac{\partial^2 \phi_h}{\partial n_e^2} \right\} \right] \, ds \right|
\lesssim \left( \sum_{e \in E_h} |e|^{-1} \left\{ \left[ \frac{\partial u_h}{\partial n_e} \right] \right\}^2_{L_2(e)} \right)^{1/2} \left( \sum_{e \in E_h} |e| \left\{ \left[ \frac{\partial^2 \phi_h}{\partial n_e^2} \right] \right\}^2_{L_2(e)} \right)^{1/2}
\lesssim \left( \sum_{e \in E_h} \eta_{e,1}^2 \right)^{1/2} \left( \sum_{T \in T_h} |\phi_h|_{H^2(T)}^2 \right)^{1/2} \leq \left( \sum_{e \in E_h} \eta_{e,1}^2 \right)^{1/2} \|\phi\|_a.
\]

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By the standard interpolation error estimates, the trace theorem with scaling and (3.36) we find

\[
\left| \sum_{e \in E_h} \frac{\sigma}{|e|} \int_{e} \left[ \left[ \frac{\partial u_h}{\partial n_e} \right] \left[ \frac{\partial \phi_h}{\partial n_e} \right] \right] ds \right|
\]

\[
= \left| \sum_{e \in E_h} \frac{\sigma}{|e|} \int_{e} \left[ \left[ \frac{\partial u_h}{\partial n_e} \right] \left[ \frac{\partial (\phi_h - \phi)}{\partial n_e} \right] \right] ds \right|
\]

\[
\lesssim \left( \sum_{e \in E_h} \frac{\sigma^2}{|e|} \left\| \left[ \frac{\partial u_h}{\partial n_e} \right] \right\|_{L^2(e)}^2 \right)^{1/2} \left| \phi \right|_{H^2(\Omega)} \lesssim \left( \sum_{e \in E_h} \eta_{e,1}^2 \right)^{1/2} \left\| \phi \right\|_a,
\]

and

\[
\left| \sum_{e \in E_h} \int_{e} \left[ \frac{\partial^2 u_h}{\partial n_e^2} \right] \left\{ \frac{\partial (\phi_h - \phi)}{\partial n_e} \right\} ds \right| \lesssim \left( \sum_{e \in E_h} \eta_{e,2}^2 \right)^{1/2} \left\| \phi \right\|_a,
\]

(3.87)

Combining (3.78), (3.81)-(3.88), we have

\[
\| u - \chi \|_a \lesssim \eta_h.
\]

(3.89)

The estimate (3.74) follows from (3.75)-(3.77) and (3.89).

The following theorem shows that the error estimator \( \eta_h \) is efficient up to terms that decay at higher orders.

**Theorem 3.15.** There exists a positive constant \( C \) depending on the shape regularity of \( T_h \) such that

\[
\eta_h \leq C \left[ \sigma^{1/2} \left[ 1 + (\beta + \gamma^{1/2})h^2 \right] \| u - u_h \|_h + \text{Osc}(f) + \text{Osc}(q) \right].
\]

(3.90)
Proof. If follows immediately from (3.11) that
\[ \sum_{e \in E} \eta_{e,1}^2 \leq \sigma \| u - u_h \|_{h}^2. \] (3.91)

On the other hand, we have, by (3.51) and (3.53)-(3.55),
\[ \sum_{T \in T_h} \eta_T^2 + \sum_{e \in E_h^1} \eta_{e,2}^2 + \sum_{e \in E_h^2} \eta_{e,3}^2 \lesssim \| u - u_h \|_{h}^2 + \text{Osc}(f)^2 + \text{Osc}(q)^2. \] (3.92)
The estimate (3.90) follows from (3.73), (3.91) and (3.92).

Remark 3.16. From (3.74) and (3.90), we have
\[ 1 \lesssim \eta_h / \| u - u_h \|_{h} \lesssim \sigma^{1/2} [1 + (\beta + \gamma^{1/2})h^2]. \]

We want \( \eta_h / \| u - u_h \|_{h} \) to be close to 1 so that the \( \eta_h \) is a good indicator of the true discretization error \( \| u - u_h \|_{h} \). Therefore, a good choice of \( \sigma \) should not be too large while maintaining positive-definiteness of \( A_h(\cdot, \cdot) \). On the other hand, \( \eta_h / \| u - u_h \|_{h} \) depends only on \( \sigma^{1/2} \) not \( \sigma \) and hence it is not difficult to obtain a good choice of \( \sigma \) in practice. Moreover, when the mesh size is small, \( \eta_h / \| u - u_h \|_{h} \) is not sensitive to \( \beta \) and \( \gamma \) either.

### 3.5 Numerical Results

In this section, we report the results of numerical tests carried out for the L-shaped domain with vertices \((0, 0), (1, 0), (1, 1), (-1, 1), (-1, -1)\) and \((0, -1)\).

For simplicity we take \( \gamma = \beta = 0 \). The exact solution in these tests is given by
\[ u = r^{4/3} \cos(\frac{2}{3} \theta)(1 - x_1^2)^2(1 - x_2)^2, \]
where \((r, \theta)\) are the polar coordinates. (So the origin is the corner \( p^* \) in the definitions of \( V^* \) and \( V_h^* \).) The penalty parameter \( \sigma \) is taken to be 5 (cf. Remark 3.16).
First we solve (3.3) on a sequence of uniform meshes, where the initial mesh (cf. Figure 3.4a) has 64 dofs. We compute the error \( e_h = \| u - u_h \|_h \) and the error estimator \( \eta_h \) defined in (3.73) for each solution and tabulated the results in Table 3.1. They confirm both the error estimate in Theorem 3.12 for the L-shaped domain where \( \alpha = 1/3 - \delta \) for any \( \delta > 0 \) (cf. Appendix A in [29]) and the reliability estimate in Theorem 3.14.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( e_h )</th>
<th>( \eta_h )</th>
<th>( \log_2(e_{2h}/e_h) )</th>
<th>( \log_2(\eta_{2h}/\eta_h) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^{-1} )</td>
<td>4.923e + 0</td>
<td>3.446e + 1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( 2^{-2} )</td>
<td>3.009e + 0</td>
<td>1.145e + 1</td>
<td>7.102e - 1</td>
<td>1.590e + 0</td>
</tr>
<tr>
<td>( 2^{-3} )</td>
<td>1.708e + 0</td>
<td>4.697e + 0</td>
<td>8.169e - 1</td>
<td>1.285e + 0</td>
</tr>
<tr>
<td>( 2^{-4} )</td>
<td>1.030e + 0</td>
<td>2.445e + 0</td>
<td>7.293e - 1</td>
<td>9.417e - 1</td>
</tr>
<tr>
<td>( 2^{-5} )</td>
<td>7.074e - 1</td>
<td>1.536e + 0</td>
<td>5.426e - 1</td>
<td>6.708e - 1</td>
</tr>
<tr>
<td>( 2^{-6} )</td>
<td>5.295e - 1</td>
<td>1.090e + 0</td>
<td>4.178e - 1</td>
<td>4.943e - 1</td>
</tr>
<tr>
<td>( 2^{-7} )</td>
<td>4.113e - 1</td>
<td>8.257e - 1</td>
<td>3.643e - 1</td>
<td>4.011e - 1</td>
</tr>
</tbody>
</table>

We then compute the solution \( u_i (i \geq 1) \) using a sequence of triangulations \( T_i (i \geq 1) \) generated adaptively from the initial mesh \( T_1 \) (cf. Figure 3.4a). The triangulation \( T_i (i \geq 2) \) is obtained from \( T_{i-1} \) by bisecting the marked triangles and edges of the mesh \( T_{i-1} \), which are marked using the \textit{a posteriori} error estimator \( \eta_h \) defined in (3.73) and the bulk criterion of Dörfler [46, 69]. The new triangulation is obtained by the longest edge bisection strategy and the hanging nodes are eliminated by the blue and green refinement [75, 76, 79] (cf. Appendix). It is clear from the meshes in Figure 3.4 that the error estimator \( \eta_h \) captures the singularity of the solution \( u \) successfully.

The numerical values of the error and error estimator associated with the adaptive meshes and the uniform meshes are plotted as functions of the number of dofs in Figure 3.5a. It is observed that the errors associated with the adaptive meshes are significantly smaller than the errors associated with the uniform meshes. Moreover the order of convergence for the solutions computed with the adaptive meshes...
appears to be optimal.

We have also plotted the efficiency index $\eta_h / \|u - u_h\|_h$ against the number of dofs in Figure 3.5b for $\sigma = 5, 10, 20, \text{ and } 50$. The asymptotic behavior of the efficiency index agrees with the efficiency estimate in Theorem 3.15 (cf. Remark 3.16).

Figure 3.4: Adaptive Meshes. A set of triangles are marked using the bulk criterion of Dörfler [46, 69]. The longest edge bisection strategy is used to obtain the new triangulation [75, 76].
Figure 3.5: Performance of the error estimator
Chapter 4

Multigrid Methods

The discrete problem (3.8) leads to a very ill-conditioned system whose condition number grows at the order of $O(h^{-4})$ [65]. Preconditioners are needed for an efficient linear solve. In this section, we present multigrid algorithms for the $C^0$ interior penalty methods.

4.1 Set-Up

In the case where $\gamma = 0$, the solutions of the continuous problem (3.3) differ by an additive constant. In Chapter 2, we looked for the solution in the subspace $V^*$ whose functions are evaluated to zero at a chosen corner. In this chapter, we will look for the solution in the zero mean subspace instead. We will consider the problem:

Find $u \in V^{**}$ such that

$$a(u, v) = (f, v) - \langle q, v \rangle \quad \forall v \in V^{**}$$  (4.1)
where $V^{**}$ is defined by

$$V^{**} = \begin{cases} V & \text{if } \gamma > 0 \\ \{ v \in V : \int \! v dx = 0 \} & \text{if } \gamma = 0 \end{cases}$$

and $V$ is defined in (3.2). In the case where $\gamma = 0$, we assume the compatibility condition (3.4) so that the continuous problem (4.1) is uniquely solvable.

Let $T_k$ $(k \geq 0)$ be a sequence of triangulation obtained by uniform refinement, i.e., $T_k$ $(k \geq 1)$ is obtained by connecting the midpoints of the edges in $T_{k-1}$ and dividing each triangle in $T_{k-1}$ into four similar triangles. Let $V_k$ be the $P_2$ Lagrange finite element spaces associated with the triangulation and $V_k^{**}$ be defined by

$$V_k^{**} = \begin{cases} V_k & \text{if } \gamma > 0 \\ \{ v \in V_k : \int \! v dx = 0 \} & \text{if } \gamma = 0 \end{cases}$$

In this section, we will consider the following discrete problem:

Find $u_k \in V_k^{**}$ such that

$$\mathcal{A}_k(u_k, v_k) = (f, v_k) - \langle q, v_k \rangle \quad \forall v_k \in V_k^{**}, \quad (4.2)$$

where $\mathcal{A}_k(\cdot, \cdot)$ is defined in (3.9).

**Remark 4.1.** The discrete bilinear form $\mathcal{A}_k(\cdot, \cdot)$ is bounded and coercive on $V_k^{**}$ (cf. Lemma 3.2 and Lemma 3.3) and hence the discrete problem (4.2) is uniquely solvable.
Remark 4.2. The dual space of $V^{**}$ and $V_k^{**}$ can be identified as

$$(V^{**})' = \begin{cases} V' & \text{if } \gamma > 0 \\ \{\zeta \in V' : \langle \zeta, \vec{1} \rangle = 0\} & \text{if } \gamma = 0 \end{cases}$$

and

$$(V_k^{**})' = \begin{cases} V_k' & \text{if } \gamma > 0 \\ \{\zeta \in V_k' : \langle \zeta, \vec{1} \rangle = 0\} & \text{if } \gamma = 0 \end{cases}$$

where $\vec{1}$ is the constant function one.

Define the operator $A_k : V_k \to V_k'$ by

$$\langle A_k v_k, w_k \rangle = A_k(v_k, w_k) \quad \forall v_k, w_k \in V_k$$

and the linear functional $\phi_k \in V_k'$ by

$$\langle \phi_k, v \rangle = (f, v) - \langle q, v \rangle \quad \forall v \in V_k.$$ 

The equation (4.2) can be rewritten as

$$A_k u_k = \phi_k.$$ 

Remark 4.3. In the case where $\gamma = 0$, $\phi_k \in (V_k^{**})'$ if the compatibility condition (3.4) is satisfied.
4.2 Algorithms

We define a mesh-dependent inner product \((\cdot, \cdot)_k\) on \(V_k\) by

\[
(v, w)_k = \frac{1}{3} \sum_{p \in \mathcal{N}_k} \left( \sum_{T \in \mathcal{T}_p} |T| \right) v(p) w(p) \quad \forall v, w \in V_k,
\]

where \(\mathcal{N}_k\) is the set of all nodes (vertices and midpoints) in \(\mathcal{T}_k\), \(\mathcal{T}_p\) is the set of the triangles sharing \(p\) as a common node and \(|T|\) denotes the area of the triangle \(T\).

**Remark 4.4.**

\[
(v, v)_k \approx \|v\|^2_{L^2(\Omega)} \quad \forall v \in V_k.
\]

**Remark 4.5.** Define \(\chi_k \in V_k\) by

\[
\chi_k(p) = \begin{cases} 
1 & \text{if } p \in \mathcal{M}_k, \\
0 & \text{otherwise},
\end{cases}
\]

where \(\mathcal{M}_k\) is the set of midpoints of all edges of the triangles in \(\mathcal{T}_k\). Then

\[
\int_{\Omega} v dx = (v, \chi_k)_k \quad \forall v \in V_k.
\]

The equality (4.9) is due to the fact that \(\int_T v dx = \frac{|T|}{3} \sum_{m \in \mathcal{M}_T} v(m)\) for arbitrary quadratic function \(v\), where \(\mathcal{M}_T\) is the set of three midpoints of \(T\).

Define \(\hat{B}_k : V_k^{**} \to (V_k^{**})'\) by

\[
\langle \hat{B}_k v, w \rangle = (v, w)_k \quad \forall v, w \in V_k^{**}.
\]

Using the operator \(\hat{B}_k\), we can write down our first smoother for (4.5) which is just the Richardson relaxation with respect to the mesh-dependent inner product (4.6).
Definition 4.6. (the first smoother)

For \( j = 1, 2, \ldots, m \),

\[
    z_j = z_{j-1} + \lambda_k \hat{B}_k^{-1}(\phi_k - A_k z_{j-1}) \tag{4.11}
\]

where \( \lambda_k \) is chosen such that \( \rho(\lambda_k \hat{B}_k^{-1} A_k) < 2 \) where \( \rho(\cdot) \) denotes the spectral radius. Note that \( \lambda_k \) is of order \( O(h_k^4) \).

Remark 4.7. In the case where \( \gamma = 0 \), \( V_k^{**} \) is the zero mean subspace of \( V_k \). However, we do not need an explicit basis for \( V_k^{**} \). The computation of \( \hat{B}_k^{-1} \) can be carried out using just the nodal basis of \( V_k \). Let \( B_k : V_k \to V'_k \) be defined by

\[
    \langle B_k w, v \rangle = (v, w)_k \quad \forall v, w \in V_k.
\]

The matrix representing the operator \( B_k \) with respect to the natural nodal basis in \( V_k \) and its dual basis in \( V'_k \) is a diagonal matrix. Let \( \hat{P}_k : V_k \to V_k^{**} \) be the projection with respect to the inner product \( (\cdot, \cdot)_k \), i.e.

\[
    \hat{P}_k v_k = v_k - ((v_k, \chi_k)_k/(\chi_k, \chi_k)_k) \chi_k, \tag{4.12}
\]

where \( \chi_k \) is defined in (4.8). Then

\[
    \hat{B}_k^{-1} \zeta_k = \hat{P}_k B_k^{-1} \zeta_k \quad \forall \zeta_k \in (V_k^{**})'. \tag{4.13}
\]

Since all eigenvalues of the operator \( \hat{B}_k \) are equivalent to \( h_k^2 \), the first smoother (4.11) is an unpreconditioned Richardson relaxation scheme and the condition number of \( \hat{B}_k^{-1} A_k \) is of the same order as that of \( A_k \) which is \( O(h_k^{-4}) \). The first smoother (4.11) is hence not effective. We will consider a different smoother next.
Define $L_k : V^{**}_k \to (V^{**}_k)'$ by

$$
\langle L_k v, w \rangle = \int_\Omega \nabla v \cdot \nabla w \, dx + \int_\Omega r v w \, dx \quad \forall v, w \in V^{**}_k, \quad (4.14)
$$

where $r = 0$ if $\gamma = 0$; otherwise, $r$ is a positive constant.

$L_k$ is a bijection from $V^{**}_k$ to $(V^{**}_k)'$ and computing $L_k^{-1}$ amounts to solving a second order problem.

For $\zeta_k \in (V^{**}_k)'$, define $\hat{S}^{-1}_k$ to be a multigrid solve [8] for the second order problem:

Find $s_k \in V^{**}_k$, such that

$$
\langle L_k s_k, w_k \rangle = \langle \zeta_k, w_k \rangle \quad \forall w_k \in V^{**}_k. \quad (4.15)
$$

It can be shown [24, 34] that if $\hat{S}^{-1}_k$ is obtained by a symmetric V-, W-cycle algorithm or a symmetric variable V-cycle algorithm. Then

$$
\langle \hat{S}_k v, v \rangle \approx \|v\|^2_{H^1(\Omega)} \quad \forall v \in V^{**}_k. \quad (4.16)
$$

**Remark 4.8.** Since the underlying finite element spaces of $C^0$ interior penalty methods are just Lagrange finite elements which are standard spaces for second order problems, $\hat{S}^{-1}_k$ can be implemented easily.

We can define the second smoother which is a preconditioned Richardson relaxation to the equation (4.5).

**Definition 4.9.** *(the second smoother)*

For $j = 1, 2, \ldots, m,$

$$
z_j = z_{j-1} + \lambda_k \hat{S}^{-1}_k (\phi_k - A_k z_{j-1}) \quad (4.17)
$$

where $\lambda_k$ is chosen such that $\rho(\lambda_k \hat{S}^{-1}_k A_k) < 2$ where $\rho(\cdot)$ denotes the spectral radius.
Note that $\lambda_k$ is of order $O(h_k^2)$.

**Remark 4.10.** It follows from (4.16) and inverse estimate that the condition number of $\hat{S}_k^{-1} A_k$ is of order $O(h_k^{-2})$ which is the same as second order problems.

Let the operator $I_{k-1}^k : V_{k-1} \rightarrow V_k$ be the natural injection and the operator $I_{k}^{k-1} : V'_{k} \rightarrow V'_{k-1}$ be the transpose of $I_{k-1}^k$ (cf. (2.27)-(2.28)). Then $I_{k-1}^k v_{k-1} \in V_{k-1}^{**}$ for all $v_{k-1} \in V_{k-1}^{**}$ and $I_{k}^{k-1} \zeta_k \in (V_{k-1}^{**})'$ for all $\zeta_k \in (V_k^{**})'$. The $V$, $W$, $F$-cycle algorithms for equation (4.5) are defined by replacing the presmoothing and postsmoothing steps in Algorithm 2.18-2.20 with the smoother defined in (4.11) or (4.17).

### 4.3 Numerical Results

Let $MG^N(k, m, z_0, \phi_k)$ (resp. $MG(k, m, z_0, \phi_k)$) be the $k^{th}$ level $V$, $W$- or $F$-cycle multigrid approximation to the discrete problem (4.5) with the first smoother (4.11) (resp. the second smoother (4.17)), $m$ presmoothing and $m$ postsmoothing steps and initial guess $z_0$. Define the norm $\| \cdot \|_A$ on $V_k^{**}$ by

$$\|v\|_A = (\langle A_k(v, v) \rangle)^{1/2} \quad \forall v \in V_k^{**}.$$  

Define the contraction numbers by $\mu_{k,m}^N = \| u_k - MG^N(k, m, z_0, \phi_k) \|_{A_k} / \| u_k - z_0 \|_{A_k}$ (resp. $\mu_{k,m} = \| u_k - MG(k, m, z_0, \phi_k) \|_{A_k} / \| u_k - z_0 \|_{A_k}$).

All experiments in this section are carried out for the singular problem $\beta = 0, \gamma = 0$. The penalty parameter $\sigma$ is taken to be 5 and the preconditioner $\hat{S}_k^{-1}$ is obtained by a $V$-cycle multigrid solve with one presmoothing step and one postsmoothing step for the second order problem (4.15).

The first set of experiments is for the unit square with the initial mesh $T_0$ containing two triangles. In this case, the index of elliptic regularity $\alpha = 1$. The
contraction numbers for the V-, W-, F-cycle algorithms with the second smoother (4.17) are reported in Table 4.1-Table 4.3. For comparison, the contraction numbers for the first smoother (4.11) are reported in Table 4.4-Table 4.6. It is observed that the multigrid algorithm with the second smoother is much more effective than with the first smoother as expected. We then plot the contraction numbers versus the number of smoothing steps \( m \) for the V-cycle algorithms with the first smoother (4.11) and the second smoother (4.17) in Figure 4.1. It looks like that the contraction numbers decrease at the rate of \( m^{-1/2} \) for the first smoother (4.11) and at the rate of \( m^{-1/2} \) for the first smoother (4.11).

The second set of experiments is for the L-shaped domain with the vertices \((0,0), (1,0), (1,1), (-1,1), (-1,-1)\) and \((0,-1)\). The initial mesh \( T_0 \) consists of six isosceles triangles sharing \((0,0)\) as a common vertex. For the L-shaped domain, the index of elliptic regularity \( \alpha = 1/3 - \delta \) for arbitrary small positive \( \delta \).

For the W-cycle algorithms with the first smoother and second smoother are reported in Table 4.7 and Table 4.8. The asymptotic behavior of the contraction numbers versus the number of smoothing steps is plotted in Figure 4.2. Again, the multigrid algorithm with the second smoother is much more effective than with the first smoother. Moreover, the contraction numbers seem to decrease at the rate of \( m^{-1/3} \) for the second smoother (4.17) and at the rate of \( m^{-1/6} \) for the first smoother (4.11).

**Remark 4.11.** Unlike the multigrid algorithms for conforming finite element methods, we do not observe contraction with only one smoothing step for the \( C^0 \) interior penalty methods. This is typical for multigrid algorithms for nonconforming finite element methods [25, 26].

**Remark 4.12.** In the case of Dirichlet boundary conditions where the boundary data \( u \) and \( \partial u / \partial n \) are given, it can be shown that the contraction numbers decrease
Figure 4.1: Asymptotic rate of decrease for the V-cycle algorithm (level $k = 7$) on the square.

Table 4.1: Contraction numbers for the V-cycle algorithm with the second smoother (4.17) on the square.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$m$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>0.212</td>
<td>0.126</td>
<td>0.0813</td>
<td>0.0594</td>
<td>0.0442</td>
<td>0.0332</td>
<td>0.0252</td>
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<td>0.0147</td>
</tr>
<tr>
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<td></td>
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<td>0.124</td>
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<td>0.0967</td>
<td>0.0861</td>
</tr>
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<td>3</td>
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<td>0.342</td>
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<td>0.234</td>
<td>0.217</td>
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</table>
Table 4.2: Contraction numbers for the W-cycle algorithm with the second smoother (4.17) on the square.

<table>
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<tbody>
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<td>0.661</td>
<td>0.368</td>
<td>0.212</td>
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<td>0.0813</td>
<td>0.0594</td>
<td>0.0442</td>
<td>0.0332</td>
<td>0.0252</td>
</tr>
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<td>0.483</td>
<td>0.360</td>
<td>0.291</td>
<td>0.241</td>
<td>0.203</td>
<td>0.172</td>
<td>0.148</td>
<td>0.128</td>
<td>0.112</td>
</tr>
<tr>
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<td>0.475</td>
<td>0.375</td>
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<td>0.282</td>
<td>0.263</td>
<td>0.229</td>
<td>0.215</td>
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<td>0.182</td>
</tr>
<tr>
<td>4</td>
<td>0.455</td>
<td>0.383</td>
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<td>0.270</td>
<td>0.256</td>
<td>0.244</td>
<td>0.233</td>
</tr>
<tr>
<td>5</td>
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<td>0.384</td>
<td>0.344</td>
<td>0.315</td>
<td>0.297</td>
<td>0.279</td>
<td>0.267</td>
<td>0.255</td>
<td>0.245</td>
</tr>
<tr>
<td>6</td>
<td>0.455</td>
<td>0.384</td>
<td>0.344</td>
<td>0.316</td>
<td>0.297</td>
<td>0.280</td>
<td>0.268</td>
<td>0.256</td>
<td>0.248</td>
</tr>
<tr>
<td>7</td>
<td>0.455</td>
<td>0.384</td>
<td>0.344</td>
<td>0.317</td>
<td>0.297</td>
<td>0.281</td>
<td>0.269</td>
<td>0.258</td>
<td>0.248</td>
</tr>
</tbody>
</table>

Table 4.3: Contraction numbers for the F-cycle algorithm with the second smoother (4.17) on the square.

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>4</th>
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<th>9</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.368</td>
<td>0.212</td>
<td>0.126</td>
<td>0.0813</td>
<td>0.0594</td>
<td>0.0442</td>
<td>0.0332</td>
<td>0.0252</td>
<td>0.0192</td>
</tr>
<tr>
<td>2</td>
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<td>0.241</td>
<td>0.203</td>
<td>0.172</td>
<td>0.148</td>
<td>0.128</td>
<td>0.112</td>
<td>0.0983</td>
</tr>
<tr>
<td>3</td>
<td>0.375</td>
<td>0.334</td>
<td>0.282</td>
<td>0.262</td>
<td>0.229</td>
<td>0.215</td>
<td>0.195</td>
<td>0.182</td>
<td>0.171</td>
</tr>
<tr>
<td>4</td>
<td>0.383</td>
<td>0.336</td>
<td>0.308</td>
<td>0.287</td>
<td>0.270</td>
<td>0.256</td>
<td>0.244</td>
<td>0.233</td>
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<tr>
<td>5</td>
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<td>0.315</td>
<td>0.297</td>
<td>0.279</td>
<td>0.267</td>
<td>0.255</td>
<td>0.245</td>
<td>0.237</td>
</tr>
<tr>
<td>6</td>
<td>0.385</td>
<td>0.344</td>
<td>0.316</td>
<td>0.297</td>
<td>0.280</td>
<td>0.268</td>
<td>0.256</td>
<td>0.248</td>
<td>0.239</td>
</tr>
<tr>
<td>7</td>
<td>0.386</td>
<td>0.345</td>
<td>0.317</td>
<td>0.297</td>
<td>0.281</td>
<td>0.269</td>
<td>0.258</td>
<td>0.248</td>
<td>0.240</td>
</tr>
</tbody>
</table>

Table 4.4: Contraction numbers for the V-cycle algorithm with the first smoother (4.11) on the square.

<table>
<thead>
<tr>
<th></th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
<th>29</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.428</td>
<td>0.410</td>
<td>0.392</td>
<td>0.376</td>
<td>0.361</td>
<td>0.346</td>
<td>0.332</td>
<td>0.320</td>
<td>0.307</td>
</tr>
<tr>
<td>2</td>
<td>0.646</td>
<td>0.614</td>
<td>0.583</td>
<td>0.555</td>
<td>0.529</td>
<td>0.504</td>
<td>0.481</td>
<td>0.459</td>
<td>0.439</td>
</tr>
<tr>
<td>3</td>
<td>0.770</td>
<td>0.728</td>
<td>0.690</td>
<td>0.654</td>
<td>0.621</td>
<td>0.591</td>
<td>0.562</td>
<td>0.535</td>
<td>0.510</td>
</tr>
<tr>
<td>4</td>
<td>0.844</td>
<td>0.797</td>
<td>0.753</td>
<td>0.713</td>
<td>0.676</td>
<td>0.641</td>
<td>0.609</td>
<td>0.579</td>
<td>0.551</td>
</tr>
<tr>
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<td>0.843</td>
<td>0.795</td>
<td>0.752</td>
<td>0.711</td>
<td>0.674</td>
<td>0.639</td>
<td>0.607</td>
<td>0.577</td>
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<tr>
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<td>0.931</td>
<td>0.876</td>
<td>0.826</td>
<td>0.780</td>
<td>0.737</td>
<td>0.697</td>
<td>0.661</td>
<td>0.627</td>
<td>0.595</td>
</tr>
<tr>
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<td>0.960</td>
<td>0.902</td>
<td>0.849</td>
<td>0.801</td>
<td>0.757</td>
<td>0.715</td>
<td>0.677</td>
<td>0.642</td>
<td>0.609</td>
</tr>
</tbody>
</table>
Table 4.5: Contraction numbers for the W-cycle algorithm with the first smoother (4.11) on the square.

<table>
<thead>
<tr>
<th>k</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.967</td>
<td>0.893</td>
<td>0.830</td>
<td>0.774</td>
<td>0.725</td>
<td>0.682</td>
<td>0.642</td>
<td>0.607</td>
<td>0.575</td>
</tr>
<tr>
<td>2</td>
<td>0.824</td>
<td>0.692</td>
<td>0.604</td>
<td>0.541</td>
<td>0.498</td>
<td>0.463</td>
<td>0.437</td>
<td>0.414</td>
<td>0.395</td>
</tr>
<tr>
<td>3</td>
<td>0.696</td>
<td>0.534</td>
<td>0.462</td>
<td>0.447</td>
<td>0.435</td>
<td>0.424</td>
<td>0.415</td>
<td>0.405</td>
<td>0.397</td>
</tr>
<tr>
<td>4</td>
<td>0.514</td>
<td>0.484</td>
<td>0.474</td>
<td>0.457</td>
<td>0.448</td>
<td>0.437</td>
<td>0.429</td>
<td>0.420</td>
<td>0.413</td>
</tr>
<tr>
<td>5</td>
<td>0.527</td>
<td>0.500</td>
<td>0.489</td>
<td>0.471</td>
<td>0.461</td>
<td>0.449</td>
<td>0.440</td>
<td>0.431</td>
<td>0.424</td>
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<tr>
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<td>0.501</td>
<td>0.490</td>
<td>0.472</td>
<td>0.463</td>
<td>0.450</td>
<td>0.442</td>
<td>0.433</td>
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</tr>
<tr>
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<td>0.532</td>
<td>0.505</td>
<td>0.492</td>
<td>0.474</td>
<td>0.464</td>
<td>0.451</td>
<td>0.442</td>
<td>0.433</td>
<td>0.425</td>
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</table>

Table 4.6: Contraction numbers for F-cycle algorithm with the first smoother (4.11) on the square.

<table>
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<th>19</th>
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<th>22</th>
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<tbody>
<tr>
<td>1</td>
<td>0.607</td>
<td>0.575</td>
<td>0.545</td>
<td>0.518</td>
<td>0.493</td>
<td>0.470</td>
<td>0.448</td>
<td>0.428</td>
<td>0.410</td>
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<tr>
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<td>0.414</td>
<td>0.395</td>
<td>0.377</td>
<td>0.362</td>
<td>0.348</td>
<td>0.335</td>
<td>0.323</td>
<td>0.312</td>
<td>0.302</td>
</tr>
<tr>
<td>3</td>
<td>0.406</td>
<td>0.398</td>
<td>0.390</td>
<td>0.382</td>
<td>0.375</td>
<td>0.369</td>
<td>0.363</td>
<td>0.357</td>
<td>0.351</td>
</tr>
<tr>
<td>4</td>
<td>0.418</td>
<td>0.413</td>
<td>0.406</td>
<td>0.400</td>
<td>0.395</td>
<td>0.390</td>
<td>0.385</td>
<td>0.380</td>
<td>0.376</td>
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<tr>
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<td>0.431</td>
<td>0.424</td>
<td>0.417</td>
<td>0.410</td>
<td>0.405</td>
<td>0.399</td>
<td>0.394</td>
<td>0.390</td>
<td>0.386</td>
</tr>
<tr>
<td>6</td>
<td>0.434</td>
<td>0.425</td>
<td>0.418</td>
<td>0.412</td>
<td>0.406</td>
<td>0.401</td>
<td>0.396</td>
<td>0.391</td>
<td>0.387</td>
</tr>
<tr>
<td>7</td>
<td>0.546</td>
<td>0.425</td>
<td>0.418</td>
<td>0.412</td>
<td>0.406</td>
<td>0.400</td>
<td>0.395</td>
<td>0.391</td>
<td>0.387</td>
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</table>

Table 4.7: Contraction numbers for the W-cycle algorithm with the second smoother (4.17) on the L-shaped domain.

<table>
<thead>
<tr>
<th>k</th>
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<th>15</th>
<th>17</th>
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<td>0.105</td>
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<td>0.0699</td>
<td>0.0614</td>
<td>0.0540</td>
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<td>0.273</td>
<td>0.206</td>
<td>0.161</td>
<td>0.139</td>
<td>0.132</td>
<td>0.125</td>
<td>0.119</td>
<td>0.113</td>
</tr>
<tr>
<td>3</td>
<td>0.390</td>
<td>0.302</td>
<td>0.238</td>
<td>0.208</td>
<td>0.182</td>
<td>0.163</td>
<td>0.152</td>
<td>0.148</td>
<td>0.144</td>
</tr>
<tr>
<td>4</td>
<td>0.386</td>
<td>0.309</td>
<td>0.271</td>
<td>0.245</td>
<td>0.224</td>
<td>0.208</td>
<td>0.193</td>
<td>0.181</td>
<td>0.170</td>
</tr>
<tr>
<td>5</td>
<td>0.384</td>
<td>0.315</td>
<td>0.279</td>
<td>0.255</td>
<td>0.237</td>
<td>0.222</td>
<td>0.209</td>
<td>0.198</td>
<td>0.189</td>
</tr>
<tr>
<td>6</td>
<td>0.384</td>
<td>0.316</td>
<td>0.281</td>
<td>0.257</td>
<td>0.240</td>
<td>0.226</td>
<td>0.213</td>
<td>0.203</td>
<td>0.193</td>
</tr>
<tr>
<td>7</td>
<td>0.387</td>
<td>0.317</td>
<td>0.281</td>
<td>0.258</td>
<td>0.240</td>
<td>0.226</td>
<td>0.214</td>
<td>0.203</td>
<td>0.194</td>
</tr>
</tbody>
</table>
Figure 4.2: Asymptotic rate of decrease for the W-cycle algorithm (level $k = 7$) on the L-shaped domain.

Table 4.8: Contraction numbers for the W-cycle algorithm with the first smoother (4.11) on the L-shaped domain.

<table>
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<th>$k$</th>
<th>$m$</th>
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<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
<th>21</th>
<th>23</th>
</tr>
</thead>
<tbody>
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<td>0.943</td>
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<td>0.680</td>
<td>0.600</td>
<td>0.537</td>
<td>0.486</td>
<td>0.443</td>
<td>0.407</td>
<td>0.375</td>
<td>0.347</td>
</tr>
<tr>
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<td>0.790</td>
<td>0.585</td>
<td>0.505</td>
<td>0.459</td>
<td>0.426</td>
<td>0.394</td>
<td>0.375</td>
<td>0.358</td>
<td>0.342</td>
<td>0.328</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.666</td>
<td>0.512</td>
<td>0.469</td>
<td>0.456</td>
<td>0.434</td>
<td>0.416</td>
<td>0.400</td>
<td>0.386</td>
<td>0.373</td>
<td>0.362</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.580</td>
<td>0.519</td>
<td>0.484</td>
<td>0.454</td>
<td>0.434</td>
<td>0.418</td>
<td>0.405</td>
<td>0.394</td>
<td>0.385</td>
<td>0.376</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0.581</td>
<td>0.527</td>
<td>0.491</td>
<td>0.465</td>
<td>0.444</td>
<td>0.427</td>
<td>0.414</td>
<td>0.402</td>
<td>0.392</td>
<td>0.384</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>0.587</td>
<td>0.531</td>
<td>0.494</td>
<td>0.467</td>
<td>0.446</td>
<td>0.429</td>
<td>0.415</td>
<td>0.404</td>
<td>0.394</td>
<td>0.386</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>0.587</td>
<td>0.530</td>
<td>0.493</td>
<td>0.467</td>
<td>0.446</td>
<td>0.429</td>
<td>0.415</td>
<td>0.404</td>
<td>0.394</td>
<td>0.386</td>
</tr>
</tbody>
</table>
at the rate of $m^{-\alpha}$ for the preconditioned Richardson smoothing scheme and at the rate of $m^{-\alpha/2}$ for the unpreconditioned Richardson smoothing scheme using the additive multigrid theory [25, 26, 34].

In the next set of experiments, we investigate the computational cost of the multigrid algorithms for (4.5). This set of experiments is for the unit square with the initial mesh $\mathcal{T}_0$ consisting of two triangles. For the second smoother, the preconditioner $\hat{S}_k^{-1}$ is obtained from a V-cycle multigrid solve with one presmoothing step and one postsmoothing step for the second order problem (4.15). On level $\mathcal{T}_9$, there are about 1 million degrees of freedom with the condition number of $10^{12}$. The timing is performed on a PC with a Tesla T10 GPU, Intel Xeon(R) E5403 2.66GHz CPU and 16G memory. The computational cost is summarized in Table 4.9.

When the GPU is enabled, sparse matrix and vector multiplication (SPMV), and vector and vector operations are performed in the GPU. The coarse level solve (LU direct solve) is performed on the CPU. Sparse matrix is stored in ELL format [12] and SPMV comes from CUSP [11]. Vector and vector operations, data transfer between CPU/GPU are taken from PETSc [6].

As shown in the Table 4.9, with the second smoother, a V-cycle $k^{th}$ level multigrid iteration with four presmoothing steps and four postsmoothing steps achieves contraction number 0.59. To achieve a similar contraction number, the V-cycle algorithm with the first smoother requires 29 presmoothing and 29 postsmoothing steps and more than twice as many floating point operations. If we want a smaller contraction number, the computational benefit using the second smoother is even larger. This is expected because contraction numbers for the multigrid algorithms with the second smoother decrease at a faster rate (cf. Remark 4.12).

Moreover, since the matrix vector multiplication accounts for a large portion of the
Table 4.9: Computational cost for V-cycle multigrid algorithms (level 9, dofs=1M).

<table>
<thead>
<tr>
<th>Methods</th>
<th>Prec.</th>
<th>flops</th>
<th>$\gamma_{k,m}$</th>
<th>w/ GPU</th>
<th>w/o GPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>V(4,4)</td>
<td>V(1,1)</td>
<td>2.1B</td>
<td>0.59</td>
<td>0.82s</td>
<td>11s</td>
</tr>
<tr>
<td>V(29,29)</td>
<td>-</td>
<td>4.8B</td>
<td>0.61</td>
<td>1.4s</td>
<td>24s</td>
</tr>
<tr>
<td>V(9,9)</td>
<td>V(1,1)</td>
<td>4.6B</td>
<td>0.41</td>
<td>1.7s</td>
<td>25s</td>
</tr>
<tr>
<td>V(175,175)</td>
<td>-</td>
<td>28B</td>
<td>0.41</td>
<td>8.1s</td>
<td>143s</td>
</tr>
</tbody>
</table>

The total computational cost in our test problem, the generic GPU implementation for SPMV can speed up the algorithms. In this experiment, the coarse level solve is performed in CPU because the initial mesh has very few degrees of freedom. If the initial mesh has a large number of degrees of freedom, it might be beneficial to perform the coarse level solve on GPUs.
Chapter 5

Applications

In this chapter, we will use the semi-implicit time stepping scheme coupled with the quadratic $C^0$ interior penalty method at each time step to solve the Cahn-Hilliard equations (1.3) and (1.4). Throughout this chapter, we choose $C_1 = 5/\epsilon$ and $C_2 = 3\lambda_0$ in equations (1.10) and (1.11). The time step $\Delta t$ and the parameter $\epsilon$ vary and are reported in each numerical example.

5.1 Phase Separation

The first experiment is for the Cahn-Hilliard equation (1.3) on the square domain $[0, 1.28] \times [0, 1.28]$. The thickness parameter $\epsilon$ is taken to be $5 \times 10^{-3}$, $\Delta t$ is taken to be $5 \times 10^{-6}$ and the penalty parameter $\sigma$ is taken to be 5. At each time step, we use multigrid methods to solve the discrete problem (3.8). The initial (level 0) mesh contains two triangles and we go up to the mesh at level 7 that contains $66K$ degrees of freedom. We use the result from the previous time step as an initial guess and keep applying V-cycle multigrid algorithm (cf. Algorithm 2.18) with four presmoothing steps and four postsmoothing steps until the $\|z^j - z^{j-1}\|_A / \|z^j\|_A \leq 10^{-3}$ where $z^{j-1}$ and $z^j$ are the outputs of two consecutive
multigrid cycles. Phase separation is observed (Figure 5.1). Note that even though the solutions of the discrete problem are computed approximately by multigrid algorithms, the mass conserves very well.

The next experiment is for the diamond-shaped domain with four vertices 
(1.28, 0.85), (0, 0.85), (0.32, 1.28), (0.96, 1.28) (1.28, 0.85). The thickness parameter $\epsilon$ is set to $5 \times 10^{-3}$ and $\sigma$ is taken to be 20. The time step $\Delta t$ is set to $5 \times 10^{-6}$ for the first 3000 steps and then switched to 0.5 for another 3000 steps. The discrete system is solved by the conjugate gradient method. Phase separation and mass conservation is observed (Figure 5.2).

### 5.2 Image Denoising

It has been shown [3, 38, 39, 74] that as $\epsilon \to 0$, the level set $u = 0.5$ evolves according to the Hele-Shaw flow with surface tension. As a result, the Cahn-Hilliard equation (1.3) asymptotically yields area preserving and curve-shortening flow [44]. In the experiment of Figure 5.3, we evolve the Cahn-Hilliard equation with the initial condition given in Figure 5.3a and finally we obtain a circle as dictated by the Hele-Shaw flow with surface tension. The parameters and the mesh in this set of experiments are the same as those in the first experiment.

This area-preserving and curve-shortening flow can be applied to image denoising. By choosing an appropriate $\epsilon$ and limiting the number of time-steps, we can smooth out the undesired features in small scale while maintaining the overall shapes in the images. We show two examples in Figure 5.4 and Figure 5.5. In these two examples we apply the two-scale scheme as in [14, 15]. We first evolve the solution with a relatively large $\epsilon$ so that the noisy boundary can be smoothed out quickly. Then, we switch to a small $\epsilon$ to obtain a sharp boundary. The input images Figure 5.4a and Figure 5.5a are taken from [37].
5.3 Image Inpainting

The modified Cahn-Hilliard equation (1.4) [14, 15] is solved for image inpainting shown in Figure 5.6 and Figure 5.7. The parameter in the fidelity term $\lambda_0$ is taken to be $10^5$, and the inpainting region is taken to be zero in the initial condition. The mesh is the same as in the first experiment. We first evolve the modified Cahn-Hilliard equation with $\epsilon = 0.1$ and $\Delta t = 0.1$. When the steady solution is reached, we switch to $\epsilon = 10^{-3}$ and $\Delta t = 10^{-5}$. A rescaling is performed such that $\|u\|_\infty = 1$ before the switch. The large $\epsilon$ ensures the topological connection and the small $\epsilon$ sharpens the boundary. Note that a different initial guess or different $\epsilon$ might lead to solutions of topologically different level sets as discussed in [14].
Figure 5.1: Phase separation on a square domain
Figure 5.2: Phase separation on a diamond-shaped domain
Figure 5.3: The evolution of a snowflake
Figure 5.4: Cahn-Hilliard equation for image denoising. We first evolve the Cahn-Hilliard equation with $\epsilon = 5 \times 10^{-3}$ for 70 steps (c). Then we switch to $\epsilon = 10^{-3}$ for another 100 steps. $\Delta t = 5 \times 10^{-6}$ for both stages. The input image (a) is taken from [37].
Figure 5.5: Cahn-Hilliard equation for image denoising. We first evolve the Cahn-Hilliard equation with $\epsilon = 5 \times 10^{-3}$ for 200 steps (d). Then we switch to $\epsilon = 10^{-3}$ for another 100 steps. $\Delta t = 5 \times 10^{-6}$ for both stages. The input image (a) is taken from [37].
Figure 5.6: Modified Cahn-Hilliard equation for image inpainting. The green region is the inpainting region. The first stage is run for 40 steps and second stage is run for 160 steps.
Figure 5.7: Modified Cahn-Hilliard equation for image inpainting. The green region is the inpainting region. The first stage is run for 400 steps and the second stage is run for 200 steps. The original image is from the trademark of Apple Inc.
Chapter 6

Conclusions

In this dissertation we have developed $C^0$ interior penalty methods for Cahn-Hilliard equations. Rigorous convergence analysis is obtained for the linearized equations based on the medius analysis [60]. A reliable and efficient *a posteriori* estimator is obtained and the performance for adaptive mesh refinement has been demonstrated. Next, multigrid methods are presented. We compare the performance of the multigrid methods based on the standard Richardson relaxation smoother with the performance of the multigrid methods based on the nonstandard smoother that uses the multigrid solve for a second order problem as a preconditioner. Finally, we apply the $C^0$ interior penalty methods to phase separation and image processing.

The rigorous analysis for the Cahn-Hilliard equations including the nonlinear term and time evolution will be a future research topic. A fully discontinuous Galerkin scheme, where the finite element functions are discontinuous, might also be investigated in the future. The discrete problem can be formulated from integration by parts formulas in a similar manner. However, the discrete bilinear form contains more terms due to the discontinuity of the finite element functions and the analysis is more complicated. Compared with $C^0$ interior penalty meth-
ods, fully discontinuous Galerkin schemes could be more effective in capturing the transition.
Bibliography


Appendix

In this appendix we will present an adaptive mesh refinement algorithm based on the error estimator \( \tilde{\eta}_T \) on each triangle \( T \) in the triangulation \( T_H \). The following algorithm will mark a set of triangle \( S \subset T_H \) such that \( \sum_{T \in S} \tilde{\eta}_T^2 \geq \theta \sum_{T \in T_H} \tilde{\eta}_T^2 \) for \( 0 < \theta < 1 \) and refine \( T_H \) based on \( S \) to obtain a new triangulation \( T_h \).

**Algorithm 1** An adaptive mesh refinement algorithm

**Input:** A triangulation \( T_H \); Error estimators \( \tilde{\eta}_T (T \in T_H) \); \( 0 < \theta < 1 \)

**Output:** A triangulation \( T_h \) which is a refinement of \( T_H \)

1: Identify the longest edge in each triangle (for isosceles triangles, choose any edge with longest length)

2: Sort: \( \tilde{\eta}_{map(1)} \geq \tilde{\eta}_{map(2)} \geq \cdots \geq \tilde{\eta}_{map(nT)} \)

\( \triangleright nT \) is the number of triangles in \( T_H \)

3: \( tsum \leftarrow 0; \)

4: \( psum \leftarrow \theta \sum_{T \in T_H} \tilde{\eta}_T^2 \)
   
   \( \text{processed}(i) \leftarrow \text{false} \) \( (i = 1, \ldots, nT) \)
   
   \( i \leftarrow 0 \)
   
   \( flag \leftarrow \text{false} \)

\( \triangleright \) to be continued on next page
Algorithm 1 An adaptive mesh refinement algorithm (continued.)

5: while $t_{sum} < p_{sum}$ do \\
6: \hspace{1em} $i \leftarrow i + 1$ \\
7: \hspace{1em} $cT \leftarrow \text{map}(i)$ \hspace{1em} \triangleright \text{pick up the triangle with next largest error estimator} \\
8: \hspace{1em} if NOT $\text{processed}(cT)$ then \\
9: \hspace{2em} flag $\leftarrow$ true \\
10: \hspace{1em} end if \\
11: \hspace{1em} while flag do \\
12: \hspace{2em} $t_{sum} = t_{sum} + \tilde{\eta}_{cT}^2$ \\
13: \hspace{2em} $\text{processed}(cT) \leftarrow$ true \\
14: \hspace{2em} $g_{Base} \leftarrow$ the longest edge in $cT$ identified in Step 1 \\
15: \hspace{2em} if $g_{Base}$ has been marked then \\
16: \hspace{3em} flag $\leftarrow$ false \\
17: \hspace{2em} else \\
18: \hspace{3em} Mark the edge $g_{Base}$ \\
19: \hspace{4em} if $g_{Base}$ is not a boundary edge then \\
20: \hspace{5em} $cT \leftarrow$ the other triangle that also have $g_{Base}$ as an edge \\
21: \hspace{5em} if $\text{processed}(cT)$ then \\
22: \hspace{6em} flag $\leftarrow$ false \\
23: \hspace{5em} end if \\
24: \hspace{3em} else \\
25: \hspace{4em} flag $\leftarrow$ false \\
26: \hspace{3em} end if \\
27: \hspace{2em} end if \\
28: \hspace{1em} end while \\
29: \hspace{1em} end while \\
30: \hspace{1em} Refine $\mathcal{T}_H$ based on the marked edges to obtain $\mathcal{T}_{h}$
Remark 6.1. In the case where the error estimator has terms associated with the edges, we need to distribute those terms to the corresponding triangles. For example, the error estimator defined in (3.73) can be distributed in the following way:

\[
\tilde{\eta}_T = \left[ \eta_T^2 + 0.5 [1 + (\beta^2 + \gamma) h^4] \sum_{e \in \mathcal{E}_h \cap \mathcal{T}_T} \eta_{e,1}^2 + [1 + (\beta^2 + \gamma) h^4] \sum_{e \in \mathcal{E}_h \cap \mathcal{T}_T} \eta_{e,1}^2 
+ 0.5 \sum_{e \in \mathcal{E}_h \cap \mathcal{T}_T} \eta_{e,2}^2 + \sum_{e \in \mathcal{E}_h \cap \mathcal{T}_T} \eta_{e,3}^2 \right]^{1/2}.
\]

Next we provide a MATLAB program that implements Algorithm 1. In the following MATLAB program, the function *refine* takes three input parameters. The first input parameter *meshIn* represents \( \mathcal{T}_H \) and *meshIn* has two fields: *node* and *T*. The field *mesh.node* is of \( nN \) rows and 2 columns with each row representing the coordinates of a vertex; The field *mesh.T* is of \( nT \) rows and 3 columns and each row contains the indices of the three vertices of a triangle. The vertex indices in each row of *mesh.T* are oriented counter-clockwise. The second input parameter *eta* is a vector of size \( nT \) and \( \eta(i) = \tilde{\eta}_T^2 \). The third input parameter *theta* is a positive number less than 1. The output *meshOut* represents the triangulation \( \mathcal{T}_h \). The output *updated* is a vector of size of the number of triangles in \( \mathcal{T}_h \) (i.e., the number of rows in *meshOut.T*). If \( updated(i) \) is a positive integer, the triangle \( i \) in \( \mathcal{T}_h \) is newly generated from the triangle \( update(i) \) in \( \mathcal{T}_H \); otherwise the triangle \( i \) in \( \mathcal{T}_h \) is not newly generated.

```matlab
function [meshOut,updated]=refine(meshIn, eta, theta)

%%%% meshIn has two fields:
%%%% meshIn.node (the vertex table) and
%%%% meshIn.T (the connectivity table)
```
eta is a vector of size of the number of rows in mesh.T and all entry in eta is nonnegative

0 < theta < 1

node=meshIn.node; T=meshIn.T;

generate lookup tables

edge=[T(:,[1 2]);T(:,[2 3]);T(:,[1 3])];
edge=unique(sort(edge,2),'rows');
nN=size(node,1); nT=size(T,1); nE=size(edge,1);
dE2t=sparse(T(:,[1 2 3]), T(:,[2 3 1]), [1:nT 1:nT 1:nT],nN,nN);
p2e=sparse(edge(:,[1 2]),edge(:,[2 1]),[1:nE,1:nE],nN,nN);

identify the longest edge in each triangle

edgeLen=zeros(nT,3);
edgeLen(:,1)=(node(T(:,2),1)-node(T(:,3),1)).^2+...
               (node(T(:,2),2)-node(T(:,3),2)).^2;
edgeLen(:,2)=(node(T(:,1),1)-node(T(:,3),1)).^2+...
               (node(T(:,1),2)-node(T(:,3),2)).^2;
edgeLen(:,3)=(node(T(:,1),1)-node(T(:,2),1)).^2+...
               (node(T(:,1),2)-node(T(:,2),2)).^2;
[tmp,locBase]=max(edgeLen,[],2);

pick the triangles to be refined and marked edges

[tmp,map]=sort(-eta);
psum=sum(eta)*theta; tsum=0;
processedT = logical(zeros(nT,1));
i = 0; flag = flag;

marker = zeros(nE,1); % used to mark edges
while (tsum < psum)
    i = i + 1;
    cT = map(i);
    if (~processedT(cT))
        flag = true;
    end
end
while (flag)
    tsum = tsum + eta(cT); processedT(cT) = true;

%%% identify the longest edge in triangle cT
switch locBase(cT)
    case 1
        v1 = T(cT, 2); v2 = T(cT, 3);
    case 2
        v1 = T(cT, 3); v2 = T(cT, 1);
    case 3
        v1 = T(cT, 1); v2 = T(cT, 2);
end
gBase = p2e(v1, v2);

if (marker(gBase) > 0)
    flag = false;

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else
    nN=nN+1;
    node(nN,:)=mean(node([v1 v2,:,:],1);
    marker(gBase)=nN;

    // go to the neighbor triangle
    cT=dE2t(v2,v1);
    if(cT==0 || processedT(cT)) flag=false; end
end
end
end

% refine the triangulation
updated=zeros(nT,1);
for i=1:nT
    if(processedT(i))
        p=T(i,:);
        for j=1:locBase(i)-1
            p=p([2 3 1]);
        end
        gBase=p2e(p(2),p(3));pro=p2e(p(1),p(2)); post=p2e(p(3),p(1));
        p=[p marker(gBase) marker(post) marker(pro)];

        if(p(5)>0 && p(6)>0) // perform red refinement
            T(i,:)=p([4 5 6]);updated(i)=i;
            T(end+1,:)=p([3 5 4]);updated(end+1)=i;
            T(end+1,:)=p([5 1 6]);updated(end+1)=i;
        end
    end
end
T(end+1,:)=p([4 6 2]); updated(end+1)=i;
else
  \%\%\% perform green or blue refinement
T(i,:)=p([4 1 2]); updated(i)=i;
T(end+1,:)=p([4 3 1]); updated(end+1)=i;
if(p(5)>0)
  T(end,:)=p([5 4 3]);
  T(end+1,:)=p([5 1 4]); updated(end+1)=i;
end
if(p(6)>0)
  T(i,:)=p([6 4 1]);
  T(end+1,:)=p([6 2 4]); updated(end+1)=i;
end
end
end
end
meshOut.node=node; meshOut.T=T;
\%\%\% the end \%\%\%
Vita

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