# A FACTORIZATION APPROACH FOR SOLVING THE HAMILTON-JACOBI-EQUATIONS IN NONLINEAR OPTIMAL CONTROL

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**Dedicated to my late grandfather**

**Dr. Aliyu Abubakar**

**May God have mercy on him**

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# TABLE OF CONTENTS

# ABSTRACT

The Hamilton-Jacobi equation (HJE) arose early in the last century in the study of the calculus of variation, classical mechanics and Hamiltonian systems. Recently, there has been a renewed interest in HJEs arising in various analysis and synthesis problems in systems theory. The HJE despite providing a necessary and sufficient condition for an optimal control, is very difficult to solve for general nonlinear systems, and therefore its application remained limited to linear systems. Yet, the HJE has been studied extensively in the literature from diverse areas of science and engineering, varying from mathematical physics, to mechanics, control theory, and to partial differential equations.

In this dissertation, some analytical approaches for solving the HJEs arising in $\mathcal{H}_\infty$, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ and $\mathcal{H}_2$ control problems for nonlinear systems are developed. Two major approaches are presented. The first approach is essentially an inversion or factorization method, and involves solving the HJE like a scalar quadratic algebraic equation with the gradient of the smooth scalar function as unknown. Since the HJE is a quadratic equation in the gradient of the unknown scalar function, we obtain two parameterized solutions which represent a parameterization of all solutions to the HJE. Thus, the problem is reduced to that of factorization of a scalar algebraic equation which we call the *discriminant equation* (or inequality). The main difficulties with this approach however are: (i) even after obtaining a solution to the discriminant equation, there is no guarantee that the gradient vector obtained subsequently represents a scalar function (i.e. represents a symmetric solution to the HJE); and (ii) there is no guarantee that the resulting solution is positive-definite. However, these difficulties can still be overcome by some additional constraints to the problem. Computational procedures for determining symmetric elementary solutions are then presented.

The second apporoach, which is a modification of the first approach, involves converting the first-order HJ partial-differential equation (PDE) to a second-order PDE. Then using a suitable parameterization, this second-order PDE is converted to a coupled system of higher-order nonlinear PDEs which can be solved using some available SYMBOLIC manipulation packages or by other methods. In general, there are no systematic procedures for solving the resulting system of higher-order PDEs, but various ad-hoc procedures can be used. This presents the most serious limitations of the approach. Both the time-varying and time-invariant systems are considered.

# CHAPTER 1

# INTRODUCTION AND LITERATURE REVIEW

In 1834-5, Hamilton found a system of ordinary differential equations (which is now called the *Hamiltonian canonical system*) equivalent to the Euler-Lagrange equation 1744. He also derived the Hamilton-Jacobi equation, which was improved/ modified by Jacobi in 1838. While in 1952, Richard Bellman developed the discrete-time equivalent of the Hamilton-Jacobi equation which is called the *dynamic programming principle*, and the name Hamilton-Jacobi-Bellman equation (HJBE) was coined (see [117] for a historical perspective). For a century now, the work of these three great mathematicians has remained the cornerstone of modern optimal control theory and analytical mechanics.

The HJBE despite providing a necessary and sufficient condition for an optimal control, is very difficult to solve for general nonlinear systems, and therefore its application remained limited to linear systems [5, 70, 117]. Yet, the HJBE has been studied extensively in the literature from diverse areas of science and engineering, varying from mathematical physics, mechanics, control theory, and partial differential equations as can be testified by the following well-edited books [16, 15, 22, 82, 117].

More recently, as early as 1990, there has been a renewed interest in the application of the HJBE to the control of nonlinear systems. This has been motivated by the successful development of the $\mathcal{H}_\infty$ control theory for linear systems and the pioneering work of Zames [118]. Under this framework, the HJBE became modified and took on a different form; essentially to account for disturbances in the system. This Hamilton-Jacobi equation was derived by Isaacs [17] from a differential game perspective. Hence the name Hamilton-Jacobi-Isaacs equation was coined, and has since been widely recognized as the nonlinear counterpart of the Riccati equation characterizing the solution of the $\mathcal{H}_\infty$ control problem for linear systems.

The biggest bottle-neck however to the practical application of the nonlinear equivalent of the $\mathcal{H}_\infty$ control theory [11, 56]-[58], [85, 107, 110] has been the difficulty in solving the Hamilton-Jacobi-Isaacs partial differential equations (or inequalities). There is no systematic numerical approach for solving them. Various attempts have however been made in this direction. Starting with the work of Luke [88], Glad [45], who proposed a polynomial approximation approach, Van der Schaft [110] improvised a recursive approach which was also refined by Isidori [58]. Since then, many other authors have proposed similar approaches to the solution of the problem [21, 30, 48, 105, 116]. Another approach using nonlinear matrix inequalities (NLMI) for a class of nonlinear systems is proposed in [85]. However, all the contributions so far in the literature are local.

The main draw-backs with the above approaches to the solution of the HJIE is that (i) they are not closed-form, and convergence of the sequence of solutions to an analytic solution cannot be guaranteed; (ii) there are no efficient methods for checking the positive definiteness of the solution; (iii) they are sensitive to uncertainties and pertubations in the system; and (iv) also sensitive to the initial condition, thus the global asymptotic stability of the closed-loop system cannot be guaranteed.

Therefore, more refined solutions that will guarantee global asymptotic stability are required if the theory of nonlinear $\mathcal{H}_\infty$ control is to yield any fruits. Thus, with this in mind, recently, Isidori and Lin [59] have shown that starting from a solution of an algebraic Riccati equation (ARE) related to the linear $\mathcal{H}_\infty$ problem, if one is free to choose a state-dependent weight of control input, it is possible to construct a global solution to the HJIE for a class of nonlinear systems in strict feedback form. A parallel approach using a backstepping procedure and inverse optimality has also been proposed in [38], and in [72, 73] for a class of strict-feedback systems.

Side-by-side with the deterministic version, the stochastic version of the HJBE (SHJBE) was first derived by Kushner [75] and has had also tremendous applications in many areas of engineering such as manufacturing, investment, financing, economic theory [100], filtering and estimation, and theoretical mechanics [7]. Invariably, there is also a major drawback in the application of the stochastic HJBE, fundamentally because it is required to admit a classical solution, i.e., a solution which is sufficiently smooth enough to the order of the derivatives involved in the equation. For controlled Markov diffusion processes which are governed by stochastic differential equations in the Ito form [40], the stochastic version also turns out to be a second-order nonlinear partial differential equation. Furthermore, classical solutions may not exist even for simple situations [40, 117]. To overcome this difficulty, Crandall and Lions introduced the so called *viscosity (or generalized) solutions* in the early 1980s [16, 82, 117] which have had wider application. Besides these approaches, there are also numerical schemes for solving the SHJBE [75]. However, due to the difficulty in solving the SHJBE, other approaches to the stabilization of nonlinear stochastic systems which avoid solving the SHJBE have also been proposed in the literature [34, 72].

In this dissertation research, we continue with the above efforts and present a factorization approach that may lead to the solutions of the HJEs arising in $\mathcal{H}_\infty$, $\mathcal{H}_2/\mathcal{H}_\infty$, and $\mathcal{H}_2$ control for a general class of affine nonlinear systems. The idea behind our approach is essentially an inversion (or factorization) approach, and involves solving the HJIE like a scalar quadratic algebraic equation with the gradient of the smooth scalar function as unknown. Since the HJE is a quadratic equation in the gradient of the unknown scalar function, we obtain two parameterized solutions which represent a parameterization of all solutions to the HJE. Thus, the problem is reduced to that of factorization of a scalar algebraic equation which we call the *discriminant equation* (or inequality). The main difficulty with this approach however are: (i) even after obtaining a solution of the discriminant equation, there is no guarantee that the gradient vector obtained subsequently represents a scalar function (i.e. represents a symmetric solution to the HJIE); and (ii) there is no guarantee that the resulting solution is positive definite. These two problems pose the most serious difficulties in using this approach. However, these difficulties can still be overcome by some additional constraints to the problem. An alternative approach which involves the solution of higher-order nonlinear partial differential equations is also considered.

The dissertation is organized as follows. In the remainder of this section, we shall introduce notations. Then in section 2, we shall give a comprehensive literature review on the problem including its origin and its status in control systems literature. Then in section 3 we shall layout our research objectives. In chapter 2, we present the factorization approach to elementary solutions of the HJIE arising in the state-feedback and measurement feedback $\mathcal{H}_\infty$ control problem for affine nonlinear systems that are either time-invariant or time-varying. We also present an approach to the solution of the associated Riccati equations in the case of linear-time-invariant systems (LTIS). In chapter 3, we present computational algorithms for finding symmetric elementary solutions to the HJIE and the associated Riccati equation. In chapter 4, we build on the results in the previous two chapters and present a factorization approach for solving the coupled HJIEs arising in the state-feeback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for affine nonlinear time-invaraint systems. We also consider the associated coupled Riccati equations in the case of LTIS.

In chapter 5, we consider the $\mathcal{H}_2$ deterministic and stochastic HJBEs, and we extend the results of the previous chapters to this case. In the case of the deterministic HJBE, the extension is straight-forward with the only modification in the gain matrix, whereas for the SHJBE, the extension is more involved since it turns out to be a second-order PDE. We also consider viscosity solutions of these HJBEs due to bounded control sets. Finally in chapter 6, we present a summary and conclusion.

**Notation:** The notation is fairly standard except where otherwise stated. Moreover, $\Re$, $\Re^n$ will denote respectively, the real line and the $n$ -dimensional real vector space, $t \in \Re$ will denote the time parameter. $M^n$, $N^n, ...$ will denote differentiable manifolds with dimension $n$, which are locally Euclidean and compact. $TM = \bigcup_{x \in M} T_x M$, $T^\star M = \bigcup_{x \in M} T_x^\star M$ will denote respectively the tangent and cotangent bundles of $M$ with dimensions $2n$. Moreover, $\pi$ and $\pi^\star$ will denote the natural projections $TM \to M$ and $T^\star M \to M$ respectively.

A $C^\infty(M)$ vector-field is a mapping $f : M \to TM$ such that $\pi \circ f = I_M$ (the identity on $M$), and $f$ has continuously differentiable partial derivatives of arbitrary order. A vector field $f$ also defines a differential equation (or a dynamic system) $\dot{x}(t) = f(x)$, $x \in M$, $x(t_0) = \hat{x}_0$. The flow (or integral curve) of the differential equation $\phi(t, \hat{x}_0)$, $t \in \Re$, is the unique solution of the differential equation for any arbitrary initial condition $\hat{x}_0$ over an open interval $I \subset \Re$. The flow of a differential equation will also be referred as the trajectory of the system and will be denoted by $x(t, \hat{x}_0)$ or $x(t)$ when the initial condition is immaterial. We shall also assume throughout this paper that the vector fields are complete, and hence the domain of the flow extends over $(-\infty, \infty)$. Furthermore, an equilibrium point of the vector field $f$ or the differential equation defined by it, is a point $\bar{x}$ such that $f(\bar{x}) = 0$ or $\phi(t, \bar{x}) = \bar{x}$ $\forall t \in \Re$. An *invariant set* for the system $\dot{x}(t) = f(x)$, is any set $\mathcal{A}$ such that, for any $\hat{x}_0 \in \mathcal{A}$, $\Rightarrow \phi(t, \hat{x}_0) \in \mathcal{A}$ for all $t \to \infty, -\infty$. A differential $k$-form, $k = 1, 2, \ldots$, $\omega_x^k$, at a point $x \in M$ is an exterior product from $T_x M$ to $\Re$, i.e., $\omega_x^k : T_x M \times \ldots \times T_x M$ ($k$ copies) $\to \Re$, which is a $k$-linear skew-symmetric function of $k$-vectors on $T_x M$. The space of all smooth $k$-forms on $M$ is denoted by $\Omega^k(M)$ or $E_k^\star(M) = \Gamma^\infty \Lambda^k(T^\star M)$, which are the smooth $C^\infty$ sections of the vector bundle $\Lambda^k(T^\star M)$. While $\mathcal{M}^{p \times q}(M)$ will denote the ring of $p \times q$ matrices over $M$.

The $\mathcal{F}$-derivative(Frèchet derivative) of a real-valued function $U : \Re^n \to \Re$ is defined as any $\nu$ such that $\lim_{v\to 0} \frac{1}{\|v\|}[U(x+v) - U(x) - \langle \nu, v\rangle] = 0$, for any $v \in \Re^n$. For a smooth function $V : \Re^n \to \Re$, $V_x = \frac{\partial V}{\partial x}$ is the row vector of first partial derivatives of $V(.)$ with respect to (wrt) $x$. $\|.\| : W \subset M \to \Re$ will denote the Euclidean norm on $W$, while $\mathcal{L}_2[0,T]$, $\mathcal{L}_2[0,\infty)$ will denote the standard Lebesgue space of vector-valued square integrable functions over $[0,T]$ and $[0,\infty)$ respectively. For a matrix $A \in \Re^{n\times n}$, $\sigma(A)$ will denote the spectral values or singular values of $A$ (depending on the context), and $A \geq B$ $(A > B)$ for an $n \times n$ matrix $B$, implies that $A - B$ is positive-semidefinite (positive-definite respectively). Lastly, $\mathcal{C}$, $\mathcal{C}_+$, $\mathcal{C}_-$ will denote the complex plane and its open right-half and left-half planes respectively.

## 1.1 Literature Review

The breakthrough in the derivation of the elegant state-space formulas for the solution of the standard linear $\mathcal{H}_\infty$ control problem in terms of two Riccati equations [35] spurred activity to derive the nonlinear counterpart of this solution. This work, unlike earlier work in the $\mathcal{H}_\infty$ theory [118, 42] that emphasized factorization of transfer functions and Nevanlinna-Pick interpolation in the frequency domain, operated exclusively in the time domain and drew strong parallels with established LQG control theory. Consequently, the nonlinear equivalent of the $\mathcal{H}_\infty$ control problem [35] has been developed by the important contributions of Basar [17], Van der Schaft, Ball [11], and Isidori [55]. Basar's dynamic differential game approach to the linear $\mathcal{H}_\infty$ control led the way to the derivation of the solution of the nonlinear problem in terms of the HJI equation which was derived by Isaacs and reported in Basar's books [17, 18]. However, the first systematic solution to the state-feedback $\mathcal{H}_\infty$-control problem for affine-nonlinear systems came from Van der Schaft [107, 110] using the theory of dissipative systems which had been laid down by Hill and Moylan [54] and Willems [112, 28]. He showed that, for time-invariant affine nonlinear systems which are smooth, the state-feedback $\mathcal{H}_\infty$ control problem is solvable by smooth feedback if there exists a smooth positive-semidefinite solution to a dissipation inequality or equivalently an infinite horizon HJB inequality. Coincidentally, this HJB inequality turned out to be the HJI inequality reported by Basar [17, 18]. An alternative solution using nonlinear matrix inequalities was also presented by Lu and Doyle [85] for a class of nonlinear systems. They also gave a parameterization of all stabilizing controllers. In subsection 1.1.1 we review the solution due to Van der schaft [110].

The solution of the output feedback problem with dynamic measurement feedback for affine nonlinear systems was presented by Ball et al. [11], Isidori and Astolfi [55, 56, 57], Lu and Doyle [85, 87], and Pavel and Fairman [96]. While the solution for a general class of nonlinear systems was presented by Isidori and Kang [58]. In subsection 1.1.1, we review the solution due to Isidori [56, 57]. At the same time, the solution of the discrete-time state and dynamic-output feedback problems were presented by Byrnes and Lin [28, 29, 78, 79, 80] and Guillard et al. [48, 49]. Another approach to the discrete-time problem using risk-sensitive control and the concept of information state for output feedback dynamic games was presented by James and Baras [62, 63, 64] for a general class of discrete-time nonlinear systems. The solution is expressed in terms of dissipation inequalities; however, the resulting controller is infinite-dimensional. A control Lyapunov function approach to the global output regulation and disturbance attenuation problem with global stability via measurement feedback for a class of nonlinear systems in which the nonlinear terms depend on the output of the system, has also been considered by Battilotti [19].

Furthermore, the solution of the problem for the continuous time-varying affine nonlinear systems was presented by Lu [84], while the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem for both continuous-time and discrete-time affine nonlinear systems using state-feedback control was solved by Lin [81]. Moreover, the robust control problem with structured uncertainties in system matrices has been extensively considered by many authors [2, 3, 20, 59, 61, 90, 91, 102, 106, 109, 114] both for the state-feedback and output-feedback problems. An inverse optimal approach to the robust control problem is also considered by Freeman and Kokotovic [43].

Finally, the filtering problem for affine nonlinear systems has also been considered by Berman and Shaked [23]; while the robust filtering problem for a class of discrete-time affine nonlinear systems has been discussed by Xi et al. [113].

A more general case of the problem is the singular nonlinear $\mathcal{H}_\infty$ control problem which has been considered by Maas and Van der Schaft [89] and Astolfi [8, 9] for continuous-time affine nonlinear systems, using both state and output feedback. Furthermore, an adaptive approach to the problem for a class of nonlinear systems in parametric strict feedback form has been presented by Zigang-Pan and Basar [93]. A fault tolerant approach is also considered by Yang et al. [115].

A more recent contribution to the literature has considered a factorization approach to the problem, which had been earlier initiated by Ball [13, 14] but discounted because of the inherent difficulties with the approach, as was in the case of the earlier approaches to the linear case that emphasised factorization and interpolation in lieu of state-space tools [118, 42]. These approaches are the $J-j$-inner-outer factorization and spectral factorization proposed by Ball and Van der schaft [12], and a chain scattering matrix approach considered by Pavel and Fairman [97]. While the former approach tries to generalize the approach in [50, 51] to the nonlinear case (the solution is only given for stable invertible continuous-time systems), the latter approach applies the method of conjugation and chain-scattering matrix developed for linear systems [69] to derive the solution of the nonlinear problem. An important outcome of the above endeavor using the factorization approach, is the derivation of state-space formulas for the coprime factorization and inner-outer factorization of nonlinear systems [101, 12] which were hitherto unavailable [13, 52, 86]. This has paved the way for employing these state-space factors in balancing, stabilization and design of reduced-order controllers for nonlinear systems [5, 92, 98, 101].

In the next few subsections, we review some of the most illustrious solutions to the problem.

### 1.1.1 Review of the Solution to the $\mathcal{H}_\infty$ Control Problem Using State and Output Measurement Feedback

In this section, we review results on the solutions to the state-feeback and output measurememt feedback nonlinear $\mathcal{H}_\infty$ control problems. For this purpose, we shall be considering affine nonlinear systems defined in local coordinates on an open subset $M$ of $\Re^n$ containing the origin $\{0\}$:

$$\begin{aligned}\Sigma \,:\, \dot{x}(t) &= f(x)+g_1(x)w(t)+g_2(x)u(t); \;\; x(0)=\hat{x}_0\\ z(t) &= h_1(x)+k_{12}(x)u(t)\\ y(t) &= h_2(x)+k_{21}(x)w(t)\end{aligned}$$

where the variables $x \in \Re^n$ is the state vector, $u:\Re_+ \to \mathcal{U}$, $w:\Re \to \mathcal{W}$ are respectively the control input and the disturbance signal which belong to some open sets $\mathcal{U} \subset \Re^k$, $\mathcal{W} \in \Re^r$, of admissible control and disturbances, $f \in C^\infty(\Re^n)$, $g_1 \in C^\infty(\Re^{n\times r})$, $g_2 \in C^\infty(\Re^{n\times k})$, while $h_1 \in C^\infty(\Re^m)$, $h_2(.) \in C^\infty(\Re^p)$, and $k_{12},\; k_{21} \in C^\infty$ have appropriate dimensions. We also assume the following.

**Assumption 1.1.1** *The origin $\{0\}$ is an equilibrium point of the system, and for simplicity*

$$\begin{gathered}h_1^T(x)k_{12}(x)=0, \;\; k_{12}^T(x)k_{12}(x)=I\\ k_{21}^T(x)g_1^T(x)=0, \;\; k_{21}(x)k_{21}^T=I.\end{gathered}$$

Before we define the problem, we introduce the following definitions.

**Definition 1.1.1** *The nonlinear system $\Sigma$ is said to have locally $\mathcal{L}_2$-gain from $w$ to $z$ in $N \subset M$, less than or equal to $\gamma$, if for the initial state $\hat{x}_0=0$ and fixed $u$, the response $z$ of the system corresponding to the trajectory $x(t,t_0,\hat{x}_0,u) \in N$ due to any $w \in \mathcal{L}_2[0,\infty)$ satisfies:*

$$\int_0^T \|z(t)\|^2 dt \le \gamma^2 \int_0^T \|w(t)\|^2 dt; \;\; \forall T>0 \tag{1.1}$$

**Definition 1.1.2** *The nonlinear system $\Sigma$ or $[f,h_1]$ is said to be locally zero-state detectable in $N$, if for $w(t)\equiv 0$, $u(t)\equiv 0$, $z(t)\equiv 0$, $\Rightarrow$ $x(t,t_0,\hat{x}_0)=0$ for all $t \ge t_0$, $\hat{x}_0 \in N$.*

Now, the state-feedback nonlinear $\mathcal{H}_\infty$ suboptimal control or local disturbance attenuation problem with internal stability, is to find for a given number $\gamma > 0$, a control action $u=\alpha(x)$ where $\alpha : N \to \Re^k$, $\alpha \in C^\infty(\Re^k)$ which renders locally the $\mathcal{L}_2$-gain of the system $\Sigma$ starting from $x(0)=0$ less or equal to $\gamma$ with internal stability of the system; where internal stability of the system is equivalent to local asymptotic stability of the state trajectories around the equilibrium point $\{0\}$. However, in the absence of the availability of the states, one might be interested in synthesizing a dynamic controller which processes the measurement $y(t)$, $t \in [0,T)$, $T \in [0,\infty)$ and generates a control action that renders locally the $\mathcal{L}_2$-gain of the system around $\{0\}$, less than or equal to $\gamma$ with internal stability. Such a controller can be represented in the form:

$$\begin{aligned}\Sigma_c : \;\; \dot{\xi}(t) &= \eta(\xi(t),y(t))\\ u(t) &= \theta(\xi(t),y(t)),\end{aligned}$$

where $\xi \in M$, $\eta : \mathcal{O} \subseteq \times \Re^p \to \Re^n$, $\theta : \mathcal{O} \times \Re^p \to \Re^k$. This problem is then known as the suboptimal $\mathcal{H}_\infty$ (or disturbance attenuation) problem with measurement feedback for the system $\Sigma$. The purpose of the control action is to achieve closed-loop stability and to attenuate the effect of the disturbance $w$ on the penalty variable or controlled output $z$. In this respect, the problem can be formulated as an optimization problem with the following cost function:

$$J(w,u) = \min_{u\in\mathcal{U}} \max_{w\in\mathcal{L}_2} \frac{1}{2}\int_0^T [\|z(\tau)\|^2 - \gamma^2\|w(\tau)\|^2]d\tau \tag{1.2}$$

subject to the dynamics $\Sigma$ and closed-loop stability of the system. It is seen that, by rendering the above cost function nonpositive, the $\mathcal{L}_2$-gain requirement can be satisfied. Then if the disturbance-free system is zero-state detectable, the closed-loop system will be internally stable [54, 111, 112].

The above cost function or performance measure also has a differential game interpretation. It constitutes a two-person zero-sum game, in which the minimizing player controls the input $u$ while the maximizing player controls the disturbance $w$. Such a game has a saddle point equilibrium solution if the value function $V(x) = \inf_{u\in\mathcal{U}} \sup_{w\in\mathcal{L}_2} \int_0^T [\|z(\tau)\|^2 - \gamma^2\|w(\tau)\|^2]dt$ is $C^1$ and satisfies the following dynamic programming equation (known as Isaac's equation or HJI equation):

$$\begin{aligned} -V_t(t,x) = \min_u \max_w \Big( & V_x(t,x)[f(x) + g_1(x)w(t) + g_2(x)u(t)] + \|z(t)\|^2 - \\ & \gamma^2\|w(t)\|^2] \Big); \quad V(T,x) = 0, \quad x \in M \end{aligned} \tag{1.3}$$

where $V_t$, $V_x$ are the row vectors of partial derivates wrt $t$ and $x$ respectively. A pair of strategies $u^\star(t,x)$, $w^\star(t,x)$ provides under feedback information pattern, a saddle point equilibrium solution to the above game if

$$J(u^\star(t,x), w(t,x)) \le J(u^\star(t,x), w^\star(t,x)) \le J(u(t,x), w^\star(t,x)) \tag{1.4}$$

However, since we are interested in the infinite-time horizon problem i.e. for a control strategy such that $\lim_{T\to\infty} J(w,u)$ remains bounded and the $\mathcal{L}_2$-gain of the system remains finite, we seek a time-independent positive-semidefinite function $V : M \to \Re$ which vanishes at $\{0\}$ and satisfies the following time-invariant HJI equation

$$\begin{aligned} & \min_u \max_w \left( V_x(x)[f(x) + g_1(x)w(t) + g_2(t,x)u(t)] + \frac{1}{2}(\|z(t)\|^2 - \gamma^2\|w(t)\|^2) \right) = 0; \\ & V(0) = 0, \quad x \in M \end{aligned} \tag{1.5}$$

Since the function on the left-hand-side (LHS) of the above equation is convex wrt $u$ and concave wrt $w$, the above optimization problem can easily be solved to get

$$u^\star = -g_2^T(x)V_x(x); \quad w^\star = \frac{1}{\gamma^2} g_1^T(x)V_x^T(x)$$

which satisfy the saddle point condition (1.4). Hence, the above feedbacks solve the state-feedback problem. The measurement feedback problem is a little more involved.

An alternative approach to the problem is through the theory of dissipative systems [54, 107, 110, 112] which we hereby introduce.

**Definition 1.1.3** *The nonlinear system $\Sigma$ is said to be locally dissipative in $N \subset M$ with respect to (wrt) the supply rate $s(w(t), z(t)) = \frac{1}{2}(\gamma^2\|w(t)\|^2 - \|z(t)\|^2)$ if there exists a positive semidefinite function (storage function) $\Psi : N \to \Re$, $\Psi(0) = 0$, such that the inequality*

$$\Psi(x(t_1)) - \Psi(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t))dt, \tag{1.6}$$

*is satisfied for all $t_1 > t_0$, $x \in N$.*

It is seen that, if the system is locally dissipative wrt to the above supply rate, then it also has locally $\mathcal{L}_2$-gain less than or equal to $\gamma$. Conversely, if the system is reachable from $\{0\}$, has an asymptotically stable equilibrium point at $x = \{0\}$ and an $\mathcal{L}_2 - gain \leq \gamma$, then the functions

$$V_a(x) = - \inf_{w \in L_2[0,T), x(0)=x} \frac{1}{2} \int_0^T (\gamma^2\|w(\tau)\|^2 - \|z(\tau)\|^2)dt, \quad T \geq 0$$

$$V_r(x) = \inf_{w \in \mathcal{L}_2(-T,0], x=x(-T)} \int_{-T}^{0} (\gamma^2\|w(\tau)\|^2 - \|z(\tau)\|^2)dt, \quad T \geq 0$$

are well-defined for all $x \in N$ and satisfy the dissipation inequality (1.6) [110]. Moreover, $V_a(0) = V_r(0) = 0$, $0 \leq V_a \leq V_r$. Thus, there exists at least one solution to the dissipation inequality. The functions $V_a$ and $V_r$ are known as the *available storage* and the *required supply* respectively. Furthermore, for any other solution $V$ to (1.6), $0 \leq V_a \leq V \leq V_r$. Additionally, if we assume that the function $\Psi$ is smooth (this is plausible if $s(w, z)$ is sufficiently smooth, but may not be necessary), then we can move from the integral version of (1.6) to its differential or infimitesimal version:

$$V_x(x)(f(x) + g_1(x)w + g_2(x)u) + \frac{1}{2}(\|z\|^2 - \gamma^2\|w\|^2 \leq 0; \quad V(0) = 0 \tag{1.7}$$

for any storage function $V$. Next, suppose the control function $u = \alpha(x)$ can be chosen such that the above inequality (1.7) is satisfied for all $w \in \mathcal{L}_2[0, \infty)$ and $x(0) = \{0\}$. Then the closed-loop system will have $\mathcal{L}_2$-gain less than or equal $\gamma$, and if additionally the system is zero-state detectable, asymptotic (internal) stability of the system can be concluded. Thus, clearly a solution to the disturbance attenuation problem can be derived from this perspective.

From whichever approach is derived, the solution to the state-feedback $\mathcal{H}_\infty$ control problem to the system $\Sigma$ can be summarized in the following theorem [107, 110, 55, 56]. We first introduce the following definitions.

**Definition 1.1.4** *A function $\psi : \Re_+ \to \Re_+$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\psi(0) = 0$.*

**Definition 1.1.5** *A function $V : \Re_+ \times M \to \Re$ is locally positive definite if (i) it is continuous, (ii)$V(t, 0) = 0 \, \forall t \geq 0$, and (iii) there exists a constant $\mu > 0$ and a function $\psi$ of class $\mathcal{K}$ such that*

$$\psi(\|x\|) \leq V(t, x), \quad \forall t \geq 0, \forall x \in B_\mu$$

*where $B_\mu = \{x \in \Re^n \; : \; \|x\| \leq \mu\}$. $V(.,.)$ is positive definite if the above inequality holds for all $x \in \Re^n$. Further, if $V$ is independent of $t$, then $V$ is positive definite (semi definite) if $V > 0 (\geq 0) \, \forall x \in M$ and $V(0) = 0$.*

**Definition 1.1.6** *A positive-definite function $V : \Re_+ \times M \to \Re_+$ is said to be proper if its support is compact i.e. for any $c > 0$, the level set $\{x \in M \; : \; 0 \leq V(t,x) \leq c, \; \forall t \in \Re_+\}$ is compact.*

**Theorem 1.1.1** *Consider the nonlinear system $\Sigma$ and the problem of designing a control law to achieve local disturbance attenuation with internal stability or equivalently locally $\mathcal{L}_2$-gain from $w$ to $z$ less or equal to $\gamma$. Suppose $\Sigma$ is locally zero-state detectable and there exists a positive-semidefinite $C^1$ function $V : N \to \Re$ defined in a neighborhood $N \subset M$ of the origin, vanishing at $\{0\}$ and satisfying the following HJIE (inequality).*

$$\begin{aligned} & V_x(x)f(x) + \frac{1}{2}V_x(x)[\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)g_2^T(x)]V_x^T(x) + \frac{1}{2}h_1^T(x)h_1(x) \leq 0; \\ & V(0) = 0 \; \forall x \in N. \end{aligned} \tag{1.8}$$

*Then the above problem is solved by the feedbacks*

$$u^\star(t) = -g_2^T(x)V_x^T(x); \;\; w^\star(t) = \frac{1}{\gamma^2}g_1^T(x)V_x^T(x) \tag{1.9}$$

*Moreover, the closed-loop system with $u(t) = u^\star(t)$ and $w(t) = 0$, $\forall t \geq 0$ is locally asymptotically stable in $M$ and globally asymptotically stable if $N = M = \Re^n$ and $V$ is proper.*

Next we summarize the solution to the measurement feedback problem. The following theorem from [57, 58] summarizes the solution to this

**Theorem 1.1.2** *Suppose that $\{f(.), \; h_1(.)\}$ is locally zero-state detectable in $N \subset M$ and there exist smooth positive semi-definite functions $V, \; W : N_1 \times N_1 \to \Re$, $N_1 \subset N$, $W(0,\xi) > 0$, $\xi \neq 0$, to the HJIE (1.8) and the HJIE:*

$$\begin{aligned} & [W_x(x,\xi) \;\; W_\xi(x,\xi)]f_e(x,\xi) + \frac{1}{2\gamma^2}[W_x(x,\xi) \; W_\xi(x,\xi)] \begin{bmatrix} g_1(x)g_1^T(x) & 0 \\ 0 & LL^T \end{bmatrix} \begin{bmatrix} W_x(x,\xi) \\ W_\xi(x,\xi) \end{bmatrix} + \\ & \frac{1}{2}h_e^T(x,\xi)h_e(x,\xi) = 0, \;\; W(0,0) = 0, \; (x,\xi) \in N_1 \times N_1 \end{aligned} \tag{1.10}$$

*respectively. Then, the problem of $\mathcal{L}_2$-disturbance attenuation ($\mathcal{H}_\infty$ control) with internal stability is solved by the output feedback*

$$\begin{aligned} \dot{\xi}(t) &= f(\xi) + g_1(\xi)\alpha_1(\xi) + g_2(\xi)\alpha_2(\xi) + L(y - h_2(\xi)) \\ u(t) &= \alpha_2(\xi) \end{aligned}$$

*for some gain matrix $L \in \Re^{n \times p}$, where $f_e(x,\xi)$, $h_e(x,\xi)$, $\alpha_1(x)$, $\alpha_2(x)$ are given by:*

$$\begin{aligned} f_e(x,\xi) &= \begin{pmatrix} f(x) + g_1(x)\alpha_1(x) + g_2(x)\alpha_2(\xi) \\ f(\xi) + g_1(\xi)\alpha_1(\xi) + g_2(\xi)\alpha_2(\xi) + L(h_2(x) - h_2(\xi)) \end{pmatrix} \\ h_e(x,\xi) &= \alpha_2(\xi) - \alpha_2(x) \\ \alpha_1(x) &= \frac{1}{\gamma^2}g_1^T(x)V_x^T(x) \\ \alpha_2(x) &= -g_2^T(x)V_x^T(x). \end{aligned}$$

Next, we discuss the solution to the time-varying problem.

### 1.1.2 Review of Solution to the $\mathcal{H}_\infty$ Control Problem for Finite-Time Horizon Problem and Time-Varying Nonlinear Systems

In this subsection, we discuss briefly the finite-time horizon and the time-varying $\mathcal{H}_\infty$ control problems for affine nonlinear systems using state-feedback only. The results for the output feedback problem can be found in references [84, 30, 55]. Since the two problems are similar and related, we shall consider the following time-varying nonlinear system

$$\Sigma_t: \quad \dot{x}(t) = f(t,x) + g_1(t,x)w(t) + g_2(t,x)u(t); \quad x(0) = \hat{x}_0 \tag{1.11}$$

$$z(t) = \begin{bmatrix} h(t,x) \\ u(t) \end{bmatrix}; \quad f(t,0) = 0, \quad h(t,0) = 0 \tag{1.12}$$

where the variables $x \in M \subset$, $u : \Re_+ \to \mathcal{U}$, $w : \Re \to \mathcal{W}$ are as defined previously, while $f : \Re \times M \to TM$, $g_1 : \Re \times M \to \mathcal{M}^{n\times r}(\Re \times M)$, $g_2 : \Re \times M \to \mathcal{M}^{n\times k}(\Re \times M)$ and $h : \Re \times M \to \Re^m$ belong to $C^\infty([0,\infty) \times M)$.

The finite-horizon state-feedback $\mathcal{H}_\infty$ control problem for the above system involves the optimization of the following functional:

$$J_t(u(t), w(t)) = \min_u \max_w \int_{t=0}^{T} \frac{1}{2}(\|z(\tau)\|^2 - \gamma^2\|w(\tau)\|^2)d\tau \tag{1.13}$$

over some finite-time interval $[0,T]$ using state-feedback controls of the form:

$$u(t) = \beta(t,x); \quad \beta(t,0) = 0. \tag{1.14}$$

The above problem represents a two-person zero-sum differential game of fixed duration $[0,T]$. A pair of strategies $u^\star(t,x)$, $w^\star(t,x)$ under feedback information pattern provides a saddle point solution to the above problem such that

$$J_t(u^\star(t,x), w(t)) \le J_t(u^\star(t,x), w^\star(t,x)) \le J_t(u(t), w^\star(t,x)) \tag{1.15}$$

if there exists a positive definite $C^1$ function $V : [0,T] \times N \to \Re_+$ satisfying the following HJIE [17, 55]:

$$-V_t(t,x) = \min_u \max_w \Big(V_x(t,x)[f(t,x) + g_1(t,x)w(t) + g_2(t,x)u(t)] + \frac{1}{2}[\|z(t)\|^2 - \gamma^2\|w(t)\|^2]\Big); \quad V(T,x) = 0, \quad x \in N. \tag{1.16}$$

It can be shown that under the assumption of feedback information pattern and the differentiability of the function $V(.,.)$, the solution to the above problem is given by the following theorem [84].

**Theorem 1.1.3** *Consider the nonlinear time-varying system $\Sigma_t$ and the problem of achieving local $\mathcal{L}_2$-gain less than or equal to $\gamma$ with internal stability for the system over a* *finite-time horizon $[0,T]$. Suppose there exists a $C^1([0,T]\times N)$ positive definite function $V:[0,T]\times N\to\Re_+$ satisfying the following time-varying Hamilton-Jacobi-Isaacs equation (inequality):*

$$V_t(t,x)+V_x(t,x)f(t,x)+\frac{1}{2}V_x(t,x)[\frac{1}{\gamma^2}g_1(t,x)g_1^T(t,x)-g_2(t,x)g_2^T(t,x)]V_x^T(t,x)+$$
$$\frac{1}{2}h^T(t,x)h(t,x)\leq 0;\quad V(T,x)\geq\chi(T)>0\;\forall x\in N,\ t\in[0,T] \tag{1.17}$$

*where $\chi$ is of class $\mathcal{K}$, then the problem is solved by the feedbacks:*

$$u^\star(t,x)=-g_2^T(t,x)V_x^T(t,x);\quad w^\star(t,x)=g_1(t,x)V_x^T(t,x).$$

*Moreover, $u^\star(t,x)$, $w^\star(t,x)$ satisfy the saddle point conditions (1.15) of the differential game (1.13), (1.11), (1.12) and $V$ is its value function, i.e.,*

$$V(t,x)=\inf_{u\in\mathcal{L}_2[0,T]}\sup_{w\in\mathcal{L}_2[0,T]}\int_0^T[\|z(t)\|^2-\gamma^2\|w(t)\|^2]d\tau. \tag{1.18}$$

Next, we discuss the solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem.

### 1.1.3 Review of the Solution to the Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem for Affine Nonlinear Systems

As in the case of the solution to the pure $\mathcal{H}_\infty$ control problem for linear systems derived by Basar [17] using the theory of differential games, one of the most elegant approach to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems for linear systems using state-feedback was derived using the theory of differential games by Limebeer et al. [77]. The problem is posed as a two-player non-zero sum differential game with two objective functions (performance measures). One performance measure is used to reflect the energy in the system, as measured by the $\mathcal{H}_2$ norm of the system, while the other objective imposes the disturbance attenuation contraint as reflected by the $\mathcal{H}_\infty$ norm of the system from the disturbance to the controlled output. Such a game is also called a Nash-game, and a solution to the problem exists if a pair of Nash equilibrium points can be found such that the two performances cannot improve beyond these points.

The nonlinear counterpart of the above approach to the linear problem was meticulously derived by Lin [80]. To summarize this solution, we consider the system $\Sigma$ with the following cost functions:

$$\min_{u\in\mathcal{U}}\max_{w\in\mathcal{W}}J_1^\infty(u,w) = \int_0^\infty(\gamma^2\|w(\tau\|^2-\|z(\tau\|^2)d\tau, \tag{1.19}$$
$$\min_{u\in\mathcal{U},w\in\mathcal{W}}J_2^\infty(u,w) = \int_0^\infty\|z(\tau)\|^2d\tau, \tag{1.20}$$

where the first cost function is associated with the $\mathcal{L}_2$-gain constraint of the system (or $\mathcal{H}_\infty$ criterion), while the second objective is related to the output energy of the system (or $\mathcal{H}_2$ criterion). Moreover, if we assume $\mathcal{W}\subset\mathcal{L}_2([0,\infty),\Re^r)$ and $\mathcal{U}\subset\mathcal{L}_2([0,\infty),\Re^k)$, then a Nash equilibrium solution to the above two-player nonzero sum game is said to exist if we can find $u^\star$, $w^\star$ such that

$$J_1^\infty(u^\star, w^\star) \leq J_1^\infty(u^\star, w) \;\; \forall w \in \mathcal{W}, \tag{1.21}$$
$$J_2^\infty(u^\star, w^\star) \leq J_2^\infty(u, w^\star) \;\; \forall u \in \mathcal{U}. \tag{1.22}$$

Furthermore, by minimizing the first objective wrt to $w$ and substituting in the second objective which is then minimized wrt to $u$, the pair of Nash equilibrium points can be found. The following theorem summarizes the solution to the state-feedback problem.

**Theorem 1.1.4** *Consider the nonlinear system $\Sigma$ and the state-feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem with internal stability. Assume that $[f, h_1]$ is zero-state detectable. Then the problem is locally solvable if there exist two $C^\omega(\omega \geq 2)$ functions $Y : N \to \Re_-$, $Y(0) = 0$, (which is locally negative-definite) and $V : N \to \Re_+$, $V(0) = 0$, (which is locally positive-definite) satisfying the following coupled HJI inequalities:*

$$Y_x(x)f(x) - \frac{1}{4}[V_x(x)g_2(x)g_2^T(x)V_x^T(x) + \frac{Y_x(x)g_1(x)g_1^T(x)Y_x^T(x)}{\gamma^2}] - \frac{1}{2}Y_x(x)g_2(x)g_2^T(x)V_x(x) - h_1^T(x)h_1(x) \geq 0, \;\; x \in N, \tag{1.23}$$

$$V_x(x)f(x) - \frac{1}{4}V_x(x)g_2(x)g_2^T(x)V_x^T(x) - \frac{V_x(x)g_1(x)g_1^T(x)Y_x^T(x)}{2\gamma^2} + h_1^T(x)h_1(x) \leq 0, \;\; x \in N. \tag{1.24}$$

*Moreover, the feedbacks $u^\star = -\frac{1}{2}g_2^T(x)V_x^T(x)$ and $w^\star = -\frac{1}{2\gamma^2}g_1^T(x)Y_x^T(x)$ are the unique optimal strategies, and the closed-loop system is locally asymptotically stable with $w = 0$.*

Next, we review previous results on the solution of HJBE and HJIE.

### 1.1.4 Review of Solutions to HJBE and HJIE

In this subsection we review some of the few approaches to the solution of the HJBE and HJIE that have been proposed in the literature. In [88] an iterative procedure for the HJBE for a class of nonlinear systems is proposed. This was further refined and generalized for affine nonlinear systems by Glad [45]. To summarize the procedure briefly, we consider the following affine nonlinear system

$$\Sigma_a : \;\; \dot{x}(t) = a(x) + b(x)u \tag{1.25}$$

under the quadratic cost function

$$\min \int_0^\infty \left[ l(x) + \frac{1}{2}u^T R u \right] dt, \tag{1.26}$$

where $a : \Re^n \to \Re^n$, $b : \Re^n \to \Re^{n \times k}$, $l : \Re^n \to \Re$, $l \geq 0$, $a$, $b$, $l \in C^\infty$, $0 < R \in \Re^{k \times k}$. Then, the HJBE corresponding to the above problem is given by

$$v_x(x)a(x) - \frac{1}{2}v_x(x)b(x)R^{-1}b^T(x)v_x^T(x) + l(x) = 0 \tag{1.27}$$

and the optimal control is given by $\tilde{u} = k(x) = -R^{-1}b^T(x)v_x^T(x)$ for some smooth $C^2(\Re^n)$ positive-semidefinite function $v : \Re^n \to \Re$. Further, assuming that there exists a positive-semidefinite solution to the algebraic Riccati equation (ARE) corresponding to the linearization of the system (1.25), then it can be shown that, there exists a real analytic solution $v$ to the HJBE (1.27) in a neighborhood $\mathcal{O}$ of the origin [88]. Hence, if we write the linearization of the system and the cost function as

$$\begin{aligned} l(x) &= \frac{1}{2}x^TQx + l_h(x) \\ a(x) &= Ax + a_h(x) \\ b(x) &= B + b_h(x) \\ v(x) &= \frac{1}{2}x^TPx + v_h(x) \end{aligned}$$

where $Q = \frac{\partial l}{\partial x}(0)$, $A = \frac{\partial a}{\partial x}(0)$, $B = b(0)$, $P = \frac{\partial v}{\partial x}(0)$ and $l_h$, $a_h$, $b_h$, $v_h$ contain higher-order terms. Substituting the above expressions in the HJBE (1.27), it splits into two parts:

$$A^TP + PA - PBR^{-1}B^TP + Q = 0 \quad (1.28)$$

$$v_{hx}(x)A_cx + v_x(x)a_h(x) - \frac{1}{2}v_{hx}(x)BR^{-1}B^Tv_{hx}^T(x) - \frac{1}{2}v_x(x)\beta_h(x)v_x^T(x) + l_h(x) = 0 \quad (1.29)$$

where

$$A_c = A - BR^{-1}B^TP, \;\; \beta_h(x) = b(x)R^{-1}b^T(x) - BR^{-1}B^T$$

and $\beta_h$ contains terms of degree 1 and higher. Equation (1.28) is the ARE of linear-quadratic control for the linearized system. Thus, if the system is stabilizable and detectable, there exists a unique solution to this equation. Hence, it represents the first-order aproximation to the solution of the HJBE (1.27). Now letting the superscript $(m)$ denote the $mth$ order terms, (1.29) can be written as

$$-v_{hx}^m(x)A_cx = [v_x(x)a_h(x) - \frac{1}{2}v_{hx}(x)BR^{-1}B^Tv_{hx}^T(x) - \frac{1}{2}v_x(x)\beta_h(x)v_x^T(x)]^m + l_h^m(x). \quad (1.30)$$

The RHS contains only $(m-1)th$, $(m-2)th, \ldots$ order terms of $v$. Therefore, the equation (1.29) defines a linear system of equations for the $mth$ order coefficients with the RHS containing previously computed terms. Thus, $v^m$ can be computed recursively. It can also be shown that the system is nonsingular as soon as $A_c$ is a stable matrix. This is satisfied vacuously if $P$ is a stabilizing solution to the ARE (1.28).

The above algorithm has been refined by Van der Schaft [110] for the HJIE. Suppose there exists a solution $P \geq 0$ to the ARE

$$F^TP + PF + P[\frac{1}{\gamma^2}G_1G_1^T - G_2G_2^T]P + H^TH = 0 \quad (1.31)$$

corresponding to the linearization of the HJIE (1.8), where $F = \frac{\partial f}{\partial x}(0)$, $G_1 = g_1(0)$, $G_2 = g_2(0)$, $H = \frac{\partial h}{\partial x}(0)$. Now let $V$ be an approximate solution to the HJIE and if we write

$$\begin{aligned} V(x) &= \frac{1}{2}x^TPx + V_h(x) && (1.32) \\ f(x) &= Fx + f_h(x) && (1.33) \\ \frac{1}{2}[\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)g_2^T(x)] &= \frac{1}{2}[\frac{1}{\gamma^2}G_1G_1^T - G_2G_2^T] + R_h(x) && (1.34) \\ \frac{1}{2}h^T(x)h(x) &= \frac{1}{2}x^TH^THx + \theta_h(x) && (1.35) \end{aligned}$$

where $V_h$, $f_h$, $R_h$, and $\theta_h(x)$ contain higher-order terms. Then the HJIE (1.8) splits into two parts: the ARE (1.31) and the higher-order equation

$$-\frac{\partial V_h(x)}{\partial x}F_\star x = \frac{\partial V(x)}{\partial x}f_h(x) + \frac{1}{2}\frac{\partial V_h(x)}{\partial x}[\frac{1}{\gamma^2}G_1G_1^T - G_2G_2^T]\frac{\partial^T V_h(x)}{\partial x}(x) + \frac{1}{2}\frac{\partial V(x)}{\partial x}R_h(x)\frac{\partial^T V(x)}{\partial x} + \theta_h(x) \tag{1.36}$$

where $F_\star \triangleq F - G_2G_2^TP + (\frac{1}{\gamma^2})G_1G_1^TP$. The $m$th order term $V^{(m)}$ of $V$ can now be computed inductively for $m = 3, 4, \ldots$. If we denote the $m$th order terms on the RHS of (1.36) by $H_m(x)$, it follows that

$$-\frac{\partial V_h^m(x)}{\partial x}F_\star x = H_m(x) \tag{1.37}$$

and thus if $P \geq 0$ is a stabilizing solution of the ARE (1.31), then we can integrate the above equation for $V_h^m$ to get, $V_h^m(x) = \int_0^\infty H_m(e^{F_\star t}x)dt$. Hence $V_h^m$ is determined by $H_m$. Further, it is easily seen that $H_m$ depends only on $V^{(m-1)}, V^{(m-2)}, \ldots, V^2 = \frac{1}{2}x^TPx$, and therefore $V^m$ can be computed inductively starting from $V^2$.

The above approximate approach to the solution of the HJBE or HJIE has a serious limitation, in the sense that, there is no guarantee that the sequence of solutions will converge to a smooth positive-semidefinite solution. Moreover, in general, the resulting solution thereby obtained cannot be guaranteed to achieve closed-loop asymptotic stability or global $\mathcal{L}_2$-gain $\leq \gamma$ since the functions $f_h$, $R_h$, $\theta_h$ are not exactly known, rather they are finite approximations from a Taylor series expansion. The procedure can also be computationally intensive as various values of $\gamma$ have to be tried. The above procedure has also been refined in [58] for the measurement feedback case; variants of the algorithm have also been proposed in other references [21, 65, 66, 116, 105].

Two algorithms that use basis function approximation are given in [21] using the Galerkin approximation and [105] using a series solution. In [105] the following series expansion for $V$ is used

$$V(x) = V^{[2]}(x) + V^{[3]}(x) + \ldots \tag{1.38}$$

where $V^{[k]}$ is a homogeneous function of order $k$, i.e. it is a linear combination of

$$N_k^n \triangleq \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$$

terms of the form $x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}$, where $i_j$ is a non-negative integer for $j = 1, \ldots, n$ and $i_1 + i_2 + \ldots + i_n = k$. Consequently, the vector whose components consist of these terms is denoted by $x^{[k]}$; for example,

$$x^{[1]} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x^{[2]} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}, \ldots .$$

Now, if we let $G(x) = [g_1(x) \;\; g_2(x)]$ and $\nu = [u \;\; w]^T$, then we can expand $f$, $G$, $\nu$ as homogeneous functions:

$$\begin{aligned} f(x) &= f^{[1]}(x) + f^{[2]}(x) + \ldots & (1.39)\\ G(x) &= G^{[0]}(x) + G^{[1]}(x) + \ldots & (1.40)\\ \nu_\star(x) &= \nu_\star^{[1]}(x) + \nu_\star^{[2]}(x) + \ldots & (1.41) \end{aligned}$$

where $\nu_\star^{[k]} = -\sum_{j=0}^{k-1}(G^{[j]})^T(V_x^{[k+1-j]})^T$, $f^{[1]}(x) = Fx$, $G^{[0]} = [G_1 \;\; G_2]$. Finally, substituting the above expansions in the HJIE (1.8), we get

$$V_x^{[m]}\bar{f}^{[1]} = -\sum_{k=1}^{m-2} V_x^{[m-k]}\bar{f}^{[k+1]} + \sum_{k=2}^{m-2}(\nu_\star^{[m-k]})^T\nu_\star^{[k]} - \bar{H}^{[m]} \tag{1.42}$$

where $\bar{f}(x) \triangleq f(x) + G(x)\nu_\star^{[1]}(x)$, $\nu_\star^{[1]}(x) = -B^T Px$, $P = P^T > 0$ is a stabilizing solution to the ARE (1.31), $\bar{H}(x) \triangleq \frac{1}{2}h^T(x)h(x)$. Now if we assume every $V^{[m]}$ is of the form $V^{[m]} = C_m x^{[m]}$, where $C_m$ is a matrix of unknown coefficients. Then substituting this in (1.8), we get a system of $N_m^n$ linear equations in the unknown entries of $C_m$. It can be shown that if $A_\star = A - BB^T P$ is stable, then the approximation $V$ is analytic and $V^{[m]}$ converges finitely.

The approach in [21] uses more or less the same approach for the HJBE with an a priori selected region of asymptotic stability.

## 1.2 Research Ojectives

We propose the following research objectives:

1. To develop a factorization approach for solving the state-feedback and measurement feedback HJIE for continuous-time affine nonlinear systems together with the associated Riccati equation.

2. To extend the above procedure to the HJIE corresponding to the state-feedback $\mathcal{H}_\infty$ control problem for time-varying continuous-time affine nonlinear systems.

3. To extend also the procedure to the continuous-time coupled HJIE arising in the state-feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for contnuous-time affine nonlinear systems.

4. To develop a similar procedure for the second-order HJBE of stochastic optimal control and associated Riccati equation for continuous-time affine nonlinear systems.

5. Finally, to apply the above results to some representative and illustrative examples of nonlinear systems; in particular, a rigid spacecraft.

# CHAPTER 2

# A FACTORIZATION APPROACH FOR SOLVING THE HAMILTON-JACOBI-ISAACS EQUATIONS IN NONLINEAR $\mathcal{H}_\infty$ CONTROL

## 2.1 Introduction

In this chapter, we present a factorization approach for solving the HJIE arising in the $\mathcal{H}_\infty$ control problem for smooth affine nonlinear systems. We shall consider both time-invariant and time-varying systems, and both the state-feedback and the output measurememt feedback HJIEs. As mentioned in the introduction, the approach is essentially an inversion approach and relies heavily on the existence of a solution to certain algebraic-differential inequalities which we refer to as the *discriminant inequalities.* An alternative approach depends on finding analytical solutions to some higher-order partial differential equations. As a by-product to the above approaches, we also obtain new approaches to the solutions of the Riccati equations arising in the $\mathcal{H}_\infty$ control problem for LTIS. The rest of the chapter is organized as follows. In section 2, we present the factorization approach that may lead to the solution of the HJIE arising in the state-feedback $\mathcal{H}_\infty$ control problem for a special class of affine nonlinear systems. In section 3, we extend the procedure to a less restricted class of affine nonlinear systems. Further, in section 4, we consider the case of general affine nonlinear systems. Then in sections 5 and 6, we consider the measurement feedback and the time-varying cases respectively. While in section 7, we discuss the relationship between the solutions we have derived and the concept of viscosity solutions of Hamilton-Jacobi-equations. Finally, in section 8, we dicuss the resolution of the discriminant inequality which is central to the solution of the problem.

## 2.2 A Factorization Approach for Solving the HJIE for a Class of Nonlinear Systems

In this section, we discuss the factorization approach for solving the HJIE for a simple class of nonlinear systems. Let us at the outset consider the following affine nonlinear state-space system $\Sigma$ defined in local coordinates over some manifold $M^n \subseteq \Re^n$ containing the origin:

$$\Sigma: \quad \dot{x}(t) = f(x) + g_1(x)w(t) + g_2(x)u(t); \quad x(0) = \hat{x}_0 \tag{2.1}$$

$$z(t) = \begin{bmatrix} h(x) \\ u \end{bmatrix}; \quad f(0) = 0, \quad h(0) = 0 \tag{2.2}$$

where $x \in M$ is the state vector, $w : \Re \to \mathcal{W} \subset L_2([0,\infty), \Re^r)$ is the disturbance (to be rejected and/or tracked), $u : \Re \to \mathcal{U} \subset \Re^k$ is the control input, and $z \in \Re^{m+k}$ is the controlled output. The functions $f : M \to TM$, $g_1 : M \to \mathcal{M}^{n\times r}(M)$, $g_2 : M \to \mathcal{M}^{n\times k}(M)$, $h : M \to \Re^m$, are smooth $C^\infty(M)$ functions of $x$. Recall from chapter 1 that the HJIE for the state-feedback problem of the above system $\Sigma$ is given by:

$$\begin{aligned} & V_x(x)f(x) + \frac{1}{2}V_x(x)[\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)g_2^T(x)]V_x^T(x) + \frac{1}{2}h^T(x)h(x) \le 0; \\ & V(0) = 0 \ \forall x \in N. \end{aligned} \tag{2.3}$$

In the sequel, we shall refer to it as equation, and the inequality will be explicitly mentioned where it is desired. Furthermore, in the remainder of this section, we shall discuss sufficiency conditions for the existence of solutions to the HJIE (2.3) which are provided by the implicit function theorem [67]. In this regard, let us write HJIE (2.3) in the form:

$$HJI(x, V_x) = 0 \tag{2.4}$$

where $HJI : T^\star M \to \Re$. Then we have the following theorem.

**Theorem 2.2.1** *Assume that $V \in C^2(M)$, and the functions $f(.)$, $g_1(.)$, $g_2(.)$, $h(.)$ are smooth $C^\infty(M)$ functions. Then $HJI(.,.)$ is continuously differentiable in an open neighborhood $\Phi \times \Psi \subset T^\star M$ of the origin. Furthermore let $(\bar{x}, \bar{V}_x)$ be a point in $\Phi \times \Psi$ such that $HJI(\bar{x}, \bar{V}_x) = 0$ and the $\mathcal{F}$-derivative of $HJI(.,.)$ with respect to $V_x$ is nonzero, i.e., $\frac{\partial}{\partial V_x}HJI(x, V_x) \neq 0$, then there exists a continuously differentiable solution :*

$$V_x(x) = \overline{HJI}(x) \tag{2.5}$$

*for some some function $\overline{HJI} : \Phi \to \Re$, of HJIE (2.3) in $\Phi \times \Psi$.*

**Proof :** The proof of the above theorem follows from standard results of the implicit function theorem [119]. This can also be shown by linearization of $HJI$ around $(\bar{x}, \bar{V}_x)$; the existence of such a point is guaranteed from the linear $\mathcal{H}_\infty$ control results [120]. Accordingly,

$$HJI(x, V_x) = HJI(\bar{x}, \bar{V}_x) + (V_x - \bar{V}_x)\frac{\partial HJI}{\partial V_x}(\bar{x}, \bar{V}_x) + (x - \bar{x})^T\frac{\partial HJI}{\partial x}(\bar{x}, \bar{V}_x) + HOT = 0 \tag{2.6}$$

where $HOT$ denote higher-order terms. Now since $HJI(\bar{x}, \bar{V}_x) = 0$, it follows from (2.6) that there exists a sphere of radius $r > 0$ centered at $(\bar{x}, \bar{V}_x)$ such that in the limit as $r \to 0$, $HOT \to 0$ and $V_x$ can be expressed in terms of $x$ in the neighborhood $\Phi \times \Psi$. □

**Remark 2.2.1** *Theorem 2.2.1 is only an existence result, and hence is not satisfactory, in the sense that, it does not guarantee the uniqueness of $V_x$. This is due to the lack of invertibility of $\frac{\partial HJI}{\partial V_x}(\bar{x}, \bar{V}_x)$ - being a vector and not a full-rank matrix. Thus, the best possible solution we might expect will have to be a minimum-norm or parameterized solution. Furthermore, the same theorem can be proven for $(\bar{x}, \bar{V}_x) = \{0\}$, but again we do not have uniqueness for $V_x$ since $\frac{\partial HJI}{\partial V_x}(\bar{x}, \bar{V}_x) = 0$. Thus, the origin becomes a bifurcation point.*

**Remark 2.2.2** *We refer to (2.5) as a solution to (2.3) by the fact that, $V$ can easily be recovered from it by carrying out the line integral $\int_0^x V_x(\sigma)d\sigma$. The integration is taken over any path joining the origin to $x$. For convenience, this is usually done along the axes as:*

$$V(x) = \int_0^{x_1} V_{x_1}(y_1, 0, \ldots, 0)dy_1 + \int_0^{x_2} V_{x_2}(x_1, y_2, \ldots, 0)dy_2 + \ldots + \int_0^{x_n} V_{x_n}(x_1, x_2, \ldots, y_n)dy_n. \quad (2.7)$$

*However, to ensure that $V_x$ is the gradient of the scalar function $V$, it is necessary and sufficient that the Hessian matrix $V_{xx}$ is symmetric for all $x \in N = \pi^\star(\Phi \times \Psi) \to M$, which is an immersed submanifold. This will be referred to as the "curl" condition. Equivalently, this also implies that the differential 1-form $dV \in E_1^\star(M)$ is "closed" or $d^2V = 0$ or $\frac{\partial V_{x_i}(x)}{\partial x_j} = \frac{\partial V_{x_j}(x)}{\partial x_i}$, $x \in N$ [6].*

Now that we have a theoretical basis for the existence of solutions to the HJIE, let us proceed in the next sections to a constructive factorization approach for solving the HJIE. Our objective is to find a positive semi-definite function $V \geq 0$ (or more desirably positive-definite) which is $C^r(N)$, $r \geq 2$, and has a *critical point* at $x = \{0\}$ (i.e $V_x(0) = 0$). Furthermore, we also require that $Det(V_{xx}(0)) \neq 0$, which makes $x = \{0\}$ a nondegenerate critical point and $V$ a *Morse function.* Thus, the following famous result leads us to expect $V$ to be locally quadratic in $M$.

**Lemma 2.2.1** *(Finite-dimensional Morse lemma [1])*
*Assume $V : N \subset M \to \Re$ is a $C^r$ map, $r \geq 2$, and $\bar{x}$ a point where (i) $\frac{\partial V}{\partial x}(\bar{x}) = 0$, (ii) $Det\left(\frac{\partial^2 V}{\partial x^2}(\bar{x})\right) \neq 0$. Then, there exists a local curvilinear coordinate system $(\xi_1, \xi_2, \ldots, \xi_n)$ for $N$ near $\bar{x}$ and an integer $0 \leq k \leq n$, such that*

$$V(x) = V(\bar{x}) - \sum_{i=1}^{k} \xi_i^2 + \sum_{k+1}^{n} \xi_i^2$$

*for all $x \in W \subset N$, a neighborhood of $\bar{x}$.*

**Remark 2.2.3** *Note that in the new coordinate system $(\xi_1, \xi_2, \ldots, \xi_n)$, the function $V$ becomes precisely quadratic; it corresponds to its second-order Taylor expansion around $\bar{x}$ (there are no first-order forms). This simple expression for $V$ is achieved at the expense of curving the coordinate system which is no longer linear.*

Combining Theorem 2.2.1 and Lemma 2.2.1, we have existence results for local solutions of the HJIE in a small neighborhood of the origin. But, for application, this is really insufficient as we require solutions over a reasonably larger domain. The question then arises: What will be the nature of the solutions in such a larger domain? To answer this question, we go back to the proof of Theorem 2.2.1. Obviously, in a larger domain, the HOT will feature in the solution and hence to account for these terms, we introduce a parameter $\zeta = \zeta(x)$ into the solution (2.5) as

$$V_x(x) = \overline{HJI}(x, \zeta).$$

Furthermore, Theorem 2.2.1 and Lemma 2.2.1 give local existence of the solutions in terms of the equation variables and the existence of one solution at a point. Then another question arises as to: What conditions in terms of system characteristics will guarantee the local existence of the solutions? Obviously, this will require some form of controllability of the system, such that a stabilizing feedback can be obtained. More specifically, the required condition is that of stabilizability of the system which we define next [86].

**Definition 2.2.1** *The nonlinear system $\Sigma$ is said to be locally smoothly stabilizable if there is a $C^2$ function $\alpha : O \subset M \to \Re^k$ such that $\dot{x} = f(x) + g_2(x)\alpha(x)$ is locally asymptotically stable about the origin $\{0\}$.*

We now state without proof the following proposition which gives necessary conditions for the existence of the solution.

**Proposition 2.2.1** *A necessary condition for the existence of a smooth positive-semidefinite solution $V \in C^\infty(O)$ to the HJIE (2.3) for the system $\Sigma$, is that, it is locally smoothly stabilizable. In addition, if the system is zero-state detectable, then $V > 0 \; \forall x \in O - \{0\}$ (i.e. positive-definite) [107].*

Again, the above proposition only gives the necessary condition implicitly in terms of a Lyapunov function which is also not guaranteed to exist. Fortunately, recourse could always be taken in terms of the linearization of the system. It is a well-known fact that, if the linear equivalent of the system $\Sigma$ about $\{0\}$ is smoothly stabilizable, then the system $\Sigma$ is locally smoothly stabilizable. And since the stabilizability of the linearized system is equivalent to some *'rank conditions'* on the system matrices [120], then the conclusion of Proposition 2.2.1 is that, the necessary condition for the existence of a local positive-semidefinite solution to the HJIE is that, the linearization of the system about $\{0\}$ is stabilizable. This assertion is also the system theoretic equivalent of Theorem 2.2.1. This issue will be further explored in Proposition 2.8.1.

Now consider the following special class of nonlinear systems:

$$\Sigma_0 : \dot{x}(t) = f(x) + w(t) + u(t) \tag{2.8}$$

$$z(t) = \begin{bmatrix} h(x) \\ u \end{bmatrix}; \; f(0) = 0, \; h(0) = 0 \tag{2.9}$$

where $f(.)$ and $h(.)$ are as defined previously in (2.1), (2.2) respectively, while $u(t) \in U \subset \Re^n$, $\forall t \in \Re$, $w(t) \in \mathcal{W} \subset \mathcal{L}_2([0, \infty), \Re^n)$.

The HJIE corresponding to $\Sigma_0$ is given by

$$V_x(x)f(x) + \frac{1}{2}\frac{(1-\gamma^2)}{\gamma^2}V_x(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) \le 0; \quad V(0) = 0 \tag{2.10}$$

Then the following theorem gives a characterization of the solutions of the above HJIE (2.10) in terms of $V_x$.

**Theorem 2.2.2** *Suppose there exists a vector-valued function (vector-field) $\zeta : N \subset M \to TN$, $\pi \circ \zeta = I_N$, such that*

$$\zeta^T(x)\zeta(x) - f^T(x)f(x) + \bar{\gamma}h^T(x)h(x) = 0, \ \forall x \in N \tag{2.11}$$

*where $\bar{\gamma} = \frac{(1-\gamma^2)}{\gamma^2}$, then*

$$V_x(x) = -\frac{1}{\bar{\gamma}}(f(x) \pm \zeta(x))^T, \ \ x \in N \tag{2.12}$$

*is a local solution of the HJIE (2.10) in $N$.*

**Proof:** Compare (2.10) with the standard quadratic equation $ax^2 + bx + c = 0$ whose solution is given by $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Thus, we get a similar solution for (2.10) provided $\zeta(.)$ exists for all $x \in N$. Direct substitution of (2.12) in (2.10) now yields(2.11).□

**Remark 2.2.4** *We shall call $\zeta$ the "discriminant factor" of the system $\Sigma_0$ or its HJIE (2.10), and (2.11) the discriminant equation. It can also be shown that both solutions (2.12) satisfy $\dot{V}(x) \leq 0$, $\forall x \in N$. Furthermore, it is clear from the discriminant equation (2.11) that the HJIE may fail to have a real-analytic solution depending on whether $f^T(x)f(x) - h^T(x)h(x) \geq 0 (\leq 0)$ or indefinite.*

Now suppose there does not exist a discriminant factor $\zeta$ for the system $\Sigma_0$ that satisfies the equality $\zeta(x)\zeta^T(x) - f^T(x)f(x) + \bar{\gamma}h^T(x)h(x) = 0$, $\forall x \in N$ as in Theorem 2.2.2. Then, the following corollary gives a parameterization of all solutions to the HJI inequality (2.10).

**Corollary 2.2.1** *Suppose instead there exists $\zeta \in \Gamma^\infty TN$ such that*

$$\zeta^T(x)\zeta(x) - f^T(x)f(x) + \bar{\gamma}h^T(x)h(x) \leq 0, \ \forall x \in N \tag{2.13}$$

*then, (2.12) solves the HJI inequality (2.10), and the set of all solutions to the inequality are given by*

$$\mathcal{V}_x^1 = \{V_x(x) \mid \zeta^T(x)\zeta(x) - f^T(x)f(x) + \bar{\gamma}h^T(x)h(x) \leq 0, \ x \in N\} \tag{2.14}$$

In the next section, we consider a more practical class of systems.

## 2.3 A Factorization Approach for Solving the HJIE for Systems with Input and Disturbance Gains

In this section, we consider elementary solutions of the HJIE for the following class of nonlinear systems which is more realistic:

$$\Sigma_1 : \dot{x}(t) = f(x) + G_1 w(t) + G_2 u(t) \tag{2.15}$$

$$z(t) = \begin{bmatrix} h(x) \\ u(t) \end{bmatrix} \tag{2.16}$$

where $G_1 \in \Re^{n\times r}$, $G_2 \in \Re^{n\times k}$ are constant matrices while the other matrices are as defined above. The HJIE corresponding to $\Sigma_1$ is given by

$$V_x(x)f(x) + \frac{1}{2}V_x(x)[\frac{1}{\gamma^2}G_1G_1^T - G_2G_2^T]V_x^T(x) + \frac{1}{2}h^T(x)h(x) \leq 0; \quad V(0) = 0, \ \forall x \in M. \tag{2.17}$$

Now define $\mathcal{Q}_\gamma = (\frac{1}{\gamma^2}G_1G_1^T - G_2G_2^T)$, and assume that $\mathcal{Q}_\gamma$ is invertible. Then the following theorem gives an elementary solution to the HJIE (2.17).

**Theorem 2.3.1** *Consider the HJIE (2.17) for the system $\Sigma_1$. Assume that $\mathcal{Q}_\gamma$ is invertible, and there exists a vector field $\zeta : N \subset M \to TN$ such that*

$$\zeta(x)^T Q_\gamma^{-1}\zeta(x) - f^T(x)\mathcal{Q}_\gamma^{-1}f(x) + h^T(x)h(x) = 0, \ \forall x \in N, \tag{2.18}$$

*then a local solution of the HJIE is given by*

$$V_x(x) = -(f(x) \pm \zeta(x))^T \mathcal{Q}_\gamma^{-1}; \ \ \forall x \in N. \tag{2.19}$$

**Proof:** By direct substitution.

The following corollary also gives a parameterization of all solutions to the HJI inequality (2.17).

**Corollary 2.3.1** *Suppose instead there exists $\zeta$ such that*

$$\zeta^T(x)\mathcal{Q}_\gamma^{-1}\zeta(x) - f^T(x)Q_\gamma^{-1}f(x) + h^T(x)h(x) \leq 0; \ \forall x \in N.$$

*Then (2.19) solves the HJI inequality (2.17) and the set of all solutions to the inequality is given by*

$$\mathcal{V}_x^2 = \{V_x(x) \mid \zeta^T(x)Q_\gamma^{-1}\zeta(x) - f^T(x)\mathcal{Q}_\gamma^{-1}f(x) + h^T(x)h(x) \leq 0, \ x \in N\}. \tag{2.20}$$

Now assume that $\mathcal{Q}_\gamma$ is not invertible. Then let us define a solution of the form:

$$V_x(x) = -(f(x) \pm \zeta(x))^T P_\gamma; \ \ x \in N \tag{2.21}$$

where $P_\gamma$ is some unknown symmetric matrix. Then substituting the above solution (2.21) in (2.17) we get

$$-f^T(x)P_\gamma f(x) + \zeta^T(x)P_\gamma f(x) + \frac{1}{2}f^T(x)P_\gamma Q_\gamma P_\gamma f(x) - \zeta^T(x)P_\gamma\mathcal{Q}_\gamma P_\gamma f(x) + \frac{1}{2}\zeta^T(x)P_\gamma\mathcal{Q}_\gamma P_\gamma\zeta(x) + \frac{1}{2}h^T(x)h(x) = 0; \ \ \forall x \in N \tag{2.22}$$

Now assume that the matrix $P_\gamma$ satisfies the condition

$$P_\gamma\mathcal{Q}_\gamma P_\gamma = P_\gamma, \tag{2.23}$$

then we get

$$\zeta^T(x)P_\gamma\zeta(x) - f^T(x)P_\gamma f(x) + h^T(x)h(x) = 0 \tag{2.24}$$

which is precisely the solvability condition (2.18) with $\mathcal{Q}_\gamma^{-1}$ replaced by $P_\gamma$. Furthermore, the condition (2.23) is one of the conditions required for the matrix $P_\gamma$ to be a generalized inverse (or Pseudo-inverse) for the matrix $\mathcal{Q}_\gamma$ denoted by $\mathcal{Q}_\gamma^+$ and satisfies the following properties [99, 20]:

$$\begin{aligned}
\mathcal{Q}_\gamma \mathcal{Q}_\gamma^+ \mathcal{Q}_\gamma &= \mathcal{Q}_\gamma \\
(\mathcal{Q}_\gamma^+ \mathcal{Q}_\gamma)^T &= \mathcal{Q}_\gamma^+ \mathcal{Q}_\gamma \\
\mathcal{Q}_\gamma^+ \mathcal{Q}_\gamma \mathcal{Q}_\gamma^+ &= \mathcal{Q}_\gamma^+ \\
(\mathcal{Q}_\gamma \mathcal{Q}_\gamma^+)^T &= \mathcal{Q}_\gamma \mathcal{Q}_\gamma^+.
\end{aligned}$$

These properties imply that $\Pi_1 = \mathcal{Q}_\gamma^+ \mathcal{Q}_\gamma$, $\Pi_2 = \mathcal{Q}_\gamma \mathcal{Q}_\gamma^+$ are orthogonal projections onto $Im(\mathcal{Q}_\gamma)$. Moreover, the matrix $\mathcal{Q}_\gamma^+$ is unique and always exists, hence we can always replace $\mathcal{Q}_\gamma^{-1}$ by $\mathcal{Q}_\gamma^+$ whenever the former does not exist. Thus, a generalized solution of (2.17) will be given by

$$V_x(x) = -(f(x) \pm \zeta(x))^T \mathcal{Q}_\gamma^+, \; x \in N \tag{2.25}$$

where $\zeta$ satisfies:

$$\zeta^T(x)\mathcal{Q}_\gamma^+(x)\zeta(x) - f^T(x)\mathcal{Q}_\gamma^+ f(x) + h^T(x)h(x) = 0; \; \forall x \in N \tag{2.26}$$

Moreover, the result of Corollary 2.3.1 still holds with $\mathcal{Q}_\gamma^{-1}$ replaced by $\mathcal{Q}_\gamma^+$.

Alternatively, we can proceed with a partial remedy of the lack of invertibility of $\mathcal{Q}_\gamma$ through the HJI inequality (2.17) instead. Since $\mathcal{Q}_\gamma$ is symmetric, then for all $x \in M$

$$\lambda_{\mathcal{Q}_\gamma min}\|x\|^2 \leq x^T \mathcal{Q}_\gamma x \leq \lambda_{\mathcal{Q}_\gamma max}\|x\|^2, \tag{2.27}$$

where $\lambda_{\mathcal{Q}_\gamma min}$, $\lambda_{\mathcal{Q}_\gamma max}$ are the minimum and maximum eigenvalues of $\mathcal{Q}_\gamma$ respectively. Thus,

$$V_x(x)\mathcal{Q}_\gamma V_x^T(x) \leq \lambda_{\mathcal{Q}_\gamma max} V_x(x)V_x^T(x); \;\; \forall x \in M$$

We can now state the following lemma.

**Lemma 2.3.1** *Consider the HJI inequality (2.17) and the following HJI inequality*

$$V_x(x)f(x) + \frac{1}{2}\lambda_{\mathcal{Q}_\gamma max}V_x(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) \leq 0; \quad V(0) = 0 \, \forall x \in M. \tag{2.28}$$

*Then, a solution for (2.28) also solves (2.17).*

**Proof:** Trivial.

Therefore, we have the following theorem regarding an elementary solution to the HJI inequality (2.17).

**Theorem 2.3.2** *Suppose there exists a $\zeta : N \to TN$ that satisfies the following inequality*

$$\zeta^T(x)\zeta(x) - f^T(x)f(x) + \lambda_{\mathcal{Q}_\gamma max}h^T(x)h(x) \leq 0; \; \forall x \in N, \tag{2.29}$$

*then*

$$V_x(x) = -\frac{1}{\lambda_{\mathcal{Q}_\gamma max}}(f(x) \pm \zeta(x))^T; \;\; \forall x \in N \tag{2.30}$$

*solves the HJI inequality (2.17).*

**Proof:** Proof follows from Lemma 2.3.1 and direct substitution.

Consequently, instead of solving the HJI inequality (2.17), we can solve HJI inequality (2.28).

**Remark 2.3.1** *Again, suppose that* $\lambda_{\mathcal{Q}_\gamma max} = 0$ *i.e.* $\mathcal{Q}_\gamma \leq 0$*, then the above procedure cannot work, but one can replace* $\lambda_{\mathcal{Q}_\gamma max}$ *by any* $\lambda > 0$ *since the inequality (2.27) continues to hold for any* $\lambda > \lambda_{max}$*, and the application of Corollary 2.3.2 should yield a solution to the inequality.*

## 2.4 A Factorization Approach for Solving the HJIE for the General Case of Affine Nonlinear Systems

In this section, we consider the general case of an affine nonlinear system $\Sigma$ and the HJIE (2.3). In this regard, let $\mathcal{Q}_\gamma(x) = (\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)g_2^T(x))$, $x \in M$, and assume that $\mathcal{Q}_\gamma(x)$ is invertible for all $x \in N \subset M$. Then the solution of (2.3) will be analogous to that of (2.17) with $\mathcal{Q}_\gamma$ replaced by $\mathcal{Q}_\gamma(x)$, $x \in N$. Moreover, all the other results will also follow analogously.

However, as in the linear case, $\mathcal{Q}_\gamma(x)$, $x \in N$ may not be invertible everywhere in $N$, or the submanifold $N$ may be too small for any practical application. Thus, to remedy this situation, we again replace $\mathcal{Q}_\gamma^{-1}(x)$ by $\mathcal{Q}_\gamma^{+}(x)$, $\forall x \in N$ to obtain the generalized solution of (2.3) as :

$$V_x(x) = -(f(x) \pm \zeta(x))^T \mathcal{Q}^+(x);\ x \in N \tag{2.31}$$

where $\zeta : N \to TN$ satisfies

$$\zeta^T(x)\mathcal{Q}_\gamma^+(x)\zeta(x) - f^T(x)\mathcal{Q}_\gamma^+(x)f(x) + h^T(x)h(x) = 0;\ \forall x \in N. \tag{2.32}$$

**Remark 2.4.1** *Since we require* $V(0) = 0$*, it implies that* $\zeta(0) = 0$*. Moreover, this is the case for any equilibrium point* $x_e$ *of the system* $\Sigma$*.*

Similarly, we proceed with the partial remedy of the singularity problem for $\mathcal{Q}_\gamma(x)$ through the HJI inequality (2.3). We first state the following nonlinear equivalent of the inequalities (2.27) for $\mathcal{Q}_\gamma(x)$, $x \in N$ [53].

**Lemma 2.4.1** *Let* $F : S \subset M \to \mathcal{M}^{n\times n}(M)$ *be a continuous symmetric matrix on a compact set* $S$*. Then, there exists a largest number* $\lambda_1 \in \Re$ *and a smallest number* $\lambda_2 \in \Re$ *such that the inequality*

$$\lambda_2\|p\|^2 \leq p^T F(x)p \leq \lambda_1\|p\|^2 \tag{2.33}$$

*holds for all* $x \in S$ *and all vectors* $0 \neq p \in \Re^n$*.*

**Remark 2.4.2** *In the above Lemma 2.4.1,* $\lambda_1$*,* $\lambda_2$ *can be chosen to be respectively, the minimum and maximum values of the function* $p^T F(x)p$ *over the set* $S$ *with* $\|p\| = 1$*.*

Thus, applying the above lemma to the matrix function $Q_\gamma(x)$, $x \in N$, we get the following HJI inequality:

$$V_x(x)f(x) + \frac{1}{2}\lambda_1 V_x(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) \le 0; \quad V(0) = 0 \; \forall x \in N. \tag{2.34}$$

Again, we have the analog of Lemma 2.3.1.

**Lemma 2.4.2** *A solution for HJI inequality (2.34) also solves (2.3).*

**Proof:** Trivial.

Similarly, we can solve HJI inequality (2.34) instead of HJI inequality (2.3). Furthermore, the solution of (2.34) is given by

$$V_x(x) = -\frac{1}{\lambda_1}(f(x) \pm \zeta(x)); \quad x \in N \tag{2.35}$$

where $\zeta$ satisfies

$$\zeta^T(x)\zeta(x) - f^T(x)f(x) + \lambda_1 h^T(x)h(x) \le 0. \tag{2.36}$$

However, care must be exercised in using Lemma 2.4.2, in the sense that, a solution may not exist at all for the HJI inequality (2.34) because of its conservative nature. This also applies to the other inequality (2.28).

Furthermore, as in the linear case, not every solution is stabilizing. Thus, the next step is to characterize the nature of the solutions (2.12), (2.19), (2.31). Also, the conditions of symmetry of the Hessian $V_{xx}$ mentioned in remark 2.2.2 must be taken into consideration. For this purpose, define the following solution sets:

$$\begin{aligned}
\mathcal{V}^- &= \{V \,|\, V_{xx}(x) = V_{xx}^T(x) \ge 0, \; (f - g_2(x)g_2^T(x)V_x^T(x)) \textit{ is l.a.s. around } \{0\}, \; \forall x \in N\}, \\
\mathcal{V}^+ &= \{V \,|\, V_{xx}(x) = V_{xx}^T(x), \; -(f - g_2(x)g_2^T(x)V_x^T(x)) \textit{ is l.a.s. around } \{0\}, \; \forall x \in N\},
\end{aligned}$$

where *l.a.s.* means locally asymptotically stable, and the Hamiltonian function $H_\gamma$ : $T^\star M \to \Re$ with canonical phase coordinates $(x, p)$:

$$H_\gamma(x,p) = p^T f(x) + p^T \frac{1}{2}[\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)g_2^T(x)]p + \frac{1}{2}h^T(x)h(x) \tag{2.37}$$

corresponding to the HJIE (2.3). Define also the corresponding Hamiltonian vector field $X_{H_\gamma} : T^\star M \to TT^\star M$ by

$$X_{H_\gamma} : \left\{ \begin{array}{lcl} \dot{x}(t) &=& \frac{\partial H_\gamma}{\partial p}(x,p) \\ \dot{p}(t) &=& -\frac{\partial H_\gamma}{\partial x}(x,p) \end{array} \right. \tag{2.38}$$

Furthermore, assume that $\Sigma$ is reachable from $\{0\}$ and $H_\gamma(x,p)$ is hyperbolic at $\{0\}$ (i.e. the Jacobian linearization of $X_{H_\gamma}$ about $\{0\}$ does not have purely imaginary eigenvalues). Let $\{0\} \in \tilde{N} \subset T^\star M$ be a closed set invariant under $X_{H_\gamma}$ (i.e. $X_{H_\gamma}(x,p) \in T_{(x,p)}\tilde{N}$, $\forall (x,p) \in \tilde{N}$) which is also projectable on $M$ under the canonical projection $\pi^\star$

(and $\pi^\star|_{\tilde{N}} : \tilde{N} \to M$ is a diffeomorphism, with $\pi^\star(\tilde{N}) = N$), then $\tilde{N}$ can be decomposed into the following direct sum of a *stable* and *unstable*-invariant submanifolds of $X_{H_\gamma}$ through $(x,p) = \{0\}$:

$$\tilde{N} = \tilde{N}^- \oplus \tilde{N}^+, \tag{2.39}$$

where $\tilde{N}^-$ and $\tilde{N}^+$ are defined by

$$\begin{aligned} \tilde{N}^- &= \{(x,p) \in \tilde{N} \mid \phi_{X_{H_\gamma}}(x,p,t) \to \{0\} \text{ } as \text{ } t \to \infty\}, & (2.40)\\ \tilde{N}^+ &= \{(x,p) \in \tilde{N} \mid \phi_{X_{H_\gamma}}(x,p,t) \to \{0\} \text{ } as \text{ } t \to -\infty\}, & (2.41)\end{aligned}$$

i.e., $X_{H_\gamma}$ restricted to $\tilde{N}^-$ is globally asymptotically stable and $-X_{H_\gamma}$ restricted to $\tilde{N}^+$ is globally asymptotically stable wrt $\{0\}$. By extension, if $\{0\}$ is a hyperbolic equilibrium point of $X_{H_\gamma}$, then there also exist global stable and unstable-invariant submanifolds of $X_{H_\gamma}$, $\tilde{M}^-$, $\tilde{M}^+$ , respectively, such that $T^\star M = \tilde{M}^- \oplus \tilde{M}^+$ and around the equilibrium point $(x,p) = \{0\}$, $\tilde{M}^-$ and $\tilde{M}^+$ can be locally parameterized as the graphs of the smooth solution $V$ of the HJIE as follows:

$$\begin{aligned} \tilde{M}^- &= \{x, \, p = V_x^-(x) | V^- = \min . \, V \in \mathcal{V}^-, \; x \in M\}, & (2.42)\\ \tilde{M}^+ &= \{x, \, p = V_x^+(x) | V^+ = max . \, V \in \mathcal{V}^+, \; x \in M\}. & (2.43)\end{aligned}$$

Moreover, if the vector field $f$ is globally asymptotically stable and $M$ is simply connected, then the above parameterization is global. However, the existence of the solution $\max V \in \mathcal{V}^+$ depends on whether the system $\Sigma$ is reachable from $\{0\}$ or not.

Furthermore, $T^\star M$ with the symplectic structure $\omega = dp \wedge dx \in E_2^\star(M)$ is a symplectic manifold, in the sense that, it preserves the symplectic structure.

**Definition 2.4.1** *A submanifold $\tilde{M} \subset T^\star M$ of dimension $n$ is called a Lagrangian submanifold if and only if, the restriction of the differential form $\omega$ to it is identically zero i.e. $\omega|_{\tilde{M}} = 0$ or $\omega(X_1(x), X_2(x)) = 0, \forall x \in \tilde{M}, \; X_1(x), \, X_2(x) \in T_x\tilde{M}$.*

The next proposition gives a characterization of the stable (resp. unstable) invariant manifolds of Hamiltonian vector fields in terms of Lagrangian submanifolds [1, 107].

**Proposition 2.4.1** *Suppose $(x,p) = \{0\}$ is a hyperbolic equilibrium point of $X_{H_\gamma}$, then the stable and unstable invariant manifolds $\tilde{M}^-$ and $\tilde{M}^+$ are Lagrangian submanifolds. Furthermore, $T_0\tilde{M}^- \subset \Lambda^-_{H_\gamma}$ and $T_0\tilde{M}^+ \subset \Lambda^+_{H_\gamma}$, where $\Lambda^+_{H_\gamma}$, $\Lambda^-_{H_\gamma}$ are the stable and unstable eigenspaces of $DX_{H_\gamma}(x,p)|_{\{0\}}$ respectively, and*

$$DX_{H_\gamma} = \begin{bmatrix} \frac{\partial^2 H}{\partial x \partial p} & \frac{\partial^2 H}{\partial p^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial p \partial x} \end{bmatrix}. \tag{2.44}$$

The important feature of a Lagrangian submanifold is that it can be locally represented as the graph of the gradient of a smooth scalar function $\mathcal{V}$. This can be shown as follows. Consider the standard differential 2-form $\omega = dp \wedge dx$ and restrict it to the submanifold $\tilde{N}$ defined by the graph of $p = \frac{\partial \mathcal{V}(x)}{\partial x}$, then

$$\omega|_{\tilde{N}} = dp \wedge dx|_{\tilde{N}} = d\left(\frac{\partial V}{\partial x}\right) \wedge dx = \frac{\partial^2 V}{\partial^2 x}(dx \wedge dx) = 0$$

In this case, the function $\mathcal{V}(.)$ is called the *generating function* of the submanifold. It also turns out that, the graph of the gradient of a smooth function $\mathcal{V}$ is always a Lagrangian submanifold, and any smooth Lagrangian manifold $\mathcal{N}^n$ can be locally represented as the graph of a smooth function $\mathcal{V}$ vice-versa. Therefore, because of this important characteristic, Lagrangian submanifolds play an important role in the study of Hamiltonian systems and Hamilton-Jacobi equations.

Let us now consider the results of sections 2.3, 2.4 applied to the following linear system:

$$\Sigma_l: \quad \dot{x}(t) = Fx(t) + G_1 w(t) + G_2 u(t) \tag{2.45}$$

$$z(t) = \begin{bmatrix} Hx(t) \\ u(t) \end{bmatrix} \tag{2.46}$$

where $F \in \Re^{n\times n}$, $G_1 \in \Re^{n\times r}$, and $G_2 \in \Re^{n\times k}$, $H \in \Re^{m\times n}$ are constant matrices. In this case, $\zeta(x) = \Gamma x$, $\Gamma \in \Re^{n\times n}$, $\mathcal{Q}_\gamma = (\frac{1}{\gamma^2}G_1G_1^T - G_2G_2^T)$. Thus, $V_x$ is given by

$$V_x(x) = -(Fx \pm \Gamma x)^T \mathcal{Q}_\gamma^+ = -x^T(F \pm \Gamma)^T \mathcal{Q}_\gamma^+ \tag{2.47}$$

where $\Gamma$ satisfies:

$$\Gamma^T \mathcal{Q}_\gamma^+ \Gamma - F^T \mathcal{Q}_\gamma^+ F + H^T H = 0 \tag{2.48}$$

Now define $\Delta = F^T \mathcal{Q}_\gamma^+ F - H^T H$. Then $\Gamma$ is given by the equation

$$\Gamma^T \mathcal{Q}_\gamma^+ \Gamma = \Delta \tag{2.49}$$

This suggests that $\Gamma$ is a coordinate transformation matrix, which in this case is called a congruence transformation between $Q_\gamma^+$ and $\Delta$.

Now, from (2.47), $V_{xx}$ is given by

$$V_{xx} = -(F \pm \Gamma)^T \mathcal{Q}_\gamma^+ \tag{2.50}$$

(cf. with the solution of the linear $H_\infty$-Riccati equation $P = X_2X_1^{-1}$, where the columns of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ span $\Lambda_{H_\gamma}^-$, the stable-eigenspace of the Hamiltonian matrix associated with the linear $H_\infty$ problem, with $X_1$ invertible [25, 120]). However, the requirement is that, $\Gamma$ must be determined such that $V_{xx}$ is symmetric and positive-semidefinite. The following proposition relates the existence of such a $\Gamma$ and the solution of the linear $H_\infty$ Riccati equation.

**Proposition 2.4.2** *Suppose there exists a $\Gamma$ that satisfies (2.49) and such that $V_{xx}$ is symmetric and positive-semidefinite, then $P = V_{xx}$ is a solution of the linear $H_\infty$ algebraic-Riccati-equation (ARE):*

$$F^T P + PF + P[\frac{1}{\gamma^2}G_1G_1^T - G_2G_2^T]P + H^T H = 0. \tag{2.51}$$

*Conversely, if the ARE (2.51) has a symmetric positive-semidefinite solution $P = X_2X_1^{-1}$ as defined above, then there exists a $\Gamma$ that satisfies (2.49).*

**Proof:** If $\Gamma$ exists such that $V_{xx} = V_{xx}^T \geq 0$, then it can be shown that $V_{xx}$ satisfies (2.51) by direct substitution. Conversely, if there exists a solution $P = P^T \geq 0$ of (2.51), then by taking $V_{xx} = P$ and substituting in (2.50), we can determine $\Gamma$.□

A direct connection between the solutions we have derived for the Riccati equation with the popular solution using Hamiltonian matrices can be shown after we state the following theorem [120] (Theorem 13.2, page 329) together with its proof which we reproduce for completeness.

**Theorem 2.4.1** *If $X \in \mathcal{C}^{n\times n}$ is a solution of the Riccati equation*

$$A^T X + XA + XRX + Q = 0 \tag{2.52}$$

*then there exist matrices $X_1,\ X_2 \in \Re^{n\times n}$, with $X_1$ invertible, such that $X = X_2 X_1^{-1}$ and the columns of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ form a basis for an $n$-dimensional invariant subspace of $H$ defined by:*

$$H \triangleq \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}.$$

**Proof:** Define $\Lambda \triangleq A + RX$. Multiplying this by X gives

$$X\Lambda = XA + XRX = -Q - A^{\star}X$$

since $X$ is a solution of (2.52). Now write the above two equalities as

$$\begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \Lambda.$$

Therefore, the columns of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ span an $n$-dimensional invariant subspace of $H$, and defining $X_1 \triangleq I$ and $X_2 \triangleq X$ completes the proofs. □

In the same spirit as in the above theorem, we now prove that the columns of $\begin{bmatrix} I \\ P \end{bmatrix} \triangleq \begin{bmatrix} I \\ -(F \pm \Gamma)^T Q_\gamma^+ \end{bmatrix}$ span an $n$-dimensional invariant subspace of the Hamiltonian matix

$$H_\gamma^l = \begin{bmatrix} F & (\frac{1}{\gamma^2} G_1 G_1^T - G_2 G_2^T) \\ -H^T H & -F^T \end{bmatrix} \triangleq \begin{bmatrix} F & \mathcal{Q}_\gamma \\ -H^T H & -F^T \end{bmatrix} \tag{2.53}$$

corresponding to the ARE (2.51).

**Theorem 2.4.2** *Suppose there exists a $\Gamma$ that satisfies (2.49) and such that $P = -(F \pm \Gamma)^T Q_\gamma^+$ is symmetric, then $P$ is a solution of ARE (2.51). Moreover, if $\mathcal{Q}_\gamma$ is nonsingular, then the columns of $\begin{bmatrix} I \\ P \end{bmatrix}$ span an $n$-dimensional invariant subspace of $H_\gamma^l$ if $\mathcal{Q}_\gamma$ is nonsingular. Otherwise, of $\begin{bmatrix} \mathcal{Q}_\gamma^+ & 0 \\ 0 & I \end{bmatrix} H_\gamma^l$.*

**Proof:** The first part of the theorem has already been shown. For the second part, using the symmetry of $P$ we have

$$\begin{aligned}\begin{bmatrix} \mathcal{Q}_\gamma^+ & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} F & \mathcal{Q}_\gamma \\ -H^T H & -F^T \end{bmatrix}\begin{bmatrix} I \\ -(F\pm\Gamma)^T\mathcal{Q}_\gamma^+ \end{bmatrix} &= \begin{bmatrix} \mathcal{Q}_\gamma^+ & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} F & \mathcal{Q}_\gamma \\ -H^T H & -F^T \end{bmatrix}\begin{bmatrix} I \\ -\mathcal{Q}_\gamma^+(F\pm\Gamma) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{Q}_\gamma^+ & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I \\ -(F\pm\Gamma)^T\mathcal{Q}_\gamma^+ \end{bmatrix}(\mp\Gamma) \end{aligned} \tag{2.54}$$

Hence, $P$ defined as above indeed spans an $n$-dimensional invariant-subspace of $H_\gamma^l$ if $Q_\gamma$ is nonsingular.□

It can also be shown [120] that $\Lambda$ in Theorem 2.4.1 is related to $X$ in the following way:

$$A + RX = X_1\Lambda X_1^{-1}, \Longrightarrow \sigma(A + RX) = \sigma(\Lambda).$$

In the same vein, from Theorem 2.4.2 and equation (2.54) we have

$$Q_\gamma^+(\mp\Gamma) = Q_\gamma^+(\mp\Gamma), \Longrightarrow \sigma\left(\mathcal{Q}_\gamma^+(\mp\Gamma)\right) = \sigma\left(\mathcal{Q}_\gamma^+(\mp\Gamma)\right)$$

Now, suppose $\mathcal{V}_p = Im\begin{bmatrix} I \\ P \end{bmatrix} \subset M^{2n}$ is the $n$-dimensional invariant subspace of $H_\gamma^l$ spanned by the columns of $\begin{bmatrix} I \\ P \end{bmatrix}$, then $\sigma(H_\gamma^l|_{\mathcal{V}_p}) = \sigma(\mp\Gamma)$. Moreover, for any solution $\tilde{P} = P_2P_1^{-1}$ of the ARE (2.51) , there exists a real $\Gamma$ such that:

$$\mp\mathcal{Q}_\gamma^+\Gamma = P_1\Lambda P_1^{-1} = \mathcal{Q}_\gamma^+ F + \mathcal{Q}_\gamma\tilde{P}$$

and $\sigma\left(Q_\gamma^+ F + \mathcal{Q}_\gamma\tilde{P}\right) = \sigma\left(\mp\mathcal{Q}_\gamma^+\Gamma\right)$.

We further introduce the following definitions from [1, 7].

**Definition 2.4.2** *An even dimensional vector space $\mathcal{S}^{2n} \subset \Re^{2n}$ together with a skew-symmetric bilinear form $\omega^2 : \mathcal{S}\times\mathcal{S} \to \Re$ which is nondegenerate (i.e. if $\omega(\nu,\mu) = 0\,\forall\mu, \Rightarrow \nu = 0$) is called a symplectic vector space.*

The form $\omega^2(\nu,\mu) = \nu^T J\mu$, where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is the $2n \times 2n$ signature matrix, is a symplectic structure on $\Re^{2n}$; Thus, $(\mathcal{S}^{2n}, \omega^2)$ forms a symplectic vector space.

**Definition 2.4.3** *Let $(\mathcal{S},\omega)$ be a symplectic vector space, and $\mathcal{F} \subset \mathcal{S}$ a subspace. The $\omega$-orthogonal complement of $\mathcal{F}$ is the subspace defined by*

$$\mathcal{F}^\perp = \{\eta \in \mathcal{S} | \omega(\eta,\eta') = 0\ \forall\eta' \in \mathcal{F}\}$$

*Thus,*

*(i) $\mathcal{F}$ is isotropic if $\mathcal{F} \subset \mathcal{F}^\perp$, i.e. $\omega(\eta,\eta') = 0\,\forall\eta,\ \eta' \in \mathcal{F}$;*

*(ii) $\mathcal{F}$ is co-isotropic if $\mathcal{F} \supset \mathcal{F}^\perp$, i.e. $\omega(\eta,\eta') = 0\ \forall\eta' \in \mathcal{F} \Rightarrow \eta \in \mathcal{F}$;*

*(iii)* $\mathcal{F}$ *is Lagrangian if* $\mathcal{F}$ *is isotropic and has an isotropic complement, i.e.,* $\mathcal{S} = \mathcal{F} \oplus \mathcal{F}'$*, and* $\mathcal{F}'$ *is isotropic.*

*(iv)* $\mathcal{F}$ *is symplectic if* $\omega$ *restricted to* $\mathcal{F} \times \mathcal{F}$ *is nondegenerate.*

The following proposition follows from the above definitions [1, 7].

**Proposition 2.4.3** *Let* $(\mathcal{S}, \omega)$ *be a symplectic vector space and* $\mathcal{F} \subset \mathcal{S}$ *a subspace. Then the following are equivalent:*

*(i)* $\mathcal{F}$ *is Lagrangian.*

*(ii)* $\mathcal{F} = \mathcal{F}^{\perp}$.

*(iii)* $\mathcal{F}$ *is isotropic and* $\dim \mathcal{F} = \frac{1}{2} \dim \mathcal{S}$

It can now be shown that the subspace $\mathcal{V}_p = Span \begin{bmatrix} I \\ P \end{bmatrix}$ with $P$ symmetric, is a Lagrangian subspace, and there is a one-to-one correspondence between the set of real-symmetric solutions of the ARE (2.51) and the set of $n$-dimensional Lagrangian $H^l_\gamma$-invariant subspaces which are complementary to $Span \begin{bmatrix} 0 \\ I \end{bmatrix}$. Moreover, if $(F, G_2)$ is controllable, then it can be shown that every $n$-dimensional Lagrangian $H^l_\gamma$-invariant subspace is complementary to $Span \begin{bmatrix} 0 \\ I \end{bmatrix}$, and hence, corresponds to a real-symmetric solution of the ARE.

Furthermore, since $\mathcal{V}_p$ is an $n$-dimensional Lagrangian $H^l_\gamma$-invariant subspace spanned by $\begin{bmatrix} I \\ P \end{bmatrix}$, by taking $X_1 = I$ and $X_2 = -(F \pm \Gamma)^T Q^+_\gamma$, the following results can be proven as in [120].

**Theorem 2.4.3** *Let* $\mathcal{V}_p = Span \begin{bmatrix} I \\ P \end{bmatrix}$ *be the n-dimensional Lagrangian invariant subspace of* $H^l_\gamma$*. If* $\lambda_i + \bar{\lambda}_j \neq 0$*,* $\forall i, j = 1, 2, \ldots, n$*,* $\lambda_i, \lambda_j \in \sigma(H^l_\gamma|_{\mathcal{V}_P})$*, then there exists a* $\Gamma$ *such that* $P = -(F \pm \Gamma)^T Q^+_\gamma$ *is hermitian.*

**Theorem 2.4.4** *Let* $\mathcal{V}_P$ *be as in the previous theorem. Then, there exists a real* $\Gamma$ *such that* $P$ *is real if and only if* $\mathcal{V}_P$ *is conjugate symmetric, i.e.* $v \in \mathcal{V}_P \Rightarrow \bar{v} \in \mathcal{V}_P$*.*

For the stabilizing solution to the ARE (2.51), we have the following proposition.

**Proposition 2.4.4** *Suppose there exists a* $\Gamma$ *such that* $P$ *is real-symmetric and* $\mp\Gamma$ *is a stability matrix (i.e.* $\sigma(\mp\Gamma) \in \mathcal{C}^-$*, then* $P$ *is a stabilizing solution to the ARE.*

The stabilizing solution belongs to a set $\mathcal{P}^- \stackrel{\Delta}{=} \{P | P = P^T \geq 0,\ \sigma(F - G_2 G_2^T P + G_1 G_1^T P) \in \mathcal{C}^-\}$ of all stabilizing solutions of the ARE (2.51). This set also contains a unique element $P^-$, the maximal solution of the ARE, such that $Span \begin{bmatrix} I \\ P^- \end{bmatrix} = \Lambda^-_{H_\gamma} \supset T_0 \tilde{M}^-$. Analogously, there exists a set of all anti-stabilizing solutions $\mathcal{P}^+ \stackrel{\Delta}{=} \{P | P = P^T,\ \sigma(-(F - G_2 G_2^T P + \gamma^{-2} G_1 G_1^T P)) \in \mathcal{C}^+\}$ of the ARE which also contains a unique element $P^+$, the minimal solution of the ARE, and such that $Span \begin{bmatrix} I \\ P^+ \end{bmatrix} = \Lambda^+_{H_\gamma} \supset T_0 \tilde{M}^+$.

We shall discuss a computational procedure for determining $\Gamma$ later.

## 2.5 A Factorization Approach for Solving the Measurement Feedback HJIEs

In this section, we consider the factorization approach for solving the HJIE arising in the a review of shall be considering

$$\begin{aligned} \Sigma_2 \ : \ \dot{x}(t) &= f(x) + g_1(x) w(t) + g_2(x) u(t) \\ z(t) &= h_1(x) + k_{12}(x) u(t) \\ y(t) &= h_2(x) + k_{21}(x) w(t) \end{aligned}$$

where the variables $x \in M^n$, $u : \Re_+ \to \mathcal{U}$, $w : \Re \to \mathcal{W}$ and the functions $f,\ g_1,\ g_2$ are as defined before; while $h_1 \in \Re^m$, $h_2(.) \in \Re^p$, and $k_{12},\ k_{21}$ have appropriate dimensions. We also assume the following for simplicity.

$$\begin{gathered} h_1^T(x) k_{12}(x) = 0, \ \ k_{12}^T(x) k_{12}(x) = I \\ k_{21}^T(x) g_1^T(x) = 0, \ \ k_{21}(x) k_{21}^T = I \end{gathered}$$

The following theorem from [58] summarizes the solution to the problem.

**Theorem 2.5.1** *Suppose that $\{f(.),\ h_1(.)\}$ are locally zero-state detectable in $N \subset M$ and there exist smooth positive semi-definite functions $V,\ W : N_1 \times N_1 \to \Re$, $N_1 \subset M$, $W(0, \xi) > 0,\ \xi \neq 0$, to the HJIE (2.3) and the HJIE:*

$$[W_x(x,\xi) \ \ W_\xi(x,\xi)] f_e(x,\xi) + \frac{1}{2}[W_x(x,\xi) \ W_\xi(x,\xi)] \begin{bmatrix} \frac{g_1(x) g_1^T(x)}{\gamma^2} & 0 \\ 0 & LL^T \end{bmatrix} \begin{bmatrix} W_x(x,\xi) \\ W_\xi(x,\xi) \end{bmatrix} + \frac{1}{2} h_e^T(x,\xi) h_e(x,\xi) = 0, \ \ W(0,0) = 0, \ (x,\xi) \in N_1 \times N_1 \tag{2.55}$$

*respectively. Then, the problem of $\mathcal{L}_2$-disturbance attenuation($\mathcal{H}_\infty$ control) with internal stability is solved by the output feedback*

$$\begin{aligned} \dot{\xi}(t) &= f(\xi) + g_1(\xi)\alpha_1(\xi) + g_2(\xi)\alpha_2(\xi) + L(y - h_2(\xi)) \\ u(t) &= \alpha_2(\xi) \end{aligned}$$

*for some gain matrix $L \in \Re^{n\times p}$, where $f_e(x,\xi)$, $h_e(x,\xi)$, $\alpha_1(x)$, $\alpha_2(x)$ are given by*

$$
\begin{aligned}
f_e(x,\xi) &= \begin{pmatrix} f(x) + g_1(x)\alpha_1(x) + g_2(x)\alpha_2(\xi) \\ f(\xi) + g_1(\xi)\alpha_1(\xi) + g_2(\xi)\alpha_2(\xi) + L(h_2(x) - h_2(\xi)) \end{pmatrix} \\
h_e(x,\xi) &= \alpha_2(\xi) - \alpha_2(x) \\
\alpha_1(x) &= \frac{1}{\gamma^2} g_1^T(x) V_x^T(x) \\
\alpha_2(x) &= -g_2^T(x) V_x^T(x).
\end{aligned}
$$

Therefore, denoting $e = (x \;\; \xi)^T$ and $W_e = (W_x \; W_\xi)$, HJIE (2.34) can be represented as

$$W_e(e)f_e(e) + \frac{1}{2}W_e(e)\Omega_\gamma(e)W_e^T(e) + \frac{1}{2}h_e^T(e)h_e(e) \leq 0 \tag{2.56}$$

with

$$\Omega_\gamma(e) = \frac{1}{\gamma^2}\begin{bmatrix} g_1(x)g_1^T(x) & 0 \\ 0 & LL^T \end{bmatrix}.$$

Thus, HJIE (2.56) can be solved as in section 2.4. By virtue of the separation properties of $\mathcal{H}_\infty$ controllers [120, 56, 57], it can be shown that the $\mathcal{H}_\infty$ nonlinear filtering HJIE [23] can also be represented in the above form and hence solved in a similar manner.

## 2.6 A Factorization Approach for Solving the HJIE for the Finite-Time Horizon Problem and Time-Varying Nonlinear Systems

In this section, we consider applying the factorization approach for solving the HJIE for the finite-time horizon problem and nonlinear time-varying systems (see also [30, 55, 84]). Since the two problems are similar and related, we shall consider the following time-varying nonlinear system

$$
\begin{aligned}
\Sigma_t: \quad \dot{x}(t) &= f(t,x) + g_1(t,x)w(t) + g_2(t,x)u(t); \quad x(0) = \hat{x}_0 &\quad (2.57)\\
z(t) &= \begin{bmatrix} h(t,x) \\ u(t) \end{bmatrix}; \quad f(t,0) = 0, \quad h(t,0) = 0 &\quad (2.58)
\end{aligned}
$$

where the variables $x \in M^n$, $u : \Re_+ \to \mathcal{U}$, $w : \Re \to \mathcal{W}$ are as defined previously, while $f : \Re \times M \to TM$, $g_1 : \Re \times M \to \mathcal{M}^{n\times r}(\Re \times M)$, $g_2 : \Re \times M \to \mathcal{M}^{n\times k}(\Re \times M)$ and $h : \Re \times M \to \Re^m$ belong to $C^\infty([0,\infty) \times M)$.

Now, the finite-horizon state-feedback $H_\infty$ control problem for the above system involves the optimization of the following functional:

$$J_t(u(t), w(t)) = \min_u \max_w \int_{t=0}^{T} (\|z(\tau)\|^2 - \gamma^2\|w(\tau)\|^2)d\tau \tag{2.59}$$

over some finite-time interval $[0, T]$ using state-feedback controls of the form:

$$u(t) = \beta(t,x); \quad \beta(t,0) = 0 \tag{2.60}$$

The above problem represents a two-person zero-sum differential game of fixed duration $[0, T]$. A pair of strategies $u^\star(t, x)$, $w^\star(t, x)$ under feedback information pattern provides a saddle point solution to the above problem such that

$$J_t(u^\star(t,x), w(t)) \le J_t(u^\star(t,x), w^\star(t,x)) \le J_t(u(t), w^\star(t,x)) \tag{2.61}$$

if there exists a positive definite $C^1$ function $V : [0, T] \times N \to \Re_+$ satisfying the following HJIE [17, 55]:

$$-V_t(t,x) = \min_u \max_w \Big( V_x(t,x)[f(t,x) + g_1(t,x)w(t) + g_2(t,x)u(t) + \|z(t)\|^2 - \gamma^2 \|w(t)\|^2] \Big); \quad V(T,x) = 0, \quad x \in N \tag{2.62}$$

where $V_t$ denotes the first partial derivative of $V$ with respect to $t$.

It can be shown that under the assumption of feedback information pattern and the differentiability of the function $V(.,.)$, the solution to the above problem is given by the following theorem [55, 84].

**Theorem 2.6.1** *Consider the nonlinear time-varying system $\Sigma_t$ and the problem of achieving local $\mathcal{L}_2$-gain less than or equal to $\gamma$ with internal stability for the system over a finite-time horizon $[0, T]$. Suppose there exists a $C^1([0,T] \times N)$ positive definite function $V$ satisfying the following time-varying Hamilton-Jacobi-Isaac equation (inequality)*

$$V_t(t,x) + V_x(t,x) f(t,x) + \frac{1}{2} V_x(t,x)[\frac{1}{\gamma^2} g_1(t,x) g_1^T(t,x) - g_2(t,x) g_2^T(t,x)] V_x^T(t,x) + \frac{1}{2} h^T(t,x) h(t,x) \le 0; \quad V(T,x) \ge \chi(T) > 0 \; \forall x \in N, \; t \in [0,T], \tag{2.63}$$

*where $\chi$ is of class $\mathcal{K}$, then the problem is solved by the feedbacks:*

$$u^\star(t,x) = -g_2^T(t,x) V_x^T(t,x); \quad w^\star(t,x) = g_1(t,x) V_x^T(t,x).$$

*Moreover, $u^\star(t,x)$, $w^\star(t,x)$ satisfy the saddle point conditions (2.61) of the differential game (2.59), (2.57) and $V$ is its value function, i.e.,*

$$V(t,x) = \inf_{u \in \mathcal{L}_2[0,T]} \sup_{w \in \mathcal{L}_2[0,T]} \int_0^T [\|z(t)\|^2 - \gamma^2 \|w(t)\|^2] dt \tag{2.64}$$

The solution of the time-varying HJIE (2.63) is the subject of our discussion in this section. Our aim is to extend the systematic procedure that we have developed in the previous sections to this case. In this respect, we shall employ the following trick. Let us redefine the time parameter $t$ as $x_0$ and $\tilde{x} = [x_0, x_1, \ldots, x_n]^T$. Thus, $[0,T] \times M \subset \Re^{n+1}$ and the equivalent system $\Sigma_t$ takes on the following form:

$$\tilde{\Sigma}_t : \quad \dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}) + \tilde{g}_1(\tilde{x}) w(t) + \tilde{g}_2(\tilde{x}) u(t); \quad \tilde{x}(0) = \tilde{\tilde{x}}_0 \tag{2.65}$$

$$\tilde{z}(t) = \begin{bmatrix} h(\tilde{x}) \\ u(x_0) \end{bmatrix}; \quad \tilde{f}(0) = (1 \;\; 0)^T, \quad \tilde{h}(0) = 0 \tag{2.66}$$

where

$$\tilde{f}(\tilde{x}) = \begin{bmatrix} 1 \\ f(\tilde{x}) \end{bmatrix}, \ \tilde{g}_1(\tilde{x}) = \begin{bmatrix} \mathbf{0}_{1\times r} \\ g_1(\tilde{x}) \end{bmatrix}, \ \tilde{g}_2(\tilde{x}) = \begin{bmatrix} \mathbf{0}_{1\times k} \\ g_2(\tilde{x}) \end{bmatrix}, \ \tilde{h}(\tilde{x}) = h(\tilde{x}).$$

Thus, the HJIE (2.63) can be represented as

$$Y_{\tilde{x}}(\tilde{x})\tilde{f}(\tilde{x}) + \frac{1}{2}Y_{\tilde{x}}(\tilde{x})\Theta(\tilde{x})Y_x^T(\tilde{x}) + \frac{1}{2}\tilde{h}^T(\tilde{x})\tilde{h}(\tilde{x}) \leq 0; \ \ Y(\tilde{x}(T)) \geq \tilde{\chi}(T) \ \forall \tilde{x} \in \bar{N} \subset M \times \Re. \tag{2.67}$$

where $\Theta(\tilde{x}) = [\frac{1}{\gamma^2}\tilde{g}_1(\tilde{x})\tilde{g}_1^T(\tilde{x}) - \tilde{g}_2(\tilde{x})\tilde{g}_2^T(\tilde{x})]$ and $Y_{\tilde{x}}(\tilde{x}) = (V_{x_0}, V_{x_1}, \ldots, V_{x_n})^T$, $\tilde{\chi}(T) > 0$ is of class $\mathcal{K}$. This HJIE is similar to the time-invariant case (2.3), therefore, we can write its solution as

$$Y_{\tilde{x}}(\tilde{x}) = -\left(\tilde{f}(\tilde{x}) \pm \tilde{\zeta}(\tilde{x})\right)^T \Theta^+; \ \tilde{x} \in \bar{N} \tag{2.68}$$

where $\tilde{\zeta} : \bar{N} \to TM$ satisfies the following algebraic inequality:

$$\tilde{\zeta}^T(\tilde{x})\Theta^+\tilde{\zeta}^T(\tilde{x}) - \tilde{f}(\tilde{x})\Theta^+\tilde{f}(\tilde{x}) + \tilde{h}^T(\tilde{x})\tilde{h}(\tilde{x}) \leq 0; \ \forall \tilde{x} \in \bar{N} \tag{2.69}$$

Therefore, if there exists a $\tilde{\zeta} \in \Gamma^\infty TM$ such that $Y_{\tilde{x}\tilde{x}}$ is symmetric and positive-definite, then (2.68) solves the time-varying HJIE (2.63). Consequently, $Y(\tilde{x}) = V(t,x)$ can be determined from $Y_{\tilde{x}}(\tilde{x})$ by carrying out the line integration $\int_0^{\tilde{x}} Y_{\tilde{x}}(\sigma)d\sigma$.

Furthermore, the Hamiltonian system (vector-field $X_{H_\gamma^t}$) corresponding to the HJIE (2.63) is defined by

$$X_{H_\gamma^t} : \begin{cases} \dot{x}(t) &= \frac{\partial H_\gamma(t,x,p)}{\partial p} \\ \dot{p}(t) &= -\frac{\partial H_\gamma(t,x,p)}{\partial x} \end{cases} \tag{2.70}$$

where $H_\gamma^t := H_\gamma(t,x,p) : T^\star M \times \Re \to \Re$ is defined by

$$H_\gamma(t,x,p) = p^T f(t,x) + \frac{1}{2}p^T[g_1(t,x)g_1^T(t,x) - g_2(t,x)g_2^T(t,x)]p + \frac{1}{2}h^T(t,x)h(t,x)$$

Moreover, the differential form on the manifold $T^\star(M \times \Re) \subset \Re^{2n+1}$ is defined by $\tilde{\omega} = dp \wedge dx - dH_\gamma \wedge dt$ and $(T^\star M \times R, \tilde{\omega})$ is a contact manifold since its dimension is odd. Similarly, if $\Sigma_t$ is reachable from $\{0\}$ $\forall t$, then the stable (resp. unstable) invariant-manifolds of the Hamiltonian vector field corresponding to $H_\gamma(t,x,p)$ through $(x,p) = \{0\}$ are locally parameterized by

$$\tilde{M}_t^- = \{x, \, p = V_x^-(t,x) | V_{xx}^-(t,x) > 0, \ X_{H_\gamma^t} \ is \ l.u.a.s. \ \forall t, \ x \ around \ \{0\}\}$$
$$\tilde{M}_t^+ = \{x, \, p = V_x^+(t,x) | V_{xx}^+(t,x), \ -X_{H_\gamma^t} \ is \ l.u.a.s. \ \forall t, \ x \ around \ \{0\}\}$$

for some $V^-(t,x)$, $V^+(t,x)$ smooth minimal and maximal solutions of the HJIE (2.63) respectively, and where *l.u.a.s.* means locally uniformly asymptotically stable. These submanifolds are also referred to as *Legendre submanifolds* and are the counterparts of the Lagrangian-submanifolds in the symplectic case. [6].

## 2.7 Relationship with Viscosity Solutions

In this section, we briefly draw a connection between the solutions approach that we have presented above and the concept of viscosity (or generalized) solutions of Hamilton-Jacobi-Bellman equations or dynamic programming principle. The idea of viscosity solutions first introduced by Crandall and Lions [16] was an attempt to take into consideration nonsmooth and possibly discontinuous Hamiltonians and value-functions of the optimal control problem.

Now, it has been shown [16] that the value function for the nonlinear $H_\infty$ control problem defined in section 2.2 is given by the functional:

$$v(x) = \inf_{u} \sup_{w \in \mathcal{L}_2} \int_0^\infty [\|z(\tau)\|^2 - \gamma^2 \|w(\tau)\|^2] dt \tag{2.71}$$

Furthermore, $v$ satisfies the the HJIE (2.3) for the case when $v \in C^1(M)$ and bounded. On the other hand, if we allow for $v \in C^0(M)$, then HJIE (2.3) takes a different form:

$$HJI(x, Dv(x)) = 0; \quad x \in M \tag{2.72}$$

where $Dv(x)$ is the gradient of $v$ at $x$. Let us now define the following sets which are respectively the *superdifferential* and *subdifferential* of $v$ at $x \in N \subset M$.

$$D^+v(x) = \left\{ p \in \Re^n \; : \; \lim_{y \to x} \sup_{y \in N} \frac{v(y) - v(x) - p.(y - x)}{\|y - x\|} \leq 0 \right\} \tag{2.73}$$

$$D^-v(x) = \left\{ q \in \Re^n \; : \; \lim_{y \to x} \inf_{y \in N} \frac{v(y) - v(x) - q.(y - x)}{\|y - x\|} \geq 0 \right\} \tag{2.74}$$

Then we have the following definitions of viscosity solutions of the HJIE (2.72).

**Definition 2.7.1** *A continuous function $v$ is a viscosity solution of HJIE (2.72) if it is both a viscosity subsolution and supersolution, i.e., it satisfies respectively the following conditions:*

$$HJI(x, p) \leq 0; \quad \forall x \in N, \; \forall p \in D^+v(x) \tag{2.75}$$

$$HJI(x, q) \geq 0; \quad \forall x \in N, \; \forall q \in D^-v(x) \tag{2.76}$$

In the context of viscosity solutions, a solution of the HJIE (2.3) that is differentiable at any $x \in N$ is referred to as a *classical solution.* The most important result we wish to mention in this section is the following [16].

**Proposition 2.7.1** *If $v \in C^0(N)$ is a classical solution of HJIE (2.3), then $v$ is a viscosity solution, and conversely if $v \in C^1(N)$ is a viscosity solution of (2.3), then $v$ is a classical solution.*

Thus, the solutions we have derived in sections 2.2, 2.3, 2.4, 2.5, 2.6 are indeed viscosity solutions of the corresponding HJIEs.

## 2.8 Resolution of the Discriminant Equation (Inequality)

Thus far, we have presented an approach to the inversion of the HJIE interms of the gradient of the smooth solution for different classes of systems and different objectives or cost functions. However, obtaining the gradient of this function also hinges on solving a discriminant equation (inequality) for the parameter $\zeta$ from which the gradient can be obatained. Therefore in this section, we consider an analytical approach for resolving the discriminant equation(inequality) and its relationship with some important matrix optimization problems arising in approximation theory for some tractable problems. We first state the following proposition which gives sufficient conditions for the local existence of solutions to the algebraic equation (2.32).

Consider the linearization of the system $\Sigma$ given by $\Sigma_l$ where in this case $F = \frac{\partial f}{\partial x}(0)$, $G_1 = g_1(0)$, $G_2 = g_2(0)$, $H = \frac{\partial h}{\partial x}(0)$. Then we can state the following.

**Proposition 2.8.1** *Consider the linearization of the system $\Sigma$ about the origin $\{0\}$ given by model $\Sigma_l$. Suppose that $[F, G_2]$ is controllable and $[H, F]$ is detectable. Then there exists a $\zeta$ that solves (2.32) and hence the HJIE (2.3) locally in a neighborhood $N_0 \subset M$ of the origin, if and only if, there exists a positive-semidefinite solution $P \geq 0$ of the algebraic Riccati equation (2.51). In other words, if the linear $\mathcal{H}_\infty$ control problem for the linearized system $\Sigma_l$ is solvable, then locally around the origin, there exists a solution of the nonlinear problem.*

**Proof:** The proof of this proposition follows from Proposition 2.4.2 by extending the solution $\Gamma$ to (2.49) to a small neighborhood $N_0$ of the origin such that $\zeta(x) = \Gamma x, \ \forall x \in N_0$. Now, the stable-eigenspace of the Hamiltonian matrix corresponding to the ARE (2.51), $H^l_\gamma$, is spanned by $\begin{bmatrix} I \\ P \end{bmatrix}$ for some symmetric positive-semidefinite matrix $P$ a solution of (2.51). By Proposition 2.4.1, $T_0T^\star M \subset span \begin{bmatrix} I \\ P \end{bmatrix}$, therefore, there exists a neighborhood $\{0\} \subset \tilde{N}_0 \subset T^\star M$ such that the projection $\pi^\star(\tilde{N}_0) \to M$ is a diffeomorphism, with $\pi^\star(\tilde{N}_0) = N_0$. Hence, there exists $\zeta : N_0 \to TN_0$ such that $\zeta(x) = \Gamma x, \ \forall x \in N_0$, for some $\Gamma \in \Re^{n \times n}$ that solves (2.32). Finally, by proposition 2.4.1, we know that $\Gamma$ exists if and only if $P \geq 0$ solves (2.51). This completes the proof. The proof of this proposition can also be obtained as a consequence of Lemma 2.2.1. Note that, by assuming the coordinate system $(\xi_1, \xi_2, \ldots, \xi_n)$ to be linear, we obtain the solution of the linearized problem. □

**Remark 2.8.1** *Thus, if the ARE corresponding to the linearization of the system is solvable, it guarantees the local existence of solutions to the algebraic equation (2.32) and hence the HJIE (2.3).*

We now consider the existence of solutions to the algebraic equation (2.32) on a larger subset $N$ of the manifold $M$. For this purpose, let $M$ be a Hilbert manifold modelled on the Euclidean space $E$ with the inner product $\langle .,. \rangle$, and let us consider the general forms of the discriminant inequalities:

$$\zeta^T(x)\mathcal{Q}_\gamma^+(x)\zeta(x) - f^T(x)\mathcal{Q}_\gamma^+(x)f(x) + h^T(x)h(x) \leq 0;\ \forall x \in N \tag{2.77}$$

$$\zeta^T(x)\zeta(x) - f^T(x)f(x) + \lambda_1 h^T(x)h(x) \leq 0;\ \ \forall x \in N \tag{2.78}$$

Further, let us first consider the equality in (2.77) which corresponds to the HJIE (2.3). Assume that there is a $\hat{\gamma} > 0$ such that $Q_{\hat{\gamma}}^+(x) \geq 0$, $\forall x \in W \subset N$, which is closed. Then, there exists a matrix-valued function $\mho_{\hat{\gamma}} \in \mathcal{M}^{n\times n}(M)$ (the square root of $Q_{\hat{\gamma}}(x)$, $x \in M$) such that $\mathcal{Q}_{\hat{\gamma}}^+(x) = \mho_{\hat{\gamma}}^T(x)\mho_{\hat{\gamma}}(x) \geq 0$, $\forall x \in W$. Then (2.77) can be written as

$$\|\mho_{\hat{\gamma}}(x)\zeta(x)\|^2 + \|h(x)\|^2 = \|\mho_{\hat{\gamma}}(x)f(x)\|^2;\ \ \forall x \in W \tag{2.79}$$

The above equation (2.79), expresses an orthogonal relationship between the vector fields $\zeta$, $f$ and the function $h$. To complete the picture, let $h^\flat \in \Re^n$ be the extension of $h$ to $\Re^n$ by filling some of the rows of $h$ with zeros. Then, it follows from above that there exists a $W_1 \subset W$ such that

$$\mho_{\hat{\gamma}}(x)f(x) = \mho_{\hat{\gamma}}(x)\zeta(x) \oplus h^\flat(x);\ \forall x \in W_1$$

if and only if

$$\zeta^T(x)\mho_{\hat{\gamma}}(x)h^\flat(x) = 0;\ \forall x \in W_1 \tag{2.80}$$

and $\zeta$ is given by

$$\zeta(x) = f(x) - \mho_{\hat{\gamma}}^+(x)h^\flat(x);\ \forall x \in W_1 \tag{2.81}$$

Thus, if we assume $\mho_{\hat{\gamma}}(x)f(x)$ to be an arbitrary vector field on $W$, then the relationship (2.79) implies that at each $x \in W_1$, $\mho_{\hat{\gamma}}(x)\zeta(x) \in T_xW_1$ and $h^\flat(x) \in T_x^\perp W_1$. This reflects the decomposition of $T_xW|_{W_1}$ into $T_xW|_{W_1} = T_xW_1 \oplus T_x^\perp W_1$, where $T_x^\perp W_1$ is the normal space to $W_1$ at $x$. Furthermore, equation (2.81) gives us yet an analytical approach for determining $\zeta$. Indeed, it narrows down our search for $\zeta$ and almost gives us an explicit solution. We summarize this result in the following proposition.

**Proposition 2.8.2** *Suppose there exists a $W \subset N$ and $\gamma = \hat{\gamma} > 0$ such that $\mathcal{Q}_\gamma^+(x)$ in (2.77) is positive-semidefinite on $W$ and there exists a $W_1 \subset W$ such that (2.80) holds. Then there exists a $\zeta$, given by (2.81), which solves (2.77) if and only if (2.80) is satisfied.*

Conversely, assume that there exists a $\tilde{W} \subset N$ such that $\mathcal{Q}_\gamma^+(x) \leq 0$, $\forall x \in \tilde{W}$. Then there exists a matrix function $\tilde{Q}_\gamma^+(x) = -\mathcal{Q}_\gamma^+(x) \geq 0$ such that (2.77) will imply

$$\zeta^T(x)\tilde{Q}_\gamma^+(x)\zeta(x) - f^T(x)\tilde{Q}_\gamma^+(x)f(x) - h^T(x)h(x) = 0;\ \forall x \in \tilde{W}, \tag{2.82}$$

which further implies that

$$\|\tilde{\mho}_\gamma(x)\zeta(x)\|^2 = \|h^\flat(x)\|^2 + \|\tilde{\mho}_\gamma(x)f(x)\|^2;\ \ \forall x \in \tilde{W} \tag{2.83}$$

where $\tilde{Q}_\gamma^+(x) = \tilde{\mho}_\gamma^T(x)\tilde{\mho}_\gamma(x)$. Equation (2.83) expresses the dual relationship between $\zeta$, $h$ and $f$. Thus, in this case, there exists a $\zeta$ such that

$$\tilde{\mho}_\gamma(x)\zeta(x) = \tilde{\mho}_\gamma(x)f(x) \oplus h^\flat(x); \quad \forall x \in \tilde{W}_1 \subset \tilde{W}_1 \tag{2.84}$$

if and only if

$$f^T(x)\mho_\gamma^T(x)h^\flat(x) = 0 \ \ \forall x \in \tilde{W}_1. \tag{2.85}$$

Conditions (2.84), (2.85) can only be satisfied if we can freely choose $h$ and consequently $h^\flat$ such that they are satisfied. Otherwise, they are meaningless. However, there could be some values of $\gamma$ and a small subset $\tilde{W}_1 \subset \tilde{W}$ for which these conditions are satisfied. In any case, this can hardly guarantee the existence of a $\zeta$ that solves the discriminant equation.

If however we consider the inequality form of equation (2.83) i.e.

$$\|\tilde{\mho}_\gamma(x)f(x)\|^2 + \|h^\flat(x)\|^2 \leq \|\tilde{\mho}_\gamma(x)\zeta(x)\|^2; \ \ \forall x \in \tilde{W}. \tag{2.86}$$

Then there always exists a $\zeta$ that satisfies it. One such solution is

$$\tilde{\mho}_\gamma(x)\zeta(x) = \tilde{\mho}_\gamma(x)f(x) + Z(x)h^\flat(x); \ \ x \in \tilde{W}_2$$

if and only if

$$f^T(x)\tilde{\mho}_\gamma^+(x)Z(x)h^\flat(x) = 0, \ \ \textit{and} \ \ Z^T(x)Z(x) \geq I \ \ \forall x \in \tilde{W}_2 \subset \tilde{W} \tag{2.87}$$

or

$$\zeta(x) = f(x) + \tilde{\mho}_\gamma^+(x)Z(x)h^\flat(x); \ \ x \in \tilde{W}_2 \tag{2.88}$$

We summarize this result in the following proposition.

**Proposition 2.8.3** *Suppose for some $\gamma > 0$ and $\tilde{W} \subset N$, $\mathcal{Q}_\gamma^+(x)$, $x \in \tilde{W}$ is negative-semidefinite, and there exists $\tilde{W}_2 \subset \tilde{W}$ such that (2.87) holds for some expansive but bounded matrix function $Z \in \mathcal{M}^{n\times n}(\tilde{W})$, then there exists a $\zeta$ that solves the discriminant inequality (2.77), and it is given by (2.88).*

**Remark 2.8.2** *Therefore, in either case when $\mathcal{Q}_\gamma^+(x)$, $x \in N$ is positive or negative semi-definite, there exists a solution to the discriminant inequality and consequently of the HJI inequality. Obviously, the critical case is when $\mathcal{Q}_\gamma^+(x)$, $x \in N$ is indefinite, and we cannot say anything about this case. The solution of the inequality (2.78) can also be persued along similar lines.*

Let us now apply the above results to the linear system $\Sigma_l$ and hence the discriminant inequality:

$$\Gamma^T\mathcal{Q}_\gamma^+\Gamma - F^T\mathcal{Q}_\gamma^+F + H^TH \leq 0 \tag{2.89}$$

If there exists a $\hat{\gamma}$ such that $\mathcal{Q}_{\hat{\gamma}}^+ \geq 0$, then there exists $\mho_{\hat{\gamma}}$ such that $\mathcal{Q}_{\hat{\gamma}}^+ = \mho_{\hat{\gamma}}^T\mho_{\hat{\gamma}} \geq 0$ and

$$\Gamma^T\mho_{\hat{\gamma}}^T\mho_{\hat{\gamma}}\Gamma - F^T\mho_{\hat{\gamma}}^T\mho_{\hat{\gamma}}F + H^TH \leq 0 \tag{2.90}$$

The above inequality can easily be related to the following matrix dilation problem [120]:

$$\min_{\Gamma} \left\| \begin{pmatrix} \Gamma\mho_{\hat{\gamma}} \\ H \end{pmatrix} \right\| \leq \sigma_{max}(\mho_{\hat{\gamma}} F) \tag{2.91}$$

where $\sigma_{max}(\mho_{\hat{\gamma}} F)$ is the spectral norm of $\mho_{\hat{\gamma}} F$. The above problem has a solution $\Gamma = Y[\sigma^2_{max}(F\mho_{\hat{\gamma}})I - H^T H]^{\frac{1}{2}} \mho^+_{\hat{\gamma}}$ for some contractive matrix $Y$ satisfying $\|Y\| \leq 1$. The limiting case where $Y = I$ corresponds to the solution of the equality.

Now assume that $\gamma$ is such that $\mathcal{Q}^+_\gamma$ is negative-semidefinite, then there exists a $\tilde{\mathcal{Q}}^+_\gamma = \tilde{\mho}^T_\gamma \tilde{\mho}_\gamma = -\mathcal{Q}^+_\gamma \geq 0$ such that

$$F^T \tilde{\mathcal{Q}}^+_\gamma F + H^T H \leq \Gamma \tilde{\mathcal{Q}}^+_\gamma \Gamma \tag{2.92}$$

which is equivalent to the the following matrix dilation problem

$$\min_{\Gamma} \left\| \begin{pmatrix} F\tilde{\mho}_\gamma \\ H \end{pmatrix} \right\| \leq \sigma_{max}(\tilde{\mho}_\gamma \Gamma) \tag{2.93}$$

which has a solution $\Gamma = \mho^+_\gamma Z[F\tilde{\mathcal{Q}}^+_\gamma F + H^T H]^{\frac{1}{2}}$ if and only if $\|Z\| \geq 1$ for some expansive matrix $Z$.

Before we close this section, let us define the following problem in relation to the above discriminant inequalities and in anology with matrix completion problems.

**Problem 2.8.1** *A vector field $\zeta \in \Gamma^\infty TM$ is a positive (semi-positive) completion of the vector field $f \in \Gamma^\infty TM$ on $N \subset M$ with respect to a matrix function $\mathcal{Q}^+ \in \mathcal{M}^{n\times n}(M)$, if the second derivative $V_{xx}$ of $V_x$ (given by 2.31)) is symmetric and positive definite (semidefinite) for all $x$ in $N$ the domain of definition of $V$.*

The above definition allows us to reduce the problem of solving the HJIE to that of finding a semi-positive completion of the vector field $f$ wrt to the matrix function $\mathcal{Q}^+_\gamma(x)$, $x \in N$.

# CHAPTER 3
# COMPUTATIONAL DETAILS

## 3.1 Introduction

In the previous chapter, we have developed a factorization approach for solving the HJIEs (inequalities) for various classes of nonlinear systems. We have systematically reduced the problem to that of a completion problem for the vector fields $f$, and distributions $g_1$ and $g_2$. The previous section has also been devoted to the problem of analytically resolving the discriminant inequalities resulting from the above reduction. These solutions are however derived blindly without taking into consideration the necessity for the solution vector $\zeta$ to be a positive (semi-positive) completion, and for the resulting gradient $V_x$ to correspond to a scalar function (dictated by the "curl-conditions"). It is therefore our aim in this chapter to address this problem, and to present a computational approach for determining positive-semidefinite solutions of the HJIE. We shall again consider the general case of the equation (2.3).

The rest of the chapter is organized as follows. In the next section, we present an analytical approach for deriving symmetric and positive-semidefinite solutions to the HJIE. This is done by adding the curl and positivity constraints to the discriminant inequality. The procedure is then illustrated by some examples in section 3. Then in section 4, we consider the computational procedure for linear systems. It is shown that for linear systems, the problem can efficiently be formulated as a convex optimization problem over a system of linear matrix inequalities (LMIs). Finally, in section 5 we present an alternative approach for computing the solution to the HJIE using higher-order PDEs.

## 3.2 Computational Procedure for Symmetric Solutions

We proceed with a procedure for determining $\zeta$ such that $V$ is a scalar positive-semidefinite function, as determined by the "curl conditions". Now,

$$V_{xx}(x) = -\left(\frac{\partial f(x)}{\partial x} \pm \frac{\partial \zeta(x)}{\partial x}\right)^T \mathcal{Q}_\gamma^+(x) - \left(I_n \otimes (f(x) \pm \zeta(x))^T\right) \frac{\partial \mathcal{Q}_\gamma^+(x)}{\partial x}, \tag{3.1}$$

where

$$\frac{\partial \mathcal{Q}_\gamma^+(x)}{\partial x} = \left[\frac{\partial \mathcal{Q}_\gamma^+(x)}{\partial x_1}, \ldots, \frac{\partial \mathcal{Q}_\gamma^+}{\partial x_n}\right]^T, \quad \frac{\partial f}{\partial x} = f_x(x) = [\frac{\partial f_1}{\partial x}, \ldots, \frac{\partial f_n}{\partial x}], \tag{3.2}$$

$$\frac{\partial \zeta}{\partial x} = \zeta_x(x) = [\frac{\partial \zeta_1}{\partial x}, \ldots, \frac{\partial \zeta_n}{\partial x}], \quad h_x(x) = \frac{\partial h}{\partial x}(x) = [\frac{\partial h_1}{\partial x}, \ldots, \frac{\partial h_m}{\partial x}], \tag{3.3}$$

and $V_{xx}(x) = V_{xx}^T(x)$ will imply

$$\left(\frac{\partial f(x)}{\partial x} \pm \frac{\partial \zeta(x)}{\partial x}\right)^T \mathcal{Q}_\gamma^+(x) + \left(I_n \otimes (f(x) \pm \zeta(x))^T\right) \frac{\partial \mathcal{Q}_\gamma^+(x)}{\partial x} =$$

$$Q_\gamma^+(x)\left(\frac{\partial f(x)}{\partial x} \pm \frac{\partial \zeta(x)}{\partial x}\right)\frac{\partial \mathcal{Q}_\gamma^+(x)}{\partial x}\left(I_n \otimes (f(x) \pm \zeta(x))\right) \quad x \in N. \tag{3.4}$$

If $\mathcal{Q}_\gamma^+(x) = \mathcal{Q}_\gamma^+$ is a constant matrix as in section 2.3, then equation (3.4) becomes

$$\left(\frac{\partial f(x)}{\partial x} \pm \frac{\partial \zeta(x)}{\partial x}\right)^T Q_\gamma^+ = Q_\gamma^+\left(\frac{\partial f(x)}{\partial x} \pm \frac{\partial \zeta(x)}{\partial x}\right), \quad x \in N. \tag{3.5}$$

Equations (3.4), (3.5) are a system of $\frac{n(n-1)}{2}$ first-order partial differential equations in $n$ unknowns, $\zeta$, which can be solved for $\zeta$ upto an arbitrary vector $\lambda \in \Gamma^\infty TN$ (cf. these conditions with the variable gradient method for finding a Lyapunov function [67]).

**Remark 3.2.1** *Under the above symmetry condition and if $\mathcal{Q}_\gamma(x)$ is invertible for all $x \in N$, then the closed-loop dynamics for the system $(\Sigma(u^\star, w^\star))$ will be given by $\dot{x} = f - g_2 g_2^T V_x^T + \frac{1}{\gamma^2} g_1 g_1^T V_x^T = \mp\zeta$ (or equivalently $\Sigma|_N = \mp\zeta$). Thus, for any $V$ to be a stabilizing solution, $\mp\zeta$ must be an asymptotically stable vector field. Moreover, if $\Sigma$ is zero-state detectable, there is also a splitting of $TM|_N$ into $TM|_N = TN^- \oplus TN^+$, where $\mp\zeta : N \to TN^-$ is asymptotically stable and $\pm\zeta : N \to TN^+$ is asymptotically stable. On the other hand, if $\mathcal{Q}_\gamma$ is singular in $N$, then the closed-loop dynamics will be governed by $\mathcal{Q}_\gamma^+(x)\Sigma|_N = \mathcal{Q}_\gamma^+(x)(\mp\zeta)$ (cf. (2.54)).*

The second requirement $V_{xx} \geq 0$ will imply that

$$\left(\frac{\partial f(x)}{\partial x} \pm \frac{\partial \zeta(x)}{\partial x}\right)^T \mathcal{Q}_\gamma^+(x) + \left(I_n \otimes (f(x) \pm \zeta(x))^T\right)\frac{\partial \mathcal{Q}_\gamma^+(x)}{\partial x} \leq 0, \quad \forall x \in N \tag{3.6}$$

which can be satisfied by a suitable selection of $\lambda$. Thus, the inequalities (3.4), (3.6) will characterize the solution of the completion problem 2.8.1. The following proposition gives sufficient conditions for the solvability of the completion problem.

**Proposition 3.2.1** *A sufficient condition for the solvability of the completion problem (Problem 2.8.1) is that there exists a real vector-field $\zeta : N_c \to TN_c$, $N_c \subseteq N$ that satisfies (3.4), (3.6) for all $x \in N_c$ and some $\gamma > 0$.*

The following corollary also gives sufficient conditions for the solvability of the the HJIE (2.3).

**Corollary 3.2.1** *Suppose in addition to the conditions in Proposition 3.2.1, the vector-field $\zeta$ satisfies (2.77) or (2.78) for some $\gamma > 0$, then the HJIE (2.3) is solvable in $N_c$.*

Finally, the following theorem summarizes the main result of chapter 2.

**Theorem 3.2.1** *Suppose there exists a smooth $C^\infty$ vector-valued function (vector-field) $\zeta : N \to TN$ that satisfies the algebraic-differential inequalities (2.32), (3.4) and (3.6), then (2.31) solves the HJIE (2.3) locally on $M$.*

## 3.3 Representative Examples

In this section, we illustrate the above solution procedure using some examples of affine nonlinear systems.

**Example 3.3.1** *Consider the following system:*

$$\begin{aligned} \dot{x}_1(t) &= -x_1^3(t) - x_2(t) \\ \dot{x}_2(t) &= x_2(t) + u(t) + w(t) \\ z(t) &= [x_2(t) \ \ u(t)]^T \end{aligned}$$

*Let* $\gamma = 2$, *then*

$$f(x) = \begin{bmatrix} -x_1^3 + x_2 \\ x_2 \end{bmatrix}; \ G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \ G_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \ \ h(x) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix};$$
$$\mathcal{Q}_\gamma = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{3}{4} \end{bmatrix}; \ \ \mathcal{Q}_\gamma^+ = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{4}{3} \end{bmatrix}.$$

*Substituting the above functions in (2.77), (3.5), (3.6), we get*

$$-4\zeta_2^2 + 4x_2^2 + 3x_2^2 \leq 0; \ \ \forall x \in N_e \tag{3.7}$$

$$\zeta_{1,x_2}(x) = -1 \ \ \forall x \in N_e \tag{3.8}$$

$$\begin{bmatrix} 0 & \frac{4}{3} - \frac{4}{3}\zeta_{1,x_2} \\ 0 & -\frac{4}{3} - \frac{4}{3}\zeta_{2,x_2} \end{bmatrix} \leq 0; \ \ \forall x \in N_e \tag{3.9}$$

*Solving the above system we get*

$$\zeta_2(x) = \pm\frac{\sqrt{7}}{2}x_2, \ \ \zeta_1(x) = -x_2 + \phi(x_1)$$

*where* $\phi(x_1)$ *is any arbitrary function.*

$$V_x(x) = -(f(x) \pm \zeta(x))^T Q^+ = (0 \quad \frac{8 \pm 4\sqrt{7}}{6}x_2)$$

*Finally, integrating the positive term in the expression for* $V_x$ *from* $0$ *to* $x$, *we get* $V(x) = \frac{2+\sqrt{7}}{3}x_2^2$ *which is positive-semidefinite.*

**Remark 3.3.1** *Note that, the above solution has been obtained with* $\mathcal{Q}_\gamma \leq 0$, *which is consistent with the results that we obtained in section 2.8. Moreover, the solution is also global, and by increasing* $\gamma > 2$, *it is possible to obtain a positive-definite solution.*

**Example 3.3.2** *In this example, we consider the model of the sattelite considered in [32, 65, 74] (and the references there in ). The equations of motion of the spinning sattelite are governed by two subsystems, namely, a kinematic model and a dynamic model. The configuration space of the satellite is a six dimensional manifold, the tangent bundle* *of $SO(3)$, $TSO(3)$, (where $SO(3)$ is the special orthogonal group), and the equations of motion are given by:*

$$\begin{aligned} \dot{R} &= RS(\omega) && (3.10) \\ J\dot{\omega} &= S(\omega)J\omega + u + Pd && (3.11) \end{aligned}$$

*where $\omega \in \Re^3$ is the angular velocity vector about a fixed inertial reference frame with three principal axes and having the origin at the center of gravity of the satellite, $R \in SO(3)$, is the orientation matrix of the satellite, $u \in \Re^3$ is the control torque input vector, and $d$ is the vector of external disturbances on the spacecraft, $P = diag\{P_1,\ P_2,\ P_3\}$, $P_i \in \Re$, $i = 1, 2, 3$, is a constant gain matrix, $J$ is the inertia matrix of the system, and $S(\omega)$ is the skew-symmetric matrix*

$$S(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

*In [32, 65, 74] the control of the full equations of motion (3.10), (3.11) were considered, while in [10] the disturbance attenuation problem for the the underactuated dynamic model is considered. In this paper, we shall consider the control of the angular velocities governed by the dynamic subsystem (3.11). By letting $J = diag\{I_1, I_2, I_3\}$, where $I_1 > 0,\ I_2 > 0,\ I_3 > 0$ and without any loss of generality we assume $I_1 \neq I_2 \neq I_3$, are the principal moments of inertia, then the subsystem (3.11) can be represented as:*

$$\begin{aligned} I_1\dot{\omega}_1(t) &= (I_2 - I_3)\omega_2(t)\omega_3(t) + u_1 + P_1 d_1(t) \\ I_2\dot{\omega}_2(t) &= (I_3 - I_1)\omega_3(t)\omega_1(t) + u_2 + P_2 d_2(t) \\ I_3\dot{\omega}_3(t) &= (I_1 - I_2)\omega_1(t)\omega_2(t) + u_3 + P_3 d_3(t) \end{aligned} \qquad (3.12)$$

*Now define*

$$A_1 = \frac{(I_2 - I_3)}{I_1},\quad A_2 = \frac{(I_3 - I_1)}{I_2},\quad A_3 = \frac{(I_1 - I_2)}{I_3} \qquad (3.13)$$

*and using Euler angles $\phi$, $\theta$, $\psi$ for the orientation $R = [r_1\ r_2, r_3]$ about a reference frame, the complete equations of motion can be explicitly represented as*

$$\begin{aligned} \dot{\omega}_1(t) &= A_1\omega_2(t)\omega_3(t) + u_1 + P_1 d_1(t) \\ \dot{\omega}_2(t) &= A_2\omega_3(t)\omega_1(t) + u_2 + P_2 d_2(t) \\ \dot{\omega}_3(t) &= A_3\omega_1(t)\omega_2(t) + u_3 + P_3 d_3(t) \end{aligned} \qquad (3.14)$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta\tan\phi & 1 & -\cos\theta\tan\phi \\ -\sin\theta\sec\phi & 0 & \cos\theta\sec\phi \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}. \qquad (3.15)$$

*We shall consider the disturbance attenuation problem for the dynamic subsystem represented as:*

$$\begin{aligned} \dot{\omega}(t) &= f(\omega) + B_1 d(t) + B_2 v(t) \\ &= \begin{bmatrix} A_1\omega_2\omega_3 \\ A_2\omega_1\omega_3 \\ A_3\omega_1\omega_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \end{aligned} \qquad (3.16)$$

*where*

$$b_1 = \frac{P_1}{I_1}, \quad b_2 = \frac{P_2}{I_2}, \quad b_3 = \frac{P_3}{I_3}$$

*and under the assumption that* $A_1 + A_2 + A_3 = 0$ *(see also [33]). In this regard, consider the output function:*

$$z = h(\omega) = \begin{bmatrix} c_1\omega_1 \\ c_2\omega_2 \\ c_3\omega_3 \end{bmatrix} \tag{3.17}$$

*where* $c_1$, $c_2$, $c_3$ *are design parameters. Then the HJIE for the system (3.14) is given*

$$V_\omega(\omega)f(\omega) + \frac{1}{2}V_\omega(\omega)\left[\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T\right]V_\omega^T(\omega) + \frac{1}{2}h^T(\omega)h(\omega) \leq 0; \quad V(0) = 0. \tag{3.18}$$

*Furthermore,*

$$Q_\gamma = (\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T) = \begin{bmatrix} \frac{b_1^2}{\gamma^2} - 1 & 0 & 0 \\ 0 & \frac{b_2^2}{\gamma^2} - 1 & 0 \\ 0 & 0 & \frac{b_3^2}{\gamma^2} - 1 \end{bmatrix}$$

*Then from section 2.3, we can obtain*

$$V_\omega(\omega) = -(f(\omega) \pm \zeta(\omega))^T \mathcal{Q}_\gamma^+ \tag{3.19}$$

*where*

$$\zeta^T(\omega)\mathcal{Q}_\gamma^+\zeta(\omega) - f^T(\omega)\mathcal{Q}_\gamma^+ f(\omega) + h^T(\omega)h(\omega) \leq 0. \tag{3.20}$$

*or*

$$\begin{aligned}(\frac{\gamma^2}{b_1^2-\gamma^2})\zeta_1^2(\omega) + (\frac{\gamma^2}{b_2^2-\gamma^2})\zeta_2^2(\omega) + (\frac{\gamma^2}{b_3^2-\gamma^2})\zeta_3^2(\omega) - (\frac{\gamma^2}{b_1^2-\gamma^2})A_1^2\omega_2^2\omega_3^2 - \\ (\frac{\gamma^2}{b_2^2-\gamma^2})A_2^2\omega_1^2\omega_3^2 - (\frac{\gamma^2}{b_3^2-\gamma^2})A_3^2\omega_1^2\omega_2^2 + c_1^2\omega_1^2 + c_2^2\omega_2^2 + c_3^2\omega_3^2 \leq 0\end{aligned} \tag{3.21}$$

*Further, from Section 3.2, we have the following additional constraints on* $\zeta$*:*

$$A_1\omega_3 + \zeta_{1,\omega_3} = \zeta_{2,\omega_1} + A_2\omega_3, \quad A_1\omega_2 + \zeta_{1,\omega_3} = A_3\omega_2 + \zeta_{3,\omega_1}, \quad A_2\omega_1 + \zeta_{3,\omega_1} = A_3\omega_1\zeta_{1,\omega_3}$$

*Now for any* $\gamma > b = \max_i\{b_i\}, i = 1,2,3,$ *and* $c_i = \sqrt{\left|(\frac{\gamma^2}{b_i^2-\gamma^2})\right|}$, $i = 1,2,3,$ *and using the condition* $A_1 + A_2 + A_3 = 0$, *we have the solution*

$$\zeta_1 = -A_1\omega_2\omega_3 + \omega_1 \tag{3.22}$$

$$\zeta_2 = -A_2\omega_1\omega_3 + \omega_2 \tag{3.23}$$

$$\zeta_3 = -A_3\omega_1\omega_2 + \omega_3. \tag{3.24}$$

*Thus,*

$$V_\omega(\omega) = -(f(\omega)+\zeta(\omega))^T Q_\gamma^+ = -\left[ (\frac{\gamma^2}{b_1^2-\gamma^2})\omega_1 \;\; (\frac{\gamma^2}{b_2^2-\gamma^2})\omega_2 \;\; (\frac{\gamma^2}{b_3^2-\gamma^2})\omega_3 \right]$$

*and integrating from* $0$ *to* $\omega$*, we get*

$$V(\omega) = -\frac{1}{2}\left( (\frac{\gamma^2}{b_1^2-\gamma^2})\omega_1^2 + (\frac{\gamma^2}{b_2^2-\gamma^2})\omega_2^2 + (\frac{\gamma^2}{b_3^2-\gamma^2})\omega_3^2 \right)$$

*which is positive definite for any* $\gamma > b$*.*

**Remark 3.3.2** *Notice that the solution of the discriminant inequality does not only give us a stabilizing feedback, but also the linearizing feedback control. Hence, However, the linearizing terms drop out in the final expression for* $V$*, and consequently in the expression for the optimal control* $u^\star = -B_2^T V_\omega(\omega)$*. This clearly shows that cancellation of the nonlinearities is not optimal, and will expend more energy from the system than is necessary.*

We consider another example. This example will illustrate a general transformation approach for handling the discriminant equation/inequality and symmetry condition.

**Example 3.3.3** *Consider the following example with the disturbance also affecting the first state equation.*

$$\begin{aligned} \dot{x}_1 &= -x_1^3 - x_2 + w_1 \\ \dot{x}_2 &= x_1 + x_2 + u + w_2 \\ z &= \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \end{bmatrix} \end{aligned}$$

*where* $Q = diag\{q_1, q_2\} \geq 0$*,* $R = r > 0$ *are weighting matrices intruduced to make the HJIE solvable. Morever, the state-feedback HJIE associated with the above system is thus given by:*

$$V_x(x)f(x) + \frac{1}{2}V_x(x)[\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)R^{-1}g_2^T(x)]V_x^T(x) + \frac{1}{2}x^TQx \leq 0, \;\; V(0)=0, \tag{3.25}$$

*with* $\mathcal{Q}_\gamma(x) = [\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)R^{-1}g_2^T(x)]$ *(see also [56]). Then*

$$f = \begin{bmatrix} -x_1^3 - x_2 \\ x_1 + x_2 \end{bmatrix}, \; G_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; G_1 = I_2, \; \mathcal{Q}_\gamma = \begin{bmatrix} \frac{1}{\gamma^2} & 0 \\ 0 & \frac{r-\gamma^2}{r\gamma^2} \end{bmatrix}, \;\; \mathcal{Q}_\gamma^+ = \begin{bmatrix} \gamma^2 & 0 \\ 0 & \frac{r\gamma^2}{r-\gamma^2} \end{bmatrix}$$

*Substituting the above functions in (2.77), (3.5), (3.6), we get*

$$\gamma^2\zeta_1^2 + \frac{r\gamma^2}{r-\gamma^2}\zeta_2^2 - \gamma^2(x_1^3+x_2)^2 - \frac{r\gamma^2}{r-\gamma^2}(x_1+x_2)^2 + q_1x_1^2 + q_2x_2^2 \;\; = \;\; 0 \tag{3.26}$$

$$\frac{r\gamma^2}{r-\gamma^2}\zeta_{2,x_1} - \zeta_{1,x_2} \;\; = \;\; \frac{\gamma^2}{r-\gamma^2} \tag{3.27}$$

$$\begin{bmatrix} \gamma^2(\zeta_{1,x_1} - 3x_1^2) & \frac{r\gamma^2}{r-\gamma^2}(\zeta_{2x_1}+1) \\ \gamma^2(\zeta_{1x_2} - 1) & \frac{r\gamma^2}{r-\gamma^2}(\zeta_{2,x_2}+1) \end{bmatrix} \;\; \leq \;\; 0 \tag{3.28}$$

*One way to handle the above algebraic-differential system is to parameterize $\zeta_1$ and $\zeta_2$ as:*

$$\zeta_1(x) = ax_1 + bx_2, \quad \zeta_2(x) = cx_1 + dx_2 + ex_1^3,$$

*where $a, b, c, d, e \in \Re$ are constants, and to try to solve for these constants. This approach may not however work for most systems. We therefore illustrate next a general procedure for handling the above system.*

*Suppose we choose $r$, $\gamma$ such that $\frac{r\gamma^2}{r-\gamma^2} > 0$ (usually we take $r >> 1$ and $\gamma < 1$). We now apply the following transformation to separate the variables:*

$$\zeta_1(x) = \frac{1}{\gamma}\rho(x)\cos\theta(x), \quad \zeta_2(x) = \sqrt{\frac{(r-\gamma^2)}{r\gamma^2}}\rho(x)\sin\theta(x),$$

*where $\rho$, $\theta : N \to \Re$. Substituting in the equation (3.26), we get*

$$\rho(x) = \pm\sqrt{\frac{r\gamma^2}{r-\gamma^2}(x_1+x_2)^2 - \gamma^2(x_1^3+x_2)^2 - q_1x_1^2 - q_2x_2^2}.$$

***Remark 3.3.3*** *Thus, for the HJIE (3.25) to be solvable, it is necessary that $\gamma$, $r$, $q_1$, $q_2$ are chosen such that the function under the square-root in the above equation is positive for all $x \in N$, so that $\rho \in \Re$, $\forall x \in N$. As a matter of fact, the above expression defines $N$, i.e., $N = \{x \mid \rho \in \Re\}$.*

***Remark 3.3.4*** *If however we choose $r, \gamma$ such that $\frac{r\gamma^2}{r-\gamma^2} < 0$, then we must paramerize $\zeta_1$, $\zeta_2$ as $\zeta_1(x) = \frac{1}{\gamma}\rho(x)\cosh\theta(x)$, $\zeta_2(x) = \sqrt{\frac{(r-\gamma^2)}{r\gamma^2}}\rho(x)\sinh\theta(x)$. A difficulty also arises when $\mathcal{Q}_\gamma^+(x)$ is not diagonal, e.g., if $\mathcal{Q}_\gamma^+ = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $a, b, c \in \Re$. Then $\zeta^T(x)\mathcal{Q}_\gamma^+\zeta(x) = i\zeta_1^2(x) + j\zeta_1(x)\zeta_2(x) + k\zeta_2^2(x)$, for some $i$, $j$, $k \in \Re$. The difficulty here is created by the cross-term $j\zeta_1\zeta_2$ as the above parameterization cannot lead to a simplification of the problem. However, we can use a completion of squares to get $\zeta^T(x)\mathcal{Q}_\gamma^+\zeta(x) = (\sqrt{i}\zeta_1(x) + \frac{j}{2\sqrt{i}}\zeta_2(x))^2 + (k - \frac{j^2}{4i})\zeta_2^2(x)$ (assuming $i > 0$, otherwise, pull-out the negative sign outside the bracket). Now we can define, $\left(\sqrt{i}\zeta_1(x) + \frac{j}{2\sqrt{i}}\zeta_2(x)\right) = \rho(x)\cos\theta(x)$, and $\zeta_2(x) = (k - \frac{j^2}{4i})^{-1/2}\rho(x)\sin\theta(x)$. Thus, in reality, $\zeta_1(x) = \rho(x)\left(\frac{1}{\sqrt{i}}\cos\theta(x) - \frac{j}{2i}(k - \frac{j^2}{4i})^{1/2}\sin\theta(x)\right)$.*

*Next, we determine $\theta(.)$ from (3.27); differentiating $\zeta_1$, $\zeta_2$ wrt $x_2$ and $x_1$ respectively, and substituting we get*

$$\beta(\rho_{x_1}(x)\sin\theta(x) + \rho(x)\theta_{x_1}(x)\cos\theta(x)) - \kappa(\rho_{x_2}(x)\cos\theta(x) - \rho(x)\theta_{x_2}(x)\sin\theta(x)) = \eta \quad (3.29)$$

*where $\beta = \frac{1}{\gamma}\sqrt{\frac{r}{(r-\gamma^2)}}$, $\kappa = \frac{1}{\gamma}$, $\eta = \frac{\gamma^2}{r-\gamma^2}$. This first-order PDE in $\theta$ can be solved using the method of* "characteristics" *[37]. However, the geometry of the problem calls for a simpler approach. Moreover, since $\theta$ is a free parameter that we have to assign to guarantee that $V_{xx}$ is symmetric and positive-(semi) definite, there are many solutions to the above PDE. One solution can be obtained as follows. Rearranging, the above equation we get*

$$(\beta\rho_{x_1}(x) + \kappa\rho(x)\theta_{x_2}(x))\sin\theta(x) + (-\kappa\rho_{x_2}(x) + \beta\rho(x)\theta_{x_1}(x))\cos\theta(x) = \eta. \quad (3.30)$$

*Now if we assign*

$$\begin{aligned}(\beta\rho_{x_1}(x)+\kappa\rho(x)\theta_{x_2}(x)) &= \frac{\eta}{2}\sin\theta(x),\\ (-\kappa\rho_{x_2}(x)+\beta\rho(x)\theta_{x_1}(x)) &= \frac{\eta}{2}\cos\theta(x),\end{aligned}$$

*then we see that (3.30) is satisfied. Further, squaring both sides of the above equations and adding, we get*

$$\frac{4}{\eta^2}(\beta\rho_{x_1}(x)+\kappa\rho(x)\theta_{x_2}(x))^2+\frac{4}{\eta^2}(-\kappa\rho_{x_2}(x)+\beta\rho(x)\theta_{x_1}(x))^2=1 \tag{3.31}$$

*which is the equation of an ellipse in the coordinates* $\theta_{x_1}$, $\theta_{x_2}$, *centered at* $\frac{\kappa\rho_{x_2}(x)}{\beta\rho(x)}$, $-\frac{\beta\rho_{x_1}(x)}{\kappa\rho(x)}$ *and radii* $\frac{\eta}{2\beta\rho(x)}$, $\frac{\eta}{2\kappa\rho(x)}$ *respectively. Thus, any point on this ellipse will give the required gradient for* $\theta$. *One point on this ellipse corresponds to the following gradients in* $\theta$:

$$\begin{aligned}\theta_{x_1}(x) &= \frac{\kappa\rho_{x_2}(x)}{\beta\rho(x)}+\frac{1}{\sqrt{2}}\left(\frac{\eta}{2\beta\rho(x)}\right)\\ \theta_{x_2}(x) &= -\frac{\beta\rho_{x_1}(x)}{\kappa\rho(x)}+\frac{1}{\sqrt{2}}\left(\frac{\eta}{2\kappa\rho(x)})\right).\end{aligned}$$

*Hence, we can finally obtain* $\theta$ *as*

$$\theta(x)=\int_{0+}^{x_1}\left(\frac{\kappa\rho_{x_2}(x)}{\beta\rho(x)}+\frac{1}{\sqrt{2}}(\frac{\eta}{2\beta\rho(x)})\right)\Bigg|_{x_2=0}dx_1+\int_{0+}^{x_2}\left(-\frac{\beta\rho_{x_1}(x)}{\kappa\rho(x)}+\frac{1}{\sqrt{2}}(\frac{\eta}{2\kappa\rho(x)})\right)dx_2 \tag{3.32}$$

*The above integral can be evaluated using MATHEMATICA or MAPLE. The result is very complicated and lengthy, so we choose not to report it here.*

***Remark 3.3.5*** *Note, any available method can also be used to solve the symmetry PDE in* $\theta$ *(3.29), as the above approach may be too restricted and might not yield the desired solution. Indeed, a general solution would be more desirable. Moreover, any solution should be checked against the positive-(semi)definite condition (3.28) to see that it is respected. Otherwise, some of the design parameters,* $r$, $\gamma$, $q_i$, *should be adjusted to see that this condition is atleast satisfied. This however will be an arduous task.*

***Remark 3.3.6*** *In the light of the preceeding remark, it should be comforting to note that, even in the linear case and the Hamiltonian matrix approach to the solution of the Riccati equation [120] and indeed many other approaches, there is no mechanism for enforcing the positive-definiteness condition; all the method seeks to find is a symmetric solution, and one has to iterate while changing* $\gamma$ *and/or the weighting matrices* $Q$ *and* $R$ *to get the desired solution. It also only happens by coincidence, that if the linear system is stabilizable and detectable, then there exists a positive definite solution to the Riccati equation, which can be obtained by the method.*

*Ultimately, we can then compute $V$ as*

$$V(x) = -\int_{0+}^{x} \left( f(x) + \rho(x) \begin{bmatrix} \cos\theta(x) \\ \sin\theta(x) \end{bmatrix} \right)^T \mathcal{Q}_\gamma^+ dx \tag{3.33}$$

**Remark 3.3.7** *It may not be necessary to compute $V$ explicitly, since the optimal control $u^\star = \alpha(x)$ is a function of $V_x$ only. What is more important is to check that the positive-(semi)definiteness condition (3.28) is locally satisfied around the origin $\{0\}$. Then the $V$ function corresponding to $V_x$ will be a candidate solution of the HJIE. However, we still cannot conclude at this point that it is a stabilizing solution. In the case of the above example, it can be seen that by setting $\zeta_1(x) = \rho(x)\cos\theta(x)$, $\zeta_2(x) = \rho(x)\sin\theta(x)$, and their derivatives equal to $0$, the inequality (3.28) is locally satisfied at the origin $\{0\}$.*

**Remark 3.3.8** *The above example has demonstrated a complete procedure for solving the HJIE for a second-order affine nonlinear system. This procedure is applicable to most second-order systems in affine form, with varying degree of complexity. It can also be easily extended to third-order and higher-order systems. In general, for a system of order $n$, one needs to solve a system of $\frac{n(n-1)}{2}$ first-oder nonlinear PDEs to arrive at the solution; and as the order of the system increases, the complexity of these equations will also increase. However, in principle, the method can be applied to a large class of affine nonlinear system.*

Next, we consider an example of a third-order LTI system.

**Example 3.3.4**

$$\dot{x}_1 = x_1 + 2x_2 + x_3 + w_1 \tag{3.34}$$
$$\dot{x}_2 = x_1 + x_2 + u_2 + w_2 \tag{3.35}$$
$$\dot{x}_3 = -x_1 + 2x_3 + u_3 + w_3 \tag{3.36}$$
$$z = \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \end{bmatrix} \tag{3.37}$$

*where $Q = diag\{q_1,\ q_2,\ q_3\} \geq 0$, $R = diag\{r_1,\ r_2,\ r_3\} > 0$. Then*

$$f = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 + x_2 \\ -x_1 + 2x_3 \end{bmatrix},\ G_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ G_1 = I_3,\ \mathcal{Q}_\gamma = \begin{bmatrix} \frac{1}{\gamma^2} & 0 & 0 \\ 0 & \frac{r_2-\gamma^2}{r_2\gamma^2} & 0 \\ 0 & 0 & \frac{r_3-\gamma^2}{r_3\gamma^2} \end{bmatrix},$$

$$\mathcal{Q}_\gamma^+ = \begin{bmatrix} \gamma^2 & 0 & 0 \\ 0 & \frac{r_2\gamma^2}{r_2-\gamma^2} & 0 \\ 0 & 0 & \frac{r_3\gamma^2}{r_3-\gamma^2} \end{bmatrix}$$

*Substituting the above functions in (2.77), (3.5), (3.6), we get*

$$\nu_1\zeta_1^2 + \nu_2\zeta_2^2 + \nu_3\zeta_3^2 - \nu_1(x_1 + 2x_2 + x_3)^2 - \nu_2(x_1 + x_2)^2 - \nu_3(-x_1 + 2x_3)^2 + q_1x_1^2 + q_2x_2^2 + q_3x_3^2 = 0 \tag{3.38}$$

$$\left.\begin{array}{rcl} \nu_1(2+\zeta_{1,x_2}) &=& \nu_2(1+\zeta_{2,x_1}) \\ \nu_1(-1+\zeta_{3,x_1} &=& \nu_3(1+\zeta_{1,x_3}) \\ \nu_2\zeta_{3,x_2} &=& \nu_3\zeta_{2,x_3} \end{array}\right\} \tag{3.39}$$

$$\left[\begin{array}{ccc} \nu_1(1+\zeta_{1,x_1}) & \nu_2(2+\zeta_{1,x_2}) & \nu_3(1+\zeta_{1,x_3}) \\ \nu_1(1+\zeta_{2,x_1}) & \nu_2(2+\zeta_{2,x_2}) & \nu_3\zeta_{2,x_3} \\ \nu_1(-1+\zeta_{3,x_1}) & \nu_3\zeta_{3,x_2} & \nu_3\zeta_{2,x_3} \end{array}\right] \leq 0 \tag{3.40}$$

*where* $\nu_1 = \gamma^2$, $\nu_2 = \frac{r_2\gamma^2}{r_2-\gamma^2}$, $\nu_3 = \frac{r_3\gamma^2}{(r_3-\gamma^2)}$. *Now, define*

$$\zeta_1 = \frac{1}{\sqrt{\nu_1}}\rho_1(x)\cos\theta(x), \quad \zeta_2(x) = \frac{1}{\sqrt{\nu_2}}\rho_2(x)\sin\theta(x), \quad \zeta_3(x) = \rho_2(x).$$

*Then, substituting in (3.38), we get*

$$\rho_1(x) + \nu_3\zeta_3^2 - \nu_1(x_1+2x_2+x_3)^2 - \nu_2(x_1+x_2)^2 - \nu_3(-x_1+2x_3)^2 + q_1x_1^2 + q_2x_2^2 + q_3x_3^2 = 0.$$

*Now take,*

$$\begin{array}{rcl} \rho_1(x) &=& \pm\sqrt{\nu_1(x_1+2x_2+x_3)^2 + \nu_2(x_1+x_2)^2 - q_1x_1^2 - q_2x_2^2}, \\ \rho_2(x) &=& \zeta_3(x) = \pm\sqrt{(-x_1+2x_3)^2 - \frac{1}{\nu_3}q_3x_3^2}, \end{array}$$

*where* $Q$, $R$, $\gamma$ *are such that* $\rho_1, \rho_2 \in \Re$. *Then, substituting the above expressions for* $\zeta_1$, $\zeta_2$, $\zeta_3$ *in (3.39), we get*

$$\nu_1(2+\frac{1}{\sqrt{\nu_1}}\rho_{1,x_2}\cos\theta - \frac{1}{\sqrt{\nu_1}}\rho_1\theta_{x_2}\sin\theta) - \nu_2(1+\frac{1}{\sqrt{\nu_2}}\rho_{1,x_1}\sin\theta + \frac{1}{\sqrt{\nu_2}}\rho_1\theta_{x_1}\cos\theta) = 0 \tag{3.41}$$

$$\nu_1(-1+\rho_{2,x_1}) - \nu_3(1+\frac{1}{\sqrt{\nu_1}}\rho_{1,x_3}\cos\theta - \frac{1}{\sqrt{\nu_1}}\rho_1\theta_{x_3}\sin\theta) = 0 \tag{3.42}$$

$$\nu_2\rho_{2,x_2} - \frac{\nu_3}{\sqrt{\nu_2}}(\rho_{1,x_3}\sin\theta + \rho_1\theta_{x_3}\cos\theta) = 0. \tag{3.43}$$

*Noting that* $\rho_{2,x_2} = 0$, *we get from the third equation*

$$\tan\theta = -\frac{\rho_1\theta_{x_3}}{\rho_{1,x_3}} \Longrightarrow \sin\theta = \frac{\rho_1\theta_{x_3}}{\sqrt{\rho_{1,x_3}^2 + \rho_1^2\theta_{x_3}^2}}, \quad \cos\theta = -\frac{\rho_{1,x_3}}{\sqrt{\rho_{1,x_3}^2 + \rho_1^2\theta_{x_3}^2}},$$

*and rearranging the first equation and substituting in the second equation above, we get*

$$(\frac{\nu_1}{\sqrt{\nu_1}}\rho_{1,x_2} + \frac{\nu_2}{\sqrt{\nu_2}}\rho_1\theta_{x_1})\cos\theta + (-\frac{\nu_1}{\sqrt{\nu_1}}\rho_1\theta_{x_2} + \frac{\nu_2}{\sqrt{\nu_2}}\rho_{1,x_1})\sin\theta = \nu_2 - 2\nu_1, \tag{3.44}$$

$$\frac{\nu_3\rho_{1,x_3}^2}{\sqrt{\nu_1(\rho_{1,x_3}^2 + \rho_1^2\theta_{x_3}^2)}} + \frac{\nu_3\rho_1^2\theta_{x_3}^2}{\sqrt{\nu_1(\rho_{1,x_3}^2 + \rho_1^2\theta_{x_3}^2)}} = \nu_3 - \nu_1(-1+\rho_{2,x_1}). \tag{3.45}$$

*From the first equation (3.44), we can now assign*

$$(\frac{\nu_1}{\sqrt{\nu_1}}\rho_{1,x_2} + \frac{\nu_2}{\sqrt{\nu_2}}\rho_1\theta_{x_1}) = \frac{\nu_2 - 2\nu_1}{2}\cos\theta, \quad and \quad (-\frac{\nu_1}{\sqrt{\nu_1}}\rho_1\theta_{x_2} + \frac{\nu_2}{\sqrt{\nu_2}}\rho_{1,x_1}) = \frac{\nu_2 - 2\nu_1}{2}\sin\theta.$$

*Finally, squaring and adding the last two equations above, we get*

$$(\frac{\nu_1}{\sqrt{\nu_1}}\rho_{1,x_2} + \frac{\nu_2}{\sqrt{\nu_2}}\rho_1\theta_{x_1})^2 + (-\frac{\nu_1}{\sqrt{\nu_1}}\rho_1\theta_{x_2} + \frac{\nu_2}{\sqrt{\nu_2}}\rho_{1,x_1})^2 = \frac{\nu_2 - 2\nu_1}{2},$$

*which is the equation of an ellipse in the* $\theta_{x_1}$, $\theta_{x_2}$ *coordinates. Therefore, any point on this ellipse gives us a candidate solution for the gradients* $\theta_{x_1}$, $\theta_{x_2}$. *While from the second equation (3.45) containing* $\theta_{x_3}$, *squaring both sides and rearranging, we get the following quartic in* $\theta_{x_3}$:

$$\rho_1^4\theta_{x_3}^4 + \rho_1^2(2\rho_{1,x_3}^2 - \nu_1\mu(x))\theta_{x_3}^2 + (\rho_{1,x_3}^4 - \nu_1\mu(x)\rho_{1,x_3}^2) = 0,$$

*where* $\mu(x) = \left[\frac{\nu_3 - \nu_1(\rho_{2,x_1} - 1)}{\nu^3}\right]^2$. *By finding any real solution to this equation for* $\theta_{x_3}$ *(this can be done by applying the quadratic formula twice), and combining with some point* $(\theta_{x_1}, \theta_{x_2})$ *on the above ellipse, we can now compute* $\theta$ *as*

$$\theta = \int_{0+}^{x_1} \theta_{x_1}\Big|_{x_2=x_3=0} dx_1 + \int_{0+}^{x_2} \theta_{x_2}\Big|_{x_3=0} dx_2 + \int_{0+}^{x_3} \theta_{x_3} dx_3. \tag{3.46}$$

*And we could compute* $V$ *as in the previous example.*

**Remark 3.3.9** *The above examples have demonstrated how the HJIE can be solved analytically for systems of orders two and three. It is now apparent how very complicated the solution can be, and there are many solutions, such that, it is hard to guess at this point which solution could result in the desired control law. The only rule of thumb one could use is to try to separate the variables as much as possible (by grouping terms of corresponding indices in the determination of* $\zeta$*) to simplify the computation and to guarantee that* $V$ *will be positive-(semi)definite after integration.*

## 3.4 Computational Procedure for Linear Systems

In this section we present a computational procedure for LTIS and the $\mathcal{H}_\infty$ Riccati equation. In the case of the linear system $\Sigma_l$, the determination of $\Gamma$ such that $V_{xx}$ is symmetric and positive-semidefinite can be formulated as an optimization problem over a system of linear matrix inequalities (LMIs) [26]:

$$OPT1: \quad \min. \; J = \gamma$$

$$s.t. \quad \begin{bmatrix} F^TQ_\gamma^+F - H^TH & \Gamma \\ \Gamma^T & Q_\gamma \end{bmatrix} \leq 0 \tag{3.47}$$

$$\begin{bmatrix} Q_\gamma^+\Gamma - \Gamma^TQ_\gamma^+ \pm (Q_\gamma^+F - F^TQ_\gamma^+) & 0 \\ 0 & -I \end{bmatrix} \leq 0 \tag{3.48}$$

$$\begin{bmatrix} \Gamma^TQ_\gamma^+ - Q_\gamma^+\Gamma \mp (Q_\gamma^+F - F^TQ_\gamma^+) & 0 \\ 0 & -I \end{bmatrix} \leq 0 \tag{3.49}$$

$$\begin{bmatrix} \Gamma^TQ_\gamma^+ \pm F^TQ_\gamma^+ & 0 \\ 0 & -I \end{bmatrix} \leq 0. \tag{3.50}$$

In the next section we consider a different approach for computing the discriminant factor using higher-order HJIE.

## 3.5 A Computational Procedure Using Higher-Order HJIE

In this subsection, we consider taking the problem one-step further by considering higher-order equations. This approach allows us to combine the constraints (3.4), (3.6) and the condition (2.77) into a single equation. In this regard, differentiating the HJIE (2.3) wrt $x$ will yield the following second-order HJIE:

$$V_{xx}(x)f(x) + f_x(x)V_x^T(x) + V_{xx}(x)Q_\gamma(x)V_x^T(x) + \frac{1}{2}(I_n \otimes V_x(x))\frac{\partial Q_\gamma(x)}{\partial x}V_x^T(x) + h_x(x)h(x) = 0 \tag{3.51}$$

Note that, in the above equation, we have dropped the inequality because it is not satisfied in general. Moreover, for the linear system $\Sigma_l$, substituting $V(x) = x^T P x$ in the above equation, results in the ARE (2.51).

Now, for simplicity and without any loss of generality, let $V_x^T(x) = f(x) + \zeta(x)$, $x \in N$. Then, $V_{xx}(x) = f_x(x) + \zeta_x(x), \zeta(0) = 0$, and since $V_{xx}$ must be symmetric, we can decompose $V_{xx}$ into its symmetric and skew-symmetric components as

$$V_{xx}(x) = V_{xx}^s(x) + V_{xx}^{sk}(x),$$

where

$$V_{xx}^s(x) = \frac{f_x(x) + f_x^T(x) + \zeta_x(x) + \zeta_x^T(x)}{2}, \quad V_{xx}^{sk}(x) = \frac{f_x(x) - f_x^T(x) + \zeta_x(x) - \zeta_x^T(x)}{2},$$

and replace $V_{xx}$ in (3.51) by $V_{xx}^s$. Thus, substituting the expressions for $V_x$ and $V_{xx}^s$ in (3.51) results in a system of $n$ first-order equations in the $n$-unknowns $\zeta_i, i = 1, \dots, n$ given by

$$\frac{1}{2}[f_x(x) + f_x^T(x) + \zeta_x(x) + \zeta_x^T(x)]f(x) + f_x(x)[f(x) + \zeta(x)] + \frac{1}{2}[f_x(x) + f_x^T(x) + \zeta_x(x) + \zeta_x^T]Q_\gamma(x)[f(x) + \zeta(x)] + \frac{1}{2}\left[I_n \otimes (f(x) + \zeta(x)\right]^T Q_x(x)\left(f(x) + \zeta(x)\right) + h_x(x)h(x) = 0, \tag{3.52}$$

or

$$[f_x(x) + f_x^T(x) + \zeta_x(x) + \zeta_x^T(x)]f(x) + [f_x(x) + f_x^T(x) + \zeta_x(x) + \zeta_x^T(x)]Q_\gamma(x) \times [f(x) + \zeta(x)] + [I_n \otimes (f(x) + \zeta(x))^T]Q_x(x)[f(x) + \zeta(x)] + 2f_x(x)[f(x) + \zeta(x)] + 2h_x(x)h(x) = 0 \tag{3.53}$$

which could be solved by the method of separation of variables or other methods [4].

Alternatively, to reduce the complexity of equation (3.53), we can satisfy the symmetry condition by taking the free variables in equation (3.51) as the diagonal elements of $V_{xx}$,

$\zeta_{i,x_i}, i = 1, \ldots, n$, while the off-diagonal elements can be chosen by inspection to make $V_{xx}$ symmetric and positive-definite.

For the case of the linear system $\Sigma_l$, taking $V_x(x) = x^T(F+\Gamma)^T$ and substituting in the above HJIE (3.53) gives the following algebraic equation

$$\frac{1}{2}[F+F^T+\Gamma+\Gamma^T]Fx+F^T[F+\Gamma]x+\frac{1}{2}[F^T+F+\Gamma^T+\Gamma]Q_\gamma[F+\Gamma]x+H^THx = 0; \;\; \forall x \in M. \tag{3.54}$$

Since the above equation holds for all $x \in M$, and defining $P = F + \Gamma$, we get the Riccati equation (2.51). However, the above equation can also be solved directly for $\Gamma$ using for example the Newton's method. By collecting terms of equal orders of $\Gamma$, we get:

$$\frac{1}{2}(\Gamma+\Gamma^T)Q_\gamma\Gamma+\frac{1}{2}(\Gamma+\Gamma^T)F++\frac{1}{2}(\Gamma+\Gamma^T)Q_\gamma F+\frac{1}{2}(F+F^T)Q_\gamma\Gamma+F^T\Gamma+\mathcal{F}(F,H) = 0, \tag{3.55}$$

where

$$\mathcal{F}(F,H) = \frac{1}{2}(F+F^T)(F+Q_\gamma F)+F^TF+H^TH$$

is the term independent of $\Gamma$.

**Remark 3.5.1** *The fact that $\Gamma$ in equation (3.55) does not have to satisfy any additional conditions, makes it easier to solve using iterative schemes. In this case, any solution $\Gamma$ will yield a symmetric solution for the linear $\mathcal{H}_\infty$ ARE (2.51). In addition, by varying $\gamma > 0$, a positive definite solution can be obtained.*

We now demonstrate how the above results can be applied using an example of a second order nonlinear system.

**Example 3.5.1** *Consider example 3.3.1 with $\gamma = 2$.*

$$f_x(x) = \begin{pmatrix} -3x_1^2 & 0 \\ 1 & 1 \end{pmatrix}, \;\; \textit{and} \;\; \zeta_x(x) = \begin{pmatrix} \zeta_{1,x_1}(x) & \zeta_{2,x_1}(x) \\ \zeta_{1,x_2}(x) & \zeta_{2,x_2}(x) \end{pmatrix}.$$

*Then*

$$V_{xx}(x) = \begin{pmatrix} \zeta_{1,x_1}(x) - 3x_1^2 & \zeta_{2,x_1} \\ \zeta_{1,x_2}+1 & 1+\zeta_{2,x_2}(x) \end{pmatrix}.$$

*Substituting in (3.51), we get*

$$\begin{pmatrix} \zeta_{1,x_1} - 3x_1^2 & \zeta_{2,x_1} \\ \zeta_{1,x_2}+1 & 1+\zeta_{2,x_2} \end{pmatrix} \cdot \begin{pmatrix} -x_1^3 + x_2 \\ \frac{1}{4}x_2 - \frac{3}{4}\zeta_2 \end{pmatrix} + \begin{pmatrix} -3x_1^2(-x_1^3+x_2+\zeta_1) \\ -x_1^3+2x_2+\zeta_1+\zeta_2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0. \tag{3.56}$$

*Multiplying out and simplifying, we get the following coupled system of partial differential equations:*

$$(-x_1^3+x_2)(\zeta_{1,x_1}(x)-3x_1^2)+(\frac{1}{4}x_2-\frac{3}{4}\zeta_2(x))\zeta_{2,x_1}(x)-3x_1^2\zeta_1(x)-6x_1^5+6x_1^2x_2 = 0, \tag{3.57}$$

$$(-x_1^3+x_2)\zeta_{1,x_2}(x)+(\frac{1}{4}x_2-\frac{3}{4}\zeta_2(x))(1+\zeta_{2,x_2})+\zeta_2(x)+\zeta_1(x)+4x_2(x)-2x_1^3 = 0. \tag{3.58}$$

*At this point, we must solve the above coupled system of partial differential equations. This itself is a very difficult problem, and there are no general procedures for solving it; some transformation approaches are however available in the literature and need to be explored [4]. However, using Mathematica, we obtain the following inplicit solutions:*

$$\begin{aligned}
\zeta_1(x_1, x_2) &= \frac{2x_2\zeta_2(x) - 3\zeta_2^2(x) + 8(x_1^6 - 2x_1^3 x_2 + \psi_1(x_2))}{8(x_1^3 - x_2)}, \\
\zeta_2(x_1, x_2) &= \frac{1}{3}\left\{x_2 \pm \sqrt{(-48x_1^3 x_2 + 52x_2^2 - 24x_1^3\zeta_1(x) + 24x_2\zeta_1(x) + \psi_2(x_1))}\right\},
\end{aligned}$$

*where $\psi_1$, $\psi_2$ are arbitrary functions. The above solution will not be symmetric since no symmetry conditions are imposed on the system of PDEs (3.59), (3.59). Thus, to obtain a symmetric solution, we must impose the following contraints on (3.56), i.e., $\zeta_{1,x_2} = -1$ and $\zeta_{2,x_1} = 0$. Hence, imposing these constraints and solving, we obtain*

$$\zeta_1(x) = \frac{1}{8(x_1^3 + x_2)}\left\{x_1(9x_1 + 4x_1^5 + 10x_2 + 8x_1^2 x_2) + 2(x_1 + x_2)\zeta_2(x) - 3\zeta_2^2(x) + 8\psi_3(x_2)\right\}, \tag{3.59}$$

$$\zeta_2(x) = \frac{1}{3}\left(x_2 \pm \sqrt{(8\int(-3x_1^3 + 10x_2 + 3\zeta_1(x))dx_2 + \Psi_4(x_1)}\right), \tag{3.60}$$

*where $\psi_3$, $\psi_4$ are arbitrary functions which must be determined such that the above constraints are satisfied. It is however obvious that, finding such functions will not be an easy task. In fact, they may not even exist!*

*On the other hand, if we use equation (3.53), we get the following system of PDEs in $\zeta$:*

$$8(-x_1^3 + x_2)\zeta_{1,x_1}(x) + (x_2 - 3\zeta_2(x))(\zeta_{1,x_2}(x) + \zeta_{2,x_1}(x)) - 3\zeta_2(x) - 24x_1^2\zeta_1(x)) + 48x_1^5 - 48x_1^2 x_2 = 0 \tag{3.61}$$

$$2(-x_1^3 + x_2)\zeta_{1,x_2}(x) + 2(-x_1^3 + x_2)\zeta_{2,x_1}(x) + (3\zeta_2(x) + x_2)\zeta_{2,x_2}(x) + 4\zeta_1(x) + \zeta_2(x) - 6x_1^3(x) + 15x_2 = 0. \tag{3.62}$$

*Unfortunately, there does not exist a general analytical solution for the above system of PDEs. Hence, an approximate solution must be sort. However, we shall not discuss this subject in this dissertation.*

**Remark 3.5.2** *A few comments are in order. First, it should be noted that the above procedure for solving the HJIE cannot be guaranteed to yield a positive-semidefinite solution, since this requirement has not been taken into account in the solution procedure. This can only be achieved by iteratively using different values of $\gamma$ until the desired solution is obtained. The solution obtained in this respect will also be global.*

# CHAPTER 4

# A FACTORIZATION APPROACH FOR SOLVING THE COUPLED HAMILTON-JACOBI-ISAACS EQUATIONS IN MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ NONLINEAR CONTROL

## 4.1 Introduction

In this chapter, we present a factorization approach that may lead to the solutions of the coupled Hamilton-Jacobi-Isaacs equations (HJIEs) arising in Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for nonlinear systems. We extend the approaches from the previous chapters for the pure $\mathcal{H}_\infty$-HJIE, and we develop also two approaches for the coupled HJIEs. Necessary and sufficient conditions for the existence of local and global solutions to the coupled HJIEs are derived and a parameterization of all solutions to the coupled HJIEs are also given. The results are then specialized to the case of linear systems and the coupled Riccati equations.

The rest of the chapter is organized as follows. In section 2, we give a review of the solution to the state-feedback nonlinear mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for affine nonlinear systems. In section 3, we review some results from chapter 2 on the solution of the HJIE and other preliminary results that are relevant to the solution of the problem. In section 3, we present the factorization approach for solving the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ coupled HJIEs. We begin with the derivation of sufficient conditions for the existence of local and global solutions to the coupled HJIEs, and then we present two methods for deriving the solutions. We also give a parameterization of all solutions to the coupled HJIEs and a characterization of the nature of the solutions. The most important tools in this endeavor are *Lagrangian-invariant manifolds* of Hamiltonian vector fields.

Then in section 5, we specialize all the results of section 4 to linear time-invariant systems (LTIS) as special cases, and we make connections with some well-known results from the theory of linear systems.

## 4.2 Review of Nonlinear Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control

We shall be considering smooth affine nonlinear systems defined in local coordinates on a manifold $M \subseteq \Re^n$:

$$\begin{aligned} \Sigma \;:\; \dot{x}(t) &= f(x) + g_1(x)w(t) + g_2(x)u(t); \quad x(0) = x_0 \\ z(t) &= h(x) + d(x)u(t) \end{aligned}$$

where the variables $x \in M$ is the state vector, $u : \Re_+ \to \mathcal{U}$, $w : \Re \to \mathcal{W}$ are respectively the control input and the disturbance signal which belong to some open sets $\mathcal{U} \subset \Re^k$, $\mathcal{W} \in \Re^r$, of admissible control and disturbances respectively, $f : M \to TM$, $f \in C^\infty(M)$, $g_1 \in C^\infty(\mathcal{M}^{n\times r}(M))$, $g_2 \in C^\infty(\mathcal{M}^{n\times k}(M))$, while $h \in C^\infty(\mathcal{M}^{m\times 1})$ and $d \in C^\infty(\mathcal{M}^{m\times k}(M))$.

**Assumption 4.2.1** *We assume the following for the system* $\Sigma$*:*

1. *The origin* $\{0\}$ *is the only equilibrium point of the system, i.e.* $f(0) = 0$, *and* $h(0) = 0$.
2. *The system is reachable from* $\{0\}$ *i.e. for any* $\bar{x} \in M$, *there exists a finite time* $\tau \geq 0$ *and a finite input* $u_{[t_0,\tau]} \in \mathcal{U}$ *such that* $x(\tau, t_0, \{0\}, u) = \bar{x}$.
3. *Also for simplicity, we shall assume that*

$$h^T(x)d(x) = 0, \;\; d^T(x)d(x) = I, \;\; \forall x \in M$$

The following definitions will also be used in the sequel.

**Definition 4.2.1** *The nonlinear system* $\Sigma$ *(or* $[f, g_2]$*) is said to be locally smoothly stabilizable in some subset* $N \subset M$ *containing* $\{0\}$, *if there exists a* $C^2$ *function* $\mathcal{F} : N \to \mathcal{U}$ *such that* $\dot{x} = f(x) + g_2(x)\mathcal{F}(x)$ *is locally asymptotically stable about* $\{0\}$. *It is smoothly stabilizable if* $N = M$.

**Definition 4.2.2** *The nonlinear system* $\Sigma$ *is said to be locally zero-state detectable if for* $w \equiv 0,\; u \equiv 0,\; z \equiv 0,\; \Rightarrow x(t, t_0, x_0, u) = 0,\; \forall t \geq 0,\; \forall x_0 \in N \subset M,\; 0 \in N$. *It is zero-state detectable if* $N = M$.

The infinite-horizon mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem for $\Sigma$ can be formulated as an optimization problem involving the following cost functions:

$$\min_{u\in\mathcal{U}} \max_{w\in\mathcal{W}} J_1^\infty(u, w) = \int_0^\infty (\gamma^2\|w(\tau)\|^2 - \|z(\tau)\|^2)d\tau \tag{4.1}$$

$$\min_{u\in\mathcal{U}, w\in\mathcal{W}} J_2^\infty(u, w) = \int_0^\infty \|z(\tau)\|^2 d\tau \tag{4.2}$$

where the first function is associated with the $\mathcal{L}_2$-gain constraint of the system (or $\mathcal{H}_\infty$ criterion), while the second objective is related to the output energy of the system (or $\mathcal{H}_2$ criterion). Moreover, if we assume $\mathcal{W} \subset \mathcal{L}_2([0,\infty), \Re^r)$ and $\mathcal{U} \subset \mathcal{L}_2([0,\infty), \Re^k)$, then a Nash equilibrium solution to the above two-player nonzero-sum game is said to exist if we can find $u^\star$, $w^\star$ such that

$$J_1^\infty(u^\star, w^\star) \leq J_1^\infty(u^\star, w) \;\; \forall w \in \mathcal{W} \tag{4.3}$$

$$J_2^\infty(u^\star, w^\star) \leq J_2^\infty(u, w^\star) \;\; \forall u \in \mathcal{U}. \tag{4.4}$$

Furthermore, by minimizing the first objective wrt to $w$ and substituting in the second objective which is then minimized wrt to $u$, the pair of Nash equilibrium points can be found. The following theorem summarizes the solution to the state-feedback problem [78].

**Theorem 4.2.1** *Consider the nonlinear system $\Sigma$ and the state-feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem with internal stability. Assume that $[f, h]$ is zero-state detectable. Then the problem is locally solvable if there exist two $C^\omega(\omega \geq 2)$ functions $Y : N \subseteq M \to \Re$, $Y(0) = 0$, (which is locally negative-semidefinite) and $V : N \to \Re$, $V(0) = 0$, (which is locally positive-semidefinite) satisfying the following coupled HJI inequalities:*

$$Y_x(x)f(x) - \frac{1}{4}[V_x(x)g_2(x)g_2^T(x)V_x^T(x) + \frac{Y_x(x)g_1(x)g_1^T(x)Y_x^T(x)}{\gamma^2}] - \frac{1}{2}Y_x(x)g_2(x)g_2^T(x)V_x^T(x) - h^T(x)h(x) \geq 0, \quad \forall x \in N \tag{4.5}$$

$$V_x(x)f(x) - \frac{1}{4}V_x(x)g_2(x)g_2^T(x)V_x^T(x) - \frac{V_x(x)g_1(x)g_1^T(x)Y_x^T(x)}{2\gamma^2} + h^T(x)h(x) \leq 0, \ \forall x \in N. \tag{4.6}$$

*Moreover, the feedbacks $u^\star = -\frac{1}{2}g_2^T(x)V_x^T(x)$ and $w^\star = -\frac{1}{2\gamma^2}g_1^T(x)Y_x^T(x)$ are the unique optimal strategies, and the closed-loop system is locally asymptotically stable with $w = 0$.*

Before we proceed, let us state and prove the following result which has not been reported in [78]. Moreover, in the sequel we shall refer to the above inequalities as equations and the inequality will be explicitly invoked where desired.

**Theorem 4.2.2** *Assume $\Sigma$ (or $[f, h]$) is zero-state detectable and there exist locally smooth solutions $Y \leq 0$, $Y(0) = 0$, $V \geq 0$, $V(0) = 0$ in $N \subset M$, $\{0\} \in N$, to the coupled HJIEs (4.5), (4.6). Then (i) $(f - \frac{1}{2}g_2g_2^TV_x^T)$ is locally asymptotically stable about $\{0\}$, so that $[f, g_2]$ is smoothly stabilizable; (ii) If $[f - \frac{g_1g_1^TY_x^T}{2\gamma^2}, h]$ is zero-state detectable, then $(f - \frac{1}{2}g_2g_2^TV_x^T - \frac{g_1g_1^TY_x^T}{2\gamma^2})$ is locally asymptotically stable.*

**Proof:**
The coupled HJIEs (4.5), (4.6) can be written as

$$\tilde{V}_x(x)[f(x) - \frac{1}{2}g_2(x)g_2^T(x)V_x^T(x)] \leq -\|\frac{1}{2}g_2^T(x)V_x^T(x)\|^2 - \gamma^2\|\frac{g_1^T(x)\tilde{V}_x^T(x)}{2\gamma^2}\|^2 - \|h(x)\|^2, \quad \forall x \in N \tag{4.7}$$

$$V_x(x)[f(x) - \frac{1}{2}g_2(x)g_2^T(x)V_x^T(x) + \frac{g_1(x)g_1^T(x)\tilde{V}_x^T(x)}{2\gamma^2}] \leq -\frac{1}{2}\|g_2^T(x)V_x^T(x)\|^2 - \|h(x)\|^2, \quad \forall x \in N, \tag{4.8}$$

where $\tilde{V} = -Y$. Thus, for any submanifold $N \subset M$ in which the solution $\tilde{V}$ exists and is positive-semidefinite, by Lyapunov's theorem and Lasalle's invariance principle [67], $(f - \frac{1}{2}g_2g_2^TV_x^T)$ is locally asymptotically stable in $N$; hence, $\Sigma$ is smoothly stabilizable. Similarly, if $[f + \frac{g_1g_1^T\tilde{V}_x}{2\gamma^2}, h]$ is zero-state detectable, then the vector-field $(f + \frac{g_1g_1^T\tilde{V}_x}{2\gamma^2} - \frac{1}{2}g_2g_2^TV_x^T)$ is locally asymptotically stable in $N$. □

**Remark 4.2.1** *The converse result to the above theorem will be the following: Suppose $\Sigma$ is smoothly stabilizable and there exist feedback strategies $u^\star(x) = F_2(x)$ and $w^\star(x) = F_1(x)$ such that the closed-loop system $\Sigma(u^\star, w^\star)$ has $\mathcal{L}_2 -$ gain $< \gamma$, $f + g_2F_2$ is locally asymptotically stable and $[f + g_1F_1, h]$ is locally zero-state detectable, then there exist* *smooth solutions $Y \leq 0$ and $V \geq 0$ to the coupled HJIEs (4.5), (4.6). We shall not prove this result here, but clearly it gives sufficient conditions for the existence of the smooth solutions $Y$ and $V$. Unfortunately this result is of little significance since it is difficult to construct any feedbacks independent of the solutions. In the next section we shall attempt to discuss sufficient conditions for the existence of local solutions to the coupled HJIEs.*

## 4.3 Review of Earlier Results for Solving the HJIE

In this subsection, we review earlier results on elementary solutions of the HJIE presented in chapter 2. For this purpose, we recall from [107, 110] that the state-feedback $\mathcal{H}_\infty$ control problem for the above system $\Sigma$ is locally solvable if there exists a smooth solution $V : N \to \Re_+$ (defined in a neighborhood $N \subset M$ of the origin, vanishing at $\{0\}$) to the following HJIE (inequality):

$$V_x(x)f(x) + \frac{1}{4}V_x(x)[\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)g_2^T(x)]V_x^T(x) + h^T(x)h(x) \leq 0; \quad V(0) = 0 \; \forall x \in N. \tag{4.9}$$

and such that $(f - \frac{1}{2}g_2g_2^TV_x^T + \frac{1}{2\gamma^2}g_1g_1^TV_x^T)$ is locally asymptotically stable. Moreover, it is shown in chapter 2 that the HJIEs associated with the measurement feedback [56, 58] and time-varying nonlinear systems [84] can also be represented in the above form. Thus, a unified procedure can be developed for solving the HJIEs. Accordingly, in chapter 2, it is shown that the above HJIE (4.9) is solvable if there exists a discriminant factor $\zeta : N \to TM$ that satisfies a system of algebraic-differential inequalities. The following theorem summarizes the approach to the HJIE.

**Theorem 4.3.1** *Suppose there exists a smooth $C^\infty$ vector-valued function (vector-field) $\zeta : N \to TM$ that satisfies the following algebraic-differential inequalities:*

$$\zeta^T(x)\mathcal{Q}_\gamma^+(x)\zeta(x) - f^T(x)\mathcal{Q}_\gamma^+(x)f(x) + h^T(x)h(x) = 0, \quad x \in N \tag{4.10}$$

$$[f_x(x) \pm \zeta_x(x)]^T\mathcal{Q}_\gamma^+(x) + [I_n \otimes (f(x) \pm \zeta(x))^T]\mathcal{Q}_{\gamma,x}^+(x) = \mathcal{Q}_\gamma^+(x)[f_x(x) \pm \zeta_x(x)] + \mathcal{Q}_{\gamma,x}^+(x)[I_n \otimes (f(x) \pm \zeta(x))], \quad x \in N \tag{4.11}$$

$$[f_x(x) \pm \zeta_x(x)]^T\mathcal{Q}_\gamma^+(x) + [I_n \otimes (f(x) \pm \zeta(x))^T]\mathcal{Q}_{\gamma,x}^+(x) \leq 0, \quad x \in N \tag{4.12}$$

*where $\mathcal{Q}_\gamma(x) = [\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)g_2^T(x)]$, $\mathcal{Q}_\gamma^+(x)$ is the generalized inverse of $Q_\gamma(x)$, $x \in N$, and $f_x$, $\zeta_x$, $\mathcal{Q}_{\gamma,x}^+$ denote the partial derivatives of the functions wrt $x$, then*

$$V_x(x) = -2[f(x) \pm \zeta(x)]^T\mathcal{Q}_\gamma^+(x); \; x \in N \tag{4.13}$$

*solves the HJIE (4.9) locally in $M$.*

Before we close this section, we state two important lemmas which will be used in the sequel. To do this, we first consider the linearization of the system $\Sigma$ at $\{0\}$ defined by

$$\bar{\Sigma}^l : \; \dot{\bar{x}} = \bar{F}\bar{x} + \bar{G}_1 w + \bar{G}_2\bar{u} \tag{4.14}$$

$$y = \bar{H}\bar{x} \tag{4.15}$$

where $\bar{F} = \frac{\partial f}{\partial x}(0)$, $\bar{G}_1 = g_1(0)$, $\bar{G}_2 = g_2(0)$, $\bar{H} = \frac{\partial h}{\partial x}(0)$. Then by the *Bounded-real-lemma* [26, 120], $\bar{\Sigma}^l$ has $\mathcal{L}_2$-gain from $u$ to $y$ less or equal to $\gamma$ if and only if there exists a solution $P = P^T \geq 0$ to the algebraic-Riccati-equation (ARE):

$$\bar{F}^T P + P\bar{F} + \frac{1}{\gamma^2} P\bar{G}_2\bar{G}_2^T P + \bar{H}^T\bar{H} = 0 \tag{4.16}$$

If in addition $\sigma\left(\bar{F} + \frac{1}{\gamma^2}\bar{G}_2\bar{G}_2^T P\right) \subset \mathcal{C}^-$, then $\bar{\Sigma}^l$ has $\mathcal{L}_2 -$ gain $< \gamma$. Moreover, the Hamiltonian matrix corresponding to (4.16) is given by

$$H_\gamma^l = \begin{bmatrix} \bar{F} & \frac{1}{\gamma^2}\bar{G}_2\bar{G}_2^T \\ -\bar{H}^T\bar{H} & -\bar{F}^T \end{bmatrix}, \tag{4.17}$$

and it is well-known [120] that $P = P^T \geq 0$ is a real solution of (4.16) if and only if (iff) $Span \begin{bmatrix} I \\ P \end{bmatrix} = \Lambda^-_{H_\gamma^l}$, the stable eigenspace of $H_\gamma^l$, and $H_\gamma^l$ does not have purely imaginary eigenvalues (i.e. it is hyperbolic).

Now consider the ARE corresponding to the LQ control of the system $\bar{\Sigma}^l$:

$$\bar{F}^T P + P\bar{F} - P\bar{G}_2\bar{G}_2^T P + \bar{H}^T\bar{H} = 0 \tag{4.18}$$

Then it can be shown that if $[\bar{F}, \bar{G}_2]$ is stabilizable and $[\bar{F}, \bar{H}]$ is detectable, the above Riccati equation (4.18) has a real-symmetric solution $P \geq 0$ such that $\sigma(\bar{F} - \bar{G}_2\bar{G}_2^T P) \subset \mathcal{C}^-$.

We now state the following lemmas.

**Lemma 4.3.1** *Suppose the nonlinear system $\Sigma$ with $w = 0$ is locally smoothly stabilizable and locally zero-state detectable in $0 \in \bar{N} \subset M$, then there exists locally a smooth solution $X \geq 0$ in $\bar{N}$ to the following Hamilton-Jacobi-inequality (HJE):*

$$X_x(x)f(x) - \frac{1}{4}X_x(x)g_2(x)g_2^T(x)X_x^T(x) + h^T(x)h(x) \leq 0, \quad X(0) = 0 \tag{4.19}$$

*Moreover $(f - \frac{1}{2}g_2g_2^T X_x)$ is locally asymptotically stable in $\bar{N}$.*

**Proof:** If $\Sigma$ is locally smoothly stabilizable in $N$, then there exists a control function $\mathcal{F} : N \to \Re^k$ such that $f + g_2\mathcal{F}$ is locally asymptotically stable. Since there is no loss of generality if the control function is synthesized optimally, then using the theory of optimal control [76], it can be shown that the control function $u = -\frac{1}{2}g_2^T(x)X_x^T(x)$, where $X \geq 0$ is a smooth solution of (4.19), is optimal with respect to (wrt) the performance index $\min_{u\in\mathcal{U}} \int_0^\infty \|z\|^2 dt$. Moreover, by the zero-state detectablity of the system and employing Lasalle's invariance principle [67], it can be shown that $X$ is a Lyapunov function for the closed-loop system. □

**Remark 4.3.1** *The proof of the above lemma also follows by linearization of the system $\Sigma$ about $\{0\}$, which is equivalent to $\bar{\Sigma}^l$, and the application of the preceeding results.*

The following lemma ([110], Theorem 10) is a nonlinear equivalent of the Bounded-real-lemma.

**Lemma 4.3.2** *Consider the system* $\Sigma$ *and its linearization* $\bar{\Sigma}^l$. *Assume* $\bar{F}$ *is asymptotically stable and the Hamiltonian vector field* $X_{H_\gamma} : T^\star M \to TT^\star M$ *defined in canonical coordinates* $(x, p)$ *by*

$$X_{H_\gamma} \quad : \quad \begin{cases} \dot{x}(t) &= \frac{\partial H_\gamma}{\partial p}(x,p) \\ \dot{p}(t) &= -\frac{\partial H_\gamma}{\partial x}(x,p) \end{cases} \tag{4.20}$$

$$H_\gamma(x,p) \quad = \quad p^T f(x) + \frac{1}{4\gamma^2} p^T g_2(x) g_2^T(x) p + h^T(x) h(x) \tag{4.21}$$

*is hyperbolic at* $\{0\}$ *(i.e.* $H_\gamma^l$ *does not have imaginary eigenvalues). Then there exists a neigborhood* $N_1 \subset M$ *of* $\{0\}$ *and a smooth solution* $Z \geq 0$ *on* $N_1$ *of the following HJE:*

$$Z_x(x) f(x) + \frac{1}{4\gamma^2} Z_x(x) g_2(x) g_2^T(x) Z_x^T(x) + h^T(x) h(x) \leq 0 \tag{4.22}$$

*and such that* $f + \frac{1}{2\gamma^2} g_2 g_2^T Z_x$ *is locally asymptotically stable on* $W$. *Furthermore, the* $\mathcal{L}_2$*-gain of the system from* $u$ *to* $y$ *is* $\leq \gamma$ *in* $N_1$.

**Proof:** Proof follows from linearization and the application of the Bounded real-lemma. A complete proof of the above lemma can also be found in [110], Theorem 10. □

## 4.4 A Factorization Appraoch for Solving the Coupled HJIEs

In this section, we extend the above procedure for the pure $\mathcal{H}_\infty$ control HJIE to the case of the coupled HJIEs of the mixed problem. We begin first with the problem of existence of the solutions to the coupled HJIEs. This problem has not been discussed in any reference, except for the linear case [44, 77, 94, 95]. Moreover, even in the linear case and in general, there are no sufficient conditions that guarantee the existence of the solutions. We propose to discuss this problem in this section, and give a proof for the local existence of solutions.

First, we note that the limiting behaviors of the coupled HJIEs are those of the pure $\mathcal{H}_\infty$ HJIE (4.9) and the pure $\mathcal{H}_2$ HJE (4.19) [77, 78]. Now, suppose there exist solutions $V = -\bar{Y} \geq 0$ and $X = \bar{V} \geq 0$ to the HJEs (4.9) and (4.19) respectively, does this then guarantee the existence of the solutions $Y \leq 0$ and $V \geq 0$ to the coupled HJIEs (4.5), (4.6)? This idea is quite logical, and with some additional conditions may provide sufficient conditions for the existence of the solutions to the coupled HJIEs. Therefore we propose the following.

**Proposition 4.4.1** *Suppose* $\Sigma$ *is smoothly stabilizable and* $[f, h]$ *is zero-state detectable, and there exist locally smooth solutions* $V = -\bar{Y} \geq 0$, $X = \bar{V} \geq 0$ *to the HJEs (4.19), (4.9) in* $0 \in N_2 \subset M$ *respectively. In addition, if* $[f - \frac{g_1 g_1^T \bar{Y}_x^T}{2\gamma^2}, h]$ *is zero-state detectable for all negative-semidefinite* $\bar{Y}$, *then there exist locally smooth solutions* $Y \leq 0$ *and* $V \geq 0$ *to the coupled HJIEs (4.5), (4.6) in* $N_2$.

**Proof:** Note that, HJIEs (4.5), (4.6) can be represented in the following form:

$$Y_x(x)\left[f(x)-\frac{1}{2}g_2(x)g_2^T(x)\bar{V}_x^T(x)\right]-\frac{Y_x(x)g_1(x)g_1^T(x)Y_x^T(x)}{4\gamma^2}-\frac{1}{4}\bar{V}_x(x)g_2(x)g_2^T(x)\bar{V}_x^T(x)- h^T(x)h(x)\geq 0 \tag{4.23}$$

$$V_x(x)\left[f(x)-\frac{g_1(x)g_1^T(x)\bar{Y}_x^T(x)}{2\gamma^2}\right]-\frac{1}{4}V_x(x)g_2(x)g_2^T(x)V_x^T(x)+h^T(x)h(x)\leq 0 \tag{4.24}$$

Now, by Theorem 4.2.2, $[f, g_2]$ stabilizable $\Rightarrow [f-\frac{g_1g_1^T\bar{Y}_x}{2\gamma^2}, g_2]$ is stabilizable. Therefore, by lemma 4.3.1 there exists a smooth solution $V\geq 0$ of the HJIE (4.24) in some neighborhood $N_3\subset N_2$ of $\{0\}$ and $(f-\frac{g_1g_1^T\bar{Y}_x}{2\gamma^2}-\frac{1}{2}g_2g_2^TV_x^T)$ is locally asymptotically stable in $N_3$. Further, $[f,h]$ zero-state detectable $\Rightarrow$

$$\left[f,\begin{pmatrix} h \\ g_2^T\bar{V}_x^T\end{pmatrix}\right] \text{ zero-state detectable} \Longrightarrow \left[f-g_2g_2^T\bar{V}_x^T,\begin{pmatrix} h \\ g_2^T\bar{V}_x^T\end{pmatrix}\right] \text{zero-state detectable.}$$

This guarantees that the Hamiltonian-vector field corresponding to (4.23) is hyperbolic. Therefore, by lemma 4.3.2, there exists a smooth solution $\tilde{Y}=-Y\geq 0$ of the HJIE (4.23) in a neighborhood $N_4\subset N_2$ of $\{0\}$ and the $\mathcal{L}_2$-gain of the system is $\leq\gamma$ in $N_4$. Hence the solutions $Y\leq 0$, $V\geq 0$ also solve the mixed $\mathcal{H}_2/H_\infty$ problem. Thus, starting with the initial solutions $Y^0=\bar{Y}$ and $V^0=\bar{V}$ in $N_3\cap N_4$ of the HJEs (4.19) and (4.9) respectively, and applying the following iterative procedure:

$$Y_x^{(i+1)}(x)\left[f(x)-\frac{1}{2}g_2(x)g_2^T(x)V_x^{(i)T}(x)\right]-\frac{Y_x^{(i+1)}(x)g_1(x)g_1^T(x)Y_x^{(i+1)T}(x)}{4\gamma^2}- \frac{1}{4}V_x^{(i)}(x)g_2(x)g_2^T(x)V_x^{(i)T}(x)-h^T(x)h(x)\geq 0 \tag{4.25}$$

$$V_x^{(i+1)}(x)\left[f(x)-\frac{g_1(x)g_1^T(x)Y_x^{(i)T}(x)}{2\gamma^2}\right]-\frac{1}{4}V_x^{(i+1)}(x)g_2(x)g_2^T(x)V_x^{(i+1)T}(x)+ h^T(x)h(x)\leq 0, \tag{4.26}$$

and if the associated Hamiltonian functions are hyperbolic, we obtain a monotone sequence of solutions $Y^0\leq Y^1\leq Y^2\leq\ldots\leq Y^i\leq\ldots 0$, and $V^0\geq V^1\geq V^2\geq\ldots\geq V^i\geq\ldots\geq 0$ of (4.25), (4.26) respectively, converging pointwise as $i\to\infty$ to smooth solutions $Y\leq 0$ and $V\geq 0$ of (4.23), (4.24) respectively, in $N_3\cap N_4$. To see that this is the case, rewrite (4.25), (4.26) as

$$Y_x^{(i+1)}(x)\left[f(x)-\frac{1}{2}g_2(x)g_2^T(x)V_x^{(i)T}(x)\right]\geq\|\frac{1}{2}g_2^T(x)V_x^{(i)T}(x)\|^2+\|\frac{g_1^T(x)Y_x^{(i+1)T}}{2\gamma}\|^2+\|h(x)\|^2$$

$$V_x^{(i+1)}(x)\left[f(x)-\frac{g_1(x)g_1^T(x)Y_x^{(i)T}(x)}{2\gamma^2}-\frac{1}{2}g_2(x)g_2^T(x)V_x^{(i+1)T}(x)\right]\leq-\|h(x)\|^2- \|\frac{1}{2}g_2^T(x)V_x^{(i+1)T}(x)\|^2.$$

Which show that $Y^i$ is nondecreasing along the trajectories of the asymptotically stable vector-field $(f-\frac{1}{2}g_2g_2^TV_x^{(i)T})$ and $V^i$ is nonincreasing along the trajectories of the asymptotically stable vector-field $(f-\frac{1}{2}g_2g_2^TV_x^{(i+1)T}-\frac{g_1(x)g_1^T(x)Y_x^{(i)T}(x)}{2\gamma^2})$ by Lyapunov's theorem.

Hence, we have local convergence to smooth solutions $Y \leq 0$ and $V \geq 0$ of the coupled HJIEs (4.5), (4.6). $\Box$

**Remark 4.4.1** *The result of the above proposition can be utilized in a numerical scheme for the computation of the solutions $Y$ and $V$. In particular, such a procedure has been applied in reference [44] for the solutions of the corresponding coupled ARE of a linear system.*

A somewhat intuitive proof to the above proposition can be persued from the point of view of differential equations. It follows that for any solution $X = \bar{V} \geq 0$ to the HJIE (4.19), we can define the following Hamiltonian function $H_{1,\gamma} : T^{\star}M \to \Re$ corresponding to the HJIE (4.23) in canonical coordinates $(x, p_1)$:

$$H_{1,\gamma}(x,p_1) = p_1^T[f(x) - \frac{1}{2}g_2(x)g_2^T(x)p_2] - \frac{p_1^T g_1(x)g_1^T(x)p_1}{4\gamma^2} - \frac{1}{4}p_2^T g_2(x)g_2^T(x)p_2 - h^T(x)h(x), \tag{4.27}$$

with $p_2 = \bar{V}_x^T(x_0)$ (initially). Similarly, for any solution $V = -\bar{Y} \geq 0$ of (4.9), we can define a corresponding Hamiltonian function $H_{2,\gamma} : T^{\star}M \to \Re$ in canonical coordinates $(x, p_2)$ for the HJIE (4.24)

$$H_{2,\gamma}(x,p_2) = p_2^T[f(x) - \frac{g_1(x)g_1^T(x)p_1}{2\gamma^2}] - \frac{1}{4}p_2^T g_2(x)g_2^T(x)p_2 + h^T(x)h(x) \tag{4.28}$$

with $p_2 = \bar{Y}_x(x_0)$ (initially). Consequently, define also the corresponding Hamiltonian vector-fields $X_{H_{1,\gamma}}, X_{H_{2,\gamma}} : T^{\star}M \to TT^{\star}M$ by

$$X_{H_{1,\gamma}} : \begin{cases} \dot{x}(t) &= \frac{\partial H_{1,\gamma}}{\partial p_1}(x,p_1),\ x(0) = x_0, \\ \dot{p}_1(t) &= -\frac{\partial H_{1,\gamma}}{\partial x}(x,p_1),\ p_1(0) = \bar{Y}_x(x_0), \end{cases} \tag{4.29}$$

$$X_{H_{2,\gamma}} : \begin{cases} \dot{x}(t) &= \frac{\partial H_{2,\gamma}}{\partial p_2}(x,p_2),\ \ x(0) = x_0, \\ \dot{p}_2(t) &= -\frac{\partial H_{2,\gamma}}{\partial x}(x,p_2),\ p_2(0) = \bar{V}_x(x_0). \end{cases} \tag{4.30}$$

Then, the above conditions represent the necessary conditions of the minimum principle for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. The zero-state detectability of $[f, h]$, $[f - \frac{g_1 g_1^T \bar{Y}_x}{2\gamma^2}, h]$ together with the other concomittant zero-state detectability conditions guarantee that the the Hamiltonian functions $H_{1,\gamma}$, $H_{2,\gamma}$ are hyperbolic. While the smooth stabilizability of $[f, g_2]$ and the other ensueing stabilizability conditions guarantee that there exists a $\gamma > 0$, such that the vector-fields $X_{H_{1,\gamma}}$ and $X_{H_{2,\gamma}}$ are stable, and $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} p_1(t), p_2(t) = 0$. Hence, there exist smooth solutions $Y \leq 0$ and $V \geq 0$ to the coupled HJIEs (4.23), (4.24).

**Remark 4.4.2** *Note that, by the preceeding proposition and if all the assumptions are satisfied, then there will exist a pair of minimal solutions $Y^-$, $V^-$ and maximal solutions $Y^+$, $V^+$ to the coupled HJIEs. These will have a geometric interpretation which we shall explain shortly.*

Now if the above solutions $Y \leq 0$, $V \geq 0$ exist, then they define the stable and unstable Lagrangian-invariant manifolds of the Hamiltonian vector fields $X_{H_{1,\gamma}}$ and $X_{H_{1,\gamma}}$ [2, 110, 1] which can be locally parameterized as follows. Define first the following solutions sets:

$$\mathcal{Y}^- = \{Y \mid (f - \frac{1}{2}g_2g_2^TV_x^T - \frac{g_1g_1^TY_x^T}{2\gamma^2}),\ V \in \mathcal{V}^-,\ \text{ is l.a.s. around } \{0\}\} \quad (4.31)$$

$$\mathcal{Y}^+ = \{Y \mid -(f - \frac{1}{2}g_2g_2^TV_x^T - \frac{g_1g_1^TY_x^T}{2\gamma^2}),\ \text{ is l.a.s. around } \{0\}\} \quad (4.32)$$

$$\mathcal{V}^- = \{V \mid (f - \frac{1}{2}g_2g_2^TV_x^T - \frac{g_1g_1^TY_x^T}{2\gamma^2}),\ Y \in \mathcal{Y}^- \text{ is l.a.s. around } \{0\}\} \quad (4.33)$$

$$\mathcal{V}^+ = \{V \mid -(f - \frac{1}{2}g_2g_2^TV_x^T - \frac{g_1g_1^TY_x^T}{2\gamma^2}) \text{ is l.a.s. around } \{0\}\} \quad (4.34)$$

where *l.a.s.* means local asymptotic stability. Then, the stable and unstable Lagrangian-invariant manifolds of $X_{H_{1,\gamma}}$ and $X_{H_{2,\gamma}}$ can be locally parameterized as

$$\mathcal{N}_1^- = \left\{x, p_1 = -Y_x^- \mid Y^- = \max.\, Y \in \mathcal{Y}^-, x \text{ around } \{0\}\right\} \quad (4.35)$$

$$\mathcal{N}_1^+ = \left\{x, p_1 = -Y_x^+ \mid Y^+ = \min.\, Y \in \mathcal{Y}^+, x \text{ around } \{0\}\right\} \quad (4.36)$$

$$\mathcal{N}_2^- = \left\{x, p_2 = V_x^- \mid V^- = \min.\, V \in \mathcal{V}^-, x \text{ around } \{0\}\right\} \quad (4.37)$$

$$\mathcal{N}_2^+ = \left\{x, p_2 = V_x^+ \mid V^+ = \max.\, V \in \mathcal{V}^+, x \text{ around } \{0\}\right\} \quad (4.38)$$

respectively.

**Remark 4.4.3** *Note that, the existence of $Y^+$, $V^+$ depend on whether the pairs $[-(f - \frac{1}{2}g_2g_2^TV_x^T), g_1]$, $[-(f - \frac{g_1g_1^TY_x^T}{2\gamma^2}), g_2]$, are smoothly stabilizable. If however $\Sigma$ is reachable as has been assumed, then both pairs of solutions $(Y^-, V^-)$ and $(Y^+, V^+)$ will exist. Moreover, it is obvious from the above parameterization that $\mathcal{N}_1^- \subseteq \mathcal{N}_2^-$ and $\mathcal{N}_1^+ \subseteq \mathcal{N}_2^+$. Also, $X_{H_{1,\gamma}}$, $X_{H_{2,\gamma}}$ restricted to $\mathcal{N}_1^-$, $\mathcal{N}_2^-$ respectively, are locally asymptotically stable, and $-X_{H_{1,\gamma}}$, $-X_{H_{2,\gamma}}$ restricted to $\mathcal{N}_1^+$, $\mathcal{N}_2^+$ respectively, are locally asymptotically stable.*

A global version of the above existence result can also be proven.

**Proposition 4.4.2** *Take the assumptions as in Proposition 4.4.1. In addition, assume that $f$ is globally asymptotically stable, $X_{H_{1,\gamma}}$, $X_{H_{2,\gamma}}$ are hyperbolic and the stable invariant manifolds $\mathcal{N}_1^-$, $\mathcal{N}_2^-$ are such that $\pi_1^\star : \mathcal{N}_1^- \to T^\star M$, $\pi_2^\star : \mathcal{N}_2^- \to T^\star M$ are diffeomorphisms. Then there exist global solutions $Y^-$ and $V^-$ to the coupled HJIEs (4.5), (4.6). Similarly, the same conclusions follow for the unstable invariant manifolds $\mathcal{N}_1^+$ and $\mathcal{N}_2^+$ (with the solutions $Y^-$, $V^-$ replaced by $Y^+$, $V^+$).*

**Proof:** We consider the stable manifolds first. Since both $\mathcal{N}_1^-$ and $\mathcal{N}_2^-$ are Lagrangian and $\pi_1^\star : \mathcal{N}_1^- \to T^\star M$, $\pi_2^\star : \mathcal{N}_2^- \to T^\star M$ are diffeomorphisms, then $\mathcal{N}_1^- = graph\, d\tilde{Y}$, and $\mathcal{N}_2^- = graph\, d\tilde{V}$ for some smooth scalar functions $\tilde{Y} : M \to \Re$, $\tilde{V} : M \to \Re$ respectively. However, by the global asymptotic stability of $f$, it follows that $M$ is diffeomorphic to $\Re^n$. Further, since the vector-fields $X_{H_{1,\gamma}}$ and $X_{H_{2,\gamma}}$ are invariant wrt to $\mathcal{N}_1^-$ and $\mathcal{N}_2^-$

(i.e. $X_{H_{1,\gamma}}(x,p_1) \in T_{(x,p)}\mathcal{N}_1, \forall (x,p_1) \in \mathcal{N}_1$ and $X_{H_{2,\gamma}}(x,p_2) \in T_{(x,p)}\mathcal{N}_2, \forall (x,p_2) \in \mathcal{N}_2$ ) respectively, then $\tilde{Y}$ and $\tilde{V}$ must be solutions of the HJIEs (4.5), (4.6) respectively [1, 107]. The result now follows by setting $\tilde{Y} = -Y^-$ and $\tilde{V} = V^-$. The same arguments carry through for the solutions $Y^+$, $V^+$ and the unstable invariant manifolds $\mathcal{N}_1^+$ and $\mathcal{N}_2^+$. □

Next, we employ Frobenius theorem to give a necessary condition for the solvability of the coupled HJIEs (4.5), (4.6).

**Theorem 4.4.1** *Consider the distribution $\Delta = Span\{X_{H_{1,\gamma}}, X_{H_{2,\gamma}}\} \subset TT^\star M$ corresponding to the coupled HJIEs (4.5), (4.6). Then, a necessary condition for the coupled HJIEs to have a solution in $N \subset M$, is that $\Delta$ be involutive for all $(x,p) \in T^\star N$.*

**Proof:** Note that, for the integral curves of the two vector fields $X_{H_{1,\gamma}}$, $X_{H_{2,\gamma}}$ to coincide on the base manifold $N$, it is necessary that $\Delta$ is involutive. The integrability of the distribution then follows from Frobenius theorem [111].□

We now present an analytical approach to the solutions of the coupled HJIEs. For this purpose, let us define the following solutions in analogy with the solution in section 4.3:

$$Y_x(x) = [f(x) + \phi(x)]^T \Phi_\gamma^+(x), \quad x \in N, \tag{4.39}$$
$$V_x(x) = [f(x) + \psi(x)]^T \Psi^+(x) \quad x \in N, \tag{4.40}$$

for some vector-valued functions $\phi,\ \psi : N \to TN$, $N \subset M$, where $\Phi(x) = \frac{1}{\gamma^2} g_1(x) g_1^T(x)$ and $\Psi(x) = g_2(x) g_2^T(x)$. Then, substituting the above solutions in the coupled HJIEs (4.5), (4.6), we get the following inequalities:

$$f^T(x)[\Phi_\gamma^+(x) - \Psi^+(x)]f(x) - \phi^T(x)\Phi_\gamma^+(x)\phi(x) - \psi^T(x)\Psi^+(x)\psi(x) - 2\psi^T(x)[\Psi^+(x) + \Phi_\gamma^+(x)]f(x) - 2\phi^T(x)\Phi_\gamma^+\psi(x) - 4h^T(x)h(x) \geq 0, \quad x \in N \tag{4.41}$$
$$f^T(x)\Psi^+(x)f(x) - \psi^T(x)\Psi^+(x)\psi(x) - 2f^T(x)\Psi^+(x)\phi(x) - 2\psi^T(x)\Psi^+(x)\phi(x) + 4h^T(x)h(x) \leq 0, \quad x \in N. \tag{4.42}$$

We shall refer to the above inequalities as the *discriminant inequalities.*

Now differentiate $Y_x$ and $V_x$ to get

$$Y_{xx}(x) = [f_x(x) + \phi_x(x)]^T \Phi_\gamma^+(x) + [I_n \otimes (f(x) + \phi(x))^T]\Phi_{\gamma,x}^+(x), \quad x \in N \tag{4.43}$$
$$V_{xx}(x) = [f_x(x) + \psi_x(x)]^T \Psi^+(x) + [I_n \otimes (f(x) + \psi(x))^T]\Psi_x^+(x), \quad x \in N \tag{4.44}$$

respectively, where

$$f_x(x) = [f_{1,x}(x)\ f_{2,x}(x) \ldots f_{n,x}(x)],\ \phi_x(x) = [\phi_{1,x}(x)\ \phi_{2,x}(x)\ \ldots \phi_{n,x}(x)],\ \psi_x = [\psi_{1,x}\ \psi_{2,x} \ldots \psi_{n,x}]$$
$$\Phi_{\gamma,x}^+(x) = \left[\Phi_{\gamma,x_1}^+(x), \ldots, \Phi_{\gamma,x_n}^+\right]^T,\ \Psi_{\gamma,x}^+(x) = \left[\Psi_{\gamma,x_1}^+(x), \ldots, \Psi_{\gamma,x_n}^+\right]^T$$

Therefore, we have the following theorem regarding the solution of the coupled HJIEs (4.5), (4.6).

**Theorem 4.4.2** *Suppose there exist two vector-valued functions $\phi, \psi : N \to TM$ satisfying the constraints (4.41), (4.42) and such that the "curl conditions [2]": $Y_{xx} = Y_{xx}^T$, $V_{xx} = V_{xx}^T$ and the "definiteness conditions": $Y_{xx}(x) \leq 0$, $V_{xx}(x) \geq 0$ are satisfied for all $x \in N$, then the coupled HJIEs (4.5), (4.6) are locally solvable in $M$.*

**Proof:** Satisfaction of the *"curl conditions"* guarantees that the gradients $Y_x$, $V_x$ correspond to scalar functions; while the *definiteness conditions* guarantee that the functions have the right signs. Finally, the conditions (4.41), (4.42) guarantee that $Y_x$, $V_x$ satisfy the coupled HJIEs (4.5), (4.6). □

**Remark 4.4.4** *The equations (4.39), (4.40) and the inequalities (4.41), (4.42) also represent a parameterization of all solutions to the coupled HJIEs. Further, $Y$ and $V$ can be obtained from $Y_x$ and $V_x$ by evaluating the line integrals $\int_0^x Y_s(s)ds$, $\int_0^x V_s(s)ds$ respectively. In particular, the "curl conditions" are those that guarantee that $N = \pi_1^\star(\mathcal{N}_1) \cap \pi_2^\star(\mathcal{N}_2)$ is the projection from the Lagrangian-invariant manifolds $\mathcal{N}_1$, $\mathcal{N}_2$ of $X_{H_{1,\gamma}}$, $X_{H_{2,\gamma}}$ respectively.*

### 4.4.1 A Higher-Dimensional Approach for Solving the Coupled HJIEs

Alternatively, we can persue a higher-dimensional approach for solving the coupled HJIEs in the following sense. Note that, HJIEs (4.5), (4.6) can be represented as

$$[Y_x(x) \;\; V_x(x)]\begin{bmatrix} f(x) \\ 0 \end{bmatrix} - \frac{1}{4}[Y_x(x) \;\; V_x(x)]\begin{bmatrix} \frac{g_1(x)g_1^T(x)}{\gamma^2} & g_2(x)g_2^T(x) \\ g_2(x)g_2^T(x) & g_2(x)g_2^T(x) \end{bmatrix}\begin{bmatrix} Y_x(x) \\ V_x(x) \end{bmatrix} - h^T(x)h(x) \geq 0 \tag{4.45}$$

$$[Y_x(x) \;\; V_x(x)]\begin{bmatrix} 0 \\ f(x) \end{bmatrix} - \frac{1}{4}[Y_x(x) \;\; V_x(x)]\begin{bmatrix} 0 & \frac{g_1(x)g_1^T(x)}{\gamma^2} \\ \frac{g_1(x)g_1^T(x)}{\gamma^2} & g_2(x)g_2^T(x) \end{bmatrix}\begin{bmatrix} Y_x(x) \\ V_x(x) \end{bmatrix} + h^T(x)h(x) \leq 0. \tag{4.46}$$

Thus, if we define $W_x(x) = [Y_x(x) \;\; V_x(x)]$, $\tilde{f}(x) = [f^T(x) \;\; \mathbf{0^T}]^\mathbf{T}$, $\bar{f}(x) = [\mathbf{0^T} \;\; \mathbf{f^T(x)}]^\mathbf{T}$ and

$$\widetilde{\Phi}_\gamma(x) = \begin{bmatrix} \frac{g_1(x)g_1^T(x)}{\gamma^2} & g_2(x)g_2^T(x) \\ g_2(x)g_2^T(x) & g_2(x)g_2^T(x) \end{bmatrix}, \quad \widetilde{\Psi}_\gamma(x) = \begin{bmatrix} 0 & \frac{g_1(x)g_1^T(x)}{\gamma^2} \\ \frac{g_1(x)g_1^T(x)}{\gamma^2} & g_2(x)g_2^T(x) \end{bmatrix}. \tag{4.47}$$

Then, HJIEs (4.45), (4.46) can be represented as

$$W_x(x)\tilde{f}(x) - \frac{1}{4}W_x(x)\widetilde{\Phi}_\gamma(x)W_x^T - h^T(x)h(x) \;\; \geq \;\; 0 \tag{4.48}$$

$$W_x(x)\bar{f}(x) - \frac{1}{4}W_x(x)\widetilde{\Psi}_\gamma(x)W_x^T + h^T(x)h(x) \;\; \leq \;\; 0 \tag{4.49}$$

The above HJEs (4.48), (4.49) are now in the form of the HJE (4.19), therefore if $[\tilde{f}, -\widetilde{\Phi}_\gamma]$ is smoothly stabilizable and $[f, h]$ is zero-state detectable, then by lemma 4.3.1 there exists a smooth solution to HJE (4.48). Similarly, if $[\bar{f}, \widetilde{\Psi}_\gamma]$ is smoothly stabilizable and $[f, h]$ is zero-state detectable, then there exists a smooth solution to the HJE (4.49). We state this in the following proposition.

**Proposition 4.4.3** *Suppose $[\tilde{f}, -\widetilde{\Phi}_\gamma]$, $[\bar{f}, \widetilde{\Psi}_\gamma]$ are smoothly stabilizable and $[f, h]$ is zero-state detectable, then there exist locally smooth solutions to the HJEs (4.48),(4.49) in $0 \in \tilde{N} \subset M$.*

Similarly, since the system $\Sigma$ is reachable from $\{0\}$, there will exist locally minimal and maximal solutions $W^-$, $W^+$ respectively, of the HJEs (4.48), (4.49) whose graphs will locally parameterize the stable and unstable Lagrangian-invariant manifolds $\widetilde{\mathcal{N}}^-$, $\widetilde{\mathcal{N}}^+$ respectively of the Hamiltonian vector-fields corresponding to (4.48) and (4.49). To do this, define the following Pseudo-Hamiltonian[1] functions $\mathcal{K}_1, \mathcal{K}_2 : M \times M \times M \to \Re$ corresponding to the HJEs (4.48), (4.49) in canonical coordinates $(x, \tilde{p}_1)$ and $(x, \tilde{p}_2)$ respectively by:

$$\mathcal{K}_{1,\gamma}(x, \tilde{p}_1) = \tilde{p}_1^T \tilde{f}(x) - \frac{1}{4}\tilde{p}_1^T \tilde{\Phi}_\gamma(x)\tilde{p}_1 - h^T(x)h(x), \quad \tilde{p}_1 \in M \times M \tag{4.50}$$

$$\mathcal{K}_{2,\gamma}(x, \tilde{p}_2) = \tilde{p}_2^T \bar{f}(x) - \frac{1}{4}\tilde{p}_2^T \tilde{\Psi}_\gamma(x)\tilde{p}_2 + h^T(x)h(x), \quad \tilde{p}_2 \in M \times M \tag{4.51}$$

and the corresponding Hamiltonian vector-fields $X_{\mathcal{K}_{1,\gamma}}, X_{\mathcal{K}_{2,\gamma}} : T^\star(M \times M) \to TT^\star(M \times M)$ by

$$X_{\mathcal{K}_{1,\gamma}} : \begin{cases} \dot{\tilde{x}}_1(t) = \frac{\partial \mathcal{K}_{1,\gamma}}{\partial \tilde{p}_1}(\tilde{x}_1, \tilde{p}_1) \\ \dot{\tilde{p}}_1 = -\frac{\partial \mathcal{K}_{1,\gamma}}{\partial \tilde{x}}(\tilde{x}_1, \tilde{p}_1) \end{cases} \tag{4.52}$$

$$X_{\mathcal{K}_{2,\gamma}} : \begin{cases} \dot{\tilde{x}}_2(t) = \frac{\partial \mathcal{K}_{2,\gamma}}{\partial \tilde{p}_2}(\tilde{x}_2, \tilde{p}_2) \\ \dot{\tilde{p}}_2(t) = -\frac{\partial \mathcal{K}_{2,\gamma}}{\partial \tilde{x}_2}(\tilde{x}_2, \tilde{p}_2) \end{cases} \tag{4.53}$$

where $\tilde{x}_1 = [x^T \ \mathbf{0}^{\mathbf{T}}_{\mathbf{n\times 1}}]^T$, $\tilde{x}_2 = [\mathbf{0}^{\mathbf{T}}_{\mathbf{n\times 1}} \ x^T]^T$. Denote also the set of stabilizing and antistabilizing solutions of the HJEs (4.48), (4.49) by

$$\mathcal{W}^- = \{W \mid (\tilde{f} + \frac{1}{2}\tilde{\Phi}W_x^T),\ (\bar{f} - \frac{1}{2}\tilde{\Psi}W_x^T),\ \text{are l.a.s. in } \tilde{N} \times \tilde{N}\} \tag{4.54}$$

$$\mathcal{W}^+ = \{W \mid -(\tilde{f} + \frac{1}{2}\tilde{\Phi}W_x^T),\ -(\bar{f} - \frac{1}{2}\tilde{\Psi}W_x^T),\ \text{are l.a.s in } \tilde{N} \times \tilde{N}\} \tag{4.55}$$

Let $X_{\mathcal{K}_1}$, $X_{\mathcal{K}_2}$ be hyperbolic at $\{0\}$, then the stable (unstable) Lagrangian-invariant submanifolds of $X_{\mathcal{K}_1}$ and $X_{\mathcal{K}_2}$ through $\{0\}$ can be locally parameterized as

$$\widetilde{\mathcal{N}}^- = \{y, \tilde{p} = W_y^- | W^- = \min .\ W \in \mathcal{W}^-,\ y \textit{ around } \{0\}\}, \tag{4.56}$$

$$\widetilde{\mathcal{N}}^+ = \{y, \tilde{p} = W_y^+ | W^+ = \max .\ W \in \mathcal{W}^+,\ y \textit{ around } \{0\}\}. \tag{4.57}$$

Again, a global existence theorem for the solution $W$ of the HJEs (4.48) and (4.49) can be stated in the following corollary.

**Corollary 4.4.1** *Take the assumptions as in Proposition 4.4.3. In addition, assume that $\tilde{f}$, $\bar{f}$ are globally asymptotically stable, $X_{\mathcal{K}_{1,\gamma}}$, $X_{\mathcal{K}_{2,\gamma}}$ are hyperbolic, and the stable invariant-manifold $\widetilde{\mathcal{N}}^-$ is such that $\tilde{\pi}^\star : \widetilde{\mathcal{N}}^- \to M$ is a diffeomorphism. Then there exist a global solution $W^-$ to the coupled HJEs (4.48), (4.49). A similar conclusion follows for the unstable invariant-manifolds $\widetilde{\mathcal{N}}^+$ (with the solution $W^-$ replaced by $W^+$).*

[1]Note that the costate vectors have twice the dimension of the state vector.

Lastly, in analogy with Theorem 4.4.1, we have the following necessary condition for the existence of solutions to the coupled HJEs (4.48), (4.49)

**Theorem 4.4.3** *Consider the distribution* $\tilde{\Delta} = Span\{X_{\mathcal{K}_{1,\gamma}}, X_{\mathcal{K}_{2,\gamma}}\} \subset TT^\star(M \times M)$, *corresponding to the coupled HJEs (4.48), (4.49), then a necessary condition for the coupled HJEs to have a solution in* $\tilde{N} \subset M$, *is that* $\tilde{\Delta}$ *be involutive for all* $(x, p) \in T^\star(\tilde{N} \times \tilde{N})$.

**Proof:** Proof follows along similar lines as Theorem 4.4.1. □

Now regarding the analytical solutions to the HJEs (4.48) and (4.49), we see that they are also in the form of the HJIE (4.9) and hence we can write the solution to these equations similarly. The following proposition therefore suggests an alternative approach to the solution to the coupled HJIEs.

**Proposition 4.4.4** *Suppose for some* $\gamma > 0$, *there exists a vector field* $\chi : \tilde{N} \to T\tilde{N} \times T\tilde{N}$, $0 \in \tilde{N} \subset M$ *that satisfies the following inequalities:*

$$\chi^T(x)\widetilde{\Phi}_\gamma^+(x)\chi(x) - \tilde{f}^T(x)\widetilde{\Phi}_\gamma^+(x)\tilde{f}(x) + h^T(x)h(x) \leq 0 \tag{4.58}$$

$$\begin{aligned} &2\tilde{f}^T(x)\widetilde{\Phi}_\gamma^+(x)\bar{f}(x) + 2\chi^T(x)\widetilde{\Phi}_\gamma^+(x)\bar{f}(x) - \tilde{f}(x)\widetilde{\Phi}_\gamma^+(x)\widetilde{\Psi}_\gamma^+(x)\widetilde{\Phi}_\gamma^+(x)\tilde{f}(x) - \\ &\quad \chi^T(x)\widetilde{\Phi}_\gamma^+(x)\widetilde{\Psi}_\gamma^+(x)\widetilde{\Phi}_\gamma^+(x)\chi(x) - 2\chi(x)\widetilde{\Phi}_\gamma^+(x)\widetilde{\Psi}_\gamma^+(x)\widetilde{\Phi}_\gamma^+(x)\tilde{f}(x) - h^T(x)h(x) \leq 0 \end{aligned} \tag{4.59}$$

$$\left[\tilde{f}_x(x) + \chi_x(x)\right]^T \widetilde{\Phi}_\gamma^+(x) + [I_n \otimes (\tilde{f}(x) + \chi(x))^T]\widetilde{\Phi}_{\gamma,x}^+(x) \geq 0 \tag{4.60}$$

and

$$[\tilde{f}_x(x)+\chi_x(x)]^T\widetilde{\Phi}_\gamma^+(x)+[I_n\otimes(\tilde{f}(x)+\chi(x))^T]\widetilde{\Phi}_{\gamma,x}^+(x) = \tilde{\Phi}_\gamma^+(x)[\tilde{f}_x(x)+\chi_x(x)]+\widetilde{\Phi}_{\gamma,x}^+(x)[I_n\otimes(\tilde{f}(x)+\chi(x))]. \tag{4.61}$$

*Then*

$$W_x(x) = 2[\tilde{f}(x) + \chi(x)]^T\widetilde{\Phi}_\gamma^+(x) \tag{4.62}$$

*is a solution of the coupled HJEs (4.48), (4.49) and equivalently of (4.5), (4.6).*

**Proof:** Inequalities (4.58), (4.59) follow by substituting (4.62) in the HJIEs (4.48), (4.49). While inequalities (4.60), (4.61) are the positive-semidefinite and symmetry conditions respectively for the Hessian matrix. □

An alternative aproach to the solution of the coupled HJEs (4.48), (4.49) can also be derived by assuming the equality and differentiating wrt $x$. Thus we obtain

$$\begin{aligned} &W_{xx}(x)\tilde{f}(x) + \tilde{f}(x)W_x^T(x) - W_{xx}(x)\widetilde{\Phi}_\gamma(x)W_x^T(x) - \frac{1}{4}[I \otimes W_x(x)]\widetilde{\Phi}_{\gamma,x}(x)W_x^T(x) - \\ &\quad 2h_x(x)h(x) = 0 \end{aligned} \tag{4.63}$$

$$\begin{aligned} &W_{xx}(x)\bar{f}(x) + \tilde{f}(x)W_x^T(x) - W_{xx}(x)\widetilde{\Psi}_\gamma(x)W_x^T(x) - \frac{1}{4}[I \otimes W_x(x)]\widetilde{\Psi}_{\gamma,x}(x)W_x^T(x) + \\ &\quad 2h_x(x)h(x) = 0 \end{aligned} \tag{4.64}$$

Now assume a solution $W_x$ to the above equations in the form

$$W_x(x) = (\tilde{f}(x) + \lambda(x))^T,\ \lambda(0) = 0 \ \text{ or } \ W_x(x) = (\bar{f}(x) + \varsigma(x))^T,\ \varsigma(0) = 0 \tag{4.65}$$

where $\lambda, \varsigma : \tilde{N} \to TM$. Then, to satisfy the symmetry condition, we decompose $W_{xx}$ into its symmetric and skew-symmetric parts:

$$W_{xx} = \frac{1}{2}[\tilde{f}_x(x) + \tilde{f}_x^T(x) + \lambda_x(x) + \lambda_x^T] + \frac{1}{2}[\tilde{f}_x(x) - \tilde{f}_x^T(x) + \lambda_x(x) - \lambda_x^T] \triangleq W_{xx}^s(x) + W_{xx}^{sk}(x),$$

and replace $W_{xx}$ in (4.63), (4.64) by the symmetric part $W_{xx}^s$. We then obtain the following coupled partial-differential-equations (PDEs) in the unknown vector $\lambda$:

$$[\tilde{f}_x(x) + \tilde{f}_x^T(x) + \lambda_x(x) + \lambda_x^T(x)]\left\{\tilde{f}(x) - \widetilde{\Phi}_\gamma(x)(\tilde{f}^T(x) + \lambda^T(x))\right\} + \left\{\tilde{f}(x) - \frac{1}{2}[I \otimes (\tilde{f}(x) + \lambda(x))]\widetilde{\Phi}_{\gamma,x}(x)\right\}(\tilde{f}(x) + \lambda(x)) - 4h_x(x)h(x) = 0 \quad (4.66)$$

$$[\tilde{f}_x(x) + \tilde{f}_x^T(x) + \lambda_x(x) + \lambda_x^T(x)]\left\{\bar{f}(x) - \widetilde{\Psi}_\gamma(x)(\tilde{f}(x) + \lambda(x))\right\} + \left\{\bar{f}(x) - \frac{1}{2}[I \otimes (\tilde{f}(x) + \lambda(x))]\widetilde{\Psi}_{\gamma,x}(x)\right\}(\tilde{f}(x) + \lambda(x)) + 4h_x(x)h(x) = 0 \quad (4.67)$$

The above system of $2n(n-1)$ nonlinear PDEs must now be solved for the unknown vector $\lambda$ from which $W_x$ can then be determined. Such a system of equations will have many solutions depending on some $2n$ arbitrary fuctions $\eta_i : M \to \Re$, $i = 1, \ldots 2n$. Hence by varying these functions and the parameter $\gamma$, the desired solution $W = [Y,\ V]$ of the coupled HJIEs can be obtained.

## 4.5 Linear Systems

In this section, we specialize the results of the preceding two sections to linear systems. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem for this class of systems has been extensively considered [24, 36, 68, 77] ,but much less has been done on the coupled Riccati equations characterizing the solution to the problem. We consider the following linear time-invariant system (LTI):

$$\Sigma^l \ : \ \dot{x}(t) = Ax(t) + G_1w(t) + G_2u(t) \quad (4.68)$$

$$z(t) = Cx(t) + Du(t) \quad (4.69)$$

where $x(t) \in \Re^n$, $u(t) \in \mathcal{U}$, $w(t) \in \mathcal{W}$ are as described previously, while $A \in \Re^{n\times n}$, $G_1 \in \Re^{n\times r}$, $G_2 \in \Re^{n\times k}$, $C \in \Re^{m\times n}$, $D \in \Re^{m\times k}$. For simplicity, we also assume that the output matrices satisfy

$$C^TD = 0; \quad D^TD = I.$$

Now, if we assume that the solutions of the coupled HJIEs (4.5), (4.6) (without any loss of generality we shall consider the equations here) are of the form $Y(x) = x^TP_1x$, $V(x) = x^TP_2x$ [77] for some symmetric positive-semidefinite matrices $P_1 \leq 0$, $P_2 \geq 0$, we get the following coupled Riccati equations.

$$A^TP_1 + P_1A - \gamma^{-2}P_1G_1G_1^TP_1 - P_1G_2G_2^TP_2 - P_2G_2G_2^TP_1 - P_2G_2G_2^TP_2 - C^TC = 0 \quad (4.70)$$

$$A^TP_2 + P_2A - \gamma^{-2}P_2G_1G_1^TP_1 - \gamma^{-2}P_1G_1G_1^TP_2 - P_2G_2G_2^TP_2 + C^TC = 0 \quad (4.71)$$

Further, the global existence of solutions to the above Riccati equations have been discussed in some references [44, 94, 95], but no sufficient conditions are given. However, if we specialize the results of Proposition 4.4.1 to the the linear system, we immediately get the following corollary.

**Corollary 4.5.1** *Consider the LTI system $\Sigma^l$ and assume that $(A, G_2)$ is stabilizable and $(C, A)$ is detectable. Assume further that there exist positive-semidefinite solutions $P \geq 0$, $K \geq 0$ to the ARE (4.18) and the ARE*

$$A^T K + KA + K[\gamma^{-2} G_1 G_1^T - G_2 G_2^T] K + C^T C = 0 \tag{4.72}$$

*respectively, (with $\bar{\Sigma}^l = \Sigma^l, D = 0$). If in addition $((A + \gamma^{-2} G_1 G_1^T K), C)$ is detectable for all positive-semidefinite $K$, then there exist solutions $P_1 \leq 0$ and $P_2 \geq 0$ to the coupled Riccati equations (4.70), (4.71).*

**Proof:** Taking $V(x) = x^T K x = -x^T \bar{P}_1 x$ and $X(x) = x^T P x = x^T \bar{P}_2 x$ in Proposition 4.4.1, the coupled ARE (4.70), (4.71) can be rearranged as

$$(A - G_2 G_2^T \bar{P}_2)^T P_1 + P_1 (A - G_2 G_2^T \bar{P}_2) - \gamma^{-2} P_1 G_1 G_1^T P_1 - \bar{P}_2 G_2 G_2^T \bar{P}_2 - C^T C = 0 \tag{4.73}$$

$$(A - \gamma^{-2} G_1 G_1^T \bar{P}_1)^T P_2 + P_2 (A - \gamma^{-2} G_1 G_1^T \bar{P}_1) - P_2 G_2 G_2^T P_2 + C^T C = 0 \tag{4.74}$$

Now $(A, G_2)$ stabilizable $\Rightarrow (A - \gamma^{-2} G_1 G_1^T \bar{P}_1, G_2)$ is stabilizable [77]. Therefore, by the results in section 4.3 (see also [120]), there exists a symmetric solution $P_2 \geq 0$ to the ARE (4.74) such that $\sigma(A - \gamma^{-2} G_1 G_1^T \bar{P}_1 - G_2 G_2^T P_2) \subset \mathcal{C}^-$. Further,

$$(A, C) \text{ detectable} \Rightarrow \left(A, \begin{bmatrix} C \\ G_2 \bar{P}_2 \end{bmatrix}\right) \text{ is detectable} \Rightarrow \left(A - G_2 G_2^T \bar{P}_2, \begin{bmatrix} C \\ G_2 \bar{P}_2 \end{bmatrix}\right) \text{ is detectable.}$$

This guarantees that the Hamiltonian matrix

$$H_{1,\gamma}^l = \begin{bmatrix} (A - G_2 G_2^T \bar{P}_2) & -\gamma^{-2} G_1 G_1^T \\ \bar{P}_2 G_2 G_2^T \bar{P}_2 + C^T C & -(A - G_2 G_2^T \bar{P}_2)^T \end{bmatrix}$$

corresponding to (4.73) is hyperbolic. Therefore, by the bounded-real lemma, there exists $\gamma < \infty$ and a symmetric solution $\tilde{P}_1 = -P_1 \geq 0$ to the Riccati equation (4.73) and such that $\sigma\left(A - G_2 G_2^T \bar{P}_2 - \gamma^2 G_1 G_1^T P_1\right) \subset \mathcal{C}^-$ and $\|T_{zw}\|_\infty < \gamma$ (where $T_{zw}$ is the transfer function from $w$ to $z$). Hence, the solutions $P_1 \leq 0$ $P_2 \geq 0$ solve the linear mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. Moreover, starting with the initial solutions $P_1^0 = \bar{P}_1$, $P_2^0 = \bar{P}_2$ and applying the following iterative procedure:

$$\begin{aligned} &(A - G_2 G_2^T P_2^i)^T P_1^{(i+1)} + P_1^{(i+1)} (A - G_2 G_2^T P_2^{(i)}) - \gamma^{-2} P_1^{(i+1)} G_1 G_1^T P_1^{(i+1)} - P_2^{(i)} G_2 G_2^T P_2^{(i)} \\ &-C^T C = 0, \end{aligned} \tag{4.75}$$

$$\begin{aligned} &(A - \gamma^{-2} G_1 G_1^T P_1^{(i)})^T P_2^{(i+1)} + P_2^{(i+1)} (A - \gamma^{-2} G_1 G_1^T P_1^{(i)}) - P_2^{(i+1)} G_2 G_2^T P_2^{(i+1)} + \\ &C^T C = 0, \end{aligned} \tag{4.76}$$

and if all the associated Hamiltonian matrices are hyperbolic, we obtain a monotone sequence of symmetric solutions $P_1^0 \leq P_1^1 \leq P_1^2 \leq \ldots \leq P_1^i \leq \ldots \leq 0$ and $P_2^0 \geq P_2^1 \geq P_2^2 \geq \ldots \geq P_1^i \geq \ldots \geq 0$ of (4.75), (4.76) respectively, converging to strong symmetric solutions $P_1 \leq 0$ and $P_2 \geq 0$ of the coupled Riccati equations (4.70), (4.71) respectively.□

**Remark 4.5.1** *One way to solve the coupled Riccati equations is then to apply the above iterative procedure [44]. It is seen that, if all the assumptions in the corollary are satisfied (which normally are), then each iteration of the algorithm involves the solution of standard Riccati equations of the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ type.*

We now specialize the solution procedures of Section 4.3 to the linear case. To do this, assume the solutions (4.39), (4.40) are of the form:

$$Y_x(x) = x^T[A+\Omega]^T\Phi_\gamma^+ \tag{4.77}$$
$$V_x(x) = x^T[A+\Xi]^T\Psi^+ \tag{4.78}$$

for some $n \times n$ matrices $\Omega$ and $\Xi$. Then

$$Y_{xx} = [A+\Omega]^T\Phi_\gamma^+ \tag{4.79}$$
$$V_{xx} = [A+\Xi]^T\Psi^+ \tag{4.80}$$

where $\Phi_\gamma = \frac{1}{\gamma^2}G_1G_1^T$, $\Psi = G_2G_2^T$, and $\Phi_\gamma^+$, $\Psi^+$ are the corresponding generalized inverses. Substituting the above expressions in the coupled HJIEs (4.5), (4.6) or the discriminant inequalities (4.41), (4.42), we get

$$A^T[\Phi_\gamma^+ - \Psi_\gamma^+]A - \Xi^T\Psi^+\Xi - \Omega^T\Phi_\gamma^+\Omega - 2\Omega^T\Phi_\gamma^+\Xi - 2\Xi^T[\Psi^+ + \Phi_\gamma^+]A - 4H^TH \geq 0, \tag{4.81}$$
$$A^T\Psi^+A - \Xi^T\Psi^+\Xi - 2A^T\Psi^+\Omega - 2\Xi^T\Psi^+\Omega + 4H^TH \leq 0. \tag{4.82}$$

Together with the symmetry (or "curl") conditions:

$$[A+\Omega]^T\Phi_\gamma^+ = \Phi_\gamma^+[A+\Omega] \tag{4.83}$$
$$[A+\Xi]^T\Psi^+ = \Psi^+[A+\Xi] \tag{4.84}$$

and the positivity conditions:

$$[A+\Omega]^T\Phi_\gamma^+ \geq 0 \tag{4.85}$$
$$[A+\Xi]^T\Psi^+ \geq 0 \tag{4.86}$$

the conditions (4.81)-(4.86) completely characterize the solution to the coupled HJIEs for the linear system $\Sigma^l$. Furthermore, an optimization routine [46] can be used to solve the above system of inequalities.

On the other hand, if we specialize the higher-dimensional procedure in Section 4.4 to the linear system $\Sigma^l$, from (4.48), (4.49) by taking $\Upsilon(y) = y^TWy$ for some $2n \times 2n$ symmetric matrix $W$ and $y \in \Re^{2n}$, we get the following standard Riccati inequalities:

$$\tilde{A}^TW + W\tilde{A} - W\tilde{\Phi}_\gamma W - \tilde{Q} \geq 0 \tag{4.87}$$
$$\bar{A}^TW + W\bar{A} - W\tilde{\Psi}_\gamma W + \bar{Q} \leq 0 \tag{4.88}$$

where $\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$, $\tilde{Q} = \begin{bmatrix} C^TC & 0 \\ 0 & 0 \end{bmatrix}$, and $\bar{Q} = \begin{bmatrix} 0 & 0 \\ 0 & C^TC \end{bmatrix}$. Then, if $\tilde{\Phi}_\gamma > 0$ and $\tilde{\Psi}_\gamma > 0$, the problem can be formulated as the following optimization problem over a system of linear-matrix-inequalities (LMIs):

$$OPT: \quad \min \gamma \ \text{ s.t. } \ W = W^T \tag{4.89}$$
$$\begin{bmatrix} -\tilde{A}^TW - W\tilde{A} + \tilde{Q} & W \\ W^T & -\tilde{\Phi}_\gamma \end{bmatrix} \leq 0 \tag{4.90}$$
$$\begin{bmatrix} \bar{A}^TW + W\bar{A} + \bar{Q} & W \\ W^T & -\tilde{\Psi}_\gamma \end{bmatrix} \leq 0 \tag{4.91}$$

Alternatively, applying the results of Proposition 4.4.4, and assuming that there exists a real matrix $\chi \in \Re^{2n\times 2n}$ such that

$$\begin{aligned} \Upsilon_y(y) &= 2y^T[\tilde{A}+\chi]^T\tilde{\Phi}_\gamma \\ \Upsilon_{yy} &= 2[\tilde{A}+\chi]^T\tilde{\Phi}_\gamma \end{aligned}$$

$$\chi^T\tilde{\Phi}_\gamma^+\chi - \tilde{A}^T\tilde{\Phi}_\gamma^+\tilde{A} + \tilde{Q} \leq 0 \tag{4.92}$$

$$2\tilde{A}^T\tilde{\Phi}_\gamma^+\bar{A} + 2\chi^T\tilde{\Phi}_\gamma^+\bar{A} - \chi^T\tilde{\Phi}_\gamma^+\tilde{\Psi}_\gamma^+\tilde{\Phi}_\gamma^+\chi - 2\chi^T\tilde{\Phi}_\gamma^+\tilde{\Psi}_\gamma^+\tilde{\Phi}_\gamma^+\tilde{A} - \tilde{A}^T\tilde{\Phi}_\gamma^+\tilde{\Psi}_\gamma^+\tilde{\Phi}_\gamma^+\tilde{A} + \bar{Q} \leq 0 \tag{4.93}$$

$$\left[\tilde{A}+\chi\right]^T\tilde{\Phi}_\gamma^+ \geq 0 \tag{4.94}$$

and

$$[\tilde{A}+\chi]^T\tilde{\Phi}_\gamma^+ = \tilde{\Phi}_\gamma^+[\tilde{A}+\chi], \tag{4.95}$$

we can solve for $\chi$ in the above inequalities, from which we can obtain $W$.

One important and popular approach for solving Riccati equations is the *invariant subspace method* for Hamiltonian matrices [120]. This approach has been applied in [94] for the solution of the coupled Riccati equations (4.70), (4.71). It can be shown that, any solutions $P_1$, $P_2$ of the coupled Riccati equations must span some invariant subspaces of the following Hamiltonian matrices corresponding to (4.70), (4.71) respectively,

$$\begin{aligned} H^l_{1,\gamma} &= \begin{bmatrix} (A-G_2G_2^TP_2) & -\gamma^{-2}G_1G_1^T \\ P_2G_2G_2^TP_2 + C^TC & -(A-G_2G_2^TP_2)^T \end{bmatrix} \\ H^l_{2,\gamma} &= \begin{bmatrix} (A-\gamma^{-2}G_1G_1^TP_1) & -G_2G_2^T \\ -C^TC & -(A-\gamma^{-2}G_1G_1^TP_1)^T \end{bmatrix}. \end{aligned}$$

Indeed,

$$\begin{bmatrix} (A-G_2G_2^TP_2) & -\gamma^{-2}G_1G_1^T \\ P_2G_2G_2^TP_2 + C^TC & -(A-G_2G_2^TP_2)^T \end{bmatrix}\begin{bmatrix} I \\ P_1 \end{bmatrix} = \begin{bmatrix} I \\ P_1 \end{bmatrix}(A-G_2G_2^TP_2-\gamma^{-2}G_1G_1^TP_1) \tag{4.96}$$

$$\begin{bmatrix} (A-\gamma^{-2}G_1G_1^TP_1) & -G_2G_2^T \\ -C^TC & -(A-\gamma^{-2}G_1G_1^TP_1)^T \end{bmatrix}\begin{bmatrix} I \\ P_2 \end{bmatrix} = \begin{bmatrix} I \\ P_2 \end{bmatrix}(A-G_2G^TP_2-\gamma^{-2}G_1G_1^TP_1). \tag{4.97}$$

Thus, if we let $\mathcal{V}_1 = Span\begin{bmatrix} I \\ P_1 \end{bmatrix} \subset \mathcal{C}^{2n}$ and $\mathcal{V}_2 = Span\begin{bmatrix} I \\ P_2 \end{bmatrix} \subset \mathcal{C}^{2n}$, then $\mathcal{V}_1$, $\mathcal{V}_2$ are $n$-dimensional $\mathcal{H}^l_{1,\gamma}$, $\mathcal{H}^l_{2,\gamma}$-invariant, and $\sigma\left(\mathcal{H}^l_{1,\gamma}|_{\mathcal{V}_1}\right) = \sigma(A-G_2G_2^TP_2-\gamma^{-2}G_1G_1^TP_1)$, $\sigma\left(\mathcal{H}^l_{2,\gamma}|_{\mathcal{V}_2}\right) = \sigma(A-G_2G_2^TP_2-\gamma^{-2}G_1G_1^TP_1)$ respectively. Moreover, if $P_1$, $P_2$ are symmetric, then the above subspaces are *Lagrangian*[1, 2]. Additionally, if $\mathcal{V}_1$, $\mathcal{V}_2$ are conjugate symmetric[2], then $P_1$, $P_2$ are real. It can further be shown that, if the above Hamiltonian matrices are hyperbolic for all $P_1$, $P_2$, then the stablizing solutions $P_1 \leq 0$, $P_2 \geq 0$ of the coupled Riccati equations correspond to those Lagrangian $H^l_{1,\gamma}$, $H^l_{2,\gamma}$-invariant subspaces

[2]A subspace $\mathcal{V}$ is conjugate symmetric if for any vector $v \in \mathcal{V}$, $\Rightarrow \bar{v} \in \mathcal{V}$

that are complementary to the $Span \begin{bmatrix} 0 \\ I \end{bmatrix}$ [25] and which also correspond to the *unstable* and *stable* eigenspaces of $H_{1,\gamma}$, $H_{2,\gamma}$ respectively.

It is also now geometrically clear how the above iterative procedure (4.75), (4.76) works. Starting with the solutions $P_1^0$, $P_2^0$, we construct a sequence of invariant-subspaces $\mathcal{V}_1^0 \subset \mathcal{V}_1^1 \subset \ldots \subset \mathcal{V}_1^\star$, $\mathcal{V}_2^0 \subset \mathcal{V}_2^1 \subset \ldots \subset \mathcal{V}_2^\star$ corresponding to the unstable and stable-eigenspaces of $H_{1,\gamma}^l$, $H_{2,\gamma}^l$ which are also complementary to $Span \begin{bmatrix} 0 \\ I \end{bmatrix}$ respectively, until we obtain the largest subspaces $\mathcal{V}_1^\star$, $\mathcal{V}_2^\star$ such that $\sigma\left(\mathcal{H}_{1,\gamma}^l|_{\mathcal{V}_1^\star}\right) = \sigma(A - G_2G_2^TP_2 - \gamma^{-2}G_1G_1^TP_1) \subset \mathcal{C}^-$ and $\sigma\left(\mathcal{H}_{2,\gamma}^l|_{\mathcal{V}_2^\star}\right) = \sigma(A - G_2G_2^TP_2 - \gamma^{-2}G_1G_1^TP_1) \subset \mathcal{C}^-$.

It also follows that, if $\Sigma^l$ is controllable, there exist minimal solutions $P_1^-$, $P_2^-$ and maximal solutions $P_1^+$, $P_2^+$ such that $\sigma\left(A - G_2G_2^TP_2^- - \gamma^{-2}G_1G_1^TP_1^+\right) \subset \mathcal{C}^-$ and $\sigma\Big(-(A - G_2G_2^TP_2^+ - \gamma^{-2}G_1G_1^TP_1^-)\Big) \subset C^+$. Furthermore, if we define $Span \begin{bmatrix} I \\ P_2^- \end{bmatrix}_{P_1^+}$ and $Span \begin{bmatrix} I \\ P_2^+ \end{bmatrix}_{P_1^-}$ as the subspaces corresponding to the solutions $P_1^\pm$, $P_2^\pm$, then $Span \begin{bmatrix} I \\ P_2^- \end{bmatrix}_{P_1^-}$ is tangent to $\mathcal{N}_1^-$, $\mathcal{N}_2^-$ at $\{0\}$ and similarly, $Span \begin{bmatrix} I \\ P_2^+ \end{bmatrix}_{P_1^+}$ is tangent to $\mathcal{N}_1^+$, $\mathcal{N}_2^+$ at $\{0\}$.

Finally, let us specialize the results of Theorem 4.4.1 to the linear system $\Sigma^l$ and the coupled Riccati equations (4.70), (4.71).

**Corollary 4.5.2** *A necessary condition for the coupled Riccati equations (4.70), (4.71) to have a solution is that the Hamiltonian matrices* $H_{1,\gamma}^l$, $H_{2,\gamma}^l$ *commute, or their Lie bracket (also commutator)* $[H_{1,\gamma}^l, H_{2,\gamma}^l] = 0$.

Similarly, if we define the following Hamiltonian matrices corresponding to the Riccati inequalities (4.87), (4.88),

$$\mathcal{K}_{1,\gamma}^l = \begin{bmatrix} \widetilde{A} & -\widetilde{\Phi}_\gamma \\ \tilde{Q} & -\widetilde{A}^T \end{bmatrix} \tag{4.98}$$

$$\mathcal{K}_{2,\gamma}^l = \begin{bmatrix} \bar{A} & -\widetilde{\Psi}_\gamma \\ -\bar{Q} & -\bar{A}^T \end{bmatrix}, \tag{4.99}$$

we can specialize the result of Theorem 4.4.3 to the linear system $\Sigma^l$ and the coupled Riccati inequalities (4.87), (4.88).

**Corollary 4.5.3** *A necessary condition for the coupled Riccati inequalities (4.87), (4.88) (equivalently (4.70), (4.71)) to have a solution is that the Hamiltonian matrices* $\mathcal{K}_{1,\gamma}^l$, $\mathcal{K}_{2,\gamma}^l$ *commute or their Lie bracket (also commutator)* $[\mathcal{K}_{1,\gamma}^l, \mathcal{K}_{2,\gamma}^l] = 0$.

We now consider an example.

**Example 4.5.1** *Consider the following system:*

$$\begin{aligned}\dot{x}_1(t) &= -x_1^3(t) + x_2(t)\\ \dot{x}_2(t) &= x_2(t) + u(t) + w(t)\\ z(t) &= [x_2(t) \;\; u(t)]^T\end{aligned}$$

*Let* $\gamma = 2$, *then*

$$f(x) = \begin{bmatrix} -x_1^3 + x_2 \\ x_2 \end{bmatrix}; \; G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \; G_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \;\; h(x) = x_2$$

$$\Phi_\gamma = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\gamma^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}; \;\; \Psi = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

*Substituting the above functions in (4.41), (4.42), we get*

$$3x_2^2 - \psi_2^2 - 4\phi_2^2 - 8\phi_2\psi_2 - 10\psi_2 x_2 - 4x_2^2 \geq 0; \;\; \forall x \in N \tag{4.100}$$

$$x_2^2 - \psi_2^2 - 2\phi_2 x_2 - 2\phi_2\psi_2 + 4x_2^2 \leq 0; \;\; \forall x \in N \tag{4.101}$$

*while the symmetry conditions give*

$$\phi_{1,x_2} = \phi_{2,x_1}, \;\; \psi_{1,x_2} = \psi_{2,x_1} \tag{4.102}$$

*Solving (4.100), (4.101) and (4.102) for* $\phi_2$ $\phi_1(x)$, *and* $\psi_2(x)$, $\psi_1$ *respectively, one viable solution is*

$$\phi_2(x) = -2.6219x_2, \;\; \phi_1(x) = c_1, \;\; \text{and } \psi_2(x) = 6.7594x_2, \; \psi_1(x) = c_2$$

*where* $c_1$, $c_2$ *are arbitrary constants. Finally, substituting in (4.39), (4.40) and taking* $c_1 = c_2 = 0$ *without any loss of generality, we get,*

$$Y_x(x) = (0 \;\; -6.48760x_2), \quad V_x(x) = (0 \;\; 7.7594x_2)$$

*which upon integration from* 0 *to some arbitrary* $x$ *give the following semi-definite solutions:*

$$Y(x) = -3.2438x_2^2, \quad V(x) = 3.8797x_2^2.$$

# CHAPTER 5

# A FACTORIZATION APPROACH FOR SOLVING THE HAMILTON-JACOBI-BELLMAN EQUATIONS IN $\mathcal{H}_2$ DETERMINISTIC AND STOCHASTIC NONLINEAR OPTIMAL CONTROL

## 5.1 Introduction

In this chapter, we present closed-form approaches for solving the Hamilton-Jacobi-Bellman equations (HJBEs) arising in $\mathcal{H}_2$ or quadratic-optimal control of nonlinear deterministic and stochastic systems. We shall extend the factorization approach that we have developed in the previous chapters to the stochastic case. It is shown that, for the deterministic HJBE of modern optimal control, the results for the HJIE of the $\mathcal{H}_\infty$ control problem generalize to this case with the only modification in the gain matrices; while for the stochastic case, the results are significantly different, since the resulting HJBE in this case is second order. Nonetheless, with some slight modifications, the ideas of the previous chapters can also be extended to this case.

Furthermore, we also consider in this chapter how to solve the HJBEs when the control set is not an open set but a bounded set. This leads to the idea of viscosity solutions of the HJBEs and the concept of nondifferentiable Hamiltonian functions. We briefly consider these cases in the chapter. Applications of the approaches developed in the chapter to other HJ-type PDEs such as conservation laws are also explored.

The rest of the chapter is organized as follows. In section 2, we give problem definition and review some important results. Then, we present an elementary closed-form solution to the HJBE for deterministic affine nonlinear systems. As a by-product, we also present a solution of the Lyapunov-type inequality arising in the factorization of affine nonlinear systems. The, in section 3, we extend the results from section 2 to the stochastic case. Both classical and viscosity solutions are considered. Finally in section 4, we discuss possible applications of the new approach to other PDEs such as conservation laws.

## 5.2 Review of Results on the $\mathcal{H}_2$ Deterministic HJBE

In this section, we introduce the deterministic $\mathcal{H}_2$ control problem and the associated HJBE, as well as some important definitions and a review of the main results from Chapter 2.

To this end, we shall consider the following affine nonlinear state-space system $\Sigma$ defined in local coordinates over some manifold $M \subseteq \Re^n$ containing the origin:

$$\Sigma: \quad \dot{x}(t) \quad = \quad f(x) + g_1(x)w(t) + g_2(x)u(t); \quad x(0) = \hat{x}_0 \tag{5.1}$$

$$z(t) = \begin{bmatrix} h(x) \\ u \end{bmatrix}; \quad f(0) = 0, \quad h(0) = 0 \tag{5.2}$$

where $x(t) \in M$ is the state vector, $u(t) \in \Re^k$ is the control input, $w(t) \in \mathcal{W}$ is the disturbance signal, and $z(t) \in \Re^{m+k}$ is the controlled output. The functions $f : M \to TM$ $g_2 : M \to \mathcal{M}^{n\times k}(M)$, $g_1 : M \to \mathcal{M}^{n\times s}(M)$, $h(.) \in \Re^m$ are smooth $C^\infty(M)$ functions of $x$. The admissible control signal $u : \Re_+ \to \mathcal{U}$ and disturbance signal $w : \Re \to \mathcal{W}$ are bounded Lebesgue measurable functions belonging to some compact sets $\mathcal{U} \subset \Re^k$, and $\mathcal{W} \subset \Re^s$ respectively.

The $\mathcal{H}_2$ control problem for the system $\Sigma$ with state feedback, is to find a control action $u = \alpha(x) \in \mathcal{U}$ such that the closed-loop system $\Sigma(u = \alpha, w = 0)$ is asymptotically stable and the $\mathcal{L}_2$-norm of the closed-loop map from $w$ to $z$, $\|\Sigma_{zw}\|_2 \in \mathcal{L}_2$ is minimized for all $w \in \mathcal{W} \subset \mathcal{L}_2$. This problem can be formulated as the following optimization problem:

$$\min_{u\in\mathcal{U}} \int_0^\infty \|z\|^2 dt \ \text{ s.t. } \ \Sigma. \tag{5.3}$$

The above problem belongs to a more general class of infinite-horizon optimal control problems with quadratic cost functions which can be formulated as:

$$\min_{u\in\mathcal{U}} J_d(x, u) = \int_0^\infty L(x, u)dt \tag{5.4}$$

subject to the dynamical equations (5.1), (5.2), where $L : M \times \mathcal{U} \to \Re$ is some suitable cost function.

Let us define the value function as the minimum value of the cost function $J(x, u)$ at each arbitrary point $x$ by

$$v(x) = \inf_{u\in\mathcal{U}} J_d(x, u), \quad x \in N \subset M. \tag{5.5}$$

Then if we assume the value function to be bounded from below, i.e., $v(x) > -\infty$, which is true since the control set $\mathcal{U}$ is compact, and if we assume also that the value function is continuously differentiable in the classical sense[1] (we shall denote it by $V$), i.e., for each point $x$ in the interior of $N$, there exists a row vector $V_x$ such that

$$\lim_{x'\to x} \frac{|V(x') - V(x) - V_x(x).(x' - x)|}{\|x' - x\|} = 0,$$

for all $x', x \in N$. It can then be shown that, a necessary and sufficient condition for the solvability of the above optimization problem is that $V \in C^1(N)$ satisfies the *dynamic programming principle* or HJBE [16, 40, 76]:

$$\inf_{u\in\mathcal{U}}\{L(x, u) + V_x(x)[f(x) + g_1(x)w + g_2(x)u]\} = 0, \quad V(0) = 0 \tag{5.6}$$

Under the above assumption of differentiability of $v$, any solution $V$ of the above equation (5.6) will be referred to as a *classical solution* if it satisfies it for all $x \in N$. In most cases

[1] The Fretchet derivative of the function exists for all $x \in M$ [119].

however, the value function $v$ fails to be differentiable at some point $x \in M$, and hence may not satisfy the HJBE (5.6) everywhere in $M$. In such cases, we would like to consider solutions that are closest to being differentiable in an extended sense. The closest such idea is that of Lipschitz continuous solutions. This leads to the concept of generalized solutions which we now define [40].

**Definition 5.2.1** *Define the Hamiltonian of the system as*

$$\mathcal{H}(x,p) = \inf_{u \in \mathcal{U}} L(x,u) + p^T[f(x) + g_2(x)u + g_1(x)w]$$

*for some $p \in \Re^n$. Then (5.6) takes the following form*

$$\mathcal{H}(x, D_x v(x)) = 0, \quad x \in N, \quad v(0) = 0 \tag{5.7}$$

*where $D_x v(x)$ denotes some derivative of $v$ at $x$, which is not necessarily a classical derivative. Now suppose $v$ is locally Lipschitz on $N$, i.e., for every compact set $O \subset N$ and $x_1$, $x_2 \in O$ there exists a constant $k_O > 0$ such that*

$$|v(x_1) - v(x_2)| \leq k_O \|x_1 - x_2\|$$

*(it is Lipschitz if $K_O = k$, independent of $O$), then $v$ is a generalized solution of (5.6) if it satisfies it for almost all $x \in N$.*

Moreover, since every locally Lipschitz function is differentiable at almost all points $x \in N$, the idea of generalized solutions indeed makes sense. However, the concept also implies the lack of uniqueness of generalized solutions. Thus, there can be infinitely many generalized solutions. In this chapter, we shall restrict ourselves to the class of generalized solutions referred to as viscosity solutions, which are unique. These are defined next. Some of the material in the chapter will also be a repetition from the previous chapters.

Assume $v$ is continuous in $N$, and define the following sets which are respectively the *superdifferential* and *subdifferential* of $v$ at $x \in N$:

$$D^+v(x) = \left\{ p \in \Re^n \; : \; \lim_{x' \to x} \sup_{x' \in N} \frac{v(x') - v(x) - p.(x' - x)}{\|x' - x\|} \leq 0 \right\} \tag{5.8}$$

$$D^-v(x) = \left\{ q \in \Re^n \; : \; \lim_{x' \to x} \inf_{x' \in N} \frac{v(x') - v(x) - q.(x' - x)}{\|x' - x\|} \geq 0 \right\} \tag{5.9}$$

$$\tag{5.10}$$

**Remark 5.2.1** *If both $D^+v(x)$ and $D^-v(x)$ are nonempty at some $x$, then $D^+v(x) = D^-v(x)$ and $v$ is differentiable at $x$. We now have the following definitions of viscosity solutions.*

**Definition 5.2.2** *A continuous function $v$ is a viscosity solution of HJBE (5.7) if it is both a viscosity subsolution and supersolution, i.e., it satisfies respectively the following conditions:*

$$\mathcal{H}(x,p) \leq 0; \;\; \forall x \in N, \; \forall p \in D^+v(x) \tag{5.11}$$

$$\mathcal{H}(x,q) \geq 0; \;\; \forall x \in N, \; \forall q \in D^-v(x) \tag{5.12}$$

*respectively.*

An alternative definition of viscosity subsolutions and supersolutions is given in terms of test functions as follows.

**Definition 5.2.3** *A continuous function $v$ is a viscosity subsolution of HJBE (5.7) if for any $\varphi \in C^1$,*

$$\mathcal{H}(x, D\varphi(x)) \leq 0$$

*at any local maximum point $x$ of $v - \varphi$. Similarly, $v$ is a viscosity supersolution if for any $\varphi \in C^1$,*

$$\mathcal{H}(x, D\varphi(x)) \geq 0$$

*at any local minimum point $x$ of $v - \varphi$.*

Finally, for the theory of viscosity solutions to be meaningful, it should be consistent with the notion of classical solutions. Thus, we have the following relationship between viscosity solutions and classical solutions [16].

**Proposition 5.2.1** *(a) If $v \in C(N)$ is a viscosity solution of (5.7), then*

$$\mathcal{H}(x, D_x v) = 0$$

*at any point $x \in N$ where $v$ is differentiable; (b)if $v$ is locally Lipschitz continuous and it is a viscosity solution of (5.7), then*

$$\mathcal{H}(x, D_x v) = 0$$

*almost everywhere in $N$.*

Lastly, the following proposition guarantees uniqueness of viscosity solutions [37, 40].

**Proposition 5.2.2** *Suppose $\mathcal{H}(x,p)$ satisfies the following Lipschitz conditions:*

$$\begin{aligned} |\mathcal{H}(x,p) - \mathcal{H}(x,q)| &\leq k\|p - q\| \\ |\mathcal{H}(x,p) - \mathcal{H}(x',p)| &\leq k\|x - x'\|(1 + \|p\|) \end{aligned}$$

*for some $k \geq 0$, $x, x', p, q \in \Re^n$, then there exists atmost one viscosity solution to the HJBE (5.7).*

In this chapter, we shall consider smooth or classical solutions of the HJBE. We shall develop a procedure for deriving elementary smooth solutions for the class of nonlinear systems represented by $\Sigma$. However, we would also like to extend such solutions, in the viscosity sense, to the case when the control set is not an open subset of $\Re^k$. Thus, we shall be mainly concerned with cost functions of the form $L(x,u) = \frac{1}{2}\|z\|^2 = \frac{1}{2}[h^T(x)h(x) + u^T u]$. If the control function $u : \Re \to \mathcal{U}$ is desired to be a state-feedback of the form:

$$u(t) = \alpha(x), \quad \alpha : N \to \mathcal{U}, \quad \alpha(0) = 0, \tag{5.13}$$

then if we assume that the disturbance signal $w(t)$ is an implulse function $w(t) = w_0\delta(t)$ with a random direction vector $w_0$ such that

$$\mathsf{E}\{w_0\} = 0, \quad \mathsf{E}\{w_0 w_0^T\} = I$$

where $\mathsf{E}$ is the mathematical expectation operator, then

$$\|\Sigma_{zw}\|_2^2 = 2\mathsf{E}\{J_d\} = \mathsf{E}\left\{\int_0^\infty \|z\|^2 dt\right\},$$

and it can be shown that the optimal feedback control that minimizes the above *2-norm* is given by:

$$u(t) = -g_2^T(x)V_x(x) \tag{5.14}$$

where $V \geq 0$ satisfies the following HJBE (or inequality):

$$V_x(x)f(x) - \frac{1}{2}V_x(x)g_2(x)g_2^T(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) \leq 0, \quad V(0) = 0 \;\; x \in N. \tag{5.15}$$

We shall also be interested in the following dual HJBE associated with some representation and factorization problems for the system $\Sigma$ [11, 101]:

$$W_x(x)f(x) + \frac{1}{2}W_x(x)g_2(x)g_2^T(x)W_x^T(x) - \frac{1}{2}h^T(x)h(x) \leq 0; \;\; W(0) = 0. \tag{5.16}$$

**Remark 5.2.2** *For $h(.) = 0$, the solution $W$ of (5.16) is the controllability grammian of the nonlinear system $\Sigma$.*

In the next few lines, we review some of the results from chapter 2 for the HJBE (5.15). In this regard, let $\mathcal{Q}(x) = -g_2(x)g_2^T(x)$ and write HJBE (5.15) as:

$$V_x(x)f(x) + \frac{1}{2}V_x(x)\mathcal{Q}(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) \leq 0; \;\; V(0) = 0 \; \forall x \in N. \tag{5.17}$$

Further, assume that $\mathcal{Q}(x)$ is invertible for all $x \in N$ or if it is not, then a generalized inverse exists for all $x \in N$. Then the following theorem is one of the main results in Chapter 2.

**Theorem 5.2.1** *Consider the HJBE (5.17) for the system $\Sigma$. Assume that a generalized inverse exists for $\mathcal{Q}(x)$ for all $x \in N \subset M$, and there exists a vector-valued function $\zeta : N \to TM$ such that*

$$\zeta^T(x)\mathcal{Q}^+(x)\zeta(x) - f^T(x)\mathcal{Q}^+(x)f(x) + h^T(x)h(x) = 0, \; \forall x \in N, \tag{5.18}$$

*then a solution of the HJBE in terms of $V_x$ is given by*

$$V_x(x) = -(f(x) \pm \zeta(x))^T\mathcal{Q}^+(x); \;\; x \in N. \tag{5.19}$$

*Moreover, $V$ can be obtained from (5.19) by carrying out the following line integral $\int_0^x V_\rho(\rho)d\rho$.*

**Remark 5.2.3** *Note however that, as in the linear case, not every solution of the Riccati equation [76] is a stabilizing solution, and in particular, the above gradient $V_x$ may not correspond to a scalar real-valued positive-definite function. Hence, it may not be a stabilizing solution to the optimal control problem of minimizing $J_d(.,.)$ with stability for the closed-loop system (5.1), (5.13). Thus, it is necessary to characterize the set of all stabilizing solutions to the HJBE. This is given by*

$$V^s = \{ V \mid \frac{\partial V_x(x)}{\partial x} = \frac{\partial^T V_x(x)}{\partial x} \geq 0,\ f(x) - g_2(x)g_2^T(x)V_x^T(x) \text{ is asymptotically stable } \forall x \in N\}. \quad (5.20)$$

*The above set defines the set of all smooth real-valued functions $V$ with $V_{xx}$ symmetric and positive-semidefinite. It is necessary for $V_x$ to be symmetric, otherwise, it will not correspond to a scalar real-valued function. This property will be referred to as the "curl" condition.*

The following corollary gives a parameterization of all solutions to the HJB inequality (5.17).

**Corollary 5.2.1** *Suppose instead there exists only $\zeta : N \to TM$ such that*

$$\zeta^T(x)\mathcal{Q}^+(x)\zeta(x) - f^T(x)\mathcal{Q}^+(x)f(x) + h^T(x)h(x) \leq 0; \quad \forall x \in N.$$

*Then (5.19) solves the HJB inequality (5.27) and the set of all solutions to the HJB inequality is given by*

$$\mathcal{V}_x^1 = \{V_x \mid \zeta^T(x)\mathcal{Q}^+(x)\zeta(x) - f^T(x)\mathcal{Q}^+(x)f(x) + h^T(x)h(x) \leq 0;\ x \in N\} \quad (5.21)$$

**Remark 5.2.4** *Clearly, the solution of the HJBE (5.16) can be similarly derived from above as*

$$W_x(x) = -(f(x) \pm \zeta(x))^T \mathcal{Q}^+(x),$$

*where $\zeta$ in this case satisfies*

$$\zeta^T(x)\mathcal{Q}^+(x)\zeta(x) - f^T(x)\mathcal{Q}^+(x)f(x) - h^T(x)h(x) \leq 0$$

*We can also derive the solution to the following Lyapunov-type equation (inequality) [101]:*

$$\frac{\partial Y}{\partial x}(x)f(x) + \frac{1}{2}h^T(x)h(x) \leq 0; \quad Y(0) = 0. \quad (5.22)$$

*If we assume a solution $Y_x(x) = -\frac{1}{2}(f(x) - \lambda(x))^T$, where $\lambda : M_1 \subset M \to M$, and substituting in the equation, then $\lambda$ must satisfy the inequality:*

$$\lambda^T(x)f(x) - f^T(x)f(x) + h^T(x)h(x) \leq 0, \quad \forall x \in M_1 \quad (5.23)$$

*together with the following additional constraints:*

$$Y_{xx} = Y_{xx}^T \Longrightarrow f_x^T(x) - \lambda_x^T(x) = f_x(x) - \lambda_x(x), \quad x \in M_1 \quad (5.24)$$

*and*

$$Y_{xx}(x) \geq 0 \Longrightarrow f_x^T(x) - \lambda_x^T(x) \leq 0, \quad \forall x \in M_1. \quad (5.25)$$

*Inequalities (5.23), (5.24), (5.25) will completely characterize the solution to the Lyapunov-type inequality (5.22) and provide necessary and sufficient conditions for its solvability for any output function $h(.)$. Hence, if there does not exist any $\lambda$ such that the conditions (5.23), (5.24) and (5.25) are satisfied, then the Lyapunov inequality (5.22) is not solvable for the output function $h(.)$. Note also that, the Lyapunov inequality is not solvable if the system $\dot{x}(t) = f(x)$ is not stable.*

**Remark 5.2.5** *Since the factorization (5.18) is not unique, and in fact is difficult to perform, in Chapter 2 we have also proposed considering the following second-order HJBE:*

$$V_{xx}(x)f(x) + f_x(x)V_x^T(x) + V_{xx}(x)\mathcal{Q}(x)V_x^T(x) + \frac{1}{2}(I_n \otimes V_x(x))\frac{\partial \mathcal{Q}(x)}{\partial x}V_x^T(x) + h_x(x)h(x) = 0 \tag{5.26}$$

*where*

$$f_x(x) = \frac{\partial f}{\partial x}(x) = [\frac{\partial f_1}{\partial x}, \ldots, \frac{\partial f_n}{\partial x}], \quad h_x(x) = \frac{\partial h}{\partial x}(x) = [\frac{\partial h_1}{\partial x}, \ldots, \frac{\partial h_m}{\partial x}].$$

*Thus, by constraining the Hessian matrix $V_{xx}$, to be symmetric, it was shown that $\zeta$ can be determined by solving a system of $\frac{n(n-1)}{2}$ PDEs in $\zeta$.*

Let us now consider the HJBE (5.6) under the constraint that the admissible control set is now $\bar{\mathcal{U}}$, a closed and bounded subset of $\Re^k$ defined by

$$\bar{\mathcal{U}} = \{u : \Re \to \Re^k, \ \ \|u\| \leq \rho\}.$$

Such problems have also been considered recently in [39] with a different criterion function. Clearly, the value function $v(x) = \inf_{u \in \bar{\mathcal{U}}} J_{\underline{d}}(x,u)$ for this optimal control problem is not differentiable at the boundaries of the set $\bar{\mathcal{U}}$, and hence the corresponding HJBE

$$D_x v(x)f(x) + \inf_{u \in \bar{\mathcal{U}}}\{L(x,u) + D_x v(x)(g_1(x)w + g_2(x)u\} = 0; \quad v(0) = 0 \tag{5.27}$$

will not have a classical solution. Any solution of (5.27) will have to be interpreted in the viscosity sense. If we now assume $L(x,u)$ to be quadratic as in the previous case, then for the period when the control $u$ is not saturated, there exists a classical solution of the HJBE (5.17) satisfying

$$V_x(x)f(x) - \frac{1}{2}V_x(x)g_2(x)g_2^T(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) \leq 0; \quad V(0) = 0, \ \ \|u\|^2 < \rho, \ \ x \in M \tag{5.28}$$

whose solution can be determined by the method outlined above for the unconstrained case with the control given by $u = -g_2^T(x)V_x(x)$. While for the period when the control is saturated, we have the following HJBE, which is actually a Lyapunov equation:

$$D_x v(x)f(x) + \frac{1}{2}h^T(x)h(x) = \frac{1}{2}\rho; \ \ \|u\|^2 = \rho, \ \ v(0) = 0. \tag{5.29}$$

A smooth solution for this Lyapunov equation can also be found. Let $v = V \in C^2(N)$ , and differentiate (5.29) wrt to $x$. Thus, we get

$$V_{xx}(x)f(x) + f_x(x)V_x^T(x) + h_x(x)h(x) = 0 \tag{5.30}$$

with $V_x$ satisfying (5.29) and $V(0) = 0$. Therefore, following the procedure as outlined in [2], a smooth solution for the above HJBE can be obtained. Finally, the combined solution of HJBEs (5.28) and (5.29) will be a viscosity solution for HJBE (5.27). If we denote the solution to HJBE (5.27) by $V_1$ and the solution to Lyapunov equation (5.29) by $V_2$, then the combined solution $v$ is given by

$$v = \begin{cases} V_1 & \text{if } \|u\|^2 < \rho \\ V_2 & \text{if } \|u\|^2 = \rho \end{cases}$$

Notice that, even though $V_1$ and $V_2$ are both smooth, $v$ is not smooth, and thus is a viscosity solution.

Our second aim in this paper is to extend the above methods of solution to the stochastic HJBE. We consider this in the next section.

## 5.3 The Stochastic HJBE

In this section, we discuss the stochastic HJBE which arises in the finite-horizon optimal control of time-invariant and/or optimal control of time-varying and stochastic systems. For this purpose, let us consider the following time-varying nonlinear stochastic system (or diffusion process) in the Ito-form defined on $M \subseteq \Re^n$:

$$\Sigma_t : \quad dx(t) = f(t,x)dt + g_1(t,x)dw(t) + g_2(t,x)u(t)dt; \quad x(t_0 = 0) = \hat{x}_0 \quad (5.31)$$

$$y(t) = h(t,x); \quad f(0) = 0, \quad h(0) = 0 \quad (5.32)$$

where the functions $f : \Re \times M \to \Re^n$, $g_2 : \Re \times M \to \Re^{n \times k}$, $g_1 : \Re \times M \to \Re^{n \times s}$, $h : \Re \times M \to \Re^m$ are smooth $C^\infty(M)$ functions, $w(t)$ is a Wiener process which is a Gaussian process with continuous (but nowhere differentiable) sample functions, with mean $\mathsf{E}\{w(t)\} = 0$ and covariance $\mathsf{E}\{w(t)w^T(s)\} = min(t,s)I$. We also make the following basic assumption for the existence and uniqueness of solutions to the stochastic differential equation.

**Assumption 5.3.1** *There exists a constant $K > 0$ such that*

*(a) (Lipschitz condition) for all $t \in [t_0, T]$, $x_1, x_2 \in M$,*

$$\|f(t,x_1) - f(t,x_2)\| + \|g_1(t,x_1) - g_1(t,x_2)\| + \|g_2(t,x_1) - g_2(t,x_2)\| \leq K\|x_1 - x_2\|$$

*(b) (Restriction on growth) for all $t \in [t_0, T]$ and $x \in \Re^n$,*

$$\|f(t,x)\|^2 + \|g_1(t,x)\|^2 + \|g_2(t,x)\|^2 \leq K^2(1 + \|x\|^2).$$

Now, the finite-horizon state-feedback stochastic optimal control problem for the above system involves the minimization of the following quadratic functional:

$$\min_{u \in \mathcal{U}} J_s(t,x,u) = \mathsf{E}\Big[\int_0^T L(t,x,u)dt\Big], \quad T > 0 \quad (5.33)$$

where $\mathsf{E}$ is the mathematical expectation operator and $L : \Re_+ \times M \times \mathcal{U} \to \Re$ is a suitable cost function, subject to the dynamics (5.31), (5.32) over some finite-time interval $[0, T]$ using state-feedback controls (possibly time-varying) of the form:

$$u(t) = \gamma(t, x); \quad \gamma(t, 0) = 0 \; \forall t \in [0, T]. \tag{5.34}$$

In this paper, we shall be mainly concerned with the case where

$$L(t, x, u) = \frac{1}{2} y^T(t) M(t, x) y(t) + \frac{1}{2} u^T(t) N(t, x) u(t)$$

is quadratic, and $M(t, x)$, $N(t, x)$ are suitable time-varying weighting matrices, and for simplicity, we shall let $M(t, x) = I$ and $N(t, x) = I$. Under this assumption, the above cost functional becomes

$$\min_{u \in \mathcal{U}} J_s(t, x, u) = \frac{1}{2} \mathsf{E}\Big[\int_0^T \|y(t)\|^2 + \|u(t)\|^2 dt\Big], \tag{5.35}$$

which is the square of the $\mathcal{H}_2$-norm of the system $\Sigma_t$ scaled by (1/2). Further, let $u^\star(t)$ denote the optimal feedback control for the above problem and $v(t, x)$ its value function, i.e., $v(t, x) = J_s^\star(t, x, u^\star(t)) = \inf_{u \in \mathcal{U}} J_s(t, x, u)$. If we assume $v(t, x) = V(t, x) \in C^2(\Re \times \mathcal{O})$, $\mathcal{O} \subset M$, then it can be shown [40, 117] that $V(t, x)$ satisfies the following SHJBE:

$$\inf_{u \in U} \{\mathcal{L}^u V(t, x) + L(t, x, u)\} = 0, \quad V(T, x) = 0 \; \forall (t, x) \in [0, T] \times \mathcal{O}, \tag{5.36}$$

where

$$\mathcal{L}^u V(t, x) = V_t(t, x) + V_x(t, x)[f(t, x) + g_2(t, x) u(t)] + \frac{1}{2} Tr\{g_1(t, x) g_1^T(t, x) V_{xx}(t, x)\}$$

is the infinitesimal generator of the process $V(t, x)$. It can further be shown that, the solution of the above stochastic optimal feedback control is given by

$$u^\star(t, x) = -g_2^T(t, x) V_x^T(t, x) \tag{5.37}$$

under which the above SHJBE takes the following form [6, 117]:

$$\begin{aligned} &V_t(t, x) + V_x(t, x) f(t, x) - \frac{1}{2} V_x(t, x) g_2(t, x) g_2^T(t, x) V_x^T(t, x) + \frac{1}{2} Tr\{g_1(t, x) g_1^T(t, x) V_{xx}(t, x)\} \\ &\quad + \frac{1}{2} h^T(t, x) h(t, x) = 0; \quad V(T, x) = 0 \; \forall x \in \mathcal{O}, \; t \in [0, T]. \end{aligned} \tag{5.38}$$

Solving the above stochastic HJBE (5.38) is the subject of discussion in this section. It should be noted however that, the above HJBE is significantly different from the equation (5.15), in the sense that, it is a second-order PDE. Therefore, its solution will be different. Our aim is to extend the procedure that we have developed in the previous sections to this case. In this regard, let us redefine the time parameter $t$ as $x_0$ and $\tilde{x} = [x_0, x_1, \ldots, x_n]^T$. Thus $[0, T] \times \mathcal{O} \subset \Re^{n+1}$ and the equivalent system $\tilde{\Sigma}_t$ takes on the following form:

$$\tilde{\Sigma}_t : \quad d\tilde{x}(t) = \tilde{f}(\tilde{x}) dx_0 + \tilde{g}_1(\tilde{x}) dw(x_0) + \tilde{g}_2(\tilde{x}) u(x_0) dx_0; \quad \tilde{x}(0) = \tilde{x}_0 \tag{5.39}$$

$$y(t) = \tilde{h}(\tilde{x}); \quad \tilde{f}(x_0, 0) = 0, \quad \tilde{h}(x_0, 0) = 0 \tag{5.40}$$

where

$$\tilde{f}(\tilde{x}) = \begin{bmatrix} 1 \\ f(\tilde{x}) \end{bmatrix}, \quad \tilde{g}_1(\tilde{x}) = \begin{bmatrix} \mathbf{0}_{1\times r} \\ g_1(\tilde{x}) \end{bmatrix}, \quad \bar{g}_2(\tilde{x}) = \begin{bmatrix} \mathbf{0}_{1\times k} \\ g_2(\tilde{x}) \end{bmatrix}, \quad \tilde{h}(\tilde{x}) = h(\tilde{x}).$$

Thus, the SHJBE (5.38) can be represented as

$$\tilde{V}_{\tilde{x}}(\tilde{x})\tilde{f}(\tilde{x}) + \frac{1}{2}\tilde{V}_{\tilde{x}}(\tilde{x})\tilde{\mathcal{Q}}(\tilde{x})\tilde{V}_x^T(\tilde{x}) + \frac{1}{2}Tr\left\{\tilde{\Theta}(\tilde{x})V_{\tilde{x}\tilde{x}}(\tilde{x})\right\} + \frac{1}{2}\tilde{h}^T(\tilde{x})\tilde{h}(\tilde{x}) = 0;\ \tilde{V}(\tilde{x}(T)) = 0\ \forall \tilde{x} \in \tilde{O} \subset \Re \times \mathcal{O} \tag{5.41}$$

where $\tilde{\mathcal{Q}}(\tilde{x}) = -\tilde{g}_2(\tilde{x})\tilde{g}_2^T(\tilde{x})$, $\tilde{\Theta}(\tilde{x}) = \tilde{g}_1(\tilde{x})\tilde{g}_1^T(\tilde{x})$, $\tilde{V}_{\tilde{x}}(\tilde{x}) = (\tilde{V}_{x_0}, \tilde{V}_{x_1}, \ldots, \tilde{V}_{x_n})$ and $\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x}) = \frac{\partial^2 \tilde{V}}{\partial \tilde{x}^2} = [\frac{\partial \tilde{V}_{x_0}}{\partial \tilde{x}}, \ldots, \frac{\partial \tilde{V}_{x_n}}{\partial \tilde{x}}]$. As in the previous section, define a solution to the SHJBE (5.41) as

$$\tilde{V}_{\tilde{x}}(x) = -(\tilde{f}(\tilde{x}) \pm \tilde{\zeta}(\tilde{x}))^T \tilde{\mathcal{Q}}^+(\tilde{x}) \tag{5.42}$$

where $\tilde{\mathcal{Q}}^+(\tilde{x})$ is the generalized inverse of $\tilde{\mathcal{Q}}(\tilde{x})$, and

$$\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x}) = -[\tilde{f}_{\tilde{x}}(\tilde{x}) \pm \tilde{\zeta}_{\tilde{x}}(\tilde{x})]^T \tilde{\mathcal{Q}}^+(\tilde{x}) - [I_n \otimes (\tilde{f}(\tilde{x}) \pm \tilde{\zeta}(\tilde{x}))^T]\frac{\partial \tilde{\mathcal{Q}}^+(\tilde{x})}{\partial \tilde{x}}$$

for some vector-valued function $\tilde{\zeta} : \tilde{O} \to \tilde{O}$, and

$$\tilde{f}_{\tilde{x}}(\tilde{x}) = \frac{\partial \tilde{f}}{\partial \tilde{x}}(\tilde{x}) = [\frac{\partial \tilde{f}_1}{\partial \tilde{x}}, \ldots, \frac{\partial \tilde{f}_n}{\partial \tilde{x}}], \quad \tilde{\zeta}_{\tilde{x}}(\tilde{x}) = \frac{\partial \tilde{\zeta}}{\partial \tilde{x}}(\tilde{x}) = [\frac{\partial \tilde{\zeta}_1}{\partial \tilde{x}}, \ldots, \frac{\partial \tilde{\zeta}_n}{\partial \tilde{x}}].$$

Then substituting the above expression in the SHJBE (5.41) we get

$$\begin{aligned} &\tilde{\zeta}^T(\tilde{x})\tilde{\mathcal{Q}}^+(\tilde{x})\tilde{\zeta}(\tilde{x}) - \tilde{f}^T(\tilde{x})\tilde{\mathcal{Q}}^+(\tilde{x})\tilde{f}(\tilde{x}) - Tr\Big\{\tilde{\Theta}(\tilde{x})[(\tilde{f}_{\tilde{x}}(\tilde{x}) \pm \tilde{\zeta}_{\tilde{x}}(\tilde{x}))^T]\tilde{\mathcal{Q}}^+(\tilde{x}) + \\ &\tilde{\Theta}(\tilde{x})[I_n \otimes (\tilde{f}(\tilde{x}) \pm \tilde{\zeta}(\tilde{x}))^T]\frac{\partial \tilde{\mathcal{Q}}^+(\tilde{x})}{\partial \tilde{x}}\Big\} + \tilde{h}^T(\tilde{x})\tilde{h}(\tilde{x}) = 0. \end{aligned} \tag{5.43}$$

Furthermore, the symmetry of $V_{\tilde{x}\tilde{x}}$ will imply that

$$[\tilde{f}_{\tilde{x}}(\tilde{x}) \pm \tilde{\zeta}_{\tilde{x}}(\tilde{x})]^T \tilde{\mathcal{Q}}^+(\tilde{x}) + [I_n \otimes (\tilde{f}(\tilde{x}) \pm \tilde{\zeta}(\tilde{x}))^T]\frac{\partial \tilde{\mathcal{Q}}^+(\tilde{x})}{\partial \tilde{x}} = \tilde{\mathcal{Q}}^+(\tilde{x})[\tilde{f}_{\tilde{x}}(\tilde{x}) \pm \tilde{\zeta}_{\tilde{x}}(\tilde{x})] + \frac{\partial^T \tilde{\mathcal{Q}}^+(\tilde{x})}{\partial \tilde{x}}[I_n \otimes (\tilde{f}(\tilde{x}) \pm \tilde{\zeta}(\tilde{x}))] \tag{5.44}$$

and lastly, $V_{xx} \geq 0$ will imply

$$[\tilde{f}_{\tilde{x}}(\tilde{x}) \pm \tilde{\zeta}_{\tilde{x}}(\tilde{x})]^T \tilde{\mathcal{Q}}^+(\tilde{x}) + [I_n \otimes (\tilde{f}(\tilde{x}) \pm \tilde{\zeta}(\tilde{x}))^T]\frac{\partial \tilde{\mathcal{Q}}^+(\tilde{x})}{\partial \tilde{x}} \leq 0 \tag{5.45}$$

The inequalities (5.43), (5.44), and (5.45) completely characterize the solutions to the SHJBE. Notice also that, there is no loss of generality if we assume the RHS of (5.43) to be $\leq 0$, in which case, we have the SHJB inequality. Moreover, if a positive-semidefinite solution of the SHJBE cannot be obtained by solving the inequalities (5.43)-(5.45), then the SHJBE is not solvable by the above method for the chosen output function $\tilde{h}(.)$, and another output function must be selected. We summarize the above result in the following theorem.

**Theorem 5.3.1** *Consider the nonlinear stochastic system $\Sigma_t$ and the HJBE (5.38) corresponding to the optimal control of the system that minimizes the cost function $J_s$. Then there exists a real $C^1$ solution of the HJBE in $\tilde{\mathcal{O}}$ if there exists a real vector-valued function $\tilde{\zeta}$ that satisfies the inequalities (5.43)-(5.45).*

Let us now apply the above results to the case of the linear stochastic system:

$$\Sigma_{st}^{l} \; : \; dx(t) \;=\; A(t)x(t)dt + B_1(t)dw(t) + B_2(t)u(t)dt; \;\; x(0) = \hat{x}_0 \tag{5.46}$$

$$y(t) \;=\; H(t)x(t) \tag{5.47}$$

where $A(t) \in \Re^{n\times n}$, $B_1(t) \in \Re^{n\times s}$, $B_2(t) \in \Re^{n\times k}$. Also, let

$$L(t,x,u) = \frac{1}{2}[y^T(t)y(t) + u^T(t)u(t)] = \frac{1}{2}[x^T(t)H^T(t)H(t)x(t) + u^T(t)u(t)].$$

Then,

$$\mathcal{L}^u V(t,x) = V_t(t,x) + V_x(t,x)[A(t)x(t) + B_2(t)u(t)] + \frac{1}{2}Tr\{B_1(t)B_1^T(t)V_{xx}(t,x)\}.$$

and the SHJBE (5.38) takes the following form:

$$V_t(t,x) + V_x(t,x)A(t)x(t) - \frac{1}{2}V_x(t,x)B_2(t)B_2^T(t)V_x^T(t,x) + \frac{1}{2}Tr\{B_1(t)B_1^T(t)V_{xx}(t,x)\} + \frac{1}{2}x^T(t)H^T(t)H(t)x(t) = 0; \;\; V(T,x) = 0, \; x \in \mathcal{O} \subset \Re^n, \; t \in [0,T]. \tag{5.48}$$

Now, rewriting the above SHJBE in the form (5.41) we get

$$\tilde{V}_{\tilde{x}}(\tilde{x})\tilde{A}(x_0)\tilde{x} + \frac{1}{2}\tilde{V}_{\tilde{x}}(\tilde{x})\tilde{\mathcal{Q}}(x_0)\tilde{V}_{\tilde{x}}^T(\tilde{x}) + \frac{1}{2}Tr\{\tilde{\Theta}(x_0)\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x})\} + \frac{1}{2}\tilde{x}^T\tilde{H}^T(x_0)\tilde{H}(x_0)\tilde{x} = 0,$$
$$\tilde{V}(\tilde{x}(T)) = 0, \; \tilde{x} \in \tilde{O} \subset \Re^{n+1}, \tag{5.49}$$

where the new variables $\tilde{A}(.)$, $\tilde{B}_1(.)$, $\tilde{B}_2(.)$, $\tilde{H}(.)$ have equivalent definitions, and

$$\tilde{\mathcal{Q}}(x_0) = -\tilde{B}_2(x_0)\tilde{B}_2^T(x_0); \;\; \tilde{\Theta}(x_0) = \tilde{B}_1(x_0)\tilde{B}_1^T(x_0).$$

Then substituting for $\tilde{V}_{\tilde{x}}$ as in the above, we get

$$\tilde{V}_{\tilde{x}}(\tilde{x}) = -\tilde{x}^T(\tilde{A}^T(x_0) \pm \tilde{\Gamma}^T(x_0))\tilde{\mathcal{Q}}^+(x_0)$$

and

$$\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x}) = -[\tilde{A}^T(x_0) \pm \tilde{\Gamma}^T(x_0)]\tilde{\mathcal{Q}}^+(x_0) - (I_n \otimes \tilde{x}^T)[\tilde{A}_{\tilde{x}}^T(x_0) \pm \tilde{\Gamma}_{\tilde{x}}^T(x_0)]\tilde{\mathcal{Q}}^+(x_0) - \Big\{I_n \otimes [\tilde{x}^T(\tilde{A}(x_0) \pm \tilde{\Gamma}(x_0))]\Big\}\tilde{\mathcal{Q}}_{\tilde{x}}^+(x_0)$$

for some $n \times n$ matrix function $\tilde{\Gamma}(x_0)$ which satisfies:

$$\tilde{x}^T\tilde{\Gamma}^T(x_0)\tilde{\mathcal{Q}}^+(x_0)\tilde{\Gamma}(x_0)\tilde{x} - \tilde{x}^T\tilde{A}^T(x_0)\tilde{\mathcal{Q}}^+(x_0)\tilde{A}(x_0)\tilde{x} + Tr\{\Theta(x_0)\tilde{V}_{\tilde{x}\tilde{x}}\} + \tilde{x}^T\tilde{H}^T(x_0)\tilde{H}(x_0)\tilde{x} = 0, \; \tilde{x} \in \tilde{O}. \tag{5.50}$$

Equating terms of corresponding powers of $\tilde{x}$, and noting that the term in the *"Trace"* is linear in $\tilde{x}$, and further that $Tr(A+B) = Tr(A) + Tr(B)$, $Tr(AB) = [vec(A^T)]^T vec(B)$ for any $n \times n$ matrices [47], we get

$$\tilde{\Gamma}^T(x_0)\tilde{\mathcal{Q}}^+(x_0)\tilde{\Gamma}(x_0) - \tilde{A}^T(x_0)\tilde{\mathcal{Q}}^+(x_0)\tilde{A}(x_0) + \tilde{H}^T(x_0)\tilde{H}(x_0) = 0 \tag{5.51}$$

and

$$Tr\Big\{\tilde{\Theta}(x_0)[\tilde{A}^T(x_0) \pm \tilde{\Gamma}^T(x_0)]\tilde{\mathcal{Q}}^+(x_0)\Big\} = 0 \tag{5.52}$$

$$Tr\Big\{(I_n \otimes \tilde{x}^T)[\tilde{A}^T_{\tilde{x}}(x_0) \pm \tilde{\Gamma}^T_{\tilde{x}}(x_0)]\tilde{\mathcal{Q}}^+(x_0)\tilde{\Theta}(x_0) + \{I_n \otimes [\tilde{x}^T(\tilde{A}(x_0) \pm \tilde{\Gamma}(x_0))]\}\tilde{\mathcal{Q}}^+_{\tilde{x}}(x_0)\tilde{\Theta})(x_0)\Big\} = 0,$$
$$\forall \tilde{x} \in \tilde{O}, \tag{5.53}$$

which can be simplified as follows. Define $\tilde{P}(x_0) = [\tilde{A}(x_0) \pm \tilde{\Gamma}(x_0)]$. Then (5.52), (5.53) will imply

$$\mathbf{vec}[\tilde{\mathcal{Q}}^+(x_0)\tilde{\Theta}(x_0)]\mathbf{vec}\,(\tilde{P}^T(x_0)) = 0 \tag{5.54}$$

$$Tr\Big\{(I_n \otimes \tilde{x}^T)[\tilde{P}^T_{\tilde{x}}(x_0)\tilde{\mathcal{Q}}^+(x_0) + (I_n \otimes \tilde{x}^T)(I_n \otimes \tilde{P}(x_0))\tilde{\mathcal{Q}}^+_{\tilde{x}}(x_0)]\tilde{\Theta}(x_0)\Big\} = 0 \;\; \forall \tilde{x} \in \tilde{O} \tag{5.55}$$

The last equation further implies

$$Tr\Big\{(I_n \otimes \tilde{x}^T)[\tilde{P}^T_{\tilde{x}}(x_0)\tilde{\mathcal{Q}}^+(x_0) + (I_n \otimes \tilde{P}(x_0))\tilde{\mathcal{Q}}^+_{\tilde{x}}(x_0)]\tilde{\Theta}(x_0)\Big\} = 0 \;\; \forall \tilde{x} \in \tilde{O},$$

or

$$[\mathbf{vec}(I_n \otimes \tilde{x})^T]^T\mathbf{vec}\Big\{[\tilde{P}^T_{\tilde{x}}(x_0)\tilde{\mathcal{Q}}^+(x_0) + (I_n \otimes \tilde{P}(x_0))\tilde{\mathcal{Q}}^+_{\tilde{x}}(x_0)]\tilde{\Theta}(x_0)\Big\} = 0 \;\; \forall \tilde{x} \in \tilde{O}, \tag{5.56}$$

which further implies

$$\mathbf{vec}\Big\{[\tilde{P}^T_{\tilde{x}}(x_0)\tilde{\mathcal{Q}}^+(x_0) + (I_n \otimes \tilde{P}(x_0))\tilde{\mathcal{Q}}^+_{\tilde{x}}(x_0)]\tilde{\Theta}(x_0)\Big\} = 0, \tag{5.57}$$

or

$$[\tilde{P}^T_{\tilde{x}}(x_0)\tilde{\mathcal{Q}}^+(x_0) + (I_n \otimes \tilde{P}(x_0))\tilde{\mathcal{Q}}^+_{\tilde{x}}(x_0)]\tilde{\Theta}(x_0) = 0. \tag{5.58}$$

The equations (5.51), (5.54), and (5.58) are the linear equivalent of the SHJBE (5.41). Finally, the remaining symmetry condition of $\tilde{V}_{\tilde{x}\tilde{x}}$ will imply

$$\tilde{P}^T(x_0)\tilde{\mathcal{Q}}^+(x_0) = \tilde{\mathcal{Q}}^+(x_0)\tilde{P}(x_0), \tag{5.59}$$

$$\tilde{P}^T_{\tilde{x}}(x_0)\tilde{\mathcal{Q}}^+(x_0) + (I_n \otimes \tilde{P}(x_0))\tilde{\mathcal{Q}}^+_{\tilde{x}}(x_0) = \tilde{\mathcal{Q}}^+(x_0)\tilde{P}(x_0) + \tilde{\mathcal{Q}}^+_{\tilde{x}}(x_0)(I_n \otimes \tilde{P}^T(x_0)). \tag{5.60}$$

In addition, the positive-semidefiniteness condition of $\tilde{V}_{\tilde{x}\tilde{x}}$ can be included in the equations (5.54), (5.58) by replacing them with the $\geq 0$ inequality. Thus, equations (5.51), (5.54), (5.58) and (5.59), (5.60) will completely characterize the solution of the linear SHJBE (5.49). A computational procedure for solving this system of equations can be developed. Moreover, when the linear system $\Sigma_{st}$ is not time-varying, the above system of inequalities simplifies considerably. We summarize the above result in the following corollary.

**Corollary 5.3.1** *Consider the SHJBE (5.49) corresponding to the linear system* $\Sigma^l_{st}$. *If there exists a matrix function* $\tilde{\Gamma} : \Re \to \Re^{n\times n}$ *that satisfies the inequalities (5.51), (5.54), (5.58)-(5.60), then there exists a real-symmetric solution to the SHJBE for the linear system* $\Sigma^{st}$.

We now illustrate the procedure that we have developed for the nonlinear case with an example.

**Example 5.3.1** *Consider the following example from [34]:*

$$\begin{aligned} dx_1(t) &= x_2(t)dt + \frac{1}{2}x_1^2(t)dw(t) \\ dx_2(t) &= u(t)dt \\ y(t) &= x_2(t) \end{aligned}$$

*Therefore,*

$$f(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} \frac{1}{2}x_1^2 \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h(x) = x_2$$

*Clearly,* $f(0) = 0$ *and* $h(0) = 0$*. Further,*

$$\mathcal{Q}(x) = -g_2(x)g_2^T(x) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \mathcal{Q}^+(x), \quad \Theta(x) = \begin{bmatrix} \frac{1}{4}x_1^4 & 0 \\ 0 & 0 \end{bmatrix}.$$

*Now, substituting the above matrices in the inequalities (5.43)-(5.45), we get the following inequalities:*

$$-\zeta_2^2 + Tr\left\{ \begin{pmatrix} \frac{1}{4}x_1^4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta_{1,x_1}(x) & \zeta_{1,x_2}(x)+1 \\ \zeta_{2,x_1}(x) & \zeta_{2,x_2}(x) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \right\} + x_2^2 = 0,$$

$$\begin{pmatrix} \zeta_{1,x_1}(x) & \zeta_{1,x_2}(x)+1 \\ \zeta_{2,x_1}(x) & \zeta_{2,x_2}(x) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \zeta_{1,x_1}(x) & \zeta_{2,x_1}(x) \\ \zeta_{1,x_2}(x)+1 & \zeta_{2,x_2}(x) \end{pmatrix},$$

$$\begin{pmatrix} \zeta_{1,x_1}(x) & \zeta_{1,x_2}(x)+1 \\ \zeta_{2,x_1}(x) & \zeta_{2,x_2}(x) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \leq 0$$

*Upon simplifying, we get*

$$\begin{aligned} -\zeta_2^2(x) + x_2^2 &= 0 \\ \zeta_{1,x_2}(x) &= -1 \\ \zeta_{2,x_2}(x) &\geq 0 \end{aligned}$$

*These give* $\zeta_2(x) = x_2$*. Further, we can integrate* $\zeta_{1,x_2}(x) = -1$ *to get* $\zeta_1(x) = -x_2 + \psi(x_1)$*, and without any loss of generality, we can assume* $\psi(x_1) = 0$*, which gives* $\zeta_1(x) = -x_2$*. Now substituting the above values in the expression for* $V_x$*, we get*

$$V_x = -\left[ \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \pm \begin{pmatrix} -x_2 \\ \pm x_2 \end{pmatrix} \right]^T \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

*and taking the '+' sign throughout, we get* $V_x(x) = (0 \;\; x_2)$*. Finally, integrating* $V_x$ *wrt* $x$*, we get* $V(x) = \frac{1}{2}x_2^2$*, which is positive-semidefinite.*

On the other hand, when the value function is not necessarily smooth, a classical solution will not exist for all $x \in N$. Thus, we can only talk about solutions in the viscosity sense which are similarly defined as in the deterministic case [16]. For this purpose, define in this case the generalized Hamiltonian function of the system by

$$G(t,x,u,p,P) = L(t,x,u) + p^T(f(t,x) + g_2(t,x)u) + \frac{1}{2}Tr\{g_1(t,x)g_1^T(t,x)P\}$$
$$\forall (t,x,u,p,P) \in [0,T] \times N \times \mathcal{U} \times \Re^n \times \Re^{n\times n} \tag{5.61}$$

for some vector $p \in \Re^n$ and $n \times n$ symmetric matrix $P$. Then the SHJBE corresponding to the minimization problem (5.33) is represented by

$$v_t + \inf_{u\in\mathcal{U}} G(t,u,v_x,v_{xx}) = 0, \;\; v(T,x) = 0, \;\; \forall (t,x) \in [0,T) \times N \tag{5.62}$$

Further, we have the following definition [40, 117]:

**Definition 5.3.1** *A function $v \in C([0,T] \times N)$ is called a viscosity subsolution of (5.62) if $v(T,x) = 0$, $\forall x \in N$, and for any function $\varphi \in C^{1,2}([0,T],N)$, whenever $v - \varphi$ attains a local maximum at $(t,x) \in [0,T) \times N$, then*

$$\varphi_t(t,x) + \inf_{u\in U} G(t,x,u,\varphi(t,x),\varphi_{xx}(t,x)) \leq 0.$$

*Similarly, $v \in C([0,T], \times M)$ is called a viscosity supersolution of (5.62) if $v(T,x) = 0, \forall x \in M$ and for any function $\varphi \in C^{1,2}([0,T],N)$,*

$$\varphi_t(t,x) + \inf_{u\in U} G(t,x,u,\varphi(t,x),\varphi_{xx}(t,x)) \geq 0$$

*at any local minimum $(t,x) \in [0,T) \times N$ of $v - \varphi$, where $C^{1,2}([0,T],N)$ is the set of functions $\Phi(t,x)$ whose first and second order partial derivatives, wrt their arguments, are continuous on $[0,T] \times N$. Furthermore, $v$ is a viscosity solution of (5.62) if it is both a viscosity subsolution and supersolution.*

Similarly, an equivalent definition of viscosity solutions for the second-order SHJBE (5.62) can be given in terms of the *second (parabolic) superdifferential and subdifferentials* of the value-function $v(t,x)$ [40, 117].

Furthermore, in contrast with the deterministic case, if the second-order stochastic HJBE is of uniformly parabolic type, i.e., for all $(t,x) \in [0,T] \times N$, and $\xi \in \Re^n$, there exists a constant $c > 0$, such that,

$$\sum_{i,j=1}^{n} \Theta_{ij}(t,x)\xi_i\xi_j \geq c\|\xi\|,$$

then the corresponding SHJBE indeed has a unique classical solution [40]. As in the deterministic case, viscosity solutions in the stochastic case typically arise when we have input contraints. Assuming for an instance that the set of admissible controls for the optimal control problem (5.33), (5.31), (5.32) belong to $\bar{\mathcal{U}}$, i.e., $u : \Re_+ \to \bar{\mathcal{U}}$. Then for the period when the control is not saturated, the optimal control is determined by the solution of the SHJBE (5.38). Assume this SHJBE has a solution $V_{1s}$. While, in the period when the control is saturated, the corresponding SHJBE is given by

$$D_t v(t,x) + D_x v(t,x) f(t,x) + \frac{1}{2} Tr\{g_1(t,x) g_1^T(t,x) D_{xx} v(t,x)\} + \frac{1}{2} h^T(t,x) h(t,x) = \frac{1}{2}\rho;$$
$$v(T,x) = 0 \; \forall x \in \mathcal{O}, \; t \in [0,T], \tag{5.63}$$

which is really a Lyapunov-type equation. The above equation can also be represented in the form of (5.41) as

$$D_{\tilde{x}} \tilde{v}(\tilde{x}) \tilde{f}(\tilde{x}) + \frac{1}{2} Tr\{\tilde{\Theta}(\tilde{x}) D_{\tilde{x}\tilde{x}} \tilde{v}(\tilde{x})\} + \frac{1}{2} \tilde{h}^T(\tilde{x}) \tilde{h}(\tilde{x}) = \frac{1}{2}\rho; \;\; \tilde{v}(\tilde{x}(T)) = 0 \;\; \forall \tilde{x} \in \tilde{\mathcal{O}}, \tag{5.64}$$

where all the new variables have the same meaning as before. A smooth solution to the above stochastic Lyapunov equation can be obtained using the parameterization as in (5.42) or using higher-order equations as in section 5.3.1. If we consider the latter approach, then assuming a smooth solution $\tilde{v} = \tilde{V} \in C^2$ and differentiating (5.64) wrt $\tilde{x}$, we get

$$\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x}) \tilde{f}(\tilde{x}) + \tilde{f}_{\tilde{x}}(\tilde{x}) \tilde{V}_{\tilde{x}}^T(\tilde{x}) + \frac{1}{2}[\mathbf{vec}(\tilde{\Theta}(\tilde{x}))_{\tilde{x}}^T \mathbf{vec}(\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x})) + \mathbf{vec}(\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x}))_{\tilde{x}}^T \mathbf{vec}(\tilde{\Theta}(\tilde{x}))] +$$
$$\tilde{h}_{\tilde{x}}(\tilde{x}) \tilde{h}(\tilde{x}) = 0, \;\; \tilde{V}(\tilde{x}(T)) = 0, \; \tilde{V}_{\tilde{x}}(0) = 0 \tag{5.65}$$

The above second-order Lyapunov-type equation can further be parameterized as in SHJBE (5.68) and then solved to get $\tilde{\xi}$ which corresponds to a smooth symmetric solution. Suppose the solution to the above equation is denoted by $V_{2s}$. Then, the combined solution to the optimal control problem (5.33), (5.31), (5.32) with $u \in \bar{\mathcal{U}}$ is characterized by the solution

$$v_s(x) = \begin{cases} V_{1s} & \text{if } \|u\|^2 < \rho \\ V_{2s} & \text{if } \|u\|^2 = \rho \end{cases}$$

### 5.3.1 An Approach for Solving the Stochastic HJBE Using Higher-Order Equations

As in the case of the deterministic HJBE, an alternate solution to the SHJBE can also be developed along the same lines as the deterministic case using higher-order equations. Accordingly, differentiating (5.41) wrt $\tilde{x}$, we get

$$\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x}) \tilde{f}(\tilde{x}) + \tilde{f}_{\tilde{x}}(\tilde{x}) \tilde{V}_{\tilde{x}}^T(\tilde{x}) + \tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x}) \tilde{\mathcal{Q}}(\tilde{x}) \tilde{V}_{\tilde{x}}^T(\tilde{x}) + \frac{1}{2}(I_n \otimes \tilde{V}_{\tilde{x}}(\tilde{x})) \frac{\partial \tilde{\mathcal{Q}}(\tilde{x})}{\partial \tilde{x}} \tilde{V}_{\tilde{x}}^T(\tilde{x}) + \frac{1}{2}[\mathbf{vec}(\tilde{\Theta}(\tilde{x}))_{\tilde{x}}^T \times$$
$$\mathbf{vec}(\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x})) + \mathbf{vec}(\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x}))_{\tilde{x}}^T \mathbf{vec}(\tilde{\Theta}(\tilde{x}))] + \tilde{h}_{\tilde{x}}(\tilde{x}) \tilde{h}(\tilde{x}) = 0; \;\; \tilde{V}(\tilde{x}(T)) = 0, \; \tilde{V}_{\tilde{x}}(0) = 0 \tag{5.66}$$

which is now a third-order equation. However, assuming $\tilde{V}_{\tilde{x}}^T = \tilde{\xi}$ for some vector-valued function $\xi : \Re \times \mathcal{O} \to \tilde{O}$, we can reduce the above equation back to a second-order equation. Futher, since $\tilde{V}_{\tilde{x}\tilde{x}} = \tilde{\xi}_{\tilde{x}}$, and must be symmetric, then by decomposing it into its components:

$$\tilde{V}_{\tilde{x}\tilde{x}}(\tilde{x}) = \tilde{V}_{\tilde{x}\tilde{x}}^s(\tilde{x}) + \tilde{V}_{\tilde{x}\tilde{x}}^{sk}(\tilde{x}) \triangleq \frac{1}{2}[\tilde{\xi}_{\tilde{x}}(\tilde{x}) + \tilde{\xi}_{\tilde{x}}^T(\tilde{x})] + \frac{1}{2}[\tilde{\xi}_{\tilde{x}}(\tilde{x}) - \tilde{\xi}_{\tilde{x}}^T(\tilde{x})],$$

we can take the symmetric component and substitute it for $\tilde{V}_{\tilde{x}\tilde{x}}$. Thus, substituting $V^s_{\tilde{x}\tilde{x}}$ in (5.66 ) we get

$$\frac{1}{2}[\tilde{\xi}_{\tilde{x}}(\tilde{x}) + \tilde{\xi}^T_{\tilde{x}}(\tilde{x})]\tilde{f}(\tilde{x}) + \tilde{f}_{\tilde{x}}(\tilde{x})\tilde{\xi}(\tilde{x}) + \frac{1}{2}[\tilde{\xi}_{\tilde{x}}(\tilde{x}) + \tilde{\xi}^T_{\tilde{x}}(\tilde{x})]\tilde{\mathcal{Q}}(\tilde{x})\tilde{\xi}(\tilde{x}) + \frac{1}{2}(I_n \otimes \tilde{\xi}^T(\tilde{x}))\frac{\partial \tilde{\mathcal{Q}}(\tilde{x})}{\partial \tilde{x}}\tilde{\xi}(\tilde{x}) +$$
$$\frac{1}{4}\mathbf{vec}(\tilde{\Theta}(\tilde{x}))^T_{\tilde{x}}\mathbf{vec}[\tilde{\xi}_{\tilde{x}}(\tilde{x}) + \tilde{\xi}^T_{\tilde{x}}(\tilde{x})] + \frac{1}{4}\mathbf{vec}[\tilde{\xi}_{\tilde{x}}(\tilde{x}) + \tilde{\xi}^T_{\tilde{x}}(\tilde{x})]^T_{\tilde{x}}\mathbf{vec}(\tilde{\Theta}(\tilde{x})) + \tilde{h}_{\tilde{x}}(\tilde{x})\tilde{h}(\tilde{x}) = 0. \quad (5.67)$$

or

$$[\tilde{\xi}_{\tilde{x}}(\tilde{x}) + \tilde{\xi}^T_{\tilde{x}}(\tilde{x})]\tilde{f}(\tilde{x}) + [\tilde{\xi}_{\tilde{x}}(\tilde{x}) + \tilde{\xi}^T_{\tilde{x}}(\tilde{x})]\tilde{\mathcal{Q}}(\tilde{x})\tilde{\xi}(\tilde{x}) + (I_n \otimes \tilde{\xi}^T(\tilde{x}))\frac{\partial \tilde{\mathcal{Q}}(\tilde{x})}{\partial \tilde{x}}\tilde{\xi}(\tilde{x}) + 2\tilde{f}_{\tilde{x}}(\tilde{x})\tilde{\xi}(\tilde{x}) +$$
$$\frac{1}{2}\mathbf{vec}(\tilde{\Theta}(\tilde{x}))^T_{\tilde{x}}\mathbf{vec}[\tilde{\xi}_{\tilde{x}}(\tilde{x}) + \tilde{\xi}^T_{\tilde{x}}(\tilde{x})] + \frac{1}{2}\mathbf{vec}[\tilde{\xi}_{\tilde{x}}(\tilde{x}) + \tilde{\xi}^T_{\tilde{x}}(\tilde{x})]^T_{\tilde{x}}\mathbf{vec}(\tilde{\Theta}(\tilde{x})) + 2\tilde{h}_{\tilde{x}}(\tilde{x})\tilde{h}(\tilde{x}) = 0. \quad (5.68)$$

Equation (5.67) or (5.68) represents a system of $\frac{n(n-1)}{2}$ second-order nonlinear PDEs in the $n$ unknowns $\xi_i, i = 1, \ldots, n$. Any solution of this equation corresponds to a solution of the SHJBE (5.38) or (5.41). Furthermore, for the case of the linear system $\Sigma^l_{st}$, the SHJBE (5.68) with $\xi(\tilde{x}) = \tilde{\Gamma}(x_0)\tilde{x}$ for some $n \times n$ matrix $\tilde{\Gamma}(x_0)$, will correspond to the following equation:

$$[\tilde{\Gamma}(x_0) + \tilde{\Gamma}^T(x_0)]\tilde{A}(x_0)\tilde{x} + [\tilde{\Gamma}(x_0) + \tilde{\Gamma}^T(x_0)]\tilde{\mathcal{Q}}(x_0)\tilde{\Gamma}(x_0)\tilde{x} + (I_n \otimes \tilde{x}^T\tilde{\Gamma}^T(x_0))\tilde{\mathcal{Q}}_{\tilde{x}}(x_0)\tilde{\Gamma}(x_0)\tilde{x} +$$
$$2\tilde{A}(x_0)\tilde{\Gamma}(x_0)\tilde{x} + \frac{1}{2}\mathbf{vec}(\tilde{\Theta}(x_0))^T_{\tilde{x}}\mathbf{vec}[\tilde{\Gamma}(x_0) + \tilde{\Gamma}^T(x_0)] + \frac{1}{2}\mathbf{vec}[\tilde{\Gamma}(x_0) + \tilde{\Gamma}(x_0)]^T_{\tilde{x}}\mathbf{vec}(\tilde{\Theta}(x_0)) +$$
$$2\tilde{H}^T(x_0)\tilde{H}(x_0)\tilde{x} = 0 \quad (5.69)$$

Again, equating terms of corresponding powers of $\tilde{x}$, we get,

$$(I_n \otimes \tilde{\Gamma}(x_0))\tilde{\mathcal{Q}}_{\tilde{x}}(x_0)\tilde{\Gamma}(x_0) = 0 \quad (5.70)$$
$$\left[\tilde{\Gamma}(x_0) + \tilde{\Gamma}^T(x_0)\right]\tilde{A}(x_0) + [\tilde{\Gamma}(x_0) + \tilde{\Gamma}^T(x_0)]\tilde{\mathcal{Q}}(x_0)\tilde{\Gamma}(x_0) + 2\tilde{A}(x_0)\tilde{\Gamma}(x_0) + 2\tilde{H}^T(x_0)\tilde{H}(x_0) = 0 \quad (5.71)$$
$$\mathbf{vec}(\tilde{\Theta}(x_0))^T_{\tilde{x}}\mathbf{vec}[\tilde{\Gamma}(x_0) + \tilde{\Gamma}^T(x_0)] + \mathbf{vec}[\tilde{\Gamma}(x_0) + \tilde{\Gamma}(x_0)]^T_{\tilde{x}}\mathbf{vec}(\tilde{\Theta}(x_0)) = 0 \quad (5.72)$$

The above system of nonlinear equations completely characterizes the solution of the linear SHJBE or equivalently the stochastic differential Riccati equation. It can be shown that the above system of equations is equivalent to the system of differential equations in [6, 117] for the linear system $\Sigma^l_{st}$. A computational procedure can be developed for solving the above system. However, the above system of equations may not offer much advantage over the system of differential equations in [6, 117]. Thus, this approach may not offer additional advantages over earlier approaches for the linear syste , however, for the nonlinear case, equation (5.67) or (5.68) certainly offers an additional leap and a step-forward in analytically solving the SHJBE.

**Example 5.3.2** *Consider Example 1 and let us use equation (5.68) to solve it. Substituting the problem data in the equation, we have*

$$\begin{bmatrix} 2\xi_{1,x_1} & \xi_{1,x_2} + \xi_{2,x_1} \\ \xi_{1,x_2} + \xi_{2,x_1} & 2\xi_{2,x_2} \end{bmatrix}\begin{bmatrix} x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2\xi_{1,x_1} & \xi_{1,x_2} + \xi_{2,x_1} \\ \xi_{1,x_2} + \xi_{2,x_1} & 2\xi_{2,x_2} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} +$$
$$2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_1^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2\xi_{1,x_1} \\ \xi_{1,x_2} + \xi_{2,x_1} \\ \xi_{1,x_2} + \xi_{2,x_1} \\ 2\xi_{2,x_2} \end{bmatrix} +$$

$$\frac{1}{2}\left[\begin{array}{cccc} 2\xi_{1,x_1x_1} & \xi_{1,x_2x_1}+\xi_{2,x_1x_1} & \xi_{1,x_2x_1}+\xi_{2,x_1x_1} & 2\xi_{2,x_2x_1} \\ 2\xi_{1,x_1x_2} & \xi_{1,x_2x_2}+2\xi_{2,x_1x_2} & \xi_{1,x_2x_2}+\xi_{2,x_1x_2} & 2\xi_{2,x_2x_2} \end{array}\right]\left[\begin{array}{c} \frac{1}{4}x^4 \\ 0 \\ 0 \\ 0 \end{array}\right]+2\left[\begin{array}{c} 0 \\ x_2 \end{array}\right]=0$$

*Upon multiplying out, we get*

$$\left[\begin{array}{c} 2x_2\xi_{1,x_1} \\ x_2(\xi_{1,x_2}+\xi_{2,x_1}) \end{array}\right]-\left[\begin{array}{c} \xi_2(\xi_{1,x_2}+\xi_{2,x_1}) \\ 2\xi_2\xi_{2,x_2} \end{array}\right]+\left[\begin{array}{c} 0 \\ 2\xi_1 \end{array}\right]+\left[\begin{array}{c} x_1^3\xi_{1,x_1} \\ 0 \end{array}\right]+\frac{1}{4}\left[\begin{array}{c} x_1^4\xi_{1,x_1x_1} \\ x_1^4\xi_{1,x_1x_2} \end{array}\right]+\left[\begin{array}{c} 0 \\ 2x_2 \end{array}\right]=0$$

or

$$\begin{aligned} x_1^4\xi_{1,x_1x_1}+(8x_2+4x_1^3)\xi_{1,x_1}-4\xi_2(\xi_{1,x_2}+\xi_{2,x_1}) &= 0, \\ x_1^4\xi_{1,x_1x_2}+4x_2(\xi_{1,x_2}+\xi_{2,x_1})-8\xi_2\xi_{2,x_2}+2\xi_1+8x_2 &= 0, \end{aligned}$$

*which is a system of coupled second-order nonlinear PDEs in the unknown parameters $\xi_1=\xi_1(x)$ and $\xi_2=\xi_2(x)$. There is no general procedure for solving such a system of PDEs, although some ad-hoc techniques are available [4]. We apply such ad-hoc technique by guessing the solution from the nature of the problem. Thus, if we take*

$$\xi_1(x)=a_1x_1+b_1x_2, \quad \xi_2(x)=a_2x_1+b_2x_2.$$

*Then, taking derivatives and substituting in the above system and collecting similar terms, we get*

$$a_1=a_2=b_1=0, \quad b_2=\pm 1$$

*Thus, we have $V_x(x)=[0 \quad x_2]$ or $V(x)=\frac{1}{2}x_2^2$ which corresponds to the previous solution.*

## 5.4 Application to Partial Differential Equations

The above procedure that we have outlined for the HJBE both deterministic and stochastic can be extended to other nonlinear PDEs, e.g. conservation laws. For this purpose, consider a Cauchy problem for a first-order system:

$$\frac{\partial Z}{\partial t}(t,x)+F(t,x,Z_x)=0; \quad Z(T,x)=0, \quad (t,x)\in\tilde{M}\subset[0,T]\times\bar{M}, \tag{5.73}$$

where $\bar{M}\subset\Re^n$, $Z:[0,T]\times\bar{M}\to\Re$ is the unknown function, and $F:[0,T]\times\bar{M}\times\Re^n\to\Re$. By redefining the time parameter $t$ as $x_0$ and letting $\tilde{x}=(x_0,x_1,\ldots,x_n)^T$, then in the new coordinates, the PDE (5.73) takes the following form:

$$\tilde{F}(\tilde{x},\tilde{Z}_{\tilde{x}})=0; \quad \tilde{Z}(\tilde{x}(T))=0 \quad \tilde{x}\in\tilde{M}. \tag{5.74}$$

We can now parameterize $\tilde{Z}_{\tilde{x}}$ in terms of some function $\varphi$ as $\tilde{Z}_{\tilde{x}}(\tilde{x})=\varphi(\tilde{x})$ such that (5.74) is solvable in terms of $\varphi$ and transforms it to an algebraic equation:

$$\tilde{F}(\tilde{x},\varphi(\tilde{x}))=0; \quad \varphi(\tilde{x}(T))=0 \tag{5.75}$$

which can be solved for $\varphi$; and finally, $\tilde{Z}$ can be recovered from $\varphi$ by carrying out the line integration $\int_0^{\tilde{x}}\varphi(\rho)d\rho$.

# CHAPTER 6
# SUMMARY AND CONCLUSION

In this thesis, we have presented a factorization approach for solving the Hamilton-Jacobi-equations (HJE) (inequalities) arising in modern and post-modern nonlinear optimal control. We have presented two methods; the first is essentially an inversion or factorization approach, and is the major approach; it involves solving the HJE like a scalar quadratic algebraic equation with the gradient of the smooth scalar function as unknown, while the second approach involves the solution of higher-order PDEs. Further, in the former approach, since the HJE is a quadratic equation in the gradient of the unknown scalar function, we obtain two parameterized solutions which represent a parameterization of all solutions to the HJE. Thus, the problem is reduced to that of factorization of a scalar algebraic equation which we called the *discriminant equation* (or inequality). In addition to this factorization problem, in order for the resulting gradient to correspond to a smooth scalar function, it is necessary that the discriminant factor satisfy some additional differential constraints, which we called the *"curl"* or symmetry conditions. These guarantee that the solution of the HJE is symmetric, and by a judicious choice of design parameters, it is possible to obtain a positive-(semi)definite solution among many candidate symmetric solutions. Moreover, we have demonstrated a computational procedure for solving the resulting discriminant equation/inequality for the case of second and third-order systems, and the procedure can easily be extended to higher-order systems with increasing complexity. Among some of the remaining challenges, is to determine a way of picking among the multitudes of the solutions to the symmetry conditions (in the free parameter $\theta$) the solutions that will result in a positive-(semi)definite $V$. Furthermore, for the case of linear systems, we have shown how the problem can be efficiently formulated as a convex optimization problem over a system of LMIs.

Also, more computational experiments are required to be performed to ascertain the practicality of the approach. As such, at this point, the method will only be of theoretical interest. Nonetheless, one utility of such a scheme would be to develop viable approximate solutions from it.

On the other hand, the second method which is a minor development over the first, involves converting the HJE into a second-order PDE. It is then possible to parameterize the solution interms of a parameter vector. This results into a system of coupled higher-order PDEs in the unknown parameter, which could be solved in some cases using SYMBOLIC manipulation software such as METHEMATICA, MATLAB, or MAPLE or other methods. Otherwise, an approximate solution must be sought. Changing the output function and/or the level of disturbance attenuation $\gamma$ may also aid in finding analytical solutions to the coupled higher-order PDEs. This method also always leads to symmetric solutions. The only short-coming of the approach is that there is no general approach for solving the coupled system of higher-order PDEs.

To recapitulate, in chapter 2, we have developed a factorization approach for solving the HJIEs arising in the state-feedback, measurement feedback and finite-horizon $\mathcal{H}_\infty$

control problem for continuous-time affine nonlinear time-invariant and time-varying systems. A parameterization of all solutions to the HJI inequality is also given. All the results for the nonlinear case are then specialized to linear time-invariant systems.

In chapter 3, we have developed some computational procedures for determining solutions using the method presented in chapter 2, and an alternative approach using higher-order PDEs is presented. Furthermore, computational results to some representative problems are presented. It is shown that, for second-order and third-order affine nonlinear systems, the computations are relatively easy and can be done by hand. However, for higher-order nonlinear systems, the compuations may become more complicated and some symbolic manipulation tools might be needed. All the results for the nonlinear case are also specialized to linear systems, and it is shown that for linear systems, the solution can be computed using mathematical programming techniques by solving optimization problems over a system of LMIs.

Computational results using the alternative method involving higher-order PDEs are also demonstrated for a second-order affine nonlinear system. However, more work needs to be done in finding an efficient method for solving the resulting system of coupled PDEs.

In chapter 4, we have extended the methods discussed above to the coupled HJIE arising in the state-feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for affine nonlinear time-invariant systems. Necessary and sufficient conditions for the existence of solutions are first derived, and then the analytical procedure is presented. Furthermore, sufficient conditions for the solvability of the coupled HJIEs in this case are given in terms of a system of coupled algebraic-differential inequalities (two coupled discriminant inequalities, two symmetry conditions and two positivity conditions). A higher-order approach for the coupled HJIEs is also presented. It is shown that, if the resulting system of inequalities and/or coupled PDEs is solvable, then a smooth solution to the coupled HJIEs can be obtained. Otherwise, it may be necessary to change the output function and/or $\gamma$ so as to obtain a solution. Computational results for a second-order system are also presented.

In chapter 5, we have again extended the above approaches to solve the HJBEs of $\mathcal{H}_2$ deterministic and stochastic optimal control. Constructive approaches that may yield smooth solutions are also presented. It is shown that, for the deterministic HJBE, the results for the HJIE of the $\mathcal{H}_\infty$ generalizes to this case with the only modification in the gain matrices; while for the stochastic case, the results are significantly different, since the resulting HJBE in this case is second-order. Nonetheless, with some slight modifications, the same procedures discussed above are extended to this case. Computational results for a second-order nonlinear system are also presented.

In addition, we have also considered in this chapter the solutions of the HJBEs when the control set is not an open set but a bounded set. This leads to the idea of generalized or viscosity solutions of the HJBEs and the concept of nondifferentiable Hamiltonian functions. We have briefly considered these cases in the chapter, and extension of the approaches to other PDEs such as conservation laws have also been discussed.

Among many other things, this dissertation has proposed a framework for elementary analysis and computational procedures for solving HJEs in $\mathcal{H}_\infty$, $\mathcal{H}_2/\mathcal{H}_\infty$ and $\mathcal{H}_2$ control of affine nonlinear systems. It has been shown how the HJE can be reduced to an algebraic equation (inequality), the solution of which with some additional side conditions, could give symmetric solutions to the HJE. However, more computational experiments will have to be performed in order to ascertain the applicability of the methods. Future work should also concentrate in developing efficient computational methods for solving the the higher-order system of coupled PDEs that result in applying the alternative approach.

Furthermore, we have also raised some interesting problems and have made connection with other problem areas, the persuit of which will lead to a more comprehensive understanding of HJEs. It is hoped that the results presented in this dissertation will facilitate the application of nonlinear $\mathcal{H}_\infty$ techniques in control systems design and analysis and will provide more insight into the solution of HJEs as applied in other areas of science and engineering, such as, analytical mechanics and mathematical physics.

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