

DAMAGE MECHANICS OF COMPOSITE MATERIALS USING FABRIC TENSORS

A Thesis

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Abstract

The major objective of this work is to relate continuum damage mechanics introduced through the concept of fabric tensors to composite materials within the framework of classical elasticity theory. A model of directional data-damage mechanics for composite materials is formulated using fabric tensors. In addition, a general hypothesis for damage mechanics is postulated. It is seen that the two available hypotheses of elastic strain equivalence and elastic energy equivalence may be obtained as special cases of the postulated general hypothesis. This general hypothesis is then used to derive the sought relationship between the damage tensor for composite materials and the fabric tensors.

Two approaches to link the fabric tensors damage effect to the behavior of composite materials are adopted. The first approach is the continuum approach, which introduces damage with fabric tensors to the composite media; where the latter is treated as a homogenized material. Properties of the constituents are homogenized before the damage with fabric tensors is introduced. The second approach is the micro-mechanical approach, where damage with fabric tensors is introduced to the constituents rather than to the homogenized material. Within the framework of classical elasticity theory, both approaches should lead to equivalent results. Thus, a comparison between the two approaches is carried out to verify their equivalency.

Damage evolution for both approaches is derived in a mathematically consistent manner that is based on sound thermodynamic principles. Numerical examples and application to the theory developed herein are presented. Micro-crack distributions in different constituents of the composite material are thoroughly investigated.

1 Introduction

Composite materials play an important role in modern industry through the design and manufacture of advanced materials capable of attaining higher stiffness/density and strength/density ratios. These ratios allow composite materials to be used in various applications where the weight and strength of the structure are highly significant design parameters (e.g. aircraft, and aerospace shuttle industries). Composite materials dramatically enhance the performance and increase the efficiency of such structures. In order to insure structural integrity and safe performance, thorough understanding of the behavior of these materials under arising loads must be established. Of particular importance is the problem of damage initiation and evolution in composites.

Damage mechanics in two-phase (matrix-fiber) composite materials is a rather complex problem that has challenged researchers during the past two decades. The literature is rich in new developments in the composite materials technology. Yet space is still available for more research to allow for a comprehensive layout of the composite material characteristics and performance under different patterns of loading.

Recently, the topic of Fabric Tensors was introduced to damage mechanics of metals (Voyiadjis and Kattan, 2005). Fabric tensors describe directional data (like micro-crack distributions) and microstructural anisotropy in the material. Micro-crack distributions within the material are used in the characterization and evaluation of damage. Fabric tensors will be used in this work to characterize damage in composite materials by describing the directional data and anisotropy of its constituents. The description of damage of composite materials to be obtained from this work by incorporating fabric tensors will provide a more realistic and physical understanding of damage.

2 Literature Review

The theory of continuum damage mechanics was introduced by Kachanov (1958) for the isotropic case of uniaxial tension and later modified for creep by Rabotnov (1969). The damage variable that they used may be interpreted as the effective surface density of micro-damages per unit volume. Kachanov (1958) pioneered the subject of continuum damage mechanics by introducing the concept of effective stress. This concept is based on considering a fictitious undamaged configuration of a body and comparing it with the actual damaged configuration. Following that, researches in different fields applied continuum damage mechanics to various fields of study. Damage mechanics has now reached the stage where it can be used in practical applications.

In contrast to fracture mechanics, which considers the process of initiation and growth of macro-cracks as a discontinuous phenomenon, continuum damage mechanics uses a continuous variable (the damage variable or tensor) which is related to the density of the defects in the material.

Based on the damage variable (or tensor), constitutive equations of evolution were developed to predict the initiation of micro-cracks for different types of phenomena. Lemaitre (1996) and Chaboche (1974) used it to solve different types of fatigue problems. Hult (1974), and Lemaitre and Chaboche (1975) used it to solve creep and creep-fatigue interaction problems. Also, it was used by Lemaitre for ductile plastic fracture (Lemaitre and Dufailly, 1977; Lemaitre 1985) and for a number of other applications (Lemaitre, 1984). In addition Voyiadjis and Kattan (1992, 1999) used continuum damage mechanics for ductile materials and metal matrix composites

(Voyiadjis and Kattan, 1990, 1996; Kattan and Voyiadjis, 1990, 1993, 2001a,b; Voyiadjis and Park, 1997,1999).

The damage variable (or tensor), based on the effective stress concept, represents average material degradation which reflects the various types of damage at the micro-scale level like nucleation and growth of voids, cracks, cavities, micro-cracks, and other microscopic defects.

For the case of isotropic damage mechanics, the damage variable is a single scalar variable and the evolution equations are easy to handle. However, it has been shown by Cauvin and Testa (1999) that two independent damage scalar variables must be used in order to accurately and consistently describe the special case of isotropic damage. It has been argued (Lemaitre, 1984) that the assumption of isotropic damage is sufficient to give good predictions of the load carrying components. However, the development of anisotropic damage has been confirmed experimentally (Hayhurst, 1972; Chow and Wang, 1987a; Lee et al., 1985) even if the virgin material is isotropic. This has prompted several researchers to investigate the general case of anisotropic damage.

The theory of anisotropic damage mechanics was developed by Sidoroff and Cordebois (Cordebois and Sidoroff, 1979; Sidoroff, 1981; Cordebois, 1983) and later used by Lee et al. (1985) and Chow and Wang (1987b, 1988) to solve simple ductile fracture problems. Prior to this development, Krajcinovic and Foneska (1981), Murakami and Ohno (1981), Murakami (1983) and Karjcinovic (1983) investigated brittle and creep fracture using appropriate anisotropic damage models. Although these models are based on a sound physical background, they lack vigorous mathematical justification and mechanical consistency. Consequently, more work needs to be done to develop a more

involved theory capable of producing results that can be used for practical applications (Krajcinovic and Foneska, 1981; Krempl 1981).

In the general case of anisotropic damage, the damage variable has been shown to be tensorial in nature (Murakami and Ohno, 1981; Leckie and Onat, 1981). This damage tensor was shown to be an irreducible even-rank tensor (Onat, 1986; Onat and Leckie 1988). Several other properties of the damage tensor have been outlined by Betten (1981, 1986) in a rigorous mathematical treatment using the theory of tensor functions.

Recently, Cauvin and Testa (1999) used an eighth-rank damage tensor and showed mathematically that this damage tensor can be reduced into a fourth-rank damage tensor within the general theory of anisotropic elasticity. They have shown that the fourth-rank damage tensor is sufficient to accurately describe anisotropic damage.

Lemaitre (1986) summarized the work done during the seventies and early eighties to describe micro-crack behavior using the theory of continuum damage mechanics. Krajcinovic (1996) summarized the work in damage mechanics relating primarily to creep damage and brittle materials. In their recent book, Voyiadjis and Kattan (1999) summarized the work done in damage mechanics in the nineties that primarily involved metals and metal matrix composites. Also, Lemaitre and Dufailly (1987) described eight different experimental methods (both direct and indirect) to measure damage according to the effective stress concept (Kachanov, 1986).

Chaboche (1986, 1988a,b) described different definitions of the damage variable based on indirect measurement procedures. Examples of these are damage variables based on the remaining life, the microstructure, and several physical parameters like

density change, resistivity change, acoustic emissions, the change in fatigue limit, and the change in mechanical behavior through the concept of effective stress.

In continuum damage mechanics usually a phenomenological approach is adopted. In this approach, the most important concept is that of the Representative Volume Element (RVE). The discontinuous and discrete elements of damage are not considered within the RVE; rather their combined effects are lumped together through the use of a macroscopic internal variable. In this way, the formulation may be derived consistently using sound mechanical and thermodynamic principles (Voyiadjis and Kattan, 2005)

The concept of fabric tensors has been formulated by Kanatani (1984a) to describe directional data and microstructural anisotropy. Fabric tensors were further elaborated upon by Lubarda and Krajcinovic (1993) to describe micro-crack distributions.

Satake (1982) applied the concept of fabric tensors to granular materials. The anisotropy due to the fabric (of the distributed data like micro-crack distributions or granular particles) is represented by a tensor in terms of the normals (to the micro-cracks or to the contact surfaces in granular materials). This tensor is usually called the fabric tensor (Satake, 1982; Kanatani, 1984a; Nemat-Nasser, 1982; Oda et al., 1982; Mehrabadi et al., 1982). The fabric tensor is usually related to the probability density function of the distributed data (micro-crack normals or contact normals).

Kanatani (1984a) formulated the concept of fabric tensors based on a rigorous mathematical treatment. He used fabric tensors to describe distributions of directional data like micro-crack distributions in a damaged material element. He applied the least square approximation (a well known statistical technique) to derive equations for the various fabric tensors he postulated. He defined three types of fabric tensors: fabric

tensors of the first kind, denoted by \mathbf{N} , fabric tensors of the second kind, denoted by \mathbf{F} , and fabric tensors of the third kind, denoted by \mathbf{D} . He derived the exact mathematical relations between these three types of fabric tensors.

Cowin (1985) made an attempt to relate the microstructure (through the use of fabric tensors) to the fourth-rank elasticity tensor. He used a normalized second-rank tensor and presented expressions for the elastic constants in terms of the invariants of the fabric tensors.

Zysset and Curnier (1995, 1996) formulated an alternative model for anisotropic elasticity based on fabric tensors. They introduced a general approach for relating the material microstructure to the fourth-rank elasticity tensors based on the Fourier series decomposition. They proposed an approximation based on a scalar and a symmetric, traceless second-rank fabric tensor. Using the representation theorem for anisotropic functions with tensorial arguments, Zysset and Curnier (1995) derived a general expression for the elastic free energy and discussed the resulting material symmetry in terms of the fabric tensors. Finally, they derived a general explicit expression for the fourth-rank elasticity tensor in terms of the fabric tensors.

Lubarda and Krajcinovic (1993) applied the definition of fabric tensors (Kanatani, 1984a) to the micro-crack density distributions. They recast the general work of Kanatani (1984a) on directional data in terms of micro-crack distributions. Lubarda and krajcinovic (1993) examined the relationship between a given, experimentally determined, distribution of micro-cracks and the scalar, second-rank, and fourth-rank fabric tensors. They employed the usual representation of experimentally measured micro-crack densities in planes with different orientations in the form of a circular

histogram (rose diagram). They then used the data contained in the circular histogram to approximate the distribution function defined on a unit sphere and centered in a material point. They solved several examples with different micro-crack distributions to illustrate this point. They assumed that one of the three types of fabric tensors is identical to the damage tensor of continuum damage mechanics.

Voyiadjis and Kattan (2005) related continuum damage mechanics of metals with the concept of fabric tensors. They applied the concept of fabric tensors introduced by Kanatani (1984a) and further elaborated upon by Lubarda and Krajcinovic (1993) as well as the work of Zysset and Curnier (1995) and Cauvin and Testa (1999) into the formulation of an elasticity tensor of damaged metallic materials. This work will be an extension of the work of Voyiadjis and Kattan (2005) to incorporate fabric tensors in the study of damage mechanics of composite materials.

3 Scope and Approaches of Introducing Damage Mechanics into Composite Materials Using Fabric Tensors

This work will illustrate an attempt to apply continuum damage mechanics introduced through the concept of fabric tensors to composite materials. Damage mechanics of composite material is more physically characterized using fabric tensors. The actual effect of the presence of micro-cracks in the composite system - or in the constituents of the composite system - on its elastic stiffness is better observed and more physically defined. Micro-crack distributions can be obtained from representative Scanning Electron Microscope (SEM) images. The orientations of these micro-cracks can be found and used to characterize damage in the composite system. An SEM image showing micro-cracks in a metal matrix composite is presented here for demonstration (Figure 3.1).

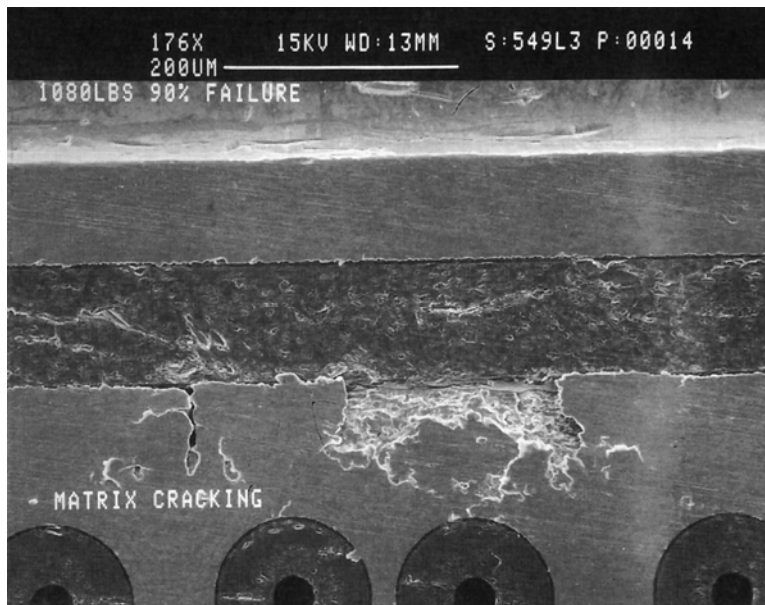


Figure 3.1: Damage in Metal Matrix Composites (Voyiadjis and Kattan, 1999)

The data from a micro-crack distribution can be then presented in the form of a circular histogram (rose diagram). This histogram can then be used to obtain distribution functions which are used to determine the values of the components of the second-rank fabric tensors. Then the components of the damage tensor are calculated using the determined values of the fabric tensor. A sample rose diagram is shown in Figure 3.2

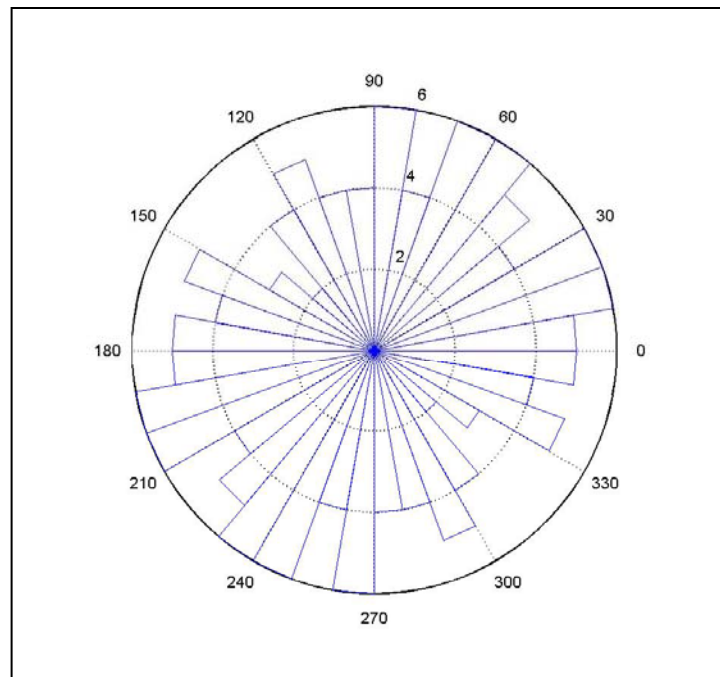


Figure 3.2: A circular histogram (rose diagram) for the micro-crack distribution data of the application.

The study will be conducted within the framework of elasticity theory. The study will involve only static analysis of composite materials. Emphasis will be given to continuous fiber reinforced composites. Fibers are assumed to be isotropic and perfectly aligned in an isotropic matrix medium. The effects of de-bonding between the matrix and the fibers will not be considered here. The composite material is assumed to consist of three distinct

phases: matrix, fiber, and interface. In addition, linear elastic analysis is assumed throughout this work.

Two approaches will be followed in this work to introduce damage mechanics with fabric tensors to composite materials. The first approach is the Continuum Approach. The second approach is the Micromechanical Approach. This chapter will provide an overview to each approach.

In the Continuum Approach, the representative volume element (RVE) of the composite medium will be homogenized before damage is introduced. The properties of the constituents will be averaged to produce an RVE with overall properties that are no longer dependent on the constituents' properties. Damage can then be introduced to the homogenized RVE. Therefore, in the Continuum Approach, damage will be incorporated into the composite system as a whole through one fourth-rank damage tensor called the overall damage tensor. Figure 3.3 shows a schematic illustration of the Continuum Approach.

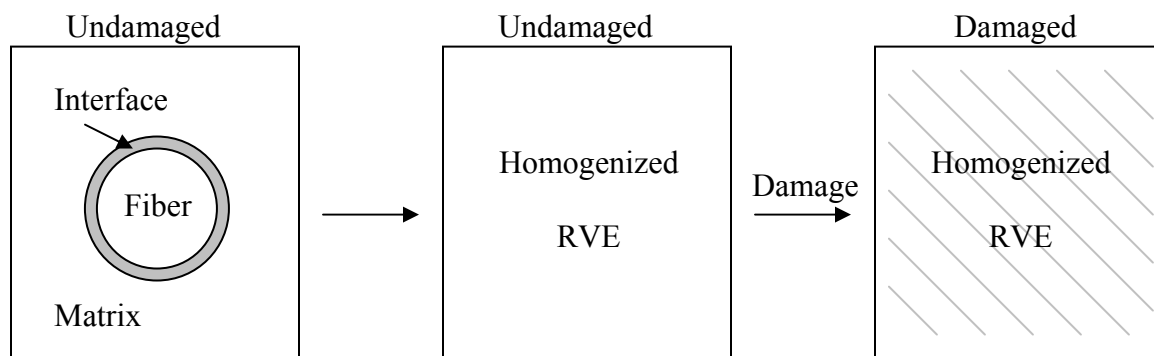


Figure 3.3 Continuum Approach

In the Micromechanical Approach, damage is introduced to the RVE at the constituent level. Therefore, three different fourth-rank damage tensors will describe damage in three different constituents of the composite medium. Then, the damaged properties of the constituents are averaged to produce the damage properties of the composite system as a whole. Figure 3.4 shows the steps involved in the Micromechanical Approach.

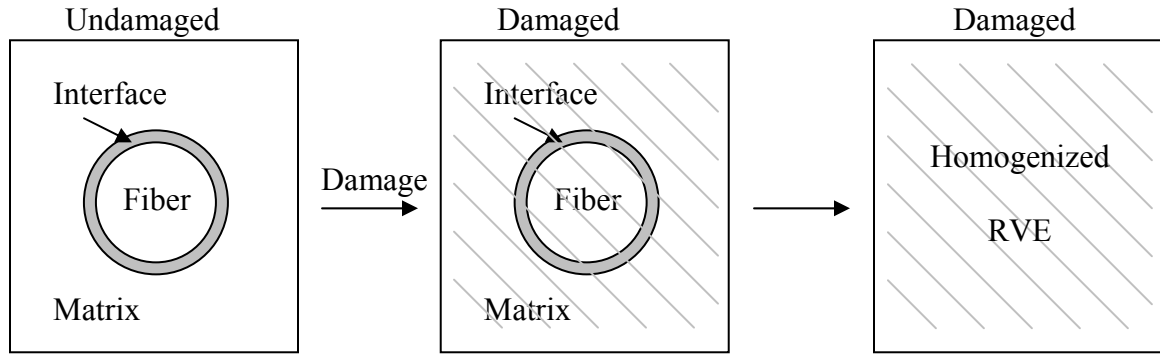


Figure 3.4 Micromechanical Approach

The tensorial notation adopted in this work as well as the tensorial operations are defined here. All vectors and tensors appear in bold type. Tensorial operations are defined as follows. For second-rank tensors **A** and **B**, the following notation is used:

$$(\mathbf{A} \pm \mathbf{B})_{ij} = \mathbf{A}_{ij} \pm \mathbf{B}_{ij},$$

$$\mathbf{A} : \mathbf{B} = \mathbf{A}_{ij} \mathbf{B}_{ij},$$

$$(\mathbf{A} \otimes \mathbf{B})_{ijkl} = \mathbf{A}_{ij} \mathbf{B}_{kl},$$

$$(\mathbf{A} \bar{\otimes} \mathbf{B})_{ijkl} = \frac{1}{2} (\mathbf{A}_{ik} \mathbf{B}_{jl} + \mathbf{A}_{il} \mathbf{B}_{jk}).$$

For fourth-rank tensors **C** and **D**,

$$(\mathbf{C} \pm \mathbf{D})_{ijkl} = \mathbf{C}_{ijkl} \pm \mathbf{D}_{ijkl},$$

$$(\mathbf{C}:\mathbf{D})_{ijkl} = \mathbf{C}_{ijmn} \mathbf{D}_{mnkl} ,$$

$$\mathbf{C}::\mathbf{D} = \mathbf{C}_{ijkl} \mathbf{D}_{ijkl} ,$$

$$(\mathbf{C} \otimes \mathbf{D})_{ijklmnpq} = \mathbf{C}_{ijkl} \mathbf{D}_{mnpq} .$$

For second-rank tensor \mathbf{A} , and fourth-rank \mathbf{C} , the following notation is used;

$$(\mathbf{C}:\mathbf{A})_{ij} = \mathbf{C}_{ijkl} \mathbf{A}_{kl} ,$$

and for fourth-rank tensor \mathbf{C} , and eight-rank tensor \mathbf{G} ,

$$(\mathbf{G}::\mathbf{C})_{ijkl} = \mathbf{G}_{ijklmnpq} \mathbf{C}_{mnpq} .$$

For damage tensors, fabric tensors, and identity tensors, a superscript with braces is used to indicate the order of the tensor. For all other tensors, the order is clear from the number of indices associated with the tensor. Indicical notation will also be used; especially when it is difficult to express a rather complex tensorial equation.

4 Review of Fabric Tensors in Damage Mechanics of Metals

The theory presented in this chapter is based on the work of Voyiadjis and Kattan (2005). They addressed damage mechanics of metals by introducing fabric tensors into the damage model formulation. Part of their effort is presented here as a foundation for the composite models to be presented in forthcoming chapters.

Here, the use of fabric tensors to describe directional data and micro-structural anisotropy is reviewed. Kanatani (1984 a,b) introduced the idea of fabric tensors with regard to the distribution of directional data. He used fabric tensors for the stereological determination of structural anisotropy. Zysset and Curnier (1995) used fabric tensors to derive an alternative model of anisotropic elasticity. They derived a new formula for the general elasticity tensor of damaged materials in terms of fabric tensors. He and Curnier (1995) formulated a more fundamental approach to damaged elastic stress-strain relations using fabric tensors. Zysset and Curnier (1996) used a damage model based on fabric tensors in the analysis of trabecular bone. Sutcliffe (1992) presented a rigorous mathematical formulation for the spectral decomposition of the general elasticity tensor.

Kanatani (1984a) formulated the concept of fabric tensors based on a rigorous mathematical treatment. He used fabric tensors to describe distributions of directional data like micro-crack distributions in a damaged material element. He applied the least square approximation (a well known statistical technique) to derive equations for the various fabric tensors he postulated. He defined three types of fabric tensors: fabric tensors of the first kind, denoted by \mathbf{N} , fabric tensors of the second kind, denoted by \mathbf{F} , and fabric tensors of the third kind, denoted by \mathbf{D} . He derived the exact mathematical relations between these three types of fabric tensors.

Zysset and Curnier (1995, 1996) formulated an alternative model for anisotropic elasticity based on fabric tensors. Actually Cowin (1985) made an attempt to relate the microstructure (through the use of fabric tensors) to the fourth-rank elasticity tensor. He used a normalized second-rank tensor and presented expressions for the elastic constants in terms of the invariants of the fabric tensors. Zysset and Curnier (1995) introduced a general approach for relating the material microstructure to the fourth-rank elasticity tensor based on the Fourier series decomposition. They proposed an approximation based on a scalar and a symmetric, traceless second-rank fabric tensor. Using the representation theorem for anisotropic functions with tensorial arguments, Zysset and Curnier (1995) derived a general expression for the elastic free energy and discussed the resulting material symmetry in terms of the fabric tensors. Finally, they derived a general explicit expression for the fourth-rank elasticity tensor in terms of the fabric tensors. This last result is very important and is used extensively here.

Lubarda and Krajcinovic (1993) applied the definitions of fabric tensors (Kanatani, 1984a) to micro-crack density distributions. They actually recast Kanatani's general work on directional data (Kanatani, 1984a) in terms of micro-crack distributions. Lubarda and Krajcinovic (1993) examined the relationship between a given, experimentally determined, distribution of micro-cracks and the scalar, second-rank and fourth-rank fabric tensors. They employed the usual representation of experimentally measured micro-crack densities in planes with different orientations in the form of a circular histogram (rose diagram). They then used the data contained in the circular histogram to approximate the distribution function defined on a unit sphere and centered in a material point. They solved several examples with different micro-crack distributions to illustrate

this point. They assumed that one of the three types of fabric tensors is identical to the damage tensor of continuum damage mechanics.

A distribution of directional data (i.e. micro-cracks) that is radially symmetric with respect to the origin is considered here. Setting $\mathbf{n}^{(\alpha)}$ to be a unit vector specifying the orientation of the micro-crack α ($\alpha = 1, \dots, N$), where N is the total number of micro-cracks, and setting the orientation distribution function as $f(\mathbf{N})$, where \mathbf{N} is the second-rank fabric tensor of the first kind presented by Kanatani (1984a) as:

$$\mathbf{N}_{ij}^{(2)} = \frac{1}{N} \sum_{\alpha=1}^N n_i^{(\alpha)} n_j^{(\alpha)} \quad (4.1)$$

then the function $f(\mathbf{N})$ can be then expanded in a convergent Fourier series as follows (Jones, 1985; Zysset and Curnier, 1995):

$$f(\mathbf{N}) = G^{(0)} \cdot 1 + \mathbf{G}^{(2)} : \mathbf{F}^{(2)}(\mathbf{N}) + \mathbf{G}^{(4)} :: \mathbf{F}^{(4)}(\mathbf{N}) + \dots \quad (4.2)$$

where $G^{(0)}$, $\mathbf{G}^{(2)}$, $\mathbf{G}^{(4)}$ are zero-rank (scalar), second-rank, and fourth-rank fabric tensors, respectively, while 1 , $\mathbf{F}^{(2)}(\mathbf{N})$, $\mathbf{F}^{(4)}(\mathbf{N})$ are zero-rank (scalar), second-rank, and fourth-rank basis functions, respectively (Kanatani, 1984 a,b; Zysset and Curnier, 1995).

The three fabric tensors $G^{(0)}$, $\mathbf{G}^{(2)}$, and $\mathbf{G}^{(4)}$ are determined using the following integrals as given by Zysset and Curnier (1995):

$$G^{(0)} = \frac{1}{4\pi} \int_S f(\mathbf{N}) da \quad (4.3)$$

$$\mathbf{G}^{(2)} = \frac{15}{8\pi} \int_S f(\mathbf{N}) \mathbf{F}^{(2)}(\mathbf{N}) da \quad (4.4)$$

$$\mathbf{G}^{(4)} = \frac{315}{32\pi} \int_S f(\mathbf{N}) \mathbf{F}^{(4)}(\mathbf{N}) da \quad (4.5)$$

where “S” is the surface of the unit sphere and “a” is the integration parameter.

Kanatani (1984 a,b) showed that the first two terms in the expansion given in equation (4.2) are enough and they can describe material anisotropy sufficiently and accurately. Therefore, we neglect the other terms in the expansion and retain only the first two terms as follows:

$$f(\mathbf{N}) = G^{(0)} + \mathbf{G}^{(2)} : \mathbf{F}^{(2)}(\mathbf{N}) \quad (4.6)$$

It is clear from the above expression that we will deal with zero-rank (scalar) and second-rank fabric tensors only - there is no need to deal with the fourth-rank fabric tensor. It should also be noted that the function f in the above approximation must remain always positive.

Lubarda and Krajcinovic (1993) assumed that the second-rank fabric tensor $\mathbf{G}^{(2)}$ is identical to the fabric tensor of the third kind $\mathbf{D}^{(2)}$ presented by Kanatani (1984a) as:

$$\mathbf{G}^{(0)} = \mathbf{D}^{(2)} = 1 \quad (4.7a)$$

$$\mathbf{G}^{(2)} = \mathbf{D}^{(2)} = \frac{15}{2}(\mathbf{N}^{(2)} - \frac{1}{3}\mathbf{I}^{(2)}) \quad (4.7b)$$

where $\mathbf{I}^{(2)}$ is the second-rank identity tensor. Equation (4.7) shows clearly a traceless tensor for a three-dimensional distribution of directional data.

The approximation of the distribution function $f(\mathbf{N})$ given in equation (4.6) characterizes anisotropy. The traceless second-rank tensor $\mathbf{G}^{(2)}$ describes orthotropy with three orthogonal planes of symmetry and all three eigenvalues being distinct. Using only the first term in equation (4.6), $f(\mathbf{N}) = G^{(0)}$, will characterize the special case of isotropy. The case of transverse isotropy is characterized if the second-rank tensor $\mathbf{G}^{(2)}$ has only two eigenvalues that are distinct (Zysset and Curnier, 1995). Therefore, we note that one

single microstructural parameter (the distribution function f) characterizes the anisotropy of the material microstructure.

Next, we write the expression of the fourth-rank constant elasticity tensor $\bar{\mathbf{E}}$ for an isotropic material as follows:

$$\bar{\mathbf{E}} = \lambda \mathbf{I}^{(2)} \otimes \mathbf{I}^{(2)} + 2\mu \mathbf{I}^{(2)} \underline{\otimes} \mathbf{I}^{(2)} \quad (4.8)$$

where λ and μ are Lamé's constants. Zysset and Curnier (1995) showed that by replacing the identity tensor $\mathbf{I}^{(2)}$ in equation (4.8) by the tensor $\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}$, we obtain the fourth-rank tensor \mathbf{E} which includes the effects of microstructural anisotropy and directional data. Thus, we have the following expression for \mathbf{E} :

$$\mathbf{E} = \lambda (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) + 2\mu (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \underline{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \quad (4.9)$$

The expression for \mathbf{E} given above provides a formula for the elasticity tensor \mathbf{E} of the damaged isotropic material in terms of the two fabric tensors $\mathbf{G}^{(0)}$ and $\mathbf{G}^{(2)}$.

Next, Voyiadjis and Kattan (2005) derived the important concepts of damage mechanics that are relevant to this work, particularly to fabric tensors. This derivation is presented within the general framework of continuum damage mechanics (Cauvin and Testa, 1999; Voyiadjis and Kattan, 1999) using a general hypothesis for strain transformation that is postulated by Voyiadjis and Kattan (2005). It is shown that general states of anisotropic damage in the material must be described by a fourth-rank damage tensor.

Murakami (1988) indicated that proper understanding of the mechanical description of the damage process of materials brought about by the internal defects are of vital importance in discussing the mechanical effects of the material deterioration on the

macroscopic behavior of materials, as well as in elucidating the process leading from these defects to the final fracture. A systematic approach to these problems of distributed defects can be provided by continuum damage mechanics (Chaboche, 1981; Hult, 1979; Kachanov, 1986; Krajcinovic, 1984; Lemaitre and Chaboche, 1978, 1985; Murakami, 1983). The fundamental notion of this theory, attributable originally to Kachanov (1958) and modified somewhat by Rabotnov (1969), is to present the damage state of the materials characterized by distributed cavities in terms of appropriate mechanical variables (internal state variables), and then to establish mechanical equations to describe their evolution and the mechanical behavior of damaged materials.

Lemaitre (1984) indicated that damage in metals is mainly the process of initiation and growth of micro-cracks and cavities. At that scale, the phenomenon is discontinuous. Kachanov (1958) was the first to introduce a continuous variable related to the density of such defects. This variable has constitutive equations of evolution, written in terms of stress or strain, which may be used in structural calculations in order to predict the initiation of macro-cracks. These constitutive equations have been formulated in the framework of thermodynamics and identified for many phenomena: dissipation and low-cycle fatigue in metals (Lemaitre, 1971), coupling between damage and creep (Leckie and Hayhurst, 1974; Hult, 1974), high-cycle fatigue (Chaboche, 1974), creep-fatigue interaction (Lemaitre and Chaboche, 1978), and ductile plastic damage (Lemaitre and Dufailly, 1977; Voyiadjis and Kattan, 1992, 1999).

In continuum damage mechanics, a micro-crack is considered to be a zone (process zone) of high gradients of rigidity and strength that has reached critical damage conditions. Thus, a major advantage of continuum damage mechanics is that it utilizes a

local approach and introduces a continuous damage variable in the process zone, while classical fracture mechanics uses more global concepts like the J-Integral and COD.

The assumption of isotropic damage is often sufficient to give a good prediction of the carrying capacity, the number of cycles, or the time to local failure in structural components. The calculations are not too difficult because of the scalar nature of the damage variable in this case. For anisotropic damage, the variable is of tensorial nature (Krajcinovic and Foneska, 1981; Murakami and Ohno, 1981; Chaboche, 1981) and the work to be done for identification of the models and for applications is much more complicated (Lemaitre, 1984; Krajcinovic and Foneska, 1981; Krempl, 1981). Nevertheless, according to Lemaitre (1984), damage mechanics has been applied since 1975 with success in several fields to evaluate the integrity of structural components and it will become one of the main tools for analyzing the strength of materials as a complement to fracture mechanics.

Let $\bar{\mathbf{E}}$ be the fourth-rank constant elasticity tensor of the virgin material and let \mathbf{E} be the elasticity tensor of the damaged material. Then, the two tensors $\bar{\mathbf{E}}$ and \mathbf{E} can be related by the following general relation (Cauvin and Testa, 1999):

$$\mathbf{E} = (\mathbf{I}^{(8)} - \boldsymbol{\phi}^{(8)}) :: \bar{\mathbf{E}} \quad (4.10)$$

where $\mathbf{I}^{(8)}$ is the eighth-rank identity tensor and $\boldsymbol{\phi}^{(8)}$ is a general eighth-rank damage tensor.

New formulations as well as a general hypothesis for strain transformation were adopted by Voyiadjis and Kattan (2005) to show that equation (4.10) can be reduced to a similar equation involving a damage tensor of rank four at most. Cauvin and Testa (1999) have shown this result only for the special case of the hypothesis of elastic strain

equivalence. Therefore, there will be no need to deal with the eighth-rank general damage tensor $\boldsymbol{\varphi}^{(8)}$ in the constitutive equations.

Kachanov (1958) and Rabotnov (1969) introduced the concept of effective stress for the case of uniaxial tension. This concept was later generalized to the three dimensional states of stress by Lemaitre (1971) and Chaboche (1981). Let $\boldsymbol{\sigma}$ be the second-rank Cauchy stress tensor and $\bar{\boldsymbol{\sigma}}$ be the corresponding effective second-rank stress tensor. The effective stress $\bar{\boldsymbol{\sigma}}$ is the stress applied to a fictitious state of the material which is totally undamaged, i.e. all damage in this state has been removed. This fictitious state is assumed to be mechanically equivalent to the actual damaged state of the material. In this regard, one of two hypotheses is usually used: (elastic strain equivalence or elastic energy equivalence). However, Voyiadjis and Kattan (2005) postulated a general hypothesis of strain transformation. They postulated that the elastic strain tensor $\boldsymbol{\varepsilon}$ in the actual damaged state is related to the effective elastic strain tensor $\bar{\boldsymbol{\varepsilon}}$ in the fictitious state by the following transformation law:

$$\bar{\boldsymbol{\varepsilon}} = \mathbf{L}(\boldsymbol{\varphi}^{(8)}) : \boldsymbol{\varepsilon} \quad (4.11)$$

where $\mathbf{L}(\boldsymbol{\varphi}^{(8)})$ is a fourth-rank tensorial function of the damage tensor $\boldsymbol{\varphi}^{(8)}$. It is noted that both hypotheses (elastic strain equivalence and elastic energy equivalence) are obtained as special cases of equation (4.11) as follows: by using $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{I}^{(4)}$, one obtains the hypothesis of elastic strain equivalence, and by using $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}^{-T}$, one obtains the hypothesis of elastic energy equivalence, where the fourth-rank tensor \mathbf{M} is the damage effect tensor used by Voyiadjis and Kattan (1999).

It can be seen that equation (4.10) may be postulated even in the absence of the concept of the effective stress space as a relation that evolves the process of degradation

of the elastic stiffness. It may be compared in form to equation (4.9). In the absence of an effective stress space concept, equation (4.11) does not exist and maybe interpreted as an identity relation.

The elastic constitutive relation is written in the actual damage state as:

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon} \quad (4.12)$$

A similar elastic constitutive relation in the fictitious state can be written as follows:

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{E}} : \bar{\boldsymbol{\varepsilon}} \quad (4.13)$$

Substituting equation (4.11) into equation (4.13), we obtain:

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{E}} : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) : \boldsymbol{\varepsilon} \quad (4.14)$$

and substituting equation (4.10) into equation (4.12), we obtain:

$$\boldsymbol{\sigma} = (\mathbf{I}^{(8)} - \boldsymbol{\varphi}^{(8)}) :: \bar{\mathbf{E}} : \boldsymbol{\varepsilon} \quad (4.15)$$

Solving equation (4.14) for $\boldsymbol{\varepsilon}$ and substituting the result into equation (4.15), we obtain:

$$\boldsymbol{\sigma} = (\mathbf{I}^{(8)} - \boldsymbol{\varphi}^{(8)}) :: \bar{\mathbf{E}} : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(8)}) : \bar{\mathbf{E}}^{-1} : \bar{\boldsymbol{\sigma}} \quad (4.16)$$

Equation (4.16) relates the actual-state stress to the fictitious-state stress. This equation can be reduced to a similar equation involving a damage tensor of rank four by using the following relation:

$$(\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}} : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(4)}) = (\mathbf{I}^{(8)} - \boldsymbol{\varphi}^{(8)}) :: \bar{\mathbf{E}} : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(8)}) \quad (4.17)$$

Then, equation (4.16) becomes:

$$\boldsymbol{\sigma} = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}} : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}}^{-1} : \bar{\boldsymbol{\sigma}} \quad (4.18)$$

and substituting \mathbf{E} for $(\mathbf{I}^{(8)} - \boldsymbol{\varphi}^{(8)}) :: \bar{\mathbf{E}}$ in equation (4.17) and rearranging, we obtain:

$$\mathbf{E} = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}} : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(4)}) : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) \quad (4.19)$$

Equation (4.19) is very important because it relates the damaged fourth-rank elasticity

tensor \mathbf{E} to the effective fourth-rank elasticity tensor $\bar{\mathbf{E}}$ through a formula that involves the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$ and fourth-rank tensorial functions of the damage tensors $\mathbf{L}^{-1}(\boldsymbol{\varphi}^{(4)})$ and $\mathbf{L}(\boldsymbol{\varphi}^{(8)})$, where the damage tensors $\boldsymbol{\varphi}^{(4)}$ and $\boldsymbol{\varphi}^{(8)}$ can be related to the fabric tensors.

Thus, it has been shown by (Voyiadjis and Kattan, 2005) that using the general hypothesis of strain transformation of equation (4.11), equation (4.10) which involves eight-rank tensors can be reduced to equation (4.19).

By comparing equation (4.19) with equation (4.9), we realize that both equations describe the same quantity \mathbf{E} . Equation (4.19) describes the elasticity tensor for the damaged material in terms of the damage tensor, while equation (4.9) describes the same elasticity tensor in terms of the fabric tensors. Equating these two equations, we obtain:

$$(\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}} : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(4)}) : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) = [\lambda (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) + 2\mu (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \underline{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})] \quad (4.20)$$

Solving the above expression for $\boldsymbol{\varphi}^{(4)}$, we obtain:

$$\boldsymbol{\varphi}^{(4)} = \mathbf{I}^{(4)} - [\lambda (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) + 2\mu (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \underline{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})] : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(8)}) : \mathbf{L}(\boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}}^{-1} \quad (4.21)$$

Equation (4.21) represents an expression for the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$ in terms of the zero-rank (scalar) fabric tensor $\mathbf{G}^{(0)}$ and the second-rank fabric tensor $\mathbf{G}^{(2)}$. Other elements appearing in the equation are constant scalars like λ and μ or constant tensors like $\mathbf{I}^{(2)}$, $\mathbf{I}^{(4)}$ and $\bar{\mathbf{E}}$. The functions $\mathbf{L}(\boldsymbol{\varphi}^{(4)})$ and $\mathbf{L}(\boldsymbol{\varphi}^{(8)})$ must be substituted for in terms of other parameters. If we substitute $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{L}(\boldsymbol{\varphi}^{(4)}) = \mathbf{I}^{(4)}$, we obtain the special case of the hypothesis of elastic strain equivalence. In this case, equation (4.21) reduces to:

$$\begin{aligned} \boldsymbol{\varphi}^{(4)} = & \mathbf{I}^{(4)} - [\lambda(\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \\ & + 2\mu(\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \underline{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})] : \bar{\mathbf{E}}^{-1} \end{aligned} \quad (4.22)$$

whereas, if we substitute $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)})$ and $\mathbf{L}(\boldsymbol{\varphi}^{(4)}) = \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)})$, we obtain the special case of the hypothesis of elastic energy equivalence. In this case, equation (4.21) becomes:

$$\begin{aligned} \boldsymbol{\varphi}^{(4)} = & \mathbf{I}^{(4)} - [\lambda(\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \\ & + 2\mu(\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \underline{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})] : \mathbf{M}^T(\boldsymbol{\varphi}^{(8)}) : \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}}^{-1} \end{aligned} \quad (4.23)$$

where \mathbf{M} is the fourth-rank damage effect tensor used by Voyiadjis and Kattan (1999).

It should be noted that equations (4.21), (4.22), and (4.23) are valid for an isotropic elastic material.

Next, we apply the spectral decomposition theorem to the second-rank fabric tensor $\mathbf{G}^{(2)}$ in order to write equation (4.9) in terms of the eigenvalues of $\mathbf{G}^{(2)}$. The second-rank fabric tensor $\mathbf{G}^{(2)}$ can be written as (Zysset and Curnier, 1995):

$$\mathbf{G}_i^{(2)} = \sum_{i=1}^3 g_i (\mathbf{g}_i \otimes \mathbf{g}_i) \quad (\text{no sum over } i) \quad (4.24)$$

where g_i ($i=1,2,3$) are the eigenvalues of $\mathbf{G}^{(2)}$ and \mathbf{g}_i ($i=1,2,3$) are the corresponding eigenvectors. The dyadic product of two eigenvectors ($\mathbf{g}_i \otimes \mathbf{g}_i$) gives rise to a second-

rank tensor. It is clear that $\sum_{i=1}^3 \mathbf{G}_i^{(2)} = \mathbf{I}^{(2)}$. Using this terminology, equation (4.9) can be

written as (Zysset and Curnier, 1995):

$$\begin{aligned} \mathbf{E}^{(4)} = & (\lambda + 2\mu)m_i^{2k} (\mathbf{G}_i^{(2)} \otimes \mathbf{G}_i^{(2)}) + \lambda m_i^k m_j^k (\mathbf{G}_i^{(2)} \otimes \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \otimes \mathbf{G}_i^{(2)}) \\ & + 2\mu m_i^k m_j^k (\mathbf{G}_i^{(2)} \underline{\otimes} \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \underline{\otimes} \mathbf{G}_i^{(2)}) \end{aligned} \quad (4.25)$$

where k is a constant scalar parameter (with a value less than zero) and m_i is given by

the following:

$$\mathbf{m}_i = \mathbf{G}^{(0)} + \mathbf{g}_i \quad (4.26)$$

In equation (4.26), we should note that $\sum_{i=1}^3 \mathbf{m}_i = \text{constant}$. It should be noted that equation (4.25) is valid for an isotropic elastic material.

The reason behind introducing the spectral decomposition theorem into the formulation is that equation (4.9) was originally introduced for granular materials. Due to the difference in the micro-structural properties between granular materials and damaged metallic materials, equation (4.9) had to be modified to equation (4.25) in order to be applicable for the study of damage in metallic materials.

Based on equation (4.25), Cauvin and Testa (1995) introduced the general 6 x 6 matrix representation of the fourth-rank elasticity tensor of the damaged material. Voyiadjis and Kattan (2005) introduced the 3 x 3 matrix representation of that tensor for the case of plane stress as follows:

$$\mathbf{E} = \frac{E}{1-\nu^2} \begin{pmatrix} m_1^{2k} & \nu m_1^k m_2^k & 0 \\ \nu m_2^k m_1^k & m_2^{2k} & 0 \\ 0 & 0 & \frac{1-\nu}{2} m_1^k m_2^k \end{pmatrix} \quad (4.27)$$

where E and ν are Young's modulus and Poisson's ratio, respectively. This equation can be compared to the well known 3 x 3 representation of the effective elasticity tensor $\bar{\mathbf{E}}$ given as:

$$\bar{\mathbf{E}} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \quad (4.28)$$

All the parameters in equation (4.27) are constants except for the fabric tensor parameters m_i which depends on the zero-rank fabric tensor $G^{(0)}$ and the eigenvalues (g_i) of the second-rank fabric tensor $G^{(2)}$. Voyiadjis and Kattan (2005) have also shown that for an isotropic material, and for the case of plane stress, the following components of the fourth-rank damage tensor $\Phi^{(4)}$ can be used to write an expression for the 3 x 3 matrix representation of the damage tensor $\Phi^{(4)}$:

$$\Phi_{1111} = 1 - \frac{m_1^k (m_1^k - v^2 m_2^k)}{1 - v^2} \quad (4.29a)$$

$$\Phi_{1212} = \frac{v m_1^k (m_1^k - m_2^k)}{1 - v^2} \quad (4.29b)$$

$$\Phi_{2222} = 1 - \frac{m_2^k (m_2^k - v^2 m_1^k)}{1 - v^2} \quad (4.29c)$$

$$\Phi_{2121} = \frac{v m_2^k (m_2^k - m_1^k)}{1 - v^2} \quad (4.29d)$$

$$\Phi_{3333} = 1 - m_1^k m_2^k \quad (4.29e)$$

where the 3 x 3 representation of $\Phi^{(4)}$ is as follows:

$$\Phi^{(4)} = \begin{bmatrix} \Phi_{1111} & \Phi_{1212} & 0 \\ \Phi_{2121} & \Phi_{2222} & 0 \\ 0 & 0 & \Phi_{3333} \end{bmatrix} \quad (4.30)$$

The equations appearing in this chapter will be used in subsequent chapters to derive the theory of damage mechanics of composite materials using fabric tensors and to solve examples of plane stress (lamina) problems.

5 The Continuum Approach

The Continuum Approach for damage mechanics with fabric tensors in composite materials will be presented in this chapter. In this approach, damage is introduced to the composite system as a whole through the damage effect tensor $\mathbf{M}(\boldsymbol{\varphi}^{(4)})$, which is assumed to be a function of the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$. The general hypothesis of strain transformation (Voyiadjis and Kattan, 2005) is also used. The composite medium will be treated as a homogenized material, i.e., properties of the constituents are homogenized before the damage with fabric tensors is introduced to the system. In this regard, the damage variable is an overall parameter describing damage in the composite system. In the formulation, a fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$ and an eighth-rank damage tensor $\boldsymbol{\varphi}^{(8)}$ are used. Eventually, however, all the equations are written in terms of the fourth-rank tensor $\boldsymbol{\varphi}^{(4)}$. A numerical example illustrating the Continuum Approach will be presented in chapter 7, section 7.1.

5.1 Elastic Constitutive Equations

Two steps are involved in the Continuum Approach (see Figure 5.1). In the first step, the elastic constitutive equations are formulated in an undamaged composite system by making use of the concept of effective stress presented first by Kachanov (1958). In the second step, damage is introduced to the composite (homogenized) system as a whole through the use of an overall damage variable. In Figure 5.1, \bar{C}^m , \bar{C}^f , and \bar{C}^i are the effective (undamaged) matrix, fiber, and interface configurations, respectively, \bar{C} is the effective (undamaged) composite configuration, while C is the damaged composite configuration.

In the first step: for elastic composites, the following linear relation is used for the constituents in their undamaged configuration \bar{C}^k :

$$\bar{\sigma}^k = \bar{E}^k : \bar{\epsilon}^k \quad (5.1)$$

where $\bar{\sigma}^k$, \bar{E}^k , and $\bar{\epsilon}^k$ are the effective constituent stress tensor, effective constituent elasticity tensor, and effective constituent strain tensor, respectively, and $k = m$ (matrix), f (fiber), and i (interface).

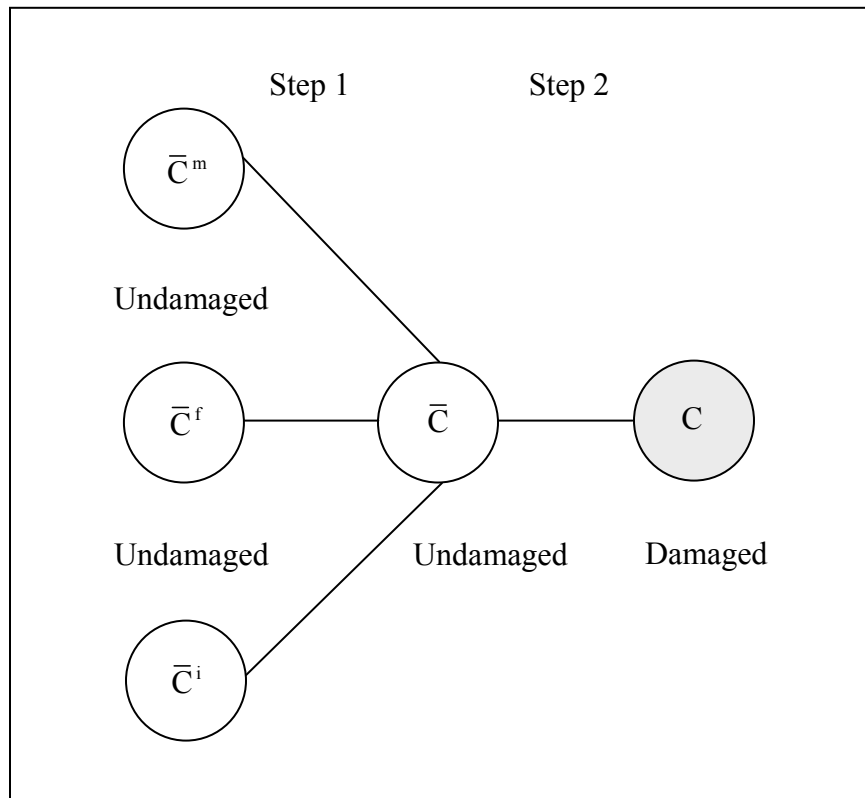


Figure 5.1: Schematic Illustration of the Continuum Approach for a Composite System

The effective constituent strain tensor $\bar{\epsilon}^k$ is related to the effective composite strain tensor $\bar{\epsilon}$ by:

$$\bar{\boldsymbol{\varepsilon}}^k = \bar{\mathbf{A}}^k : \bar{\boldsymbol{\varepsilon}} \quad (5.2)$$

where $\bar{\mathbf{A}}^k$ is the fourth-rank effective strain concentration tensor. This tensor is usually determined through the use of a homogenization technique such as the Voigt model, Reuss model, or Mori-Tanaka model (Voyiadjis and Kattan, 1999). See also the numerical examples in chapter 7 for more details about these models.

In the effective composite configuration $\bar{\mathbf{C}}$, the following linear relation describes the elastic response:

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{E}} : \bar{\boldsymbol{\varepsilon}} \quad (5.3)$$

where $\bar{\mathbf{E}}$ is the fourth-rank constant effective elasticity tensor. Applying equations (5.1), (5.2), and (5.3) into the following rule of mixtures:

$$\bar{\boldsymbol{\sigma}} = \sum_k \bar{c}^k \boldsymbol{\sigma}^k \quad (5.4)$$

where \bar{c}^k is the effective constituent's volume fraction satisfying $\sum_k \bar{c}^k = 1$, and $\bar{\boldsymbol{\sigma}}$ is the composite effective stress tensor, one obtains the following expression for $\bar{\mathbf{E}}$:

$$\bar{\mathbf{E}} = \sum_k \bar{c}^k \bar{\mathbf{E}}^k : \bar{\mathbf{A}}^k \quad (5.5)$$

where $\bar{\mathbf{E}}$ is the composite effective elasticity tensor.

In the second step: Damage is now introduced to the composite system as a whole through a general hypothesis of strain transformation (Voyiadjis and Kattan, 2005). It is postulated that the elastic strain tensor $\boldsymbol{\varepsilon}$ in the actual damaged state is related to the effective elastic strain tensor $\bar{\boldsymbol{\varepsilon}}$ in the fictitious state by the following transformation law:

$$\bar{\boldsymbol{\varepsilon}} = \mathbf{L}(\boldsymbol{\varphi}^{(8)}) : \boldsymbol{\varepsilon} \quad (5.6)$$

where $\mathbf{L}(\boldsymbol{\varphi}^{(8)})$ is a fourth-rank tensorial function of the eighth-rank damage tensor $\boldsymbol{\varphi}^{(8)}$. It is noted that the two hypotheses (elastic strain equivalence and elastic energy equivalence) are obtained as special cases of equation (5.6). By using $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{I}^{(4)}$, we obtain the hypothesis of elastic strain equivalence, and by using $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)})$, we obtain the hypothesis of elastic energy equivalence, where the fourth-rank tensor $\mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)})$ is the damage effect tensor used by Voyiadjis and Kattan (1999).

Next, the fourth-rank damage effect tensor $\mathbf{M}(\boldsymbol{\varphi}^{(4)})$, used by Voyiadjis and Kattan (1999), is introduced as:

$$\bar{\boldsymbol{\sigma}} = \mathbf{M}(\boldsymbol{\varphi}^{(4)}) : \boldsymbol{\sigma} \quad (5.7)$$

In order to incorporate fabric tensors in this work, the fourth-rank damage effect tensor will be defined here as follows:

$$\mathbf{M}(\boldsymbol{\varphi}^{(4)}) = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)})^{-1} \quad (5.8)$$

where $\mathbf{I}^{(4)}$ is the fourth-rank identity tensor.

In the composite damaged (actual) configuration, the following linear elastic relation holds:

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon} \quad (5.9)$$

where it is emphasized that \mathbf{E} is a variable fourth-rank elasticity tensor that depends on the state of damage, i.e. $\boldsymbol{\varphi}^{(4)}$. Substituting equation (5.6) into equation (5.3), one obtains:

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{E}} : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) : \boldsymbol{\varepsilon} \quad (5.10)$$

Solving equation (5.10) for $\boldsymbol{\varepsilon}$ and substituting the result into equation (5.9), one obtains the following relation:

$$\boldsymbol{\sigma} = \mathbf{E} : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(8)}) : \bar{\mathbf{E}}^{-1} : \bar{\boldsymbol{\sigma}} \quad (5.11)$$

Comparing equation (5.11) with equation (5.7), it can be seen that:

$$\mathbf{M}^{-1}(\boldsymbol{\varphi}^{(4)}) = \mathbf{E} : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(8)}) : \bar{\mathbf{E}}^{-1} \quad (5.12)$$

By rearranging the terms in equation (5.12), one obtains an expression for the fourth-rank elasticity tensor for the composite system in the actual (damaged) configuration:

$$\mathbf{E} = \mathbf{M}^{-1}(\boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}} : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) \quad (5.13)$$

Equation (5.13) illustrates the exact dependence of \mathbf{E} on the damage state. By substituting the result obtained for $\bar{\mathbf{E}}$ (equation (5.5)) into equation (5.13), one obtains the following:

$$\mathbf{E} = \mathbf{M}^{-1}(\boldsymbol{\varphi}^{(4)}) : (\bar{c}^m \bar{\mathbf{E}}^m : \bar{\mathbf{A}}^m + \bar{c}^f \bar{\mathbf{E}}^f : \bar{\mathbf{A}}^f + \bar{c}^i \bar{\mathbf{E}}^i : \bar{\mathbf{A}}^i) : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) \quad (5.14)$$

Equation (5.14) is the general transformation relation for the elasticity tensor. Next, the following two special cases are obtained:

1. Substituting $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{I}^{(4)}$ into equation (5.13), one obtains the special case of elastic strain equivalence:

$$\mathbf{E} = \mathbf{M}^{-1}(\boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}} \quad (5.15)$$

2. Substituting $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)})$ into equation (5.13), one obtains the special case of elastic energy equivalence:

$$\mathbf{E} = \mathbf{M}^{-1}(\boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}} : \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)}) \quad (5.16)$$

The expression for the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$ is given by Voyiadjis and Kattan (2005) for an isotropic elastic material. Modifying their expression to include general orthotropic behavior described by Zysset and Curnier (1995) gives the following definition of the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$:

$$\begin{aligned}
\boldsymbol{\varphi}^{(4)} = & \mathbf{I}^{(4)} - [\lambda_{ij} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j + \lambda_{ij}^* \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \} + 2\mu_{ij} \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \}] : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(8)}) : \mathbf{L}(\boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}}^{-1}
\end{aligned} \tag{5.17}$$

where λ_{ij} ($i = j$), λ_{ij}^* ($i < j$), and μ_{ij} ($i < j$) are Lamé's constants for an orthotropic material, $\mathbf{G}^{(0)}$, $\mathbf{G}^{(2)}$ are the zero-rank, and second-rank fabric tensors, and the notation $(\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i$ ($i = 1, 2, 3$) is defined later in equation (5.24).

Applying equation (5.17) into equation (5.8), the following expression for $\mathbf{M}(\boldsymbol{\varphi}^{(4)})$ is obtained:

$$\begin{aligned}
\mathbf{M}(\boldsymbol{\varphi}^{(4)}) = & ([\lambda_{ij} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j + \lambda_{ij}^* \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \} + 2\mu_{ij} \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \}] : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(8)}) : \mathbf{L}(\boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}}^{-1})^{-1}
\end{aligned} \tag{5.18}$$

Substituting equation (5.18) into equation (5.13), a general expression for the elasticity tensor of the composite system in the damaged configuration is obtained:

$$\begin{aligned}
\mathbf{E} = & [\lambda_{ij} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j + \lambda_{ij}^* \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \} + 2\mu_{ij} \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \}] : \mathbf{L}^{-1}(\boldsymbol{\varphi}^{(8)}) : \mathbf{L}(\boldsymbol{\varphi}^{(4)}) : \mathbf{L}(\boldsymbol{\varphi}^{(8)})
\end{aligned} \tag{5.19}$$

Although $\bar{\mathbf{E}}$ does not appear in equation (5.19), the elastic properties of the composite material are already presented by Lamé's constants. Furthermore, $\bar{\mathbf{E}}$ will appear when we make the substitutions for $\mathbf{L}(\boldsymbol{\varphi}^{(8)})$. Equation (5.19) is the general expression for the damaged elasticity tensor \mathbf{E} in terms of fabric tensors and damage tensors. Next the following two special cases are obtained:

1. For the special case of elastic strain equivalence, $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{L}(\boldsymbol{\varphi}^{(4)}) = \mathbf{I}^{(4)}$, one obtains:

$$\begin{aligned}
\mathbf{E} = & [\lambda_{ij} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j + \lambda_{ij}^* \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \} + 2\mu_{ij} \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \}]
\end{aligned} \tag{5.20}$$

2. For the special case of elastic energy equivalence, $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)})$ and $\mathbf{L}(\boldsymbol{\varphi}^{(4)}) = \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)})$, where both $\mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)})$ and $\mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)})$ are fourth-rank tensors, one obtains:

$$\begin{aligned}
\mathbf{E} = & [\lambda_{ij} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j + \lambda_{ij}^* \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \} + 2\mu_{ij} \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \}] : \mathbf{M}^T(\boldsymbol{\varphi}^{(8)}) : \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)}) : \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)})
\end{aligned} \tag{5.21}$$

If we further simplify the equation above by adopting $\mathbf{M}(\boldsymbol{\varphi}^{(8)})$ such that $\mathbf{M}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}(\boldsymbol{\varphi}^{(4)})$, we get:

$$\begin{aligned}
\mathbf{E} = & [\lambda_{ij} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j + \lambda_{ij}^* \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \} + 2\mu_{ij} \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \}] : \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)})
\end{aligned} \tag{5.22}$$

Substituting equation (5.18) into equation (5.22) while using the above mentioned assumption of elastic energy equivalence, $\mathbf{L}(\boldsymbol{\varphi}^{(4)}) = \mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)})$, one obtains:

$$\begin{aligned}
\mathbf{E} = & [\lambda_{ij} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j + \lambda_{ij}^* \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \} + 2\mu_{ij} \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \}] : \bar{\mathbf{E}}^{-T} : [\lambda_{ij} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \\
& + \lambda_{ij}^* \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \} \\
& + 2\mu_{ij} \{ (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \bar{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \}]^T
\end{aligned} \tag{5.23}$$

Equation (5.23) represents an expression for the elasticity tensor of the composite system in the damaged configuration in terms of the fabric tensors. Therefore, if a micro-crack distribution of the composite material is obtained, this elasticity tensor can be determined by making use of the homogenized properties $(\lambda_{ij} (i = j), \lambda_{ij}^* (i < j) \text{ and } \mu_{ij} (i < j))$, the fabric tensors $\mathbf{G}^{(0)}$ and $\mathbf{G}^{(2)}$, and the constant elasticity tensor $\bar{\mathbf{E}}$. However, it should be noted that equation (5.23) is valid only for the case of the hypothesis of elastic energy equivalence.

Next, we apply the spectral decomposition theorem to the second-rank fabric tensor $\mathbf{G}^{(2)}$ (Zysset and Curnier, 1995):

$$\mathbf{G}_i^{(2)} = \sum_{i=1}^3 g_i (\mathbf{g}_i \otimes \mathbf{g}_i) \quad (\text{no sum over } i) \quad (5.24)$$

where $g_i (i=1,2,3)$ are the eigenvalues of $\mathbf{G}^{(2)}$ and $\mathbf{g}_i (i=1,2,3)$ are the corresponding eigenvectors. The dyadic product of two eigenvectors $(\mathbf{g}_i \otimes \mathbf{g}_i)$ gives rise to a second-rank tensor $\mathbf{G}_i^{(2)}$. It is clear that $\sum_{i=1}^3 \mathbf{G}_i^{(2)} = \mathbf{I}^{(2)}$. Using this new terminology (Zysset and

Curnier, 1995), we can write equation (5.23) as follows:

$$\begin{aligned} \mathbf{E} = & [\lambda_{ij} m_p^{2k} (\mathbf{G}_i^{(2)} \otimes \mathbf{G}_j^{(2)}) + \lambda_{ij}^* m_p^k m_q^k (\mathbf{G}_i^{(2)} \otimes \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \otimes \mathbf{G}_i^{(2)}) \\ & + 2\mu_{ij} m_p^k m_q^k (\mathbf{G}_i^{(2)} \bar{\otimes} \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \bar{\otimes} \mathbf{G}_i^{(2)})] : \bar{\mathbf{E}}^{-T} : [\lambda_{ij} m_p^{2k} (\mathbf{G}_i^{(2)} \otimes \mathbf{G}_j^{(2)}) \\ & + \lambda_{ij}^* m_p^k m_q^k (\mathbf{G}_i^{(2)} \otimes \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \otimes \mathbf{G}_i^{(2)}) + 2\mu_{ij} m_p^k m_q^k (\mathbf{G}_i^{(2)} \bar{\otimes} \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \bar{\otimes} \mathbf{G}_i^{(2)})]^T \end{aligned} \quad (5.25)$$

where k is a constant scalar parameter with a value less than zero and m_p and m_q (p and q are subscripts not indices) correspond to two parameters given by the following (Zysset and Curnier, 1995):

$$m_i = G^{(0)} + g_i \quad (i = 1, 2) \quad (5.26)$$

where $\sum_{i=1}^2 m_i = \text{constant}$.

5.2 Damage Evolution

Next, indicial notation as well as tensorial notation will be used to derive the required damage evolution equation. The reason for using indicial notation is that some of the formulas will be much easier to handle when they are in the indicial form. In the derivation of damage evolution, we adopt the hypothesis of elastic energy equivalence and the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$.

We start with the elastic strain energy function U defined as:

$$U = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \quad \text{or} \quad U = \frac{1}{2} \boldsymbol{\sigma}_{ij} \boldsymbol{\varepsilon}_{ij} \quad (5.27a,b)$$

Using equations (5.9) and (5.13), U can be written as:

$$U = \frac{1}{2} \mathbf{M}^{-1}(\boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}} : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) : \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \quad (5.28a)$$

or

$$U = \frac{1}{2} \mathbf{M}_{ijkl}^{-1}(\boldsymbol{\varphi}^{(4)}) \bar{\mathbf{E}}_{klmn} \mathbf{L}_{mnpq}(\boldsymbol{\varphi}^{(8)}) \boldsymbol{\varepsilon}_{pq} \boldsymbol{\varepsilon}_{ij} \quad (5.28b)$$

Substituting for $\mathbf{M}^{-1}(\boldsymbol{\varphi}^{(4)})$ by taking the inverse of equation (5.8), and applying the hypothesis of elastic energy equivalence $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)})$ as well as adopting $\mathbf{M}(\boldsymbol{\varphi}^{(8)})$ such that $\mathbf{M}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}(\boldsymbol{\varphi}^{(4)})$, one obtains:

$$U = \frac{1}{2} (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)}) : \bar{\mathbf{E}} : (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)})^T : \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \quad (5.29a)$$

or in indicial notation:

$$U = \frac{1}{2} (\mathbf{I} - \boldsymbol{\varphi})_{ijkl} \bar{\mathbf{E}}_{klmn} (\mathbf{I} - \boldsymbol{\varphi})_{pqmn} \boldsymbol{\varepsilon}_{pq} \boldsymbol{\varepsilon}_{ij} \quad (5.29b)$$

Next, we differentiate the elastic energy function U of equation (5.29) with respect to the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$ to obtain the fourth-rank thermodynamic force \mathbf{Y} associated with the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$:

$$\mathbf{Y} = \frac{\partial U}{\partial \boldsymbol{\varphi}^{(4)}} = -\boldsymbol{\varepsilon}^T \otimes \left[\bar{\mathbf{E}} : (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)})^T : \boldsymbol{\varepsilon} \right] \quad (5.30a)$$

or

$$\mathbf{Y}_{rstu} = \frac{\partial U}{\partial \boldsymbol{\varphi}_{rstu}} = -(\mathbf{I} - \boldsymbol{\varphi})_{pqmn} \boldsymbol{\varepsilon}_{pq} \boldsymbol{\varepsilon}_{rs} \bar{\mathbf{E}}_{tumn} \quad (5.30b)$$

As can be seen from equations (5.30a,b), the thermodynamic force \mathbf{Y} is a function of the fourth-rank damage tensor and the second-rank strain tensor. Therefore, if we use the chain rule to differentiate the thermodynamic force \mathbf{Y} , we obtain the following:

$$d\mathbf{Y}_{rstu} = \frac{\partial \mathbf{Y}_{rstu}}{\partial \boldsymbol{\varphi}_{ijkl}} d\boldsymbol{\varphi}_{ijkl} + \frac{\partial \mathbf{Y}_{rstu}}{\partial \boldsymbol{\varepsilon}_{ab}} d\boldsymbol{\varepsilon}_{ab} \quad (5.31)$$

If we differentiate the thermodynamic force \mathbf{Y} with respect to the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$, we obtain:

$$\frac{\partial \mathbf{Y}}{\partial \boldsymbol{\varphi}^{(4)}} = \boldsymbol{\varepsilon}^T \otimes \bar{\mathbf{E}} \otimes \boldsymbol{\varepsilon} \quad \text{or} \quad \frac{\partial \mathbf{Y}_{rstu}}{\partial \boldsymbol{\varphi}_{ijkl}} = \boldsymbol{\varepsilon}_{ij} \boldsymbol{\varepsilon}_{rs} \bar{\mathbf{E}}_{tukl} \quad (5.32a,b)$$

and the derivative of \mathbf{Y} with respect to the second-rank strain tensor gives:

$$\frac{\partial \mathbf{Y}_{rstu}}{\partial \boldsymbol{\varepsilon}_{ab}} = -(\mathbf{I} - \boldsymbol{\varphi})_{pqmn} \left[\delta_{pa} \delta_{qb} \boldsymbol{\varepsilon}_{rs} + \delta_{ra} \delta_{sb} \boldsymbol{\varepsilon}_{pq} \right] \bar{\mathbf{E}}_{tumn} \quad (5.33)$$

Substituting equations (5.32) and (5.33) back into equation (5.31), the incremental equation of the thermodynamic force \mathbf{Y} is obtained as follows:

$$d\mathbf{Y}_{rstu} = \boldsymbol{\varepsilon}_{ij} \boldsymbol{\varepsilon}_{rs} \bar{\mathbf{E}}_{tukl} d\boldsymbol{\varphi}_{ijkl} - (\mathbf{I} - \boldsymbol{\varphi})_{pqmn} \left[\delta_{pa} \delta_{qb} \boldsymbol{\varepsilon}_{rs} + \delta_{ra} \delta_{sb} \boldsymbol{\varepsilon}_{pq} \right] \bar{\mathbf{E}}_{tumn} d\boldsymbol{\varepsilon}_{ab} \quad (5.34)$$

Next, we introduce a generalized damage criterion $g(\mathbf{Y}, L)$. This damage criterion is a function of the thermodynamic force \mathbf{Y} associated with the fourth-rank damage tensor and the damage strengthening parameter $L(\ell)$ which in turn is a function of the overall scalar damage parameter ℓ . The function $g(\mathbf{Y}, L)$ is given in indicial notation as:

$$g = \sqrt{\frac{1}{2} \mathbf{Y}_{klij} \mathbf{J}_{klmn} \mathbf{Y}_{mnij}} - \ell_o - L(\ell) \leq 0 \quad (5.35)$$

where \mathbf{J} is a constant fourth-rank tensor given in Appendix A, and ℓ_o is the initial threshold of damage.

In order to derive a normality rule for the evolution of damage, we start with the power of dissipation Π which is given by:

$$\Pi = -\mathbf{Y} : d\boldsymbol{\phi}^{(4)} - L d\ell \quad (5.36)$$

The problem here is to extremize Π subject to the condition $g = 0$. Using the mathematical theory of functions of several variables, we introduce the Lagrangian multiplier $d\lambda$ and form the objective function $\Psi(\mathbf{Y}, L)$ such that:

$$\Psi = \Pi - d\lambda . g \quad (5.37)$$

The problem now reduces to extremizing the function Ψ . To do so, the two necessary

conditions are $\frac{\partial \Psi}{\partial \mathbf{Y}^{(4)}} = 0$ and $\frac{\partial \Psi}{\partial L} = 0$. Using these conditions, along with equations (5.36)

and (5.37), we obtain:

$$d\boldsymbol{\phi}^{(4)} = -d\lambda \frac{\partial g}{\partial \mathbf{Y}} \quad \text{or} \quad d\phi_{mnpq} = -d\lambda \frac{\partial g}{\partial Y_{mnpq}} \quad (5.38a,b)$$

and

$$d\ell = d\lambda \quad (5.39)$$

In order to solve the differential equation given by equation (5.38), we must first find an expression for the Lagrangian multiplier $d\lambda$. This can be done by invoking the consistency condition $dg = 0$. Applying the chain rule of differentiation to equation (5.35), we obtain:

$$\frac{\partial g}{\partial \mathbf{Y}_{ijkl}} d\mathbf{Y}_{ijkl} + \frac{\partial g}{\partial L} dL = 0 \quad (5.40)$$

The derivative of g with respect to the thermodynamic force tensor \mathbf{Y} is given as: (from equation (5.35))

$$\frac{\partial g}{\partial \mathbf{Y}_{ijkl}} = \frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnkl}}{2\sqrt{\frac{1}{2} \mathbf{Y}_{rspq} \mathbf{J}_{rstu} \mathbf{Y}_{tupq}}} \quad (5.41)$$

Observing that $(\partial g / \partial L) = -1$, and $dL = (\partial L / \partial \ell) d\ell$, equation (5.40) can be solved for $d\ell$ as follows:

$$d\ell = \frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnkl} d\mathbf{Y}_{ijkl}}{2(\partial L / \partial \ell) \sqrt{\frac{1}{2} \mathbf{Y}_{rspq} \mathbf{J}_{rstu} \mathbf{Y}_{tupq}}} \quad (5.42)$$

Substituting equation (5.42) back into equation (5.38) with the use of equation (5.39), we obtain the following:

$$d\phi_{rstu} = \frac{-\mathbf{J}_{ijmn} \mathbf{Y}_{mnkl} d\mathbf{Y}_{ijkl} \mathbf{J}_{rsvw} \mathbf{Y}_{vwtu}}{2(\partial L / \partial \ell) (\mathbf{Y}_{cdab} \mathbf{J}_{cdef} \mathbf{Y}_{efab})} \quad (5.43)$$

Substituting for $d\mathbf{Y}_{ijkl}$ from equation (5.34) and rearranging the terms to factor out common parameters, we obtain:

$$\left[\delta_{er} \delta_{fs} \delta_{vt} \delta_{wu} + \frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnl} \boldsymbol{\varepsilon}_{ef} \boldsymbol{\varepsilon}_{ij} \bar{\mathbf{E}}_{klvw} \mathbf{J}_{rscd} \mathbf{Y}_{cdtu}}{2(\partial L / \partial \ell)(\mathbf{Y}_{\delta\gamma\alpha\beta} \mathbf{J}_{\delta\gamma\mu\theta} \mathbf{Y}_{\mu\theta\alpha\beta})} \right] d\boldsymbol{\varphi}_{efvw} =$$

$$= \left[\frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnl} (\mathbf{I} - \boldsymbol{\varphi})_{pqgh} (\delta_{pa} \delta_{qb} \boldsymbol{\varepsilon}_{ij} + \delta_{ia} \delta_{jb} \boldsymbol{\varepsilon}_{pq}) \bar{\mathbf{E}}_{klgh} d\boldsymbol{\varepsilon}_{ab} \mathbf{J}_{rscd} \mathbf{Y}_{cdtu}}{2(\partial L / \partial \ell)(\mathbf{Y}_{\delta\gamma\alpha\beta} \mathbf{J}_{\delta\gamma\mu\theta} \mathbf{Y}_{\mu\theta\alpha\beta})} \right] \quad (5.44)$$

which can be rewritten as:

$$\mathbf{A}_{efvwrstu} d\boldsymbol{\varphi}_{efvw} = \mathbf{B}_{rstuab} d\boldsymbol{\varepsilon}_{ab} \quad (5.45)$$

where

$$\mathbf{A}_{efvwrstu} = \left[\delta_{er} \delta_{fs} \delta_{vt} \delta_{wu} + \frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnl} \boldsymbol{\varepsilon}_{ef} \boldsymbol{\varepsilon}_{ij} \bar{\mathbf{E}}_{klvw} \mathbf{J}_{rscd} \mathbf{Y}_{cdtu}}{2(\mathbf{Y}_{\delta\gamma\alpha\beta} \mathbf{J}_{\delta\gamma\mu\theta} \mathbf{Y}_{\mu\theta\alpha\beta})(\partial L / \partial \ell)} \right] \quad (5.46)$$

and

$$\mathbf{B}_{rstuab} = \left[\frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnl} (\mathbf{I} - \boldsymbol{\varphi})_{pqgh} (\delta_{pa} \delta_{qb} \boldsymbol{\varepsilon}_{ij} + \delta_{ia} \delta_{jb} \boldsymbol{\varepsilon}_{pq}) \bar{\mathbf{E}}_{klgh} \mathbf{J}_{rscd} \mathbf{Y}_{cdtu}}{2(\mathbf{Y}_{\delta\gamma\alpha\beta} \mathbf{J}_{\delta\gamma\mu\theta} \mathbf{Y}_{\mu\theta\alpha\beta})(\partial L / \partial \ell)} \right] \quad (5.47)$$

Further rearrangement of equation (5.44) will give the following:

$$d\boldsymbol{\varphi}_{efvw} = \mathbf{A}_{efvwrstu}^{-1} \mathbf{B}_{rstuab} d\boldsymbol{\varepsilon}_{ab} \quad (5.48)$$

where it should be noted that $\mathbf{A}(\mathbf{Y}, \boldsymbol{\varepsilon})$ and $\mathbf{B}(\mathbf{Y}, \boldsymbol{\varepsilon})$ are tensorial functions of \mathbf{Y} , and $\boldsymbol{\varepsilon}$.

Equation (5.48) represents the general evolution equation for the damage tensor. The

above evolution equation applies for the general case of elastic deformation and damage.

Equation (5.48) represents the damage evolution equation, i.e., an equation relating the

increment of the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)}$ to the increment of the strain tensor $\boldsymbol{\varepsilon}_{ab}$.

Next, we discuss a special case of damage evolution. For the case of a one dimensional problem, assuming that the Poisson's ratio $\nu = 0$, and that all the components of $\boldsymbol{\varphi}^{(4)}$ are zeros except $\boldsymbol{\varphi}_{1111}^{(4)}$, which we denote here by φ , and also noting

that all the components of \mathbf{J} vanishes except $\mathbf{J}_{1111} = 1$, equation (5.44) reduces to the following:

$$\left(\frac{\partial L}{\partial \ell}\right) d\varphi = \bar{E} \varepsilon \left(d\varepsilon(1-\varphi) - \frac{1}{2} \varepsilon d\varphi \right) \quad (5.49)$$

where φ and ε are the scalar damage and strain variables, and \bar{E} is Young's modulus for the virgin material.

This damage evolution equation can be solved easily by the simple change of variables $x = \frac{1}{2} \varepsilon^2 (1-\varphi)$ where $dx = \frac{\partial x}{\partial \varphi} d\varphi + \frac{\partial x}{\partial \varepsilon} d\varepsilon$, and noting that the expression on the right-hand side of equation (5.49) is nothing but $\bar{E} dx$. Performing the integration with the initial condition that $\varphi=0$ when $\varepsilon=0$, we obtain:

$$\varphi = \frac{\bar{E}(\partial L / \partial \ell) \varepsilon^2}{2 + \bar{E}(\partial L / \partial \ell) \varepsilon^2} \quad (5.50)$$

where in the range of the elastic strain of an isotropic medium, Figure 5.2 shows the relation between φ and ε using the material constants $\bar{E} = 2.756e3$ MPa and $(\partial L / \partial \ell) = 0.0005$.

Next, we relate damage evolution to the evolution of the fabric tensors. If we take equation (5.17) and apply the hypothesis of elastic energy equivalence to it, along with the assumption that $\mathbf{M}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}(\boldsymbol{\varphi}^{(4)})$, and then taking its time derivative, we obtain:

$$\begin{aligned} d\boldsymbol{\varphi}^{(4)} = & -[2\lambda_{ij} \{(\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (d\mathbf{G}^{(0)} \mathbf{I}^{(2)} + d\mathbf{G}^{(2)})_j \\ & + 2\lambda_{ij}^* \{(\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \otimes (d\mathbf{G}^{(0)} \mathbf{I}^{(2)} + d\mathbf{G}^{(2)})_j + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \otimes (d\mathbf{G}^{(0)} \mathbf{I}^{(2)} + d\mathbf{G}^{(2)})_i\} \\ & + 4\mu_{ij} \{(\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_i \bar{\otimes} (d\mathbf{G}^{(0)} \mathbf{I}^{(2)} + d\mathbf{G}^{(2)})_j + (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})_j \bar{\otimes} (d\mathbf{G}^{(0)} \mathbf{I}^{(2)} + d\mathbf{G}^{(2)})_i\} \\ &]: \bar{\mathbf{E}}^{-1} \end{aligned} \quad (5.51a)$$

where $dG^{(0)}$, and $dG^{(2)}$ are the increments of the fabric tensors representing their evolution. Using the spectral decomposition theorem of equation (5.25), we obtain:

$$\begin{aligned}
d\phi^{(4)} = & -[2k\lambda_{ij}m_p^{2k-1}dm_p(\mathbf{G}_i^{(2)} \otimes \mathbf{G}_j^{(2)}) + \lambda_{ij}m_p^{2k}(\mathbf{G}_i^{(2)} \otimes d\mathbf{G}_j^{(2)} + d\mathbf{G}_j^{(2)} \otimes \mathbf{G}_i^{(2)}) \\
& + k\lambda_{ij}^*m_p^{k-1}m_q^kdm_p(\mathbf{G}_i^{(2)} \otimes \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \otimes \mathbf{G}_i^{(2)}) \\
& + k\lambda_{ij}^*m_p^km_q^{k-1}dm_q(\mathbf{G}_i^{(2)} \otimes \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \otimes \mathbf{G}_i^{(2)}) \\
& + \lambda_{ij}^*m_p^km_q^k(\mathbf{G}_i^{(2)} \otimes d\mathbf{G}_j^{(2)} + d\mathbf{G}_i^{(2)} \otimes \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \otimes d\mathbf{G}_i^{(2)} + d\mathbf{G}_j^{(2)} \otimes \mathbf{G}_i^{(2)}) \\
& + 2k\mu_{ij}m_p^{k-1}m_q^kdm_p(\mathbf{G}_i^{(2)} \underline{\otimes} \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \underline{\otimes} \mathbf{G}_i^{(2)}) \\
& + 2k\mu_{ij}m_p^km_q^{k-1}dm_q(\mathbf{G}_i^{(2)} \underline{\otimes} \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \underline{\otimes} \mathbf{G}_i^{(2)}) \\
& + 2\mu_{ij}m_p^km_q^k(\mathbf{G}_i^{(2)} \underline{\otimes} d\mathbf{G}_j^{(2)} + d\mathbf{G}_i^{(2)} \underline{\otimes} \mathbf{G}_j^{(2)} + \mathbf{G}_j^{(2)} \underline{\otimes} d\mathbf{G}_i^{(2)} + d\mathbf{G}_j^{(2)} \underline{\otimes} \mathbf{G}_i^{(2)})]
\end{aligned}
\tag{5.51b}$$

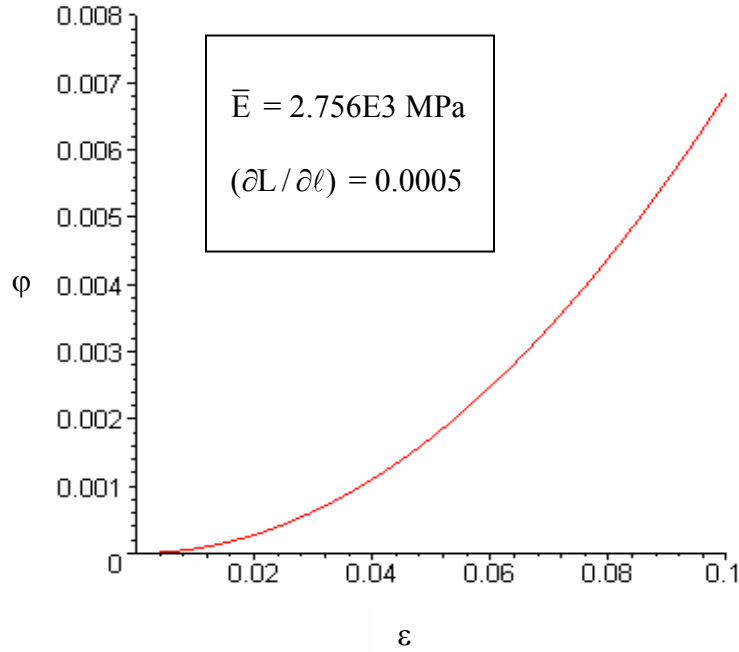


Figure 5.2: Damage Evolution of an Isotropic Elastic Material (ϕ - ϵ)

Equations (5.51a,b) represent the general evolution relation for the damage tensor in terms of the evolution of the fabric tensor. It should be noted that the evolution equations

could be derived in terms of fabric tensors from the beginning, but this is beyond the scope of this work.

6 The Micromechanical Approach

The Micromechanical Approach for damage mechanics with fabric tensors in composite materials will be presented in this chapter. In this approach, and in contrast to the Continuum Approach, damage mechanics is introduced separately to the constituents of the composite material through different constituents' damage effect tensors $\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k})$, where ($k = m, f, i$) refers to the constituent ($m =$ matrix, $f =$ fiber, and $i =$ interface). It is assumed that for each constituent k , the constituent damage effect tensor is a function of the constituent fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)k}$. The general hypothesis of strain transformation (Voyiadjis and Kattan, 2005) will be used here. The damaged composite medium will be treated as a system of damaged constituents where the total damage can be calculated in terms of the damage of these constituents. In the formulation, a fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)k}$ and an eighth-rank damage tensor $\boldsymbol{\varphi}^{(8)k}$ are used for each constituent k . Eventually, however, all the equations will be written in terms of the fourth-rank tensor $\boldsymbol{\varphi}^{(4)k}$ for the constituent k . Numerical examples illustrating the Micromechanical Approach will be presented in chapter 7, section 7.2.

6.1 Elastic Constitutive Equations

Two steps are involved in the Micromechanical Approach (see Figure 6.1). In the first step, we start with a Representative Volume Element (RVE) that contains the undamaged constituents. Damage is introduced into the formulation using separate damage tensors for the constituents $\boldsymbol{\varphi}^{(4)k}$. In the second step, the damaged properties of the constituents will be accounted for in calculating the total damage of the composite system. The effects of delamination will not be considered here because we deal with one single lamina.

However, the effects of debonding may be represented through the damage tensor $\boldsymbol{\varphi}^{(4)i}$ of the interface. In Figure 6.1, \bar{C}^m , \bar{C}^f , and \bar{C}^i are the effective (undamaged) matrix, fiber, and interface configurations, respectively, C^m , C^f , and C^i are the damaged matrix, fiber, and interface configurations, respectively, C is the damaged composite configuration, and \mathbf{A}^m , \mathbf{A}^f , and \mathbf{A}^i are the strain concentration tensors in the actual (damaged) configuration for the matrix, fiber, and interface, respectively. The method to calculate \mathbf{A}^k ($k = m, f, i$) is shown later in this chapter.

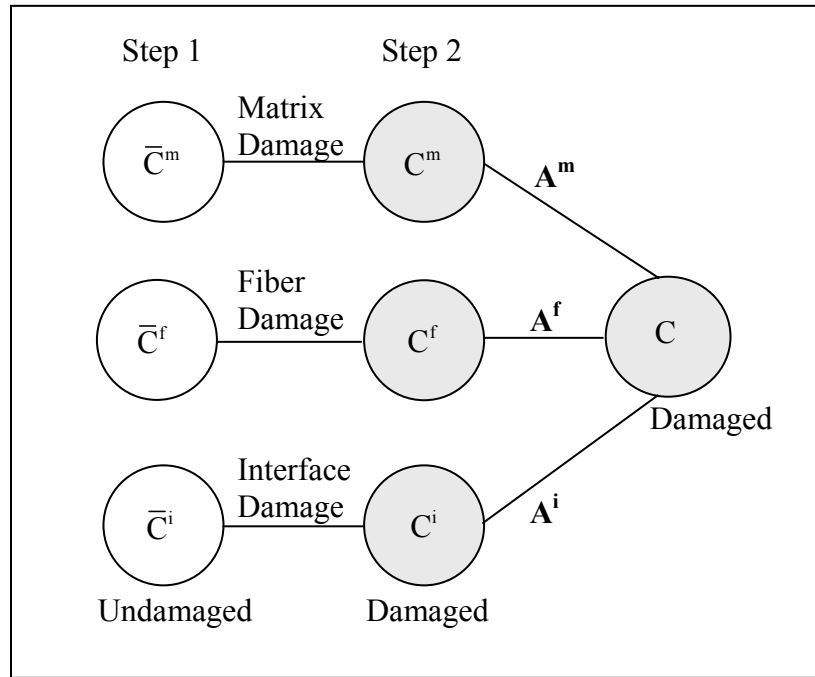


Figure 6.1 Schematic Illustration of the Micromechanical Approach for a Composite System

In the first step: the following relation can be written on the constituent's level to introduce the constituents' fourth-rank damage effect tensors $\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k})$ as follows:

$$\bar{\boldsymbol{\sigma}}^k = \mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) : \boldsymbol{\sigma}^k \quad , \quad k = m, f, i \quad (6.1)$$

The above equation represents the damage transformation equation for each constituent stress tensor, where $\bar{\boldsymbol{\sigma}}^k$, $\boldsymbol{\sigma}^k$, and $\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k})$ are the effective constituent stress tensor, the actual (damaged) constituent stress tensor, and the constituent damage effect tensor, respectively. This formula is obtained by modifying the following formula given by Voyiadjis and Kattan (1999) to include the constituent identifier k:

$$\bar{\boldsymbol{\sigma}} = \mathbf{M}(\boldsymbol{\varphi}^{(4)}) : \boldsymbol{\sigma} \quad (6.2)$$

In order to derive a similar transformation equation for the constituent strain tensor, the general hypothesis of strain transformation is used for each constituent k as follows:

$$\bar{\boldsymbol{\varepsilon}}^k = \mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) : \boldsymbol{\varepsilon}^k \quad , \quad k = m, f, i \quad (6.3)$$

where $\bar{\boldsymbol{\varepsilon}}^k$, $\boldsymbol{\varepsilon}^k$, and $\mathbf{L}^k(\boldsymbol{\varphi}^{(8)k})$ are the effective constituent strain tensor, the actual (damaged) constituent strain tensor, and the constituent general fourth-rank strain transformation function of the eighth-rank damage tensor $\boldsymbol{\varphi}^{(8)k}$, respectively.

The following linear relation is used for the constituents in their undamaged configuration $\bar{\mathbf{C}}^k$ to obtain the constituent effective stress as follows:

$$\bar{\boldsymbol{\sigma}}^k = \bar{\mathbf{E}}^k : \bar{\boldsymbol{\varepsilon}}^k \quad , \quad k = m, f, i \quad (6.4)$$

where $\bar{\mathbf{E}}^k$ is the constituent fourth-rank effective constant elasticity tensor. Applying equations (6.3) and (6.4) into equation (6.1) and rearranging terms, one obtains:

$$\boldsymbol{\sigma}^k = \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1} : \bar{\mathbf{E}}^k : \mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) : \boldsymbol{\varepsilon}^k \quad , \quad k = m, f, i \quad (6.5)$$

from which the following relation is obtained:

$$\boldsymbol{\sigma}^k = \mathbf{E}^k : \boldsymbol{\varepsilon}^k \quad , \quad k = m, f, i \quad (6.6)$$

where \mathbf{E}^k is given by

$$\mathbf{E}^k = \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1} : \bar{\mathbf{E}}^k : \mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) \quad , \quad k = m, f, i \quad (6.7)$$

Equation (6.7) represents the constituent fourth-rank actual (damaged) elasticity tensor which is clearly a variable that depends on the state of damage.

In the second step: we start by relating the effective constituent strain tensor $\bar{\boldsymbol{\varepsilon}}^k$ to the effective composite strain tensor $\bar{\boldsymbol{\varepsilon}}$ through the effective constituent strain concentration tensor $\bar{\mathbf{A}}^k$ as follows:

$$\bar{\boldsymbol{\varepsilon}}^k = \bar{\mathbf{A}}^k : \bar{\boldsymbol{\varepsilon}} \quad , \quad k = m, f, i \quad (6.8)$$

Using equation (6.8) and the general hypothesis of strain transformation at the composite level given by the following equation:

$$\bar{\boldsymbol{\varepsilon}} = \mathbf{L}(\boldsymbol{\varphi}^{(8)}) : \boldsymbol{\varepsilon} \quad (6.9)$$

where $\bar{\boldsymbol{\varepsilon}}$, $\boldsymbol{\varepsilon}$, and $\mathbf{L}(\boldsymbol{\varphi}^{(8)})$ are the composite effective strain tensor, the composite actual (damaged) strain tensor, and the composite strain transformation function, respectively, and using equation (6.3), one obtains:

$$\boldsymbol{\varepsilon}^k = \left(\mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) \right)^{-1} : \bar{\mathbf{A}}^k : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) : \boldsymbol{\varepsilon} \quad , \quad k = m, f, i \quad (6.10)$$

from which the following relation is obtained:

$$\boldsymbol{\varepsilon}^k = \mathbf{A}^k : \boldsymbol{\varepsilon} \quad , \quad k = m, f, i \quad (6.11)$$

where \mathbf{A}^k is given by:

$$\mathbf{A}^k = \left(\mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) \right)^{-1} : \bar{\mathbf{A}}^k : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) \quad , \quad k = m, f, i \quad (6.12)$$

Next, we find an expression for the fourth-rank composite elasticity tensor \mathbf{E} in terms of the constituents' properties. Introducing the law of mixtures in the damaged configuration:

$$\boldsymbol{\sigma} = \sum_k c^k \boldsymbol{\sigma}^k \quad , \quad k=m, f, i \quad (6.13)$$

where c^k and $\boldsymbol{\sigma}^k$ are the constituent's actual (damaged) volume fraction and the composite actual (damaged) stress tensor, and using the following equation relating the actual composite stress tensor $\boldsymbol{\sigma}$ to the actual composite strain tensor $\boldsymbol{\varepsilon}$ through the fourth-rank actual composite elasticity tensor \mathbf{E} :

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon} \quad (6.14)$$

along with equations (6.6), and (6.11), one obtains:

$$\mathbf{E} : \boldsymbol{\varepsilon} = \left(\sum_k c^k \mathbf{E}^k : \mathbf{A}^k \right) : \boldsymbol{\varepsilon} \quad (6.15)$$

Post multiplying both sides by $\boldsymbol{\varepsilon}^{-1}$, we obtain:

$$\mathbf{E} = \sum_k c^k \mathbf{E}^k : \mathbf{A}^k \quad (6.16)$$

Substituting for \mathbf{A}^k from equation (6.12), we get:

$$\mathbf{E} = \sum_k c^k \mathbf{E}^k : \left(\mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) \right)^{-1} : \bar{\mathbf{A}}^k : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) \quad (6.17)$$

and since $\mathbf{L}(\boldsymbol{\varphi}^{(8)})$ is common to all terms in the above equation, we can write

$$\mathbf{E} = \left(\sum_k c^k \mathbf{E}^k : \left(\mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) \right)^{-1} : \bar{\mathbf{A}}^k \right) : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) \quad (6.18)$$

Equation (6.18) gives (in general terms) the elasticity tensor in the damaged composite system according to the Micromechanical Approach.

Next, we introduce fabric tensors into our formulation. On the constituent level, and after adding the constituent identifier (k) to all the parameters in the equation, the equation given by Voyiadjis and kattan (2005) for $\boldsymbol{\varphi}^{(4)}$ can be used along with their

definition of the damage effect tensor, $\mathbf{M}(\boldsymbol{\varphi}^{(4)}) = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)})^{-1}$, to obtain the following equation:

$$\begin{aligned} \mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) = & ([\lambda^k (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \otimes (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \\ & + 2\mu^k (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \underline{\otimes} (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k})] \\ & : (\mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}))^{-1} : \mathbf{L}^k(\boldsymbol{\varphi}^{(4)k}) : (\bar{\mathbf{E}}^k)^{-1})^{-1}, \quad k=m, f, i \end{aligned} \quad (6.19)$$

where λ^k and μ^k are Lamé's constants for the k -th constituent, $\mathbf{G}^{(0)k}$ and $\mathbf{G}^{(2)k}$ are the k^{th} constituent zero-rank and second-rank fabric tensors, and $\mathbf{I}^{(2)}$ is the second-rank identity tensor.

Two special cases can be obtained from equations (6.18) and (6.19) as follows:

1. The special case of elastic strain equivalence is obtained by setting each of $\mathbf{L}(\boldsymbol{\varphi}^{(8)})$ and $\mathbf{L}^k(\boldsymbol{\varphi}^{(8)k})$ to be equal to the fourth-rank identity tensor $\mathbf{I}^{(4)}$. In this case, equation (6.18) becomes:

$$\mathbf{E} = \sum_k c^k \mathbf{E}^k : \bar{\mathbf{A}}^k \quad (6.20)$$

and equation (6.19) becomes:

$$\begin{aligned} \mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) = & ([\lambda^k (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \otimes (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \\ & + 2\mu^k (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \underline{\otimes} (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k})] : (\bar{\mathbf{E}}^k)^{-1})^{-1} \end{aligned} \quad (6.21)$$

where $k=m, f, i$

2. The special case of elastic energy equivalence is obtained by setting

$\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)})$ and $\mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) = (\mathbf{M}^k(\boldsymbol{\varphi}^{(8)k}))^{-T}$. In this case, equation (6.18)

becomes:

$$\mathbf{E} = \left(\sum_k c^k \mathbf{E}^k : (\mathbf{M}^k(\boldsymbol{\varphi}^{(8)k}))^T : \bar{\mathbf{A}}^k \right) : \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)}) \quad (6.22)$$

and equation (6.19) becomes:

$$\begin{aligned} \mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) = & ([\lambda^k (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \otimes (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \\ & + 2\mu^k (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \underline{\otimes} (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k})] \\ & : (\mathbf{M}^k(\boldsymbol{\varphi}^{(8)k}))^T : (\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}))^{-T} : (\bar{\mathbf{E}}^k)^{-1})^{-1} \quad , \quad k=m, f, i \end{aligned} \quad (6.23)$$

If we make a further simplifying assumption that $\mathbf{M}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}(\boldsymbol{\varphi}^{(4)})$ and

$\mathbf{M}^k(\boldsymbol{\varphi}^{(8)k}) = \mathbf{M}^k(\boldsymbol{\varphi}^{(4)k})$, then equation (6.22) can be written for the case of elastic energy

equivalence as:

$$\mathbf{E} = \left(\sum_k c^k \mathbf{E}^k : (\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}))^T : \bar{\mathbf{A}}^k \right) : \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)}) \quad (6.24)$$

and equation (6.23) can be written as:

$$\begin{aligned} \mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) = & ([\lambda^k (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \otimes (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \\ & + 2\mu^k (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \underline{\otimes} (\mathbf{G}^{(0)k} \mathbf{I}^{(2)} + \mathbf{G}^{(2)k})] : (\bar{\mathbf{E}}^k)^{-1})^{-1} \end{aligned} \quad (6.25)$$

where $k=m, f, i$. Substituting equation (6.25) and the equation given by Voyiadjis and

Kattan (2005) for the composite damage effect tensor equation, $\mathbf{M}(\boldsymbol{\varphi}^{(4)}) = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)})^{-1}$,

into equation (6.24), we obtain:

$$\begin{aligned} \mathbf{E} = & \left\{ \sum_k c^k \mathbf{E}^k : ([\lambda^k (\mathbf{G}^{k(0)} \mathbf{I}^{(2)} + \mathbf{G}^{k(2)}) \otimes (\mathbf{G}^{k(0)} \mathbf{I}^{k(2)} + \mathbf{G}^{k(2)}) \right. \\ & + 2\mu^k (\mathbf{G}^{k(0)} \mathbf{I}^{k(2)} + \mathbf{G}^{k(2)}) \underline{\otimes} (\mathbf{G}^{k(0)} \mathbf{I}^{k(2)} + \mathbf{G}^{k(2)})] : (\bar{\mathbf{E}}^k)^{-1})^{-T} \\ & : \bar{\mathbf{A}}^k \} : \bar{\mathbf{E}}^{-T} : [\lambda (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \otimes (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \\ & + 2\mu (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \underline{\otimes} (\mathbf{G}^{(0)} \mathbf{I}^{(2)} + \mathbf{G}^{(2)})]^T \quad , \quad k=m, f, i \end{aligned} \quad (6.26)$$

Equation (5.26) represents the elasticity tensor in the damaged composite system

according to the Micromechanical Approach.

There is an alternative way to obtain an expression for $\mathbf{M}(\boldsymbol{\varphi}^{(4)})$ that appears in equation (6.24). Applying the following equation for the constituent effective stress tensor:

$$\bar{\boldsymbol{\sigma}}^k = \bar{\mathbf{B}}^k : \bar{\boldsymbol{\sigma}} \quad , k = m, f, i \quad (6.27)$$

where $\bar{\mathbf{B}}^k$ is the constituent effective stress concentration tensor, into the law of mixtures in the undamaged configuration:

$$\bar{\boldsymbol{\sigma}} = \sum_k \bar{c}^k \bar{\boldsymbol{\sigma}}^k \quad (6.28)$$

where \bar{c}^k is the constituent's effective volume fraction, we obtain:

$$\bar{\boldsymbol{\sigma}} = \left(\sum_k \bar{c}^k \bar{\mathbf{B}}^k \right) : \bar{\boldsymbol{\sigma}} \quad \text{or} \quad \mathbf{I}^{(4)} = \sum_k \bar{c}^k \bar{\mathbf{B}}^k \quad (6.29)$$

and applying equations (6.1) and (6.2) into equation (6.27), and rearranging terms, we obtain:

$$\boldsymbol{\sigma}^k = \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1} : \bar{\mathbf{B}}^k : \mathbf{M}(\boldsymbol{\varphi}^{(4)}) : \boldsymbol{\sigma} \quad , k = m, f, i \quad (6.30)$$

which can be written as

$$\boldsymbol{\sigma}^k = \mathbf{B}^k : \boldsymbol{\sigma} \quad , k = m, f, i \quad (6.31)$$

where

$$\mathbf{B}^k = \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1} : \bar{\mathbf{B}}^k : \mathbf{M}(\boldsymbol{\varphi}^{(4)}) \quad , k = m, f, i \quad (6.32)$$

Next, using equation (6.31) and the law of mixtures in the damaged configuration given by equation (6.13), we obtain:

$$\boldsymbol{\sigma} = \left(\sum_k c^k \mathbf{B}^k \right) : \boldsymbol{\sigma} \quad \text{or} \quad \mathbf{I}^{(4)} = \sum_k c^k \mathbf{B}^k \quad (6.33)$$

Using equations (6.33) and (6.32), one obtains the following:

$$\mathbf{I}^{(4)} = \left(\sum_k \mathbf{c}^k \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1} : \bar{\mathbf{B}}^k \right) : \mathbf{M}(\boldsymbol{\varphi}^{(4)}) \quad (6.34)$$

and upon rearranging the terms to get an expression for $\mathbf{M}(\boldsymbol{\varphi}^{(4)})$, we get:

$$\mathbf{M}(\boldsymbol{\varphi}^{(4)}) = \left(\sum_k \mathbf{c}^k \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1} : \bar{\mathbf{B}}^k \right)^{-1} \quad (6.35)$$

Therefore, equation (6.24) can be written as:

$$\mathbf{E} = \left(\sum_k \mathbf{c}^k \mathbf{E}^k : \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^T : \bar{\mathbf{A}}^k \right) : \left(\sum_k \mathbf{c}^k \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1} : \bar{\mathbf{B}}^k \right)^T \quad (6.36)$$

Equation (6.36) is an alternative expression that represents the elasticity tensor in the damaged composite system according to the Micromechanical Approach.

6.2 Damage Evolution

Next, indicial notation as well as tensorial notation will be used to derive the required damage evolution equation for the constituent k , then to relate the damage evolution of the constituents to the overall damage evolution of the composite system. The reason for using indicial notation is that some of the formulas will be much easier to handle when they are in the indicial form. In the derivation of damage evolution, we adopt the hypothesis of elastic energy equivalence and the fourth-rank constituent damage tensor $\boldsymbol{\varphi}^{(4)k}$.

We start with the elastic strain energy function U^k for the constituent k , defined as:

$$U^k = \frac{1}{2} \boldsymbol{\sigma}^k : \boldsymbol{\varepsilon}^k \quad \text{or} \quad U^k = \frac{1}{2} \boldsymbol{\sigma}_{ij}^k \boldsymbol{\varepsilon}_{ij}^k \quad (6.37a,b)$$

Using equations (6.6) and (6.7), U^k can be written as:

$$U^k = \frac{1}{2} \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1} : \bar{\mathbf{E}}^k : \mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) : \boldsymbol{\varepsilon}^k : \boldsymbol{\varepsilon}^k \quad (6.38a)$$

or

$$U^k = \frac{1}{2} \left(\mathbf{M}_{ijkl}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1} \bar{\mathbf{E}}_{klmn}^k \mathbf{L}_{mnpq}^k(\boldsymbol{\varphi}^{(8)k}) \boldsymbol{\varepsilon}_{pq}^k \boldsymbol{\varepsilon}_{ij}^k \quad (6.38b)$$

Substituting for $\left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1}$ from the following equation given by Voyiadjis and Kattan (1999) and modified here to include the constituent identifier k :

$$\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})^{-1} \quad (6.39)$$

Applying the hypothesis of elastic energy equivalence $\mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) = \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(8)k}) \right)^{-T}$ as well as adopting $\mathbf{M}^k(\boldsymbol{\varphi}^{(8)k})$ such that $\mathbf{M}^k(\boldsymbol{\varphi}^{(8)k}) = \mathbf{M}^k(\boldsymbol{\varphi}^{(4)k})$, one obtains:

$$U^k = \frac{1}{2} (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k}) : \bar{\mathbf{E}}^k : (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})^T : \boldsymbol{\varepsilon}^k : \boldsymbol{\varepsilon}^k \quad (6.40a)$$

or in indicial notation:

$$U^k = \frac{1}{2} (\mathbf{I} - \boldsymbol{\varphi}^k)_{ijkl} \bar{\mathbf{E}}_{klmn}^k (\mathbf{I} - \boldsymbol{\varphi}^k)_{pqmn} \boldsymbol{\varepsilon}_{pq}^k \boldsymbol{\varepsilon}_{ij}^k \quad (6.40b)$$

Next, we differentiate the elastic energy function U^k with respect to the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)k}$ to obtain the thermodynamic force \mathbf{Y}^k associated with the fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)k}$:

$$\mathbf{Y}^k = \frac{\partial U^k}{\partial \boldsymbol{\varphi}^{(4)k}} = -(\boldsymbol{\varepsilon}^k)^T \otimes \left[\bar{\mathbf{E}}^k : (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})^T : \boldsymbol{\varepsilon}^k \right] \quad (6.41a)$$

or

$$\mathbf{Y}_{rstu}^k = \frac{\partial U^k}{\partial \boldsymbol{\varphi}_{rstu}^k} = -(\mathbf{I} - \boldsymbol{\varphi}^k)_{pqmn} \boldsymbol{\varepsilon}_{pq}^k \boldsymbol{\varepsilon}_{rs}^k \bar{\mathbf{E}}_{tumn}^k \quad (6.41b)$$

As can be seen from equations (6.41a,b), the constituent thermodynamic force \mathbf{Y}^k is a function of the constituent fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)k}$ and the constituent second-

rank strain tensor $\boldsymbol{\varepsilon}^k$. Therefore, if we use the chain rule to differentiate the constituent thermodynamic force \mathbf{Y}^k , we obtain the following:

$$d\mathbf{Y}_{rstu}^k = \frac{\partial \mathbf{Y}_{rstu}^k}{\partial \boldsymbol{\varphi}_{ijkl}^k} d\boldsymbol{\varphi}_{ijkl}^k + \frac{\partial \mathbf{Y}_{rstu}^k}{\partial \boldsymbol{\varepsilon}_{ab}^k} d\boldsymbol{\varepsilon}_{ab}^k \quad (6.42)$$

If we differentiate the constituent thermodynamic force \mathbf{Y}^k with respect to the constituent fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)k}$, we obtain:

$$\frac{\partial \mathbf{Y}^k}{\partial \boldsymbol{\varphi}^{(4)k}} = (\boldsymbol{\varepsilon}^k)^T \otimes \bar{\mathbf{E}}^k \otimes \boldsymbol{\varepsilon}^k \quad \text{or} \quad \frac{\partial \mathbf{Y}_{rstu}^k}{\partial \boldsymbol{\varphi}_{ijkl}^k} = \boldsymbol{\varepsilon}_{ij}^k \boldsymbol{\varepsilon}_{rs}^k \bar{\mathbf{E}}_{tukl}^k \quad (6.43a,b)$$

and the derivative of \mathbf{Y}^k with respect to the constituent second-rank strain tensor $\boldsymbol{\varepsilon}^k$ gives:

$$\frac{\partial \mathbf{Y}_{rstu}^k}{\partial \boldsymbol{\varepsilon}_{ab}^k} = -(\mathbf{I} - \boldsymbol{\varphi}^k)_{pqmn} \left[\boldsymbol{\delta}_{pa} \boldsymbol{\delta}_{qb} \boldsymbol{\varepsilon}_{rs}^k + \boldsymbol{\delta}_{ra} \boldsymbol{\delta}_{sb} \boldsymbol{\varepsilon}_{pq}^k \right] \bar{\mathbf{E}}_{tumn}^k \quad (6.44)$$

Substituting equations (6.43) and (6.44) back into equation (6.42), the incremental equation of the thermodynamic force \mathbf{Y} is obtained as follows:

$$d\mathbf{Y}_{rstu}^k = \boldsymbol{\varepsilon}_{ij}^k \boldsymbol{\varepsilon}_{rs}^k \bar{\mathbf{E}}_{tukl}^k d\boldsymbol{\varphi}_{ijkl}^k - (\mathbf{I} - \boldsymbol{\varphi}^k)_{pqmn} \left[\boldsymbol{\delta}_{pa} \boldsymbol{\delta}_{qb} \boldsymbol{\varepsilon}_{rs}^k + \boldsymbol{\delta}_{ra} \boldsymbol{\delta}_{sb} \boldsymbol{\varepsilon}_{pq}^k \right] \bar{\mathbf{E}}_{tumn}^k d\boldsymbol{\varepsilon}_{ab}^k \quad (6.45)$$

Next, we introduce a generalized damage criterion at the constituent level $g^k(\mathbf{Y}^k, L^k)$. This damage criterion is a function of the constituent thermodynamic force \mathbf{Y}^k associated with the constituent fourth-rank damage tensor $\boldsymbol{\varphi}^{(4)k}$ and the constituent damage strengthening parameter $L^k(\ell^k)$ which in turn is a function of the constituent overall scalar damage parameter ℓ^k . The function $g^k(\mathbf{Y}^k, L^k)$ is given in indicial notation as:

$$g^k = \sqrt{\frac{1}{2} \mathbf{Y}_{klj}^k \mathbf{J}_{klmn} \mathbf{Y}_{mnij}^k} - \ell_o^k - L^k(\ell^k) \leq 0 \quad (6.46)$$

where \mathbf{J} is a constant fourth-rank tensor given in Appendix A, and ℓ_o^k is the initial threshold of damage for the constituent k .

In order to derive a normality rule for the evolution of damage, we start with the power of dissipation Π^k at the constituent level which is given by:

$$\Pi^k = -\mathbf{Y}^k : d\boldsymbol{\phi}^{(4)k} - L^k d\ell^k \quad (6.47)$$

The problem here is to extremize Π^k subject to the condition $\mathbf{g}^k = 0$. Using the mathematical theory of functions of several variables, we introduce the Lagrangian

multiplier $d\lambda^k$ and form the objective function $\Psi^k(\mathbf{Y}^k, L^k)$ such that:

$$\Psi^k = \Pi^k - d\lambda^k \cdot \mathbf{g}^k \quad (6.48)$$

The problem now reduces to extremizing the function Ψ^k . To do so, the two necessary

conditions are $\frac{\partial \Psi^k}{\partial \mathbf{Y}^{(4)k}} = 0$ and $\frac{\partial \Psi^k}{\partial L^k} = 0$. Using these conditions, along with equations

(6.47) and (6.48), we obtain:

$$d\boldsymbol{\phi}^{(4)k} = -d\lambda^k \frac{\partial \mathbf{g}^k}{\partial \mathbf{Y}^k} \quad \text{or} \quad d\boldsymbol{\phi}_{mnpq}^k = -d\lambda^k \frac{\partial \mathbf{g}^k}{\partial \mathbf{Y}_{mnpq}^k} \quad (6.49a,b)$$

and

$$d\ell^k = d\lambda^k \quad (6.50)$$

In order to solve the differential equation given by equation (6.49), we must first find an expression for the constituent Lagrangian multiplier $d\lambda^k$. This can be done by invoking the consistency condition $d\mathbf{g}^k = 0$ for the constituent k . Applying the chain rule of differentiation to equation (6.46), we obtain:

$$\frac{\partial \mathbf{g}^k}{\partial \mathbf{Y}_{ijkl}^k} d\mathbf{Y}_{ijkl}^k + \frac{\partial \mathbf{g}^k}{\partial L^k} dL^k = 0 \quad (6.51)$$

The derivative of g^k with respect to the thermodynamic force tensor \mathbf{Y}^k is given as: (from equation (6.46))

$$\frac{\partial g^k}{\partial \mathbf{Y}_{ijkl}^k} = \frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnl}^k}{2\sqrt{\frac{1}{2} \mathbf{Y}_{rspq}^k \mathbf{J}_{rstu} \mathbf{Y}_{tupq}^k}} \quad (6.52)$$

Observing that $(\partial g^k / \partial L^k) = -1$, and $dL^k = (\partial L^k / \partial \ell^k) d\ell^k$, equation (6.51) can be solved for $d\ell^k$ as follows:

$$d\ell^k = \frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnl}^k d\mathbf{Y}_{ijkl}^k}{2(\partial L^k / \partial \ell^k) \sqrt{\frac{1}{2} \mathbf{Y}_{rspq}^k \mathbf{J}_{rstu} \mathbf{Y}_{tupq}^k}} \quad (6.53)$$

Substituting equation (6.53) back into equation (6.49) with the use of equation (6.50), we obtain the following:

$$d\phi_{rstu}^k = \frac{-\mathbf{J}_{ijmn} \mathbf{Y}_{mnl}^k d\mathbf{Y}_{ijkl}^k \mathbf{J}_{rsvw} \mathbf{Y}_{vwtu}^k}{2(\partial L^k / \partial \ell^k) (\mathbf{Y}_{cdab}^k \mathbf{J}_{cdef} \mathbf{Y}_{efab}^k)} \quad (6.54)$$

Substituting for $d\mathbf{Y}_{ijkl}^k$ from equation (6.45) and rearranging the terms to factor out common parameters, we obtain:

$$\begin{aligned} & \left[\delta_{er} \delta_{fs} \delta_{vt} \delta_{wu} + \frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnl}^k \boldsymbol{\varepsilon}_{ef}^k \boldsymbol{\varepsilon}_{ij}^k \bar{\mathbf{E}}_{klvw}^k \mathbf{J}_{rscd} \mathbf{Y}_{cdtu}^k}{2(\partial L^k / \partial \ell^k) (\mathbf{Y}_{\delta\gamma\alpha\beta}^k \mathbf{J}_{\delta\gamma\mu\theta} \mathbf{Y}_{\mu\theta\alpha\beta}^k)} \right] d\phi_{efvw}^k = \\ & = \left[\frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnl}^k (\mathbf{I} - \boldsymbol{\phi}^k)_{pqgh} (\delta_{pa} \delta_{qb} \boldsymbol{\varepsilon}_{ij}^k + \delta_{ia} \delta_{jb} \boldsymbol{\varepsilon}_{pq}^k) \bar{\mathbf{E}}_{klgh}^k d\boldsymbol{\varepsilon}_{ab}^k \mathbf{J}_{rscd} \mathbf{Y}_{cdtu}^k}{2(\partial L^k / \partial \ell^k) (\mathbf{Y}_{\delta\gamma\alpha\beta}^k \mathbf{J}_{\delta\gamma\mu\theta} \mathbf{Y}_{\mu\theta\alpha\beta}^k)} \right] \end{aligned} \quad (6.55)$$

which can be rewritten as:

$$\mathbf{C}_{efvwrstu}^k d\phi_{efvw}^k = \mathbf{D}_{rstuab}^k d\boldsymbol{\varepsilon}_{ab}^k \quad (6.56)$$

where

$$\mathbf{C}_{efvwrstu}^k = \left[\delta_{er} \delta_{fs} \delta_{vt} \delta_{wu} + \frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnl}^k \boldsymbol{\varepsilon}_{ef}^k \boldsymbol{\varepsilon}_{ij}^k \bar{\mathbf{E}}_{klvw}^k \mathbf{J}_{rscd} \mathbf{Y}_{cdtu}^k}{2(\mathbf{Y}_{\delta\gamma\alpha\beta}^k \mathbf{J}_{\delta\gamma\mu\theta} \mathbf{Y}_{\mu\theta\alpha\beta}^k) (\partial L^k / \partial \ell^k)} \right] \quad (6.57)$$

and

$$\mathbf{D}_{rstuab}^k = \left[\frac{\mathbf{J}_{ijmn} \mathbf{Y}_{mnkl}^k (\mathbf{I} - \boldsymbol{\phi}^k)_{pqgh} (\delta_{pa} \delta_{qb} \boldsymbol{\varepsilon}_{ij}^k + \delta_{ia} \delta_{jb} \boldsymbol{\varepsilon}_{pq}^k) \bar{\mathbf{E}}_{klgh}^k \mathbf{J}_{rscd} \mathbf{Y}_{cdtu}^k}{2(\mathbf{Y}_{\delta\gamma\alpha\beta}^k \mathbf{J}_{\delta\gamma\mu\theta} \mathbf{Y}_{\mu\theta\alpha\beta}^k)(\partial L^k / \partial \ell^k)} \right] \quad (6.58)$$

Further rearrangement of equation (6.55) will give the following:

$$d\boldsymbol{\phi}_{efvw}^k = \left(\mathbf{C}_{efvwrstu}^k \right)^{-1} \mathbf{D}_{rstuab}^k d\boldsymbol{\varepsilon}_{ab}^k \quad (6.59)$$

where it should be noted that $\mathbf{C}^k(\mathbf{Y}^k, \boldsymbol{\varepsilon}^k)$ and $\mathbf{D}^k(\mathbf{Y}^k, \boldsymbol{\varepsilon}^k)$ are tensorial functions of \mathbf{Y}^k and $\boldsymbol{\varepsilon}^k$. Equation (6.59) represents the general evolution equation for the damage tensor at the constituent level. The above evolution equation applies for the general case of elastic deformation and damage.

Next, we relate the damage evolution equations of the constituents to obtain a damage evolution equation of the composite medium. We start with a one dimensional case and then generalize our results for a multi-dimensional case (see Figure 6.2).

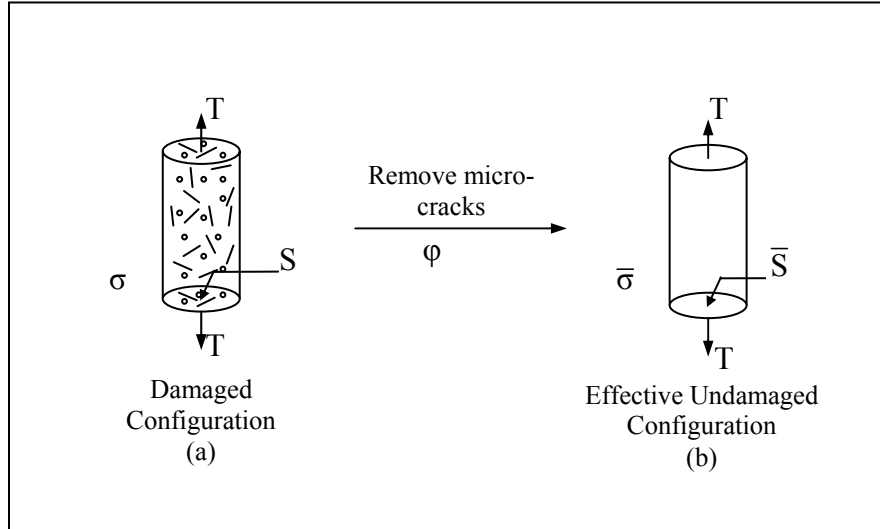


Figure 6.2: Isotropic Damage in Uniaxial Tension (Concept of Effective Stress)

Consider the concept of effective stress in an RVE of uniform thickness where the damage variable ϕ^k (scalar) of a constituent k is defined as the ratio of the net cross-sectional (undamaged) area \bar{S}^k of a RVE (one dimensional problem) to the total (damaged) cross-sectional area S^k , as follows:

$$\phi^k = \frac{S^k - \bar{S}^k}{S^k} \quad (6.60a)$$

Rearranging the terms we obtain:

$$\phi^k S^k = S^k - \bar{S}^k \quad (6.60b)$$

and for a composite system made from k constituents, we have:

$$\sum_k \phi^k S^k = \sum_k (S^k - \bar{S}^k) \quad (6.61a)$$

or

$$\sum_k \phi^k S^k = \sum_k S^k - \sum_k \bar{S}^k \quad (6.61b)$$

where $\sum_k S^k = S$ and $\sum_k \bar{S}^k = \bar{S}$, which gives the following:

$$\sum_k \phi^k S^k = S - \bar{S} \quad (6.61c)$$

where S and \bar{S} are the composite damaged and the composite undamaged cross-sectional areas. Dividing equation (6.62c) by (S) , we obtain:

$$\sum_k \phi^k \frac{S^k}{S} = \frac{S - \bar{S}}{S} \quad (6.62a)$$

where $\frac{S^k}{S}$ is the volume fraction c^k of the constituent k , and $\frac{S - \bar{S}}{S}$ is the overall damage

variable of the composite system, ϕ . Therefore, equation (6.62a) becomes:

$$\boldsymbol{\varphi} = \sum_k \mathbf{c}^k \boldsymbol{\varphi}^k \quad (6.62b)$$

Generalizing equation (6.62b) for the three dimensional case, we obtain:

$$\boldsymbol{\varphi}^{(4)} = \sum_k \mathbf{c}^k \boldsymbol{\varphi}^{(4)k} \quad \text{or} \quad \boldsymbol{\varphi}_{ijkl} = \sum_k \mathbf{c}^k \boldsymbol{\varphi}_{ijkl}^k \quad (6.63a,b)$$

and taking the derivative of equation (6.63), we obtain the following incremental relation:

$$d\boldsymbol{\varphi}^{(4)} = \sum_k \mathbf{c}^k d\boldsymbol{\varphi}^{(4)k} \quad \text{or} \quad d\boldsymbol{\varphi}_{ijkl} = \sum_k \mathbf{c}^k d\boldsymbol{\varphi}_{ijkl}^k \quad (6.64a,b)$$

Equation (6.64) is a damage evolution equation that relates the increment of the overall damage tensor $d\boldsymbol{\varphi}^{(4)}$ of the composite material to the increments of the constituents' damage tensors $d\boldsymbol{\varphi}^{(4)k}$. Substituting equation (6.59) into equation (6.64b), we obtain the following:

$$d\boldsymbol{\varphi}_{abcd} = \sum_k \mathbf{c}^k \left(\mathbf{C}_{abcdefgh}^k \right)^{-1} \mathbf{D}_{efghij}^k d\boldsymbol{\varepsilon}_{ij}^k \quad (6.65)$$

In the following step, an expression for $d\boldsymbol{\varepsilon}_{ij}^k$ is sought. We start with equation (6.11) and we take its derivative to obtain the following:

$$d\boldsymbol{\varepsilon}^k = d\mathbf{A}^k : \boldsymbol{\varepsilon} + \mathbf{A}^k : d\boldsymbol{\varepsilon} \quad \text{or} \quad d\boldsymbol{\varepsilon}_{ij}^k = d\mathbf{A}_{ijkl}^k \boldsymbol{\varepsilon}_{kl} + \mathbf{A}_{ijkl}^k d\boldsymbol{\varepsilon}_{kl} \quad (6.66a,b)$$

where \mathbf{A}^k is given in equation (6.12). As can be seen from equation (6.66), an expression for the derivative of \mathbf{A}^k is required. By applying the hypothesis of elastic energy equivalence: $\mathbf{L}^k(\boldsymbol{\varphi}^{(8)k}) = (\mathbf{M}^k(\boldsymbol{\varphi}^{(8)k}))^{-T}$ and $\mathbf{L}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(8)})$ to equation (6.12), as well as adopting $\mathbf{M}^k(\boldsymbol{\varphi}^{(8)k})$ and $\mathbf{M}(\boldsymbol{\varphi}^{(8)})$ such that $\mathbf{M}^k(\boldsymbol{\varphi}^{(8)k}) = \mathbf{M}^k(\boldsymbol{\varphi}^{(4)k})$ and $\mathbf{M}(\boldsymbol{\varphi}^{(8)}) = \mathbf{M}(\boldsymbol{\varphi}^{(4)})$, one obtains:

$$\mathbf{A}^k = \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^T : \bar{\mathbf{A}}^k : \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)}) \quad (6.67)$$

substituting for $\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k})$ from equation (6.39) and for $\mathbf{M}(\boldsymbol{\varphi}^{(4)})$ from equation (6.35), we have:

$$\mathbf{A}^k = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})^{-T} : \bar{\mathbf{A}}^k : \left(\sum_J \mathbf{c}^J (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)J}) : \bar{\mathbf{B}}^J \right)^T \quad (6.68a)$$

or in indicial notations:

$$\mathbf{A}_{ijkl}^k = (\mathbf{I} - \boldsymbol{\varphi}^k)_{ijmn}^{-T} \bar{\mathbf{A}}_{mnpq}^k \left(\sum_J \mathbf{c}^J (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)J}) : \bar{\mathbf{B}}^J \right)_{pqkl}^T \quad (6.68b)$$

and by eliminating the transpose symbol, we have:

$$\mathbf{A}_{ijkl}^k = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnij}^{-1} \bar{\mathbf{A}}_{mnpq}^k \left(\sum_J \mathbf{c}^J (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)J}) : \bar{\mathbf{B}}^J \right)_{klpq} \quad (6.68c)$$

and by expanding the last term in the right hand side, we obtain:

$$\mathbf{A}_{ijkl}^k = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnij}^{-1} \bar{\mathbf{A}}_{mnpq}^k \left(\sum_J \mathbf{c}^J (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)J})_{klrs} \bar{\mathbf{B}}_{rspq}^J \right) \quad (6.68d)$$

By taking the derivative of \mathbf{A}_{ijkl}^k with respect to the variables $\boldsymbol{\varphi}^{(4)k}$ and $\boldsymbol{\varphi}^{(4)J}$, we obtain

an expression for $d\mathbf{A}_{ijkl}^k$ (required in equation (6.66)):

$$\begin{aligned} d\mathbf{A}_{ijkl}^k = & d \left[(\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnij}^{-1} \right] \bar{\mathbf{A}}_{mnpq}^k \left(\sum_J \mathbf{c}^J (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)J})_{klrs} \bar{\mathbf{B}}_{rspq}^J \right) + \\ & + (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnij}^{-1} \bar{\mathbf{A}}_{mnpq}^k \left(\sum_J \mathbf{c}^J (-d\boldsymbol{\varphi}^{(4)J})_{klrs} \bar{\mathbf{B}}_{rspq}^J \right) \end{aligned} \quad (6.69)$$

where the derivative $d \left[(\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnij}^{-1} \right]$ is shown in Appendix B to be equal to

$$d \left[(\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnij}^{-1} \right] = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnpq}^{-1} d\boldsymbol{\varphi}_{pqkl}^{(4)k} (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{klij}^{-1} \quad (6.70)$$

Therefore, equation (6.69) becomes:

$$\begin{aligned}
d\mathbf{A}_{ijkl}^k &= (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnpq}^{-1} d\boldsymbol{\varphi}_{pqvw}^{(4)k} (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{vwij}^{-1} \bar{\mathbf{A}}_{mntu}^k \left(\sum_J \mathbf{c}^J (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)J})_{klrs} \bar{\mathbf{B}}_{rstu}^J \right) \\
&\quad - (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnij}^{-1} \bar{\mathbf{A}}_{mnpq}^k \left(\sum_J \mathbf{c}^J d\boldsymbol{\varphi}_{klrs}^{(4)J} \bar{\mathbf{B}}_{rspq}^J \right)
\end{aligned} \tag{6.71}$$

Now that we have an expression for $d\mathbf{A}_{ijkl}^k$, we can write an expression for $d\boldsymbol{\varepsilon}_{ij}^k$. Applying equations (6.68d) and (6.71) into equation (6.66b), we obtain:

$$\begin{aligned}
d\boldsymbol{\varepsilon}_{ij}^k &= [(\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnpq}^{-1} d\boldsymbol{\varphi}_{pqvw}^{(4)k} (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{vwij}^{-1} \bar{\mathbf{A}}_{mntu}^k \left(\sum_J \mathbf{c}^J (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)J})_{klrs} \bar{\mathbf{B}}_{rstu}^J \right) \\
&\quad - (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnij}^{-1} \bar{\mathbf{A}}_{mnpq}^k \left(\sum_J \mathbf{c}^J d\boldsymbol{\varphi}_{klrs}^{(4)J} \bar{\mathbf{B}}_{rspq}^J \right)] \boldsymbol{\varepsilon}_{kl} \\
&\quad + [(\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnpq}^{-1} \bar{\mathbf{A}}_{mnpq}^k \left(\sum_J \mathbf{c}^J (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)J})_{klrs} \bar{\mathbf{B}}_{rspq}^J \right)] d\boldsymbol{\varepsilon}_{kl}
\end{aligned} \tag{6.72}$$

Equation (6.72) gives an expression for $d\boldsymbol{\varepsilon}_{ij}^k$ appearing in equation (6.65). Substituting equation (6.72) into equation (6.65), we obtain the following:

$$d\boldsymbol{\varphi}_{abcd} = \mathbf{G}_{abcdkl} \boldsymbol{\varepsilon}_{kl} + \mathbf{H}_{abcdkl} d\boldsymbol{\varepsilon}_{kl} \tag{6.73}$$

where the two sixth-rank tensors \mathbf{G} and \mathbf{H} are given as:

$$\begin{aligned}
\mathbf{G}_{abcdkl} &= \sum_k \mathbf{c}^k \left(\mathbf{C}_{abcdefgh}^k \right)^{-1} \mathbf{D}_{efghij}^k (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnpq}^{-1} d\boldsymbol{\varphi}_{pqvw}^{(4)k} (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{vwij}^{-1} \bar{\mathbf{A}}_{mntu}^k \\
&\quad \left(\sum_J \mathbf{c}^J (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)J})_{klrs} \bar{\mathbf{B}}_{rstu}^J \right) - (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnij}^{-1} \bar{\mathbf{A}}_{mnpq}^k \left(\sum_J \mathbf{c}^J d\boldsymbol{\varphi}_{klrs}^{(4)J} \bar{\mathbf{B}}_{rspq}^J \right)
\end{aligned} \tag{6.74}$$

and

$$\mathbf{H}_{abcdkl} = \sum_k \mathbf{c}^k \left(\mathbf{C}_{abcdefgh}^k \right)^{-1} \mathbf{D}_{efghij}^k (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})_{mnij}^{-1} \bar{\mathbf{A}}_{mnpq}^k \left(\sum_J \mathbf{c}^J (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)J})_{klrs} \bar{\mathbf{B}}_{rspq}^J \right) \tag{6.75}$$

As can be seen from equation (6.73), the increment of the composite damage tensor $d\boldsymbol{\varphi}_{abcd}$ is a function of not only the increment of the composite strain tensor $d\boldsymbol{\varepsilon}_{kl}$, but it is also a function of the composite strain tensor $\boldsymbol{\varepsilon}_{kl}$ itself. This equation is nonlinear and its solution requires an iterative procedure that involves solving a set of nonlinear simultaneous equations, even for a simple problem of uniaxial tension.

Next, we illustrate damage evolution process in a one dimensional bar made of a composite material using a simple damage evolution approach (see Figure 6.3). In this illustration, this approach allows us to obtain the relation between the composite damage variable φ_x and the composite strain ε_x by solving a quadratic equation. The constituents of the one dimensional bar are assumed to be a polyimide matrix and a graphite fiber only, i.e. the volume fraction of the interface region is negligible and assumed to be equal to zero. The properties of the composite material are as follows:

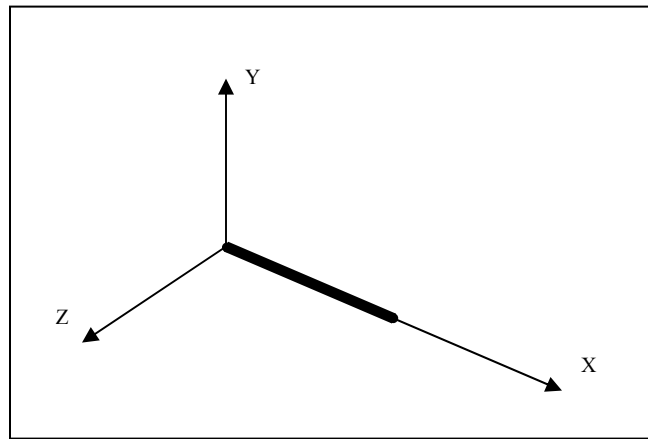


Figure 6.3 Illustration of a One Dimensional Bar

For the matrix: $\bar{E}^m = 2.756 * 10^3$ MPa, $\nu^m = 0.33$, $c^m = 60\%$

For the fiber: $\bar{E}^f = 2.756 * 10^5$ MPa, $\nu^f = 0.2$, $c^f = 40\%$

For the composite: $\bar{E} = c^m \bar{E}^m + c^f \bar{E}^f = 1.119 * 10^5$ MPa

It has been shown in Chapter 5 that for an isotropic elastic material in a one dimensional problem, the following relation holds:

$$\varphi = \frac{\bar{E}(\partial L / \partial \ell) \varepsilon^2}{2 + \bar{E}(\partial L / \partial \ell) \varepsilon^2} \quad (6.76)$$

and by assuming elastic isotropic constituents, we can write the following equation relating the constituent damage variable φ^k to the constituent strain ε^k as follows (based on equation (6.76)):

$$\varphi^k = \frac{\bar{E}^k (\partial L^k / \partial \ell^k) (\varepsilon^k)^2}{2 + \bar{E}^k (\partial L^k / \partial \ell^k) (\varepsilon^k)^2} \quad (6.77)$$

We will solve the problem by assuming that the elastic strain range of a material is up to 10%. At each step, the matrix strain ε^m will be assigned; incremented from zero% to 10% by a 1% increment. When the elastic strain of the matrix ε^m is known, φ^m can be obtained from equation (6.77). Then we need to solve for the fiber strain ε^f and the fiber damage variable φ^f in order to be able to obtain an expression for the composite damage variable φ , where the latter can be plotted against the composite strain ε to illustrate damage evolution in the composite system. For each matrix strain, we proceed as follows:

By observing that for a one dimensional problem the definition of $\mathbf{M}^k(\varphi^{(4)k})$ and $\mathbf{M}(\varphi^{(4)})$ reduces to $M^k(\varphi^k) = (1 - \varphi^k)^{-1}$ and $M(\varphi) = (1 - \varphi)^{-1}$ (scalars), equation (6.67) reduces to the following:

$$A^k = (1 - \varphi^k)^{-1} \bar{A}^k (1 - \varphi) \quad \text{or} \quad A^k = \bar{A}^k \frac{(1 - \varphi)}{(1 - \varphi^k)} \quad (6.78a,b)$$

Now that we have an expression for A^k , we can relate the constituent strain ε^k to the composite strain ε by reducing equation (6.11) to the one dimensional case. Assuming

the Poisson's ratio $\nu = 0$ and that all the damage tensor components $\phi_{ijkl}^{(4)k}$ vanish except

$\phi_{1111}^{(4)k}$ which is denoted here by φ , we obtain:

$$\varepsilon^k = \bar{A}^k \frac{(1-\varphi)}{(1-\varphi^k)} \varepsilon \quad (6.79a)$$

Re-writing equation (6.79a) for the matrix constituent and the fiber constituent, we obtain:

$$\varepsilon^m = \bar{A}^m \frac{(1-\varphi)}{(1-\varphi^m)} \varepsilon \quad (6.79b)$$

and

$$\varepsilon^f = \bar{A}^f \frac{(1-\varphi)}{(1-\varphi^f)} \varepsilon \quad (6.79c)$$

Note that equations (6.79b) and (6.79c) have the composite strain ε as a common factor.

Dividing equation (6.79c) by equation (6.79b), a relation between the matrix strain and the fiber strain can be obtained as follows:

$$\varepsilon^f = \frac{\bar{A}^f}{\bar{A}^m} \frac{(1-\varphi^m)}{(1-\varphi^f)} \varepsilon^m \quad (6.80a)$$

and by rearranging terms to obtain an expression for φ^f , we obtain:

$$\varphi^f = 1 - \left(\frac{\bar{A}^f}{\bar{A}^m} (1-\varphi^m) \frac{\varepsilon^m}{\varepsilon^f} \right) \quad (6.80b)$$

and by writing equation (6.77) in terms of the fiber, we have:

$$\varphi^f = \frac{\bar{E}^f (\partial L^f / \partial \ell^f) (\varepsilon^f)^2}{2 + \bar{E}^f (\partial L^f / \partial \ell^f) (\varepsilon^f)^2} \quad (6.81)$$

By realizing that equations (6.80b) and (6.81) are expressions for the same variable φ^f ,

equating the two equations will give us a quadratic expression for the fiber strain ε^f :

$$\frac{\bar{E}^f (\partial L^f / \partial \ell^f) (\varepsilon^f)^2}{2 + \bar{E}^f (\partial L^f / \partial \ell^f) (\varepsilon^f)^2} + \left(\frac{\bar{A}^f}{\bar{A}^m} (1 - \varphi^m) \frac{\varepsilon^m}{\varepsilon^f} \right) - 1 = 0 \quad (6.82)$$

Solving equation (6.82) gives two values of the fiber strain ε^f , one of which is discarded because it is meaningless. By adopting the correct value of ε^f , the fiber strain φ^f can be obtained from equation (6.81), and the composite strain ε can be obtained from equation (6.79). Substituting ε^m and ε^f back into equation (6.62b), we obtain a value for the composite damage variable φ . Repeating these steps for each increment of the matrix strain, we can have enough values of the composite damage variable φ and the composite strain ε to plot a graph showing damage evolution.

In order to proceed with the solution of equation (6.82), the effective constituent strain concentration factor \bar{A}^k must be known as well as the material properties $\alpha^m = (\partial L^m / \partial \ell^m)$ and $\alpha^f = (\partial L^f / \partial \ell^f)$. Therefore, a composite material model(s) need to be adopted and values for α^m and α^f need be assigned. In this work, we will adopt both the Voigt model and the Reuss model. The reason for adopting these models in particular is that they define the upper and lower bounds for all other composite materials models. The values for α^m and α^f will be taken as $5 \cdot 10^{-4}$ and $1 \cdot 10^{-4}$, respectively.

In the Voigt model, the effective constituent strain $\bar{\varepsilon}^k$ is assumed to be equal to the effective composite strain $\bar{\varepsilon}$; i.e. the strains are constant throughout the composite. Therefore, equation (6.8) in a one dimensional case indicates that $\bar{A}^k = 1$ according to the Voigt model. The damage evolution equations then become as follows (equations (6.78,b) through equation (6.82), equation (6.81) remains unchanged):

$$A^k = \frac{(1 - \varphi)}{(1 - \varphi^k)} \quad (6.83)$$

$$\varepsilon^k = \frac{(1-\varphi)}{(1-\varphi^k)} \varepsilon \quad (6.84)$$

$$\varepsilon^f = \frac{(1-\varphi^m)}{(1-\varphi^f)} \varepsilon^m \quad (6.85)$$

$$\varphi^f = 1 - \frac{\varepsilon^m}{\varepsilon^f} (1-\varphi^m) \quad (6.86)$$

$$\frac{\bar{E}^f (\partial L^f / \partial \ell^f) (\varepsilon^f)^2}{2 + \bar{E}^f (\partial L^f / \partial \ell^f) (\varepsilon^f)^2} + \frac{\varepsilon^m}{\varepsilon^f} (1-\varphi^m) - 1 = 0 \quad (6.87)$$

Incrementing the matrix strain ε^m from 0% to 10% using the Voigt model with increments of 1%, we obtain the following graph for damage evolution between the composite damage variable φ and the composite strain ε :

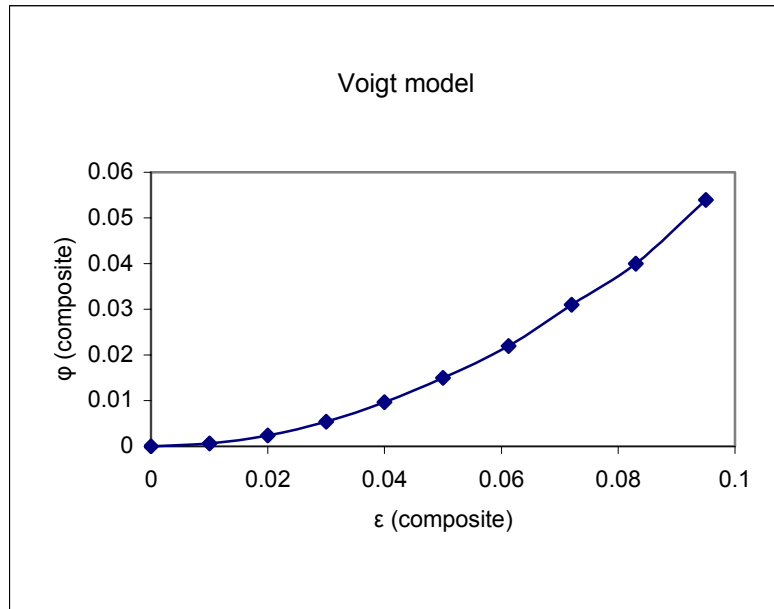


Figure 6.4 Damage Evolution (Voigt Model)

In the Reuss model, the effective constituent stress $\bar{\sigma}^k$ is assumed to be equal to the

effective composite stress $\bar{\sigma}$. Therefore, equation (6.27) in a one dimensional case indicates that $\bar{B}^k = 1$ and $\bar{\sigma} = \bar{\sigma}^k$ according to the Reuss model. Substituting $\bar{\sigma} = \bar{E}\bar{\varepsilon}$ and $\bar{\sigma}^k = \bar{E}^k\bar{\varepsilon}^k$, rearranging terms and comparing the result to equation (6.8), we obtain:

$$\bar{A}^k = \frac{\bar{E}}{\bar{E}^k} \quad (6.88)$$

The damage evolution equations then become as follows (equations (6.78,b) through equation (6.82), equation (6.81) remains unchanged):

$$A^k = \frac{\bar{E}}{\bar{E}^k} \frac{(1-\varphi)}{(1-\varphi^k)} \quad (6.89)$$

$$\varepsilon^k = \frac{\bar{E}}{\bar{E}^k} \frac{(1-\varphi)}{(1-\varphi^k)} \varepsilon \quad (6.90)$$

$$\varepsilon^f = \frac{\bar{E}^m}{\bar{E}^f} \frac{(1-\varphi^m)}{(1-\varphi^f)} \varepsilon^m \quad (6.91)$$

$$\varphi^f = 1 - \left(\frac{\bar{E}^m}{\bar{E}^f} (1-\varphi^m) \frac{\varepsilon^m}{\varepsilon^f} \right) \quad (6.92)$$

$$\frac{\bar{E}^f (\partial L^f / \partial \ell^f) (\varepsilon^f)^2}{2 + \bar{E}^f (\partial L^f / \partial \ell^f) (\varepsilon^f)^2} + \left(\frac{\bar{E}^m}{\bar{E}^f} (1-\varphi^m) \frac{\varepsilon^m}{\varepsilon^f} \right) - 1 = 0 \quad (6.93)$$

Incrementing the matrix strain ε^m using the Reuss model, and using the rule of mixture to obtain the composite strain, we obtain a graph for damage evolution between the composite damage variable φ and the composite strain ε (see Figure 6.5):

Next, we present the damage evolution curves for the Voigt model and the Reuss model on the same graph in order to compare the results obtained by each model. As can be seen from Figure 6.6, the values of the damage variable obtained using the Voigt

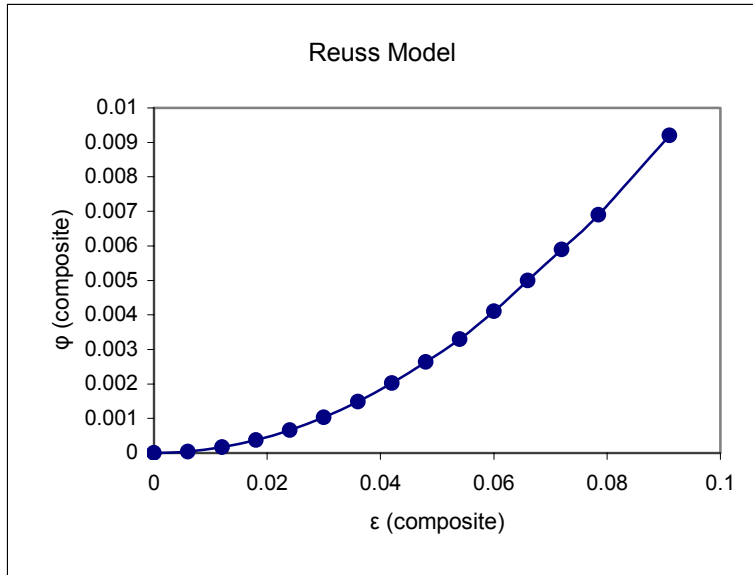


Figure 6.5 Damage Evolution (Reuss Model)

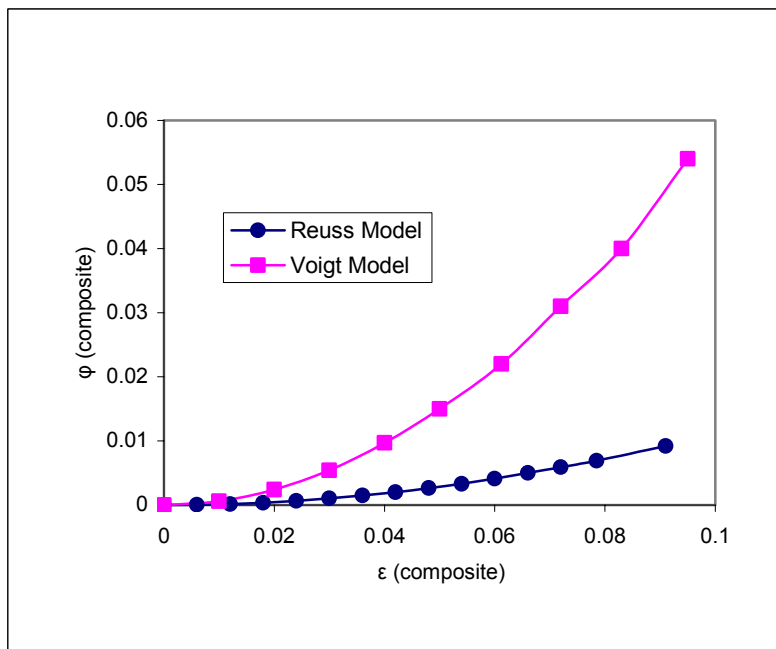


Figure 6.6 Voigt and Reuss Models

model are bigger than those obtained using the Reuss model. Other models of composite

materials should have their damage curves somewhere in between the region bounded by the curves of the Voigt and the Reuss Models.

It should also be noted that the values of the damage variable of a composite material obtained by both models (Voigt and Reuss) are higher than those obtained in Chapter 5 for an isotropic elastic material.

We conclude this chapter by writing the equation of composite damage evolution, using the Micromechanical Approach, in terms of the fabric tensors. Voyiadjis and Kattan (2005) have shown that for an isotropic elastic material, the increment of the material damage tensor $d\boldsymbol{\phi}^{(4)}$ can be related to the increment of the materials fabric tensors $d\mathbf{G}^{(0)}$ and $d\mathbf{G}^{(2)}$ through the following relation:

$$\begin{aligned} d\boldsymbol{\phi}^{(4)} = & -2[\lambda(\mathbf{G}^{(0)}\mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \otimes (d\mathbf{G}^{(0)}\mathbf{I}^{(2)} + d\mathbf{G}^{(2)}) \\ & + 2\mu(\mathbf{G}^{(0)}\mathbf{I}^{(2)} + \mathbf{G}^{(2)}) \bar{\otimes} (d\mathbf{G}^{(0)}\mathbf{I}^{(2)} + d\mathbf{G}^{(2)})] : \bar{\mathbf{E}}^{-1} \end{aligned} \quad (6.94)$$

Applying this equation to the isotropic constituents of the composite material, we obtain:

$$\begin{aligned} d\boldsymbol{\phi}^{(4)k} = & -2[\lambda^k (\mathbf{G}^{(0)k}\mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \otimes (d\mathbf{G}^{(0)k}\mathbf{I}^{(2)} + d\mathbf{G}^{(2)k}) \\ & + 2\mu^k (\mathbf{G}^{(0)k}\mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \bar{\otimes} (d\mathbf{G}^{(0)k}\mathbf{I}^{(2)} + d\mathbf{G}^{(2)k})] : (\bar{\mathbf{E}}^k)^{-1} \end{aligned} \quad (6.95)$$

substituting equation (6.95) into equation (6.64a) which relates the increment of the composite damage tensor $d\boldsymbol{\phi}^{(4)}$ to the increments of the constituents damage tensors $d\boldsymbol{\phi}^{(4)k}$, we have the following:

$$\begin{aligned} d\boldsymbol{\phi}^{(4)} = & \sum_k c^k \{ -2[\lambda^k (\mathbf{G}^{(0)k}\mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \otimes (d\mathbf{G}^{(0)k}\mathbf{I}^{(2)} + d\mathbf{G}^{(2)k}) \\ & + 2\mu^k (\mathbf{G}^{(0)k}\mathbf{I}^{(2)} + \mathbf{G}^{(2)k}) \bar{\otimes} (d\mathbf{G}^{(0)k}\mathbf{I}^{(2)} + d\mathbf{G}^{(2)k})] : (\bar{\mathbf{E}}^k)^{-1} \} \end{aligned} \quad (6.96)$$

Equation (6.96) is an alternative damage evolution equation where the increment of the composite damage tensor $d\boldsymbol{\varphi}^{(4)}$ is obtained by relating it to the equation of the fabric tensors of the constituents $\mathbf{G}^{(0)k}$ and $\mathbf{G}^{(2)k}$.

6.3 Equivalence of the Micromechanical Approach and the Continuum Approach

In this section, we show the equivalence of the Micromechanical Approach and the Continuum Approach. This can be done by showing that both elasticity tensors given in Chapters 5 and 6 are equal to each other.

It was shown in Chapter 5 that, using the general strain transformation hypothesis (Voyiadjis and Kattan, 2005), the damaged elasticity tensor \mathbf{E} corresponding to the Continuum Approach is given as follows:

$$\mathbf{E} = \mathbf{M}^{-1}(\boldsymbol{\varphi}^{(4)}) : \left(\sum_k \bar{c}^k \bar{\mathbf{E}}^k : \bar{\mathbf{A}}^k \right) : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) \quad \text{where } k = m, f, i \quad (6.97)$$

In Chapter 6, it was shown that the damaged elasticity tensor \mathbf{E} corresponding to the Micromechanical Approach is given in equation (6.18).

By applying equations (6.1) and (6.2) into the law of mixtures in the undamaged configuration, equation (6.28), and comparing the results with the law of mixtures in the damaged configuration, equation (6.13), we obtain the following transformation equation for the volume fractions:

$$c^k \mathbf{I}^{(4)} = \bar{c}^k \mathbf{M}^{-1}(\boldsymbol{\varphi}^{(4)}) : \mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \quad (6.98)$$

Substituting equation (6.7) and (6.12) into equation (6.16) and rearranging terms, we obtain:

$$\mathbf{E} = \left(\sum_k c^k \left(\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) \right)^{-1} : \bar{\mathbf{E}}^k : \bar{\mathbf{A}}^k \right) : \mathbf{L}(\boldsymbol{\varphi}^{(8)}) \quad (6.99)$$

and substituting equation (6.99) into equation (6.98), we obtain equations (6.97), which is the damaged elasticity tensor \mathbf{E} obtained using the Continuum Approach. Thus, the equivalence between the Continuum Approach and the Micromechanical Approach is proved within the framework of the theory of elasticity.

7 Numerical Examples

In this chapter, we will present examples to illustrate the applicability of the theory developed in the earlier chapters. Three examples will be presented in this chapter. One example will be presented using the Continuum Approach, where damage is introduced to the composite system as a homogenized system that does not depend on the constituents' properties (after homogenization). Another two examples will be presented to illustrate the use of the Micromechanical Approach, where damage is introduced to the constituents of the composite system separately, and then by homogenizing the damaged properties of the constituents, damage of the composite system as a whole is obtained. In all the examples, a single lamina is used where the condition of plane stress is assumed.

7.1 Example 1: Using the Continuum Approach

In this section, we present an application of the Continuum Approach to damage with fabric tensors of composites for the case of a parallel micro-crack distribution. Consider a two-dimensional parallel micro-crack distribution in a composite lamina as shown in Figure 7.1. The Representative Volume Element (RVE) shown is assumed to be isolated from a cross section, of a fiber-reinforced composite material, perpendicular to the direction of load application, i.e., micro-cracks will grow in a direction perpendicular to the direction of the load. These micro-cracks are thus oriented such that their normals are at an angle $\theta = 90^\circ$ (Voyiadjis and Kattan, 2005).

The composite lamina will be assumed to be made of graphite epoxy (GY70/339) with the following properties: $E_{11} = 2.89 * 10^5$ MPa, $E_{22} = 6.063 * 10^3$ MPa, $G_{12} = 4.134 * 10^3$ MPa, $\nu_{12} = 0.31$, and $\nu_{21} = 0.0065$.

Because of the nature of the Continuum Approach, identification of micro-cracks growing in different constituents will not be considered. The fabric tensors and the

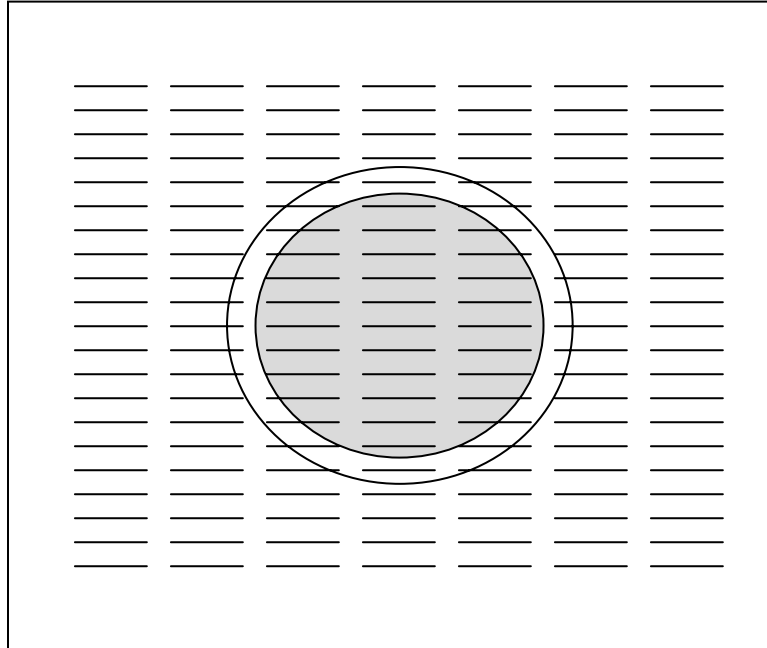


Figure 7.1: RVE of the Composite System Showing a Set of Parallel Micro-cracks in a lamina (Example 1)

damage tensor will correspond to the composite system as a whole. The analysis of micro-crack distributions in the matrix and fibers separately will be dealt with using the Micromechanical Approach. We will calculate the fabric tensors and the damage tensor for this set of micro-cracks.

It should be pointed out that the number of these parallel micro-cracks is immaterial in this case since we are considering an RVE. We will obtain the same fabric tensors irrespective of the number of these parallel micro-cracks. First we calculate the second-rank tensor $\mathbf{G}^{(2)}$. The fabric tensor $\mathbf{G}^{(2)}$ is taken to correspond to the fabric tensor of the third kind $\mathbf{D}^{(2)}$ introduced by Kanatani (1984a), where $\mathbf{D}^{(2)}$ is given as (see chapter 4):

$$\mathbf{G}^{(2)} = \mathbf{D}^{(2)} = \frac{15}{2}(\mathbf{N}^{(2)} - \frac{1}{3}\mathbf{I}^{(2)}) \quad (7.1)$$

where $\mathbf{I}^{(2)}$ is the second-rank identity tensor and $\mathbf{N}^{(2)}$ is the second-rank fabric tensor of the first kind given by Kanatani (1984a) as:

$$\mathbf{N}_{ij}^{(2)} = \frac{1}{N} \sum_{\alpha=1}^N n_i^{(\alpha)} n_j^{(\alpha)} \quad (7.2)$$

where N is the total number of micro-cracks, and according to Voyiadjis and Kattan, (2005), by letting $\theta^{(\alpha)}$ be the orientation angle of the normal to the micro-crack α ($\alpha=1, \dots, N$), then the components of the normals $n_i^{(\alpha)}$ ($i=1,2$) are given by:

$$n_1^{(\alpha)} = \cos \theta^{(\alpha)} \quad (7.3a)$$

$$n_2^{(\alpha)} = \sin \theta^{(\alpha)} \quad (7.3b)$$

and the components of the second-rank fabric tensor of the first kind $\mathbf{N}^{(2)}$ then become:

$$\mathbf{N}_{11}^{(2)} = \frac{1}{N} \sum_{\alpha=1}^N (\cos \theta^{(\alpha)})^2 \quad (7.4a)$$

$$\mathbf{N}_{22}^{(2)} = \frac{1}{N} \sum_{\alpha=1}^N (\sin \theta^{(\alpha)})^2 \quad (7.4b)$$

$$\mathbf{N}_{12}^{(2)} = \mathbf{N}_{21}^{(2)} = \frac{1}{N} \sum_{\alpha=1}^N \sin \theta^{(\alpha)} \cos \theta^{(\alpha)} \quad (7.4c)$$

while the rest of the components are equal to zero.

For this example, the second-rank fabric tensor of the first kind $\mathbf{N}^{(2)}$ then becomes:

$$\mathbf{N}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (7.5)$$

and by applying equation (7.5) into equation (7.1), we obtain the second-rank fabric tensor of the third kind $\mathbf{D}^{(2)}$ (which is the same as the second-rank fabric tensor $\mathbf{G}^{(2)}$) for this example as:

$$\mathbf{G}^{(2)} = \begin{bmatrix} -2.5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2.5 \end{bmatrix} \quad (7.6)$$

The zero-rank fabric tensor (scalar) $G^{(0)}$ is taken to correspond to the zero-rank fabric tensor of the third kind D used by Kanatani (1984a). He defined the zero-rank fabric tensor of the third kind $D^{(0)}$ to be equal to unity.

$$G^{(0)} = D^{(0)} = 1 \quad (7.7)$$

Next, it is obvious that the two eigenvalues of $\mathbf{G}^{(2)}$ are $g_1 = -2.5$ and $g_2 = 5$. Substituting these eigenvalues as well as the value of $G^{(0)}$ into the following equation given by Zysset and Curnier (1995):

$$m_i = G^{(0)} + g_i \quad (7.8)$$

we obtain the values for the fabric tensor parameters:

$$m_1 = -1.5 \quad (7.9a)$$

and

$$m_2 = 6 \quad (7.9b)$$

The general state of damage is described by the general fourth-rank damage tensor $\boldsymbol{\phi}^{(4)}$, whereas in this example of plane stress, it is represented by the following general 3 x 3 matrix:

$$\boldsymbol{\Phi}^{(4)} = \begin{pmatrix} \Phi_{1111} & \Phi_{1212} & \Phi_{1313} \\ \Phi_{2121} & \Phi_{2222} & \Phi_{2323} \\ \Phi_{3131} & \Phi_{3232} & \Phi_{3333} \end{pmatrix} \quad (7.10)$$

In this case, and using the hypothesis of elastic energy equivalence (see Chapter 4), the following equation can be written, in matrix form, to represent the damage elasticity tensor \mathbf{E} .

$$[\mathbf{E}] = ([\mathbf{I}^{(4)}] - [\boldsymbol{\Phi}^{(4)}])[\bar{\mathbf{E}}] \quad (7.11)$$

where the fourth-rank identity tensor represented by the following 3 x 3 identity matrix:

$$\mathbf{I}^{(4)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (7.12)$$

The effective constant elasticity tensor $\bar{\mathbf{E}}$ of a composite material is given as follows for the case of plane stress:

$$\bar{\mathbf{E}} = \begin{pmatrix} \frac{E_{11}}{1 - \nu_{12}\nu_{21}} & \frac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}} & \frac{E_{22}}{1 - \nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{pmatrix} \quad (7.13)$$

Substituting the properties of the graphite epoxy (GY70/339), we obtain:

$$\bar{\mathbf{E}} = \begin{pmatrix} 289.6 & 1.882 & 0 \\ 1.882 & 6.075 & 0 \\ 0 & 0 & 4.134 \end{pmatrix} \text{GPa} \quad (7.14)$$

Next, by comparing the damaged and the undamaged elasticity tensors of an elastic isotropic material given in Chapter 4, the matrix representation of the elasticity tensor \mathbf{E}

of the damaged composite material treated as a homogenized system may be written as follows:

$$\mathbf{E} = \begin{pmatrix} \frac{E_{11}m_1^{2k}}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_{11}m_1^k m_2^k}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{21}E_{11}m_2^k m_1^k}{1-\nu_{12}\nu_{21}} & \frac{E_{22}m_2^{2k}}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12}m_1^k m_2^k \end{pmatrix} \quad (7.15)$$

Upon substituting for m_1 , m_2 , and $k = -0.2$ (Voyiadjis and Kattan, 2005) into equation (7.15), we obtain:

$$\mathbf{E} = \begin{pmatrix} 246.2 & 1.213 & 0 \\ 1.213 & 2.967 & 0 \\ 0 & 0 & 2.020 \end{pmatrix} \text{GPa} \quad (7.16)$$

Equation (7.16) can now be compared to equation (7.14) to observe the reduction in the elasticity tensor components attributed to damage.

The components of the damage tensor $\phi^{(4)}$ in the case of plane stress can be found by applying the procedure given by Voyiadjis and Kattan (2005). Substituting equations (7.10), (7.12), and (7.13) into equation (7.11) and comparing the result with equation (7.15), we obtain nine linear simultaneous algebraic equations in the damage tensor components ϕ_{ijkl} . Four of these nine algebraic equations are readily solved to give the following: $\phi_{1313} = \phi_{2323} = \phi_{3131} = \phi_{3232} = 0$. Therefore, four of the damage tensor components ϕ_{ijkl} vanish in the case of plane stress. This leaves us the following system of five simultaneous algebraic equations.

$$1 - \phi_{1111} - \nu_{21}\phi_{1212} = m_1^{2k} \quad (7.17a)$$

$$v_{21} - v_{21}\phi_{1111} - \phi_{1212} = v_{21}m_1^k m_2^k \quad (7.17b)$$

$$v_{21} - \phi_{2121} - v_{21}\phi_{2222} = v_{21}m_1^k m_2^k \quad (7.17c)$$

$$1 - v_{21}\phi_{2121} - \phi_{2222} = m_2^{2k} \quad (7.17d)$$

$$1 - \phi_{3333} = m_1^k m_2^k \quad (7.17e)$$

where ϕ_{3333} is the out-of-plane damage tensor component; which indicates that the case of plane stress does not imply a case of plane damage (Voyiadjis and Kattan, 2005). This component is readily available from equation (7.17e) if the values of m_i are known. The remaining four damage tensor components ϕ_{1111} , ϕ_{2222} , ϕ_{1212} , and ϕ_{2121} can be obtained by solving the remaining four implicit equations (7.17a-d) simultaneously to obtain:

$$\phi_{1111} = 1 - \frac{m_1^k (m_1^k - v_{21}^2 m_2^k)}{1 - v_{21}^2} \quad (7.18a)$$

$$\phi_{1212} = \frac{v_{21} m_1^k (m_1^k - m_2^k)}{1 - v_{21}^2} \quad (7.18b)$$

$$\phi_{2222} = 1 - \frac{m_2^k (m_2^k - v_{21}^2 m_1^k)}{1 - v_{21}^2} \quad (7.18c)$$

$$\phi_{2121} = \frac{v_{21} m_2^k (m_2^k - m_1^k)}{1 - v_{21}^2} \quad (7.18d)$$

Using the values of material parameters, $v_{21} = 0.0065$ and $k = -0.2$, as well as the values for the fabric tensor parameters $m_1 = -1.5$ and $m_2 = 6$, the damage tensor $\phi^{(4)}$ can be obtained as (using its principal values):

$$\phi^{(4)} = \begin{pmatrix} 0.150 & 0 & 0 \\ 0 & 0.512 & 0 \\ 0 & 0 & 0.356 \end{pmatrix} \quad (7.19)$$

Note that the values of the components of $\boldsymbol{\varphi}^{(4)}$ are relatively high when compared to the range given by Lemaitre and Chaboche (1994): $0.2 \leq \varphi \leq 0.8$. The reason is that the RVE of the composite system adopted in this example is assumed to be infested with micro-cracks (see Figure 7.1). Therefore, the composite material damage will be high. On the other hand, the elasticity tensor components, which are a measure of the material stiffness or ability to resist external loads, will considerably decrease. The relation between the components of the Damage tensor $\boldsymbol{\varphi}^{(4)}$ and the second-rank fabric tensor parameters m_1 and m_2 can be obtained by following the procedure outlined in Appendix D.

7.2 Examples 2 and 3: Using the Micromechanical Approach

In this section, we first present an application of the Micromechanical Approach to damage with fabric tensors of composites for the case of a parallel micro-crack distribution in each constituent of a single composite lamina. In the Micromechanical Approach, damage will be introduced to each constituent separately. The overall damage of the composite system can then be calculated based on the constituents' individual damage tensors.

Consider a composite lamina that is composed of two elastic isotropic constituents, matrix and fibers; (the volume fraction of the interface constituent is assumed to be zero in this example). In each constituent, consider a two-dimensional parallel micro-crack distribution as shown in Figure 7.2. The RVE shown is assumed to be isolated from a cross section, of a fiber-reinforced composite material, perpendicular to the direction of load application, i.e., micro-cracks will grow in a direction perpendicular to the direction of the load. These micro-cracks, in each constituent, are thus oriented such that their normals are at an angle $\theta = 90^\circ$ (Voyiadjis and Kattan, 2005).

The composite material constituents are assumed to be a polyimide matrix and graphite fibers, with the following properties:

$$\bar{E}^m = 2.756 * 10^3 \text{ MPa}, \nu^m = 0.33, c^m = 60\%,$$

$$\bar{E}^f = 2.756 * 10^5 \text{ MPa}, \nu^f = 0.2, c^m = 40\%$$

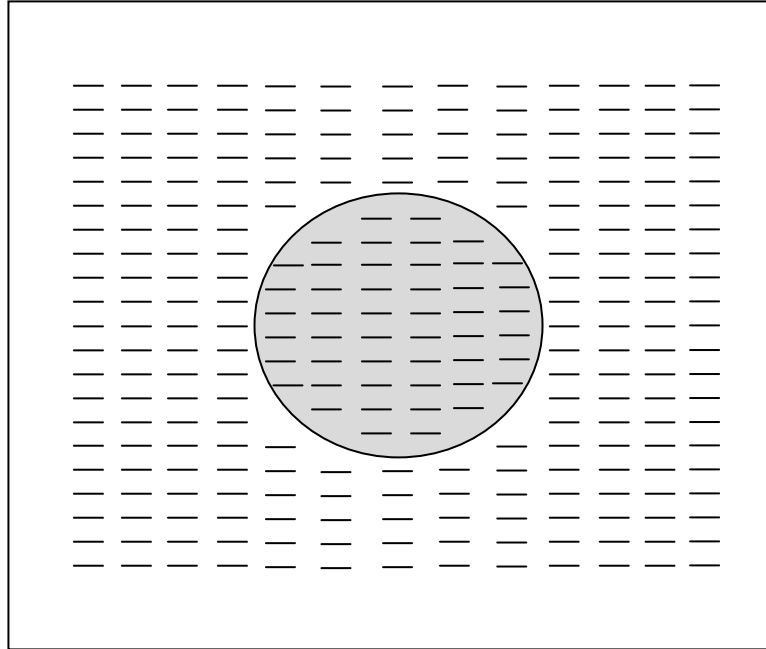


Figure 7.2: RVE of the Composite System Showing Two Sets of Parallel Micro-cracks in a lamina (Example 2)

In the Micromechanical Approach, we will need the expression for the damaged isotropic elasticity tensor \mathbf{E} given by Voyiadjis and Kattan (2005). The 3 x 3 representation of the tensor \mathbf{E} is given as (see Chapter 4):

$$\mathbf{E} = \frac{E}{1-\nu^2} \begin{pmatrix} m_1^{2k} & \nu m_1^k m_2^k & 0 \\ \nu m_2^k m_1^k & m_2^{2k} & 0 \\ 0 & 0 & \frac{1-\nu}{2} m_1^k m_2^k \end{pmatrix} \quad (7.20)$$

where m_i ($i=1,2$) are the second-rank fabric tensor parameters, given in equation (7.8).

We start our calculations for the matrix constituent.

For the matrix:

Using equations (7.3a,b), (7.2), and (7.1), we obtain the matrix second-rank fabric tensor of the first and second kinds $\mathbf{N}^{(2)m}$ and $\mathbf{D}^{(2)m}$ to be equal to the matrices given in equations (7.5) and (7.6), respectively.

Therefore, the matrix second-rank fabric tensor $\mathbf{G}^{(2)m}$ will be equal to:

$$\mathbf{G}^{(2)m} = \begin{bmatrix} -2.5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2.5 \end{bmatrix} \quad (7.21)$$

where it is obvious that the eigenvalues of matrix $\mathbf{G}^{(2)m}$ are $g_1^m = -2.5$ and $g_2^m = 5$. The matrix zero-rank fabric tensor of the third kind \mathbf{D} will be taken to be equal to unity, thus:

$$\mathbf{G}^{(0)m} = 1 \quad (7.22)$$

Using equation (7.8) the matrix second-rank fabric tensor parameters m_i^m ($i=1,2$) can be given as follows:

$$m_1^m = -1.5 \quad (7.23a)$$

and

$$m_2^m = 6 \quad (7.23b)$$

Applying the values obtained in equations (7.23a,b) into equation (7.20), along with $\bar{E}^m = 2.756 * 10^3$ MPa, $\nu^m = 0.33$, and $k = -0.2$, we obtain the matrix second-rank damaged elasticity tensor \mathbf{E}^m as follows:

$$\mathbf{E}^m = \frac{2.756 * 10^3}{1 - 0.33^2} \begin{pmatrix} 0.85 & 0.213 & 0 \\ 0.213 & 0.49 & 0 \\ 0 & 0 & 0.216 \end{pmatrix} \text{MPa} \quad (7.24)$$

Using the following equations (see Chapter 4) that are similar to equations (7.17e) and (7.18a-d), except that they are used to find the components of the damage tensor $\boldsymbol{\phi}^{(4)}$ for an isotropic elastic material:

$$\phi_{1111} = 1 - \frac{m_1^k (m_1^k - v^2 m_2^k)}{1 - v^2} \quad (7.25a)$$

$$\phi_{1212} = \frac{v m_1^k (m_1^k - m_2^k)}{1 - v^2} \quad (7.25b)$$

$$\phi_{2222} = 1 - \frac{m_2^k (m_2^k - v^2 m_1^k)}{1 - v^2} \quad (7.25c)$$

$$\phi_{2121} = \frac{v m_2^k (m_2^k - m_1^k)}{1 - v^2} \quad (7.25d)$$

$$\phi_{3333} = 1 - m_1^k m_2^k \quad (7.25e)$$

the matrix damage tensor $\boldsymbol{\phi}^{(4)m}$ can be expressed in principal values as follows:

$$\boldsymbol{\phi}^{(4)m} = \begin{bmatrix} 0.125 & 0 & 0 \\ 0 & 0.530 & 0 \\ 0 & 0 & 0.356 \end{bmatrix} \quad (7.26)$$

For the fibers:

Due to the fact that the set of micro-cracks in the fibers has the same angle of orientation as that of the matrix, the fabric tensors to be obtained for the fibers will be the same as those obtained for the matrix. Using equations (7.3a,b), (7.2), and (7.1), we obtain the fibers second-rank fabric tensor of the first kind and second kind $\mathbf{N}^{(2)f}$ and $\mathbf{D}^{(2)f}$ to be equal to the matrices given in equations (7.5) and (7.6), respectively.

Therefore, the fibers second-rank fabric tensor $\mathbf{G}^{(2)f}$ will be equal to:

$$\mathbf{G}^{(2)f} = \begin{bmatrix} -2.5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2.5 \end{bmatrix} \quad (7.27)$$

where it is obvious that the eigenvalues of matrix $\mathbf{G}^{(2)f}$ are $g_1^f = -2.5$ and $g_2^f = 5$. The fibers zero-rank fabric tensor of the third kind D will be taken to be equal to unity, thus:

$$G^{(0)f} = 1 \quad (7.28)$$

Using equation (7.8) the fibers second-rank fabric tensor parameters m_i^f ($i=1,2$) can be given as follows:

$$m_1^f = -1.5 \quad (7.29a)$$

and

$$m_2^f = 6 \quad (7.29b)$$

Applying the values obtained in equations (7.29a,b) into equation (7.20), along with

$\bar{E}^f = 2.756 \cdot 10^5$ MPa, $\nu^f = 0.2$, and $k = -0.2$, we obtain the fibers second-rank damaged elasticity tensor \mathbf{E}^f as follows:

$$\mathbf{E}^f = \frac{2.756 \cdot 10^5}{1 - 0.2^2} \begin{pmatrix} 0.85 & 0.129 & 0 \\ 0.129 & 0.49 & 0 \\ 0 & 0 & 0.258 \end{pmatrix} \text{MPa} \quad (7.30)$$

and again, using equations (7.25a-e), the fibers damage tensor $\boldsymbol{\phi}^{(4)f}$ can be expressed in principal values as follows:

$$\boldsymbol{\phi}^{(4)f} = \begin{bmatrix} 0.141 & 0 & 0 \\ 0 & 0.480 & 0 \\ 0 & 0 & 0.356 \end{bmatrix} \quad (7.31)$$

One way to obtain the overall damage tensor of the composite material is accomplished by using the following equation (derived in Chapter 6):

$$\boldsymbol{\varphi}^{(4)} = \sum_k c^k \boldsymbol{\varphi}^{(4)k} \quad , k = m, f \quad (7.32a)$$

where in this example, equation (7.32a) becomes:

$$\boldsymbol{\varphi}^{(4)} = c^m \boldsymbol{\varphi}^{(4)m} + c^f \boldsymbol{\varphi}^{(4)f} \quad (7.32b)$$

Therefore, using equations (7.26) and (7.31), as well as the volume fractions of the constituents, the composite damage tensor $\boldsymbol{\varphi}^{(4)}$ can be given as:

$$\boldsymbol{\varphi}^{(4)} = \begin{bmatrix} 0.131 & 0 & 0 \\ 0 & 0.510 & 0 \\ 0 & 0 & 0.356 \end{bmatrix} \quad (7.33)$$

Next, we make use of the following expression (derived in Chapter 6) in order to find the composite damaged elasticity tensor \mathbf{E} :

$$\mathbf{E} = \left(\sum_k c^k \mathbf{E}^k : (\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}))^T : \bar{\mathbf{A}}^k \right) : \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)}) \quad k = m, f \quad (7.34)$$

The terms appearing in the equation above can be calculated as follows: the constituent volume fraction c^k and the constituent damaged elasticity tensors \mathbf{E}^k are already known.

The constituent damage effect tensor $\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k})$ can be obtained as follows (see Chapter 6):

$$\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}) = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)k})^{-1} \quad , k = m, f \quad (7.35)$$

where the constituent damage tensor $\boldsymbol{\varphi}^{(4)k}$ has already been calculated in equations (7.26) and (7.31) and $\mathbf{I}^{(4)}$ is given in equation (7.12).

The composite damage effect tensor $\mathbf{M}(\boldsymbol{\varphi}^{(4)})$ can be calculated using the following equation:

$$\mathbf{M}(\boldsymbol{\varphi}^{(4)}) = (\mathbf{I}^{(4)} - \boldsymbol{\varphi}^{(4)})^{-1} \quad (7.36)$$

where the composite damage tensor $\boldsymbol{\varphi}^{(4)}$ is given by equation (7.33).

The constituent effective strain concentration tensor $\bar{\mathbf{A}}^k$ can be obtained using different composite material models. In this example, $\bar{\mathbf{A}}^k$ will be calculated twice, using the Voigt model and the Reuss model. We start with the Voigt Model:

Voigt model:

In the Voigt model, the effective constituent strain $\bar{\boldsymbol{\varepsilon}}^k$ is assumed to be equal to the effective composite strain $\bar{\boldsymbol{\varepsilon}}$; i.e. the strains are constant throughout the composite. Therefore, $\bar{\mathbf{A}}^k = \mathbf{I}^{(4)}$ according to the Voigt model, where $\mathbf{I}^{(4)}$ is the fourth-rank identity tensor given in matrix form by equation (7.12). Therefore, equation (7.34) becomes:

$$\mathbf{E} = \left(\sum_k c^k \mathbf{E}^k : (\mathbf{M}^k(\boldsymbol{\varphi}^{(4)k}))^T \right) : \mathbf{M}^{-T}(\boldsymbol{\varphi}^{(4)}) \quad k = m, f \quad (7.37)$$

upon substitution of all terms in the equation above, we obtain the following equation for the composite damaged elasticity tensor \mathbf{E} :

$$\mathbf{E} = \begin{bmatrix} 100 & 15 & 0 \\ 15 & 54 & 0 \\ 0 & 0 & 30 \end{bmatrix} \text{ GPa} \quad (7.38)$$

Equation (7.38) can be compared to the composite undamaged elasticity tensor $\bar{\mathbf{E}}$, which can be found using the law of mixtures applied to the effective stresses:

$$\bar{\boldsymbol{\sigma}} = \bar{c}^m \bar{\boldsymbol{\sigma}}^m + \bar{c}^f \bar{\boldsymbol{\sigma}}^f \quad (7.39)$$

Substituting $\bar{\boldsymbol{\sigma}} = \bar{\mathbf{E}} : \bar{\boldsymbol{\varepsilon}}$, $\bar{\boldsymbol{\sigma}}^m = \bar{\mathbf{E}}^m : \bar{\mathbf{A}}^m : \bar{\boldsymbol{\varepsilon}}$, and $\bar{\boldsymbol{\sigma}}^f = \bar{\mathbf{E}}^f : \bar{\mathbf{A}}^f : \bar{\boldsymbol{\varepsilon}}$, and making use of the fact that, in the Voigt model, $\bar{\mathbf{A}}^k = \mathbf{I}^{(4)}$, we obtain:

$$\bar{\mathbf{E}} = \bar{c}^m \bar{\mathbf{E}}^m + \bar{c}^f \bar{\mathbf{E}}^f \quad (7.40)$$

where $\bar{\mathbf{E}}^k$ is given as follows: (isotropic elastic constituents, see Chapter 4)

$$\bar{\mathbf{E}}^k = \frac{E^k}{1-(\nu^k)^2} \begin{pmatrix} 1 & \nu^k & 0 \\ \nu^k & 1 & 0 \\ 0 & 0 & \frac{1-\nu^k}{2} \end{pmatrix} \quad k = m, f \quad (7.41)$$

Substituting values for E^k and ν^k into equation (7.40) for the matrix and the fibers, we have:

$$\bar{\mathbf{E}}^m = \frac{2.756 * 10^3}{1-0.33^2} \begin{pmatrix} 1 & 0.33 & 0 \\ 0.33 & 1 & 0 \\ 0 & 0 & 0.34 \end{pmatrix} \text{ GPa} \quad (7.42)$$

$$\bar{\mathbf{E}}^f = \frac{2.756 * 10^5}{1-0.2^2} \begin{pmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0 \\ 0 & 0 & 0.4 \end{pmatrix} \text{ GPa} \quad (7.43)$$

Equations (7.24) can be compared to equation (7.42) and equation (7.30) can be compared to equation (7.43) to observe the effect of damage in the matrix and the fibers on the elastic stiffness of the matrix and the fibers, respectively.

Using the volume fractions, equation (7.40) will give the following result for the composite undamaged elasticity tensor $\bar{\mathbf{E}}$:

$$\bar{\mathbf{E}} = \begin{bmatrix} 117 & 24 & 0 \\ 24 & 117 & 0 \\ 0 & 0 & 47 \end{bmatrix} \text{ GPa} \quad (7.44)$$

Equation (7.38) can be compared to equation (7.44) to observe the effect of the presence of micro-cracks in the composite system on its elastic stiffness based on the Voigt model.

Reuss model:

In the Reuss model, the stresses in the constituents are assumed to be equal to the composite stress, i.e. the stresses are assumed to be constant throughout the composite system. Therefore, $\bar{\mathbf{B}}^k = \mathbf{I}^{(4)}$ according to the Reuss model, where $\mathbf{I}^{(4)}$ is the fourth-rank identity tensor given in matrix form by equation (7.12). Substituting $\bar{\mathbf{B}}^k = \mathbf{I}^{(4)}$ into the following relation for the constituents in their effective configuration:

$$\bar{\boldsymbol{\sigma}}^k = \bar{\mathbf{B}}^k : \bar{\boldsymbol{\sigma}} \quad (7.45)$$

and substituting $\bar{\boldsymbol{\sigma}}^k = \bar{\mathbf{E}}^k : \bar{\boldsymbol{\varepsilon}}^k$ and $\bar{\boldsymbol{\sigma}} = \bar{\mathbf{E}} : \bar{\boldsymbol{\varepsilon}}$ into equation (7.45), and comparing the result to the following equation:

$$\bar{\boldsymbol{\varepsilon}}^k = \bar{\mathbf{A}}^k : \bar{\boldsymbol{\varepsilon}} \quad (7.46)$$

we obtain:

$$\bar{\mathbf{A}}^k = (\bar{\mathbf{E}}^k)^{-1} : \bar{\mathbf{E}} \quad (7.47)$$

where $\bar{\mathbf{E}}^k$ has already been calculated in equations (7.42) and (7.43), and $\bar{\mathbf{E}}$ (using the Reuss model) can be found by applying the law of mixtures to the effective strains:

$$\bar{\boldsymbol{\varepsilon}} = \bar{c}^m \bar{\boldsymbol{\varepsilon}}^m + \bar{c}^f \bar{\boldsymbol{\varepsilon}}^f \quad (7.48)$$

substituting $\bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{E}}^{-1} : \bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}^k = (\bar{\mathbf{E}}^k)^{-1} : \bar{\boldsymbol{\sigma}}^k$, and knowing that the stresses are constant throughout the composite, we obtain:

$$\bar{\mathbf{E}}^{-1} = \bar{c}^m (\bar{\mathbf{E}}^m)^{-1} + \bar{c}^f (\bar{\mathbf{E}}^f)^{-1} \quad (7.49)$$

Applying equations (7.42) and (7.43), and the constituents' volume fractions into equation (7.49), and taking its inverse, we obtain the effective composite elasticity tensor $\bar{\mathbf{E}}$ as follows:

$$\bar{\mathbf{E}} = \begin{bmatrix} 5.1 & 1.7 & 0 \\ 1.7 & 5.1 & 0 \\ 0 & 0 & 1.7 \end{bmatrix} \text{ GPa} \quad (7.50)$$

Substituting equation (7.50) into equation (7.47), expressions for $\bar{\mathbf{A}}^m$ and $\bar{\mathbf{A}}^f$ can be obtained (see Appendix C). Now that all the parameters in equation (7.34) have been calculated, an expression for the composite damaged elasticity tensor can be obtained as (using the Reuss model):

$$\mathbf{E} = \begin{bmatrix} 4.4 & 1.1 & 0 \\ 1.1 & 2.5 & 0 \\ 0 & 0 & 1.1 \end{bmatrix} \text{ GPa} \quad (7.51)$$

Equation (7.51) can be compared with equation (7.50) to observe the effect of the presence of micro-cracks on the elastic stiffness of a composite system based on the Reuss model. Equation (7.51) can also be compared with equation (7.44) to see the difference between the composite damaged elasticity tensor obtained by the Voigt model and the composite damaged elasticity tensor obtained by the Reuss model. It is well known that the Voigt model gives an upper bound for the stiffness coefficients while the Reuss model gives a lower bound.

Next, we present Example 3, which is the second example using the Micromechanical Approach. Consider an RVE of a composite material that has two elastic isotropic constituents: matrix and fibers, i.e. the volume fraction of the interface layer is assumed to be zero in this example. Each constituent has a set of micro-cracks oriented in such a way that the normal to half of these micro-cracks has an angle $\theta = 90^\circ$ while the normal to the other half has an angle $\theta = 0^\circ$ (see Figure 7.3).

The RVE shown is assumed to be isolated from a cross section, of a fiber-reinforced composite material, perpendicular to the direction of load application, i.e., micro-cracks will grow in a direction perpendicular to the direction of the load. The composite material constituents are assumed to be a polyimide matrix and graphite fibers (the same constituents used in the first example using the Micromechanical Approach, i.e. Example 2).

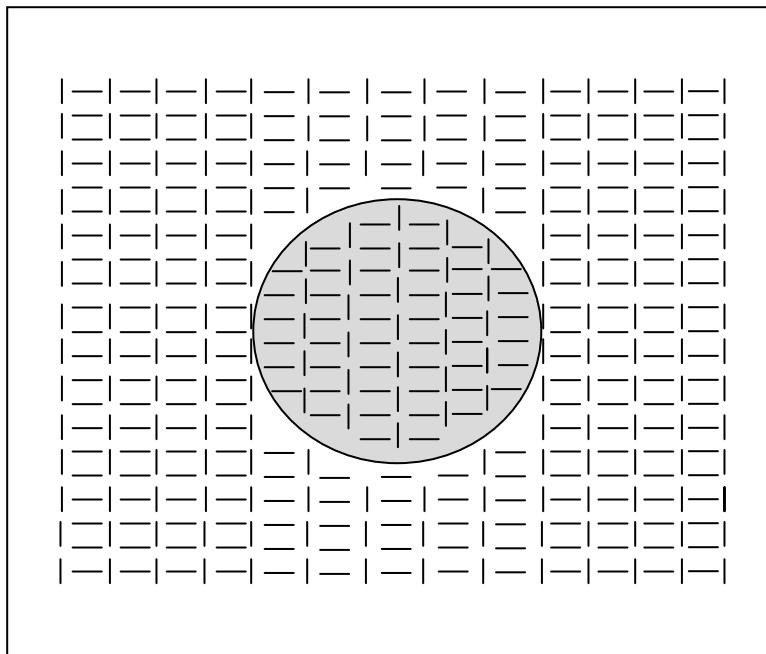


Figure 7.3: RVE of the Composite System Showing Two Sets of Micro-cracks in a lamina (Example 3)

The procedure followed to solve this example is identical to that of the second example. We start our calculations for the matrix constituent.

For the matrix:

We start by calculating the components of the matrix second-rank fabric tensor of the

first kind $\mathbf{N}^{(2)m}$, given by equations (7.4a-c). As an illustration, the $\mathbf{N}_{11}^{(2)m}$ component will be calculated here:

$$\mathbf{N}_{11}^{(2)m} = \frac{1}{N} \sum_{\alpha=1}^N (\cos \theta^{(\alpha)})^2 = \frac{1}{N} \left(\frac{N}{2} (\cos 0)^2 + \frac{N}{2} (\cos 90)^2 \right) = \frac{1}{2} \quad (7.52)$$

The matrix second-rank fabric tensor of the first kind $\mathbf{N}^{(2)m}$ is therefore given as:

$$\mathbf{N}^{(2)m} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (7.53)$$

The matrix second-rank fabric tensor of the second kind $\mathbf{D}^{(2)m}$, and thus $\mathbf{G}^{(2)m}$ are obtained using equation (7.1) as follows:

$$\mathbf{G}^{(2)m} = \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 1.25 & 0 \\ 0 & 0 & -2.5 \end{bmatrix} \quad (7.54)$$

where it is obvious that the eigenvalues of matrix $\mathbf{G}^{(2)m}$ are $g_1^m = 1.25$ and $g_2^m = 1.25$. The matrix zero-rank fabric tensor of the third kind \mathbf{D} will be taken to be equal to unity, thus:

$$\mathbf{G}^{(0)m} = 1 \quad (7.55)$$

Using equation (7.8) the matrix second-rank fabric tensor parameters m_i^m ($i=1,2$) are given as follows:

$$m_1^m = 2.25 \quad (7.56a)$$

and

$$m_2^m = 2.25 \quad (7.56b)$$

Applying the values obtained in equations (7.56a,b) into equation (7.20), along with $\bar{E}^m = 2.756 * 10^3$ MPa, $\nu^m = 0.33$, and $k = -0.2$, we obtain the matrix second-rank damaged elasticity tensor \mathbf{E}^m as follows:

$$\mathbf{E}^m = \frac{2.756 * 10^3}{1 - 0.33^2} \begin{pmatrix} 0.72 & 0.24 & 0 \\ 0.24 & 0.72 & 0 \\ 0 & 0 & 0.24 \end{pmatrix} \text{MPa} \quad (7.57)$$

and using equations (7.25a-e), the matrix damage tensor $\boldsymbol{\phi}^{(4)m}$ can be expressed in principal values as follows:

$$\boldsymbol{\phi}^{(4)m} = \begin{bmatrix} 0.277 & 0 & 0 \\ 0 & 0.277 & 0 \\ 0 & 0 & 0.277 \end{bmatrix} \quad (7.58)$$

For the fibers:

Due to the fact that the set of micro-cracks in the fibers has the same angles of orientation as that of the matrix, the fabric tensors to be obtained for the fibers will be the same as those obtained for the matrix. Using equations (7.3a,b), (7.2), and (7.1), we obtain the fibers second-rank fabric tensor of the first kind and second kind $\mathbf{N}^{(2)f}$ and $\mathbf{D}^{(2)f}$ to be equal to the matrices given in equations (7.53) and (7.54), respectively.

Therefore, the fibers second-rank fabric tensor $\mathbf{G}^{(2)f}$ will be equal to:

$$\mathbf{G}^{(2)f} = \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 1.25 & 0 \\ 0 & 0 & -2.5 \end{bmatrix} \quad (7.59)$$

where it is obvious that the eigenvalues of matrix $\mathbf{G}^{(2)f}$ are $g_1^f = 1.25$ and $g_2^f = 1.25$. The fibers zero-rank fabric tensor of the third kind $D^{(0)}$ will be taken to be equal to unity, thus: $G^{(0)f} = 1$ (7.60)

Using equation (7.8) the fibers second-rank fabric tensor parameters m_1^f can be given as follows:

$$m_1^f = 2.25 \quad (7.61a)$$

and

$$m_2^f = 2.25 \quad (7.61b)$$

Applying the values obtained in equations (7.61a,b) into equation (7.20), along with

$\bar{E}^f = 2.756 \cdot 10^5$ MPa, $\nu^f = 0.2$, and $k = -0.2$, we obtain the fibers second-rank damaged elasticity tensor \mathbf{E}^f as follows:

$$\mathbf{E}^f = \frac{2.756 \cdot 10^5}{1 - 0.2^2} \begin{pmatrix} 0.72 & 0.14 & 0 \\ 0.14 & 0.72 & 0 \\ 0 & 0 & 0.29 \end{pmatrix} \text{MPa} \quad (7.62)$$

and again, using equations (7.25a-e), the fibers damage tensor $\boldsymbol{\phi}^{(4)f}$ can be expressed in principal values as follows:

$$\boldsymbol{\phi}^{(4)f} = \begin{bmatrix} 0.277 & 0 & 0 \\ 0 & 0.277 & 0 \\ 0 & 0 & 0.277 \end{bmatrix} \quad (7.63)$$

Following the same procedure presented in Example 2, only the results thereof will be shown here:

The composite damage tensor $\boldsymbol{\phi}^{(4)}$ can be given as:

$$\boldsymbol{\phi}^{(4)} = \begin{bmatrix} 0.277 & 0 & 0 \\ 0 & 0.277 & 0 \\ 0 & 0 & 0.277 \end{bmatrix} \quad (7.64)$$

Voigt model:

The equation for the composite damaged elasticity tensor \mathbf{E} :

$$\mathbf{E} = \begin{bmatrix} 84 & 16 & 0 \\ 16 & 84 & 0 \\ 0 & 0 & 34 \end{bmatrix} \text{ GPa} \quad (7.65)$$

Equation (7.65) can be compared to the composite undamaged elasticity tensor $\bar{\mathbf{E}}$:

$$\bar{\mathbf{E}} = \begin{bmatrix} 117 & 24 & 0 \\ 24 & 117 & 0 \\ 0 & 0 & 47 \end{bmatrix} \text{ GPa} \quad (7.66)$$

The effect of the presence of micro-cracks in the composite system on its elastic stiffness is well observed from equation (7.65) and (7.66).

Reuss model:

Following the same procedure used in example two, we obtain the effective composite elasticity tensor $\bar{\mathbf{E}}$ using the Reuss model as follows (the same as in example two):

$$\bar{\mathbf{E}} = \begin{bmatrix} 5.1 & 1.7 & 0 \\ 1.7 & 5.1 & 0 \\ 0 & 0 & 1.7 \end{bmatrix} \text{ GPa} \quad (7.67)$$

Using equation (7.34), an expression for the composite damaged elasticity tensor can be obtained as (using the Reuss model):

$$\mathbf{E} = \begin{bmatrix} 3.7 & 1.2 & 0 \\ 1.2 & 3.7 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \text{ GPa} \quad (7.68)$$

Equation (7.68) can be compared with equation (7.67) to observe the effect of the presence of micro-cracks on the elastic stiffness of a composite system. Equation (7.68) can also be compared with equation (7.65) to see the difference between the composite damaged elasticity tensor obtained by the Voigt model and the composite damaged elasticity tensor obtained by the Reuss model. It is well known that the Voigt model gives an upper bound for the stiffness coefficients while the Reuss model gives a lower bound.

Detailed calculations for the examples presented in this chapter using the Micromechanical Approach are shown in Appendix C. The software MAPLE is used to facilitate the process. The first example presented in this chapter using the Continuum Approach is self explanatory and requires no further elaboration.

8 Conclusions

- In this work, continuum damage mechanics with fabric tensors is introduced to the elastic theory of fiber-reinforced composite materials. The use of fabric tensors is important in accounting for the qualitative and quantitative effects of micro-cracks (material defects) on the elastic stiffness of a composite material.
- Two approaches are used to introduce continuum damage mechanics with fabric tensors into the composite material. The first approach is the Continuum Approach, where damage in the composite material is obtained by considering the composite medium as a whole. Effective properties of the constituents are homogenized to give the effective properties of the composite system before damage is introduced. The second approach is the Micromechanical Approach, where damage is introduced to each constituent separately before damage of the whole composite system is described.
- In the Continuum Approach, damage with fabric tensors of the composite material is introduced through one single fourth-rank damage effect tensor $\mathbf{M}(\boldsymbol{\phi}^{(4)})$ that is a function of the composite fourth-rank damage tensor $\boldsymbol{\phi}^{(4)}$. No reference is made to damage in the constituents because the composite material is treated as a homogenized medium.
- The evolution of damage in the composite system is found using the Continuum Approach, relating the increment of the composite damage tensor $d\boldsymbol{\phi}_{ijkl}$ to the increment of the composite elastic strain tensor $d\boldsymbol{\varepsilon}_{ab}$. For the one dimensional case, it is shown that the composite damage will increase as the composite elastic

strain increases. The composite strain is incremented up to a value of 10%, which is assumed to represent the range of elastic deformation of a composite material. The values of the composite damage obtained are in the range 0-0.008 (see Chapter 5) which are reasonable because damage under elastic static range of loading is usually small.

- Using the Continuum Approach, the composite actual (damaged) fourth-rank elasticity tensor \mathbf{E} is calculated for an RVE of a composite material including a set of parallel micro-cracks oriented such that their normal is at an angle 90° (plane stress problem). The zero-rank and the second-rank fabric tensors ($\mathbf{G}^{(0)}$ and $\mathbf{G}^{(2)}$) are calculated for the RVE based on the given micro-crack distribution. Using the parameters obtained from these fabric tensors, the composite actual (damaged) fourth-rank elasticity tensor \mathbf{E} and the composite fourth-rank damage tensor $\boldsymbol{\phi}^{(4)}$ (using their 3×3 matrix representation) are obtained. Comparing the composite damaged fourth-rank elasticity tensor \mathbf{E} to the composite effective (undamaged) elasticity tensor $\bar{\mathbf{E}}$, the effect of the presence of micro-cracks in the RVE on the elastic properties of the composite material is well observed.
- In the Micromechanical Approach, damage with fabric tensors is introduced to the composite medium at the constituent level. For each constituent, a constituent fourth-rank damage effect tensor $\mathbf{M}^k(\boldsymbol{\phi}^{(4)k})$ is introduced, where $\boldsymbol{\phi}^{(4)k}$ is the constituent fourth-rank damage tensor.
- The evolution of damage in the composite system is found using the Micromechanical Approach, relating the increment of the composite damage

tensor $d\phi_{ijkl}$ to the increment of the composite elastic strain tensor $d\epsilon_{ab}$ and the composite elastic strain tensor ϵ_{ab} itself. The damage evolution equation obtained is a nonlinear equation; the solution of which requires solving a set of nonlinear equations even for the simple case of uniaxial tension. For the one dimensional case, it is shown (using another procedure) that the composite damage will increase as the composite elastic strain increases using two different models, namely, the Voigt model and the Reuss model. The composite elastic strain is incremented up to a value of 10%, which is assumed to represent the range of elastic deformation of a composite material. The values of the composite damage variable obtained are in the range 0-0.06 (see Chapter 6) which are reasonable because damage under the elastic static range of loading is usually small.

- Using the Micromechanical Approach, the composite actual (damaged) fourth-rank elasticity tensor \mathbf{E} is calculated twice. First it is calculated for an RVE of a composite material including a set of parallel micro-cracks oriented such that their normal is at an angle 90° (plane stress problem), then it is calculated again for an RVE of a composite material including a set of micro-cracks oriented such that the normal to half of these micro-cracks has an angle of 0° while the normal to the other half has an angle of 90° . The constituent zero-rank and second-rank fabric tensors ($\mathbf{G}^{(0)k}$ and $\mathbf{G}^{(2)k}$) are calculated for both RVEs based on the given micro-crack distributions. Using the parameters obtained from these constituents' fabric tensors, the constituent actual (damaged) fourth-rank elasticity tensor \mathbf{E}^k and the constituent fourth-rank damage tensor $\phi^{(4)k}$ (using their 3×3 matrix representation) are obtained. Applying the Voigt and the Reuss models, the

composite fourth-rank actual (damaged) elasticity tensor \mathbf{E} is found and then compared to the composite effective fourth-rank elasticity tensor $\bar{\mathbf{E}}$ to observe the effect of the presence of micro-cracks in the RVE on the elastic properties of the composite material.

- It should be noted that the values of damage variable obtained from damage evolution for both approaches (Chapters 5 and 6) can not be compared to the values of damage obtained from the examples in Chapter 7. Damage evolution in both approaches showed the relation between the composite damage variable and the composite elastic strain of an initially undamaged composite materials. Whereas, in the examples of Chapter 7, the RVEs of the composite materials are assumed to be infested with micro-cracks, which means that these composite materials are initially damaged. That's why the values of the components of the damage tensors in the examples of Chapter 7 are relatively high.

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Appendix A

Matrix Representation of the Fourth-Rank Tensor \mathbf{J}

The 6 x 6 matrix representation of the symmetric and isotropic fourth-rank tensor \mathbf{J} is given by Voyiadjis and Kattan (1999), as follows:

$$[\mathbf{J}] = \begin{bmatrix} 1 & \mu & \mu & 0 & 0 & 0 \\ \mu & 1 & \mu & 0 & 0 & 0 \\ \mu & \mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1-\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1-\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1-\mu) \end{bmatrix}$$

where μ is a material constant satisfying $-1/2 \leq \mu \leq 1$.

Appendix B

Relevant Derivation

The procedure used to obtain the derivative $d[(\mathbf{I}^{(4)} - \boldsymbol{\Phi}^{(4)k})_{mnij}^{-1}]$ is explained here.

We start with a general arbitrary fourth-rank tensor $\mathbf{A}^{(4)}$. It is well known that the product of a fourth-rank tensor and its fourth-rank inverse yields the fourth-rank identity tensor $\mathbf{I}^{(4)}$, therefore, we have the following equation:

$$\mathbf{A}_{ijmn} \mathbf{A}_{mnlk}^{-1} = \mathbf{I}_{ijkl} \quad (\text{B.1})$$

Taking the derivative of both sides, and recognizing that the derivative of the fourth-rank identity tensor $\mathbf{I}^{(4)}$ vanishes, we obtain:

$$d\mathbf{A}_{ijmn} \mathbf{A}_{mnlk}^{-1} + \mathbf{A}_{ijmn} d\mathbf{A}_{mnlk}^{-1} = \mathbf{0} \quad (\text{B.2a})$$

which can be rewritten as:

$$\mathbf{A}_{ijmn} d\mathbf{A}_{mnlk}^{-1} = -d\mathbf{A}_{ijmn} \mathbf{A}_{mnlk}^{-1} \quad (\text{B.2b})$$

Multiplying both sides by \mathbf{A}_{pqij}^{-1} , we obtain:

$$\mathbf{A}_{pqij}^{-1} \mathbf{A}_{ijmn} d\mathbf{A}_{mnlk}^{-1} = -\mathbf{A}_{pqij}^{-1} d\mathbf{A}_{ijmn} \mathbf{A}_{mnlk}^{-1} \quad (\text{B.3a})$$

By noting that $\mathbf{A}_{pqij}^{-1} \mathbf{A}_{ijmn} = \mathbf{I}_{pqmn}$, we have:

$$\mathbf{I}_{pqmn} d\mathbf{A}_{mnlk}^{-1} = -\mathbf{A}_{pqij}^{-1} d\mathbf{A}_{ijmn} \mathbf{A}_{mnlk}^{-1} \quad (\text{B.3b})$$

and using the definition of the fourth-rank identity tensor $\mathbf{I}^{(4)}$ given as follows (Cauvin and Testa, 1999):

$$\mathbf{I}_{pqmn} = \frac{1}{2} (\boldsymbol{\delta}_{pm} \boldsymbol{\delta}_{qn} + \boldsymbol{\delta}_{pn} \boldsymbol{\delta}_{qm}) \quad (\text{B.4})$$

equation (A.3b) then becomes:

$$d\mathbf{A}_{pqkl}^{-1} = -\mathbf{A}_{pqij}^{-1} d\mathbf{A}_{ijmn} \mathbf{A}_{mnkl}^{-1} \quad (\text{B.5a})$$

and by changing indices, equation (B.5a) becomes as follows:

$$d\mathbf{A}_{mnij}^{-1} = -\mathbf{A}_{mnpq}^{-1} d\mathbf{A}_{pqkl} \mathbf{A}_{klij}^{-1} \quad (\text{B.5b})$$

Next, by setting $\mathbf{A}_{pqkl} = (\mathbf{I}^{(4)} - \boldsymbol{\phi}^{(4)k})_{pqkl}^{-1}$ and realizing that $d\mathbf{A}_{pqkl} = -d\boldsymbol{\phi}_{pqkl}^{(4)k}$, then

$d[(\mathbf{I}^{(4)} - \boldsymbol{\phi}^{(4)k})_{mnij}^{-1}]$ can be given according to equation (B.5b) as follows:

$$d[(\mathbf{I}^{(4)} - \boldsymbol{\phi}^{(4)k})_{mnij}^{-1}] = (\mathbf{I}^{(4)} - \boldsymbol{\phi}^{(4)k})_{mnpq}^{-1} d\boldsymbol{\phi}_{pqkl}^{(4)k} (\mathbf{I}^{(4)} - \boldsymbol{\phi}^{(4)k})_{klij}^{-1} \quad (\text{B.6})$$

Appendix C

MAPLE Files

Micromechanical Approach, Example 1, Voigt Model:

```
> restart;  
> with(linalg):  
> E_m:=(2.756e3/(1-  
0.33^2))*matrix(3,3,[0.85,0.213,0,0.213,0.49,0,0,0,0.216]);
```

$$E_m := 3092.806643 \begin{bmatrix} 0.85 & 0.213 & 0 \\ 0.213 & 0.49 & 0 \\ 0 & 0 & 0.216 \end{bmatrix}$$

```
> phi_m:=matrix(3,3,[0.125,0,0,0,0.53,0,0,0,0.356]);
```

$$\phi_m := \begin{bmatrix} 0.125 & 0 & 0 \\ 0 & 0.53 & 0 \\ 0 & 0 & 0.356 \end{bmatrix}$$

```
> E_f:=(2.756e5/(1-  
0.2^2))*matrix(3,3,[0.85,0.129,0,0.129,0.49,0,0,0,0.258]);
```

$$E_f := 287083.3333 \begin{bmatrix} 0.85 & 0.129 & 0 \\ 0.129 & 0.49 & 0 \\ 0 & 0 & 0.258 \end{bmatrix}$$

```
> phi_f:=matrix(3,3,[0.141,0,0,0,0.48,0,0,0,0.356]);
```

$$\phi_f := \begin{bmatrix} 0.141 & 0 & 0 \\ 0 & 0.48 & 0 \\ 0 & 0 & 0.356 \end{bmatrix}$$

```
> phi:=matadd(phi_m,phi_f,0.6,0.4);
```

$$\phi := \begin{bmatrix} 0.1314 & 0. & 0. \\ 0. & 0.510 & 0. \\ 0. & 0. & 0.3560 \end{bmatrix}$$

```
> Iden:=matrix(3,3,[1,0,0,0,1,0,0,0,1]);
```

$$Iden := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> MT_m:=transpose(inverse(matadd(Iden,-phi_m)));
```

$$MT_m := \begin{bmatrix} 1.142857143 & -0. & 0. \\ -0. & 2.127659574 & -0. \\ 0. & -0. & 1.552795031 \end{bmatrix}$$

```
> MT_f:=transpose(inverse(matadd(Iden,-phi_f)));
```

$$MT_f := \begin{bmatrix} 1.164144354 & -0. & 0. \\ -0. & 1.923076923 & -0. \\ 0. & -0. & 1.552795031 \end{bmatrix}$$

```
> M_tran_inv:=matadd(Iden,-phi);
```

$$M_{tran_inv} := \begin{bmatrix} 0.8686 & 0. & 0. \\ 0. & 0.490 & 0. \\ 0. & 0. & 0.6440 \end{bmatrix}$$

```
>
```

```
E_MPa:=multiply(0.6*multiply(E_m,MT_m)+0.4*multiply(E_f,MT_f),M_tran_inv);
```

$$E_{MPa} := \begin{bmatrix} 100264.9774 & 14370.95529 & 0. \\ 15371.42214 & 53970.06147 & 0. \\ 0. & 0. & 30027.82772 \end{bmatrix}$$

```
> E_GPa:=matrix(3,3,[E_MPa[1,1]*1e-3,E_MPa[1,2]*1e-3,0,E_MPa[2,1]*1e-3,E_MPa[2,2]*1e-3,0,0,0,E_MPa[3,3]*1e-3]);
```

$$E_{GPa} := \begin{bmatrix} 100.2649774 & 14.37095529 & 0 \\ 15.37142214 & 53.97006147 & 0 \\ 0 & 0 & 30.02782772 \end{bmatrix}$$

Comparing the result to the effective overall elastic modulus

```
> Ebar_m:=(2.756e3/(1-0.33^2))*matrix(3,3,[1,0.33,0,0.33,1,0,0,0,0.335]);
```

$$Ebar_m := 3092.806643 \begin{bmatrix} 1 & 0.33 & 0 \\ 0.33 & 1 & 0 \\ 0 & 0 & 0.335 \end{bmatrix}$$

```
> Ebar_f:=(2.756e5/(1-0.2^2))*matrix(3,3,[1,0.2,0,0.2,1,0,0,0,0.4]);
```

$$Ebar_f := 287083.3333 \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$$

```
> Ebar_GPa:=matadd(1e-3*Ebar_m,1e-3*Ebar_f,0.6,0.4);
```

$$Ebar_{GPa} := \begin{bmatrix} 116.6890173 & 23.57904238 & 0. \\ 23.57904238 & 116.6890173 & 0. \\ 0. & 0. & 46.55498746 \end{bmatrix}$$

Micromechanical Approach, Example 1, Reuss Model:

```
> restart;
```

```

> with(linalg):
> Ebar_m:=(2.756e3/(1-
0.33^2))*matrix(3,3,[1,0.33,0,0.33,1,0,0,0,0.335]);
      Ebar_m := 3092.806643  $\begin{bmatrix} 1 & 0.33 & 0 \\ 0.33 & 1 & 0 \\ 0 & 0 & 0.335 \end{bmatrix}$ 

> Ebar_f:=(2.756e5/(1-
0.2^2))*matrix(3,3,[1,0.2,0,0.2,1,0,0,0,0.4]);
      Ebar_f := 287083.3333  $\begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$ 

> Ebar_m_inv:=inverse(Ebar_m);
      Ebar_m_inv :=  $\begin{bmatrix} 0.0003628447025 & -0.0001197387518 & 0. \\ -0.0001197387518 & 0.0003628447025 & -0. \\ 0. & -0. & 0.0009651669089 \end{bmatrix}$ 

> Ebar_f_inv:=inverse(Ebar_f);
      Ebar_f_inv :=  $\begin{bmatrix} 0.3628447025 \cdot 10^{-5} & -0.7256894050 \cdot 10^{-6} & 0. \\ -0.7256894050 \cdot 10^{-6} & 0.3628447025 \cdot 10^{-5} & -0. \\ 0. & -0. & 0.8708272861 \cdot 10^{-5} \end{bmatrix}$ 

> Ebar_inv:=matadd(Ebar_m_inv,Ebar_f_inv,0.6,0.4);
      Ebar_inv :=  $\begin{bmatrix} 0.0002191582003 & -0.00007213352684 & 0. \\ -0.00007213352684 & 0.0002191582003 & -0. \\ 0. & -0. & 0.0005825834544 \end{bmatrix}$ 

> Ebar:=inverse(Ebar_inv);
      Ebar :=  $\begin{bmatrix} 5117.282009 & 1684.297455 & 0. \\ 1684.297455 & 5117.282009 & -0. \\ 0. & -0. & 1716.492276 \end{bmatrix}$ 

> Abar_m:=multiply(Ebar_m_inv,Ebar);
      Abar_m :=  $\begin{bmatrix} 1.655102993 & -0.0015985514 & 0. \\ -0.0015985514 & 1.655102993 & -0. \\ 0. & 0. & 1.656701544 \end{bmatrix}$ 

> Abar_f:=multiply(Ebar_f_inv,Ebar);
      Abar_f :=  $\begin{bmatrix} 0.01734550986 & 0.002397826754 & 0. \\ 0.002397826754 & 0.01734550986 & -0. \\ 0. & 0. & 0.01494768310 \end{bmatrix}$ 

> E_m:=(2.756e3/(1-
0.33^2))*matrix(3,3,[0.85,0.213,0,0.213,0.49,0,0,0,0.216]);

```

$$E_m := 3092.806643 \begin{bmatrix} 0.85 & 0.213 & 0 \\ 0.213 & 0.49 & 0 \\ 0 & 0 & 0.216 \end{bmatrix}$$

```
> E_f := (2.756e5 / (1 - 0.2^2)) * matrix(3, 3, [0.85, 0.129, 0, 0.129, 0.49, 0, 0, 0, 0.258]);
```

$$E_f := 287083.3333 \begin{bmatrix} 0.85 & 0.129 & 0 \\ 0.129 & 0.49 & 0 \\ 0 & 0 & 0.258 \end{bmatrix}$$

```
> MT_m := matrix([[1.142857143, -0., 0.], [-0., 2.127659574, -0.], [0., -0., 1.552795031]]);
```

$$MT_m := \begin{bmatrix} 1.142857143 & -0. & 0. \\ -0. & 2.127659574 & -0. \\ 0. & -0. & 1.552795031 \end{bmatrix}$$

```
> MT_f := matrix([[1.164144354, -0., 0.], [-0., 1.923076923, -0.], [0., -0., 1.552795031]]);
```

$$MT_f := \begin{bmatrix} 1.164144354 & -0. & 0. \\ -0. & 1.923076923 & -0. \\ 0. & -0. & 1.552795031 \end{bmatrix}$$

```
> M_tran_inv := matrix([[.8686, 0., 0.], [0., .490, 0.], [0., 0., .6440]]);
```

$$M_{tran_inv} := \begin{bmatrix} 0.8686 & 0. & 0. \\ 0. & 0.490 & 0. \\ 0. & 0. & 0.6440 \end{bmatrix}$$

```
> E_MPa := multiply(0.6 * multiply(E_m, MT_m, Abar_m) + 0.4 * multiply(E_f, MT_f, Abar_f), M_tran_inv);
```

$$E_{MPa} := \begin{bmatrix} 4361.703251 & 1056.255221 & 0. \\ 1131.916361 & 2508.604507 & 0. \\ 0. & 0. & 1106.906944 \end{bmatrix}$$

Comparing the result to the effective overall elastic modulus

```
> Ebar_MPa := matrix([[5117.282009, 1684.297455, 0.], [1684.297455, 5117.282009, -0.], [0., -0., 1716.492276]]);
```

$$Ebar_{MPa} := \begin{bmatrix} 5117.282009 & 1684.297455 & 0. \\ 1684.297455 & 5117.282009 & -0. \\ 0. & -0. & 1716.492276 \end{bmatrix}$$

Micromechanical Approach, Example 2, Voigt Model:

```
> restart;
> with(linalg):
```

```
> E_m:=(2.756e3/(1-0.33^2))*matrix(3,3,[0.72,0.24,0,0.24,0.72,0,0,0,0.24]);
```

$$E_m := 3092.806643 \begin{bmatrix} 0.72 & 0.24 & 0 \\ 0.24 & 0.72 & 0 \\ 0 & 0 & 0.24 \end{bmatrix}$$

```
> phi_m:=matrix(3,3,[0.277,0,0,0,0.277,0,0,0,0.277]);
```

$$\phi_m := \begin{bmatrix} 0.277 & 0 & 0 \\ 0 & 0.277 & 0 \\ 0 & 0 & 0.277 \end{bmatrix}$$

```
> E_f:=(2.756e5/(1-0.2^2))*matrix(3,3,[0.72,0.14,0,0.14,0.72,0,0,0,0.29]);
```

$$E_f := 287083.3333 \begin{bmatrix} 0.72 & 0.14 & 0 \\ 0.14 & 0.72 & 0 \\ 0 & 0 & 0.29 \end{bmatrix}$$

```
> phi_f:=matrix(3,3,[0.277,0,0,0,0.277,0,0,0,0.277]);
```

$$\phi_f := \begin{bmatrix} 0.277 & 0 & 0 \\ 0 & 0.277 & 0 \\ 0 & 0 & 0.277 \end{bmatrix}$$

```
> phi:=matadd(phi_m,phi_f,0.6,0.4);
```

$$\phi := \begin{bmatrix} 0.2770 & 0. & 0. \\ 0. & 0.2770 & 0. \\ 0. & 0. & 0.2770 \end{bmatrix}$$

```
> Iden:=matrix(3,3,[1,0,0,0,1,0,0,0,1]);
```

$$Iden := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> MT_m:=transpose(inverse(matadd(Iden,-phi_m)));
```

$$MT_m := \begin{bmatrix} 1.383125864 & -0. & 0. \\ -0. & 1.383125864 & -0. \\ 0. & -0. & 1.383125864 \end{bmatrix}$$

```
> MT_f:=transpose(inverse(matadd(Iden,-phi_f)));
```

$$MT_f := \begin{bmatrix} 1.383125864 & -0. & 0. \\ -0. & 1.383125864 & -0. \\ 0. & -0. & 1.383125864 \end{bmatrix}$$

```
> M_tran_inv:=matadd(Iden,-phi);
```

$$M_{tran_inv} := \begin{bmatrix} 0.7230 & 0. & 0. \\ 0. & 0.7230 & 0. \\ 0. & 0. & 0.7230 \end{bmatrix}$$

```
> E_MPa:=multiply(0.6*multiply(E_m,MT_m)+0.4*multiply(E_f,MT_
f),M_tran_inv);
```

$$E_MPa := \begin{bmatrix} 84016.09245 & 16522.03081 & 0. \\ 16522.03081 & 84016.09245 & 0. \\ 0. & 0. & 33747.03081 \end{bmatrix}$$

```
> E_GPa:=matrix(3,3,[E_MPa[1,1]*1e-3,E_MPa[1,2]*1e-
3,0,E_MPa[2,1]*1e-3,E_MPa[2,2]*1e-3,0,0,0,E_MPa[3,3]*1e-
3]);
```

$$E_GPa := \begin{bmatrix} 84.01609245 & 16.52203081 & 0 \\ 16.52203081 & 84.01609245 & 0 \\ 0 & 0 & 33.74703081 \end{bmatrix}$$

Comparing the result to the effective overall elastic modulus

```
> Ebar_m:=(2.756e3/(1-
0.33^2))*matrix(3,3,[1,0.33,0,0.33,1,0,0,0,0.335]);
```

$$Ebar_m := 3092.806643 \begin{bmatrix} 1 & 0.33 & 0 \\ 0.33 & 1 & 0 \\ 0 & 0 & 0.335 \end{bmatrix}$$

```
> Ebar_f:=(2.756e5/(1-
0.2^2))*matrix(3,3,[1,0.2,0,0.2,1,0,0,0,0.4]);
```

$$Ebar_f := 287083.3333 \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$$

```
> Ebar_GPa:=matadd(1e-3*Ebar_m,1e-3*Ebar_f,0.6,0.4);
```

$$Ebar_GPa := \begin{bmatrix} 116.6890173 & 23.57904238 & 0. \\ 23.57904238 & 116.6890173 & 0. \\ 0. & 0. & 46.55498746 \end{bmatrix}$$

Micromechanical Approach, Example 2, Reuss Model:

```
> restart;
```

```
> with(linalg):
```

```
> Ebar_m:=(2.756e3/(1-
0.33^2))*matrix(3,3,[1,0.33,0,0.33,1,0,0,0,0.335]);
```

$$Ebar_m := 3092.806643 \begin{bmatrix} 1 & 0.33 & 0 \\ 0.33 & 1 & 0 \\ 0 & 0 & 0.335 \end{bmatrix}$$

```
> Ebar_f:=(2.756e5/(1-
0.2^2))*matrix(3,3,[1,0.2,0,0.2,1,0,0,0,0.4]);
```

$$Ebar_f := 287083.3333 \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$$

> **Ebar_m_inv:=inverse(Ebar_m);**

$$Ebar_m_inv := \begin{bmatrix} 0.0003628447025 & -0.0001197387518 & 0. \\ -0.0001197387518 & 0.0003628447025 & -0. \\ 0. & -0. & 0.0009651669089 \end{bmatrix}$$

> **Ebar_f_inv:=inverse(Ebar_f);**

$$Ebar_f_inv := \begin{bmatrix} 0.3628447025 \cdot 10^{-5} & -0.7256894050 \cdot 10^{-6} & 0. \\ -0.7256894050 \cdot 10^{-6} & 0.3628447025 \cdot 10^{-5} & -0. \\ 0. & -0. & 0.8708272861 \cdot 10^{-5} \end{bmatrix}$$

> **Ebar_inv:=matadd(Ebar_m_inv,Ebar_f_inv,0.6,0.4);**

$$Ebar_inv := \begin{bmatrix} 0.0002191582003 & -0.00007213352684 & 0. \\ -0.00007213352684 & 0.0002191582003 & -0. \\ 0. & -0. & 0.0005825834544 \end{bmatrix}$$

> **Ebar:=inverse(Ebar_inv);**

$$Ebar := \begin{bmatrix} 5117.282009 & 1684.297455 & 0. \\ 1684.297455 & 5117.282009 & -0. \\ 0. & -0. & 1716.492276 \end{bmatrix}$$

> **Abar_m:=multiply(Ebar_m_inv,Ebar);**

$$Abar_m := \begin{bmatrix} 1.655102993 & -0.0015985514 & 0. \\ -0.0015985514 & 1.655102993 & -0. \\ 0. & 0. & 1.656701544 \end{bmatrix}$$

> **Abar_f:=multiply(Ebar_f_inv,Ebar);**

$$Abar_f := \begin{bmatrix} 0.01734550986 & 0.002397826754 & 0. \\ 0.002397826754 & 0.01734550986 & -0. \\ 0. & 0. & 0.01494768310 \end{bmatrix}$$

> **E_m := 3092.806643*matrix([[.72, .24, 0], [.24, .72, 0], [0, 0, .24]]);**

$$E_m := 3092.806643 \begin{bmatrix} 0.72 & 0.24 & 0 \\ 0.24 & 0.72 & 0 \\ 0 & 0 & 0.24 \end{bmatrix}$$

> **E_f := 287083.3333*matrix([[.72, .14, 0], [.14, .72, 0], [0, 0, .29]]);**

$$E_f := 287083.3333 \begin{bmatrix} 0.72 & 0.14 & 0 \\ 0.14 & 0.72 & 0 \\ 0 & 0 & 0.29 \end{bmatrix}$$

```
> MT_m := matrix([[1.383125864, -0., 0.], [-0., 1.383125864, -0.], [0., -0., 1.383125864]]);
```

$$MT_m := \begin{bmatrix} 1.383125864 & -0. & 0. \\ -0. & 1.383125864 & -0. \\ 0. & -0. & 1.383125864 \end{bmatrix}$$

```
> MT_f := matrix([[1.383125864, -0., 0.], [-0., 1.383125864, -0.], [0., -0., 1.383125864]]);
```

$$MT_f := \begin{bmatrix} 1.383125864 & -0. & 0. \\ -0. & 1.383125864 & -0. \\ 0. & -0. & 1.383125864 \end{bmatrix}$$

```
> M_tran_inv := matrix([[.7230, 0., 0.], [0., .7230, 0.], [0., 0., .7230]]);
```

$$M_{tran_inv} := \begin{bmatrix} 0.7230 & 0. & 0. \\ 0. & 0.7230 & 0. \\ 0. & 0. & 0.7230 \end{bmatrix}$$

```
> E_MPa := multiply(0.6*multiply(E_m,MT_m,Abar_m)+0.4*multiply(E_f,MT_f,Abar_f),M_tran_inv);
```

$$E_{MPa} := \begin{bmatrix} 3683.334524 & 1212.098031 & 0. \\ 1212.098031 & 3683.334524 & 0. \\ 0. & 0. & 1235.618245 \end{bmatrix}$$

Comparing the result to the effective overall elastic modulus

```
> Ebar_MPa := matrix([[5117.282009, 1684.297455, 0.], [1684.297455, 5117.282009, -0.], [0., -0., 1716.492276]]);
```

$$Ebar_{MPa} := \begin{bmatrix} 5117.282009 & 1684.297455 & 0. \\ 1684.297455 & 5117.282009 & -0. \\ 0. & -0. & 1716.492276 \end{bmatrix}$$

Appendix D

Relation between the Components of the Fourth-Rank Damage Tensor and the Second-Rank Fabric Tensor Parameters m_1 and m_2

The procedure used to obtain the relation between the components of the fourth-rank damage tensor ϕ_{ijkl} and the second-rank fabric tensor parameters m_1 alone and m_2 alone is illustrated here. Using equations (7.18a-d) and equation (7.17e) presented in Chapter 7, and noting that the summation of the second-rank fabric tensor parameters m_1 and m_2 is equal to a constant denoted here by C (see Chapter 4), we obtain the following equations relating the components of the fourth-rank damage tensor ϕ_{ijkl} to the second-rank fabric tensor parameter m_1 alone or m_2 alone:

$$\phi_{1111} = 1 - \frac{m_1^k (m_1^k - v_{21}^2 (C - m_1)^k)}{1 - v_{21}^2} \quad (\text{D.1a})$$

$$\phi_{1111} = 1 - \frac{(C - m_2)^k ((C - m_2)^k - v_{21}^2 m_2^k)}{1 - v_{21}^2} \quad (\text{D.1b})$$

$$\phi_{1212} = \frac{v_{21} m_1^k (m_1^k - (C - m_1)^k)}{1 - v_{21}^2} \quad (\text{D.2a})$$

$$\phi_{1212} = \frac{v_{21} (C - m_2)^k ((C - m_2)^k - m_2^k)}{1 - v_{21}^2} \quad (\text{D.2b})$$

$$\phi_{2222} = 1 - \frac{(C - m_1)^k ((C - m_2)^k - v_{21}^2 m_1^k)}{1 - v_{21}^2} \quad (\text{D.3a})$$

$$\phi_{2222} = 1 - \frac{m_2^k (m_2^k - v_{21}^2 (C - m_2)^k)}{1 - v_{21}^2} \quad (\text{D.3b})$$

$$\Phi_{2121} = \frac{v_{21}(C - m_1)^k ((C - m_1)^k - m_1^k)}{1 - v_{21}^2} \quad (D.4a)$$

$$\Phi_{2121} = \frac{v_{21}m_2^k (m_2^k - (C - m_2)^k)}{1 - v_{21}^2} \quad (D.4b)$$

$$\Phi_{3333} = 1 - m_1^k (C - m_1)^k \quad (D.5a)$$

$$\Phi_{3333} = 1 - (C - m_2)^k m_2^k \quad (D.5b)$$

Equations (D.1-5) are plotted in Figures D.1 and D.2 to show the relation between the component of $\Phi^{(4)}$ and m_1 alone or m_2 alone. In the plots, the constant C and v_{21} are taken to be equal to 4.5 and 0.0065, respectively.

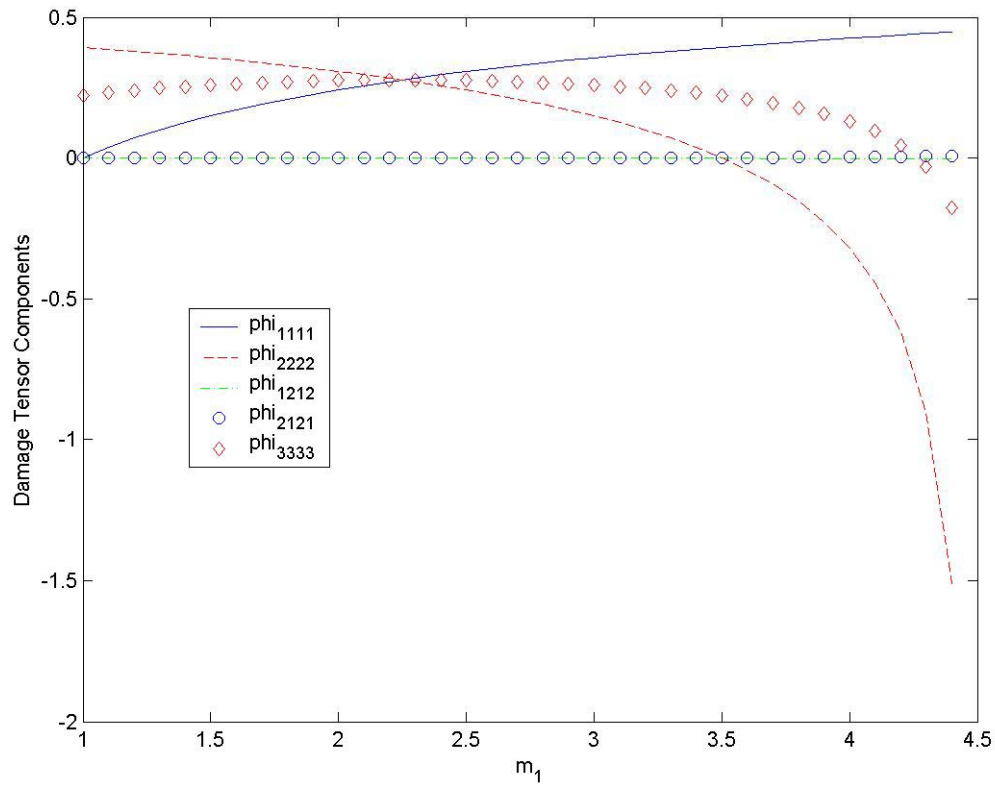


Figure D.1: Relation between Φ_{ijkl} and m_1

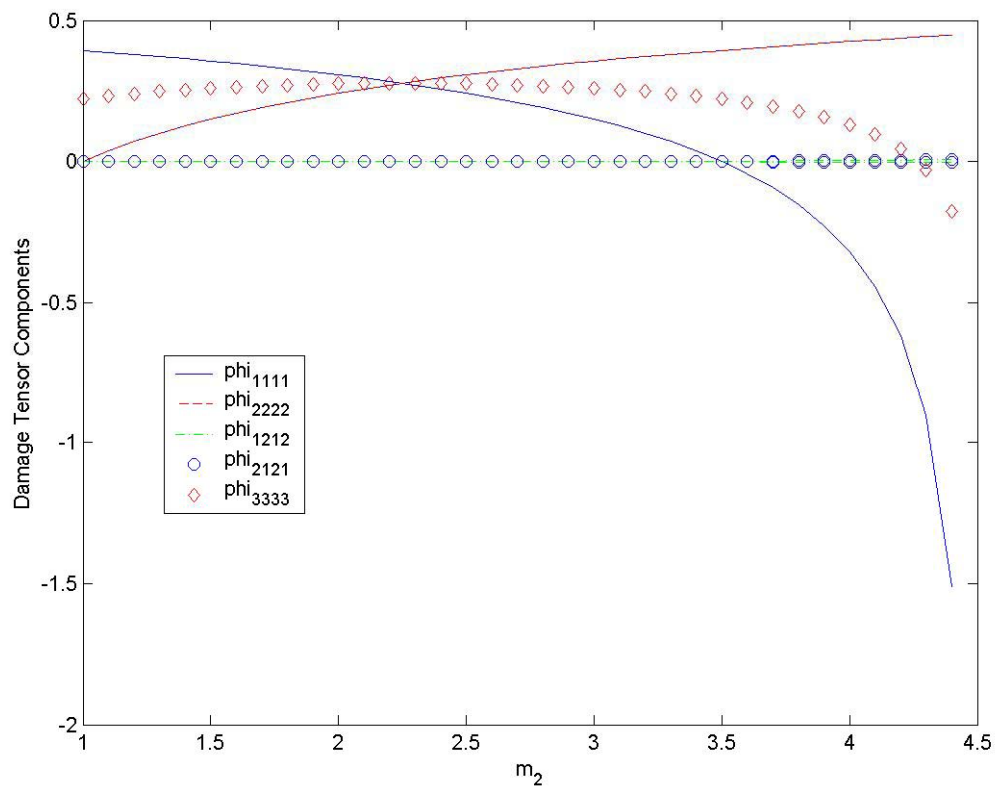


Figure D.2: Relation between ϕ_{ijkl} and m_2

Vita

Ziad Nouredin Taqieddin was born in Amman - Jordan, on May 18, 1978. He attended the Applied Science University (ASU), Amman - Jordan, during the period 1996-2001. He was honored upon completing his Bachelor of Science degree in Civil Engineering for ranking first among his class and rating excellent. He applied to join Louisiana State University in the spring semester of 2003. He is currently enrolled at Louisiana State University and is a candidate for the Master of Science in Civil Engineering degree.