FILIPPOV'S OPERATOR AND DISCONTINUOUS DIFFERENTIAL EQUATIONS

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Dedicated to:
My parents
my wife Fouziyah, and
my friends Waleed & Ibraheem

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Notation

R^n	Euclidean n-dimensional space
$\rho(R^n)$	The family of all subsets of R^n
$\\|\cdot\\|$	Euclidean norm
$\langle\cdot,\cdot\rangle$	Euclidean inner product
B_n	The closed unit ball in R^n
$B_r(x_0)$	Ball with radius r and center in x_0
$d_A(x)$	Distance from point x to A
$d_H(A,B)$	Hausdorff distance between sets A and B
$H(\cdot,\cdot)$	The support function (Hamiltonian)
L^∞	Space of all bounded measurable functions
$m(A)$	Lebesgue measure of set A
C^1	Class of continuously differentiable function
$\mathcal{F}[f]$	Filippov operator
$\mathcal{K}[f]$	Krasovskij operator
$\mathcal{N}$	Newton (classical) solution
$\mathcal{C}$	Caratheodory solution
$\mathcal{F}$	Filippov solution
$\mathcal{K}$	Krasovskig solution
$\mathcal{H}$	Hermes solution
$\mathcal{E}$	Euler solution

Abstract

This thesis is mainly concerned about properties of the so-called Filippov operator that is associated with a differential inclusion

$$\dot{x}(t) \in F\Big(x(t)\Big) \text{ a.e. } t \in [0, T],$$

where $F : R^n \rightrightarrows R^n$ is given set-valued map. The operator $\mathcal{F}$ produces a new set-valued map $\mathcal{F}[F]$, which in effect regularizes F so that $\mathcal{F}[F]$ has nicer properties. After presenting its definition, we show that $\mathcal{F}[F]$ is always upper-semicontinuous as a map from R^n to the metric space of compact subsets of R^n endowed with the Hausdorff metric. Our main approach is to study the operator via its support function, which we show is an upper semicontinuous function. We show that the support function can be used to characterize the operator, and prove a new result that characterizes those set-valued maps that are fixed by $\mathcal{F}$; this result was previously known to hold only in dimension one. We also generalize to higher dimensions a known result that characterizes those set-valued maps that are almost everywhere singleton-valued (that is, $F(x) = \{f(x)\}$ where $f : R^n \to R^n$ is an ordinary function).

The latter part of the thesis introduces four generalized solution concepts of discontinuous differential equations. These are known as the Filippov, Krasovskij, Hermes, and Euler solution concepts. We study the relations among these solution concepts, and in particular prove that the Euler and Hermes solutions in the autonomous case coincide.

Introduction

The classical notion of an ordinary differential equation has the form

$$\frac{dx}{dt} = f\Big(t, x(t)\Big), \tag{0.1}$$

where $f : R \times R^n \to R^n$ is a given function and the solution $x(t)$ is a differentiable function whose derivative satisfies the equation everywhere (or almost everywhere if f is not continuous) in t on a given interval $[0, T]$. There is often an initial condition $x(0) = x_0$ that is also given. A well-known existence theorem due to Peano says that a solution always exists if T is small enough and f is continuous. When f is not continuous, however, solutions may not exist, as can be seen from the following simple example.

Example *$\dot{x} = sgn t$. For $t < 0$, we have $\dot{x} = -1$, the solution being given by $x = -t + c_1$; for $t > 0$, we have $\dot{x} = 1$, the solution being $x = t + c_2$. Proceeding from the requirement of solution continuity for $t = 0$, we obtain*

$$x(0) = \lim_{t \to 0^-} (-t + c_1) = \lim_{t \to 0^+} (t + c_2), \quad x(0) = c_1 = c_2.$$

Consequently, the solution is expressed by the formula $x(t) = |t| + c$. For $t = 0$, the derivative $\dot{x}(t)$ does not exist (Figure 1).

However, we face discontinuous differential equations in many applications. A large number of problems from mechanics and electrical engineering leads to differential equations with discontinuous right-hand sides because many physical laws are expressed by discontinuous functions, for example, a dry friction force of some electronic devices. Many differential equations appearing in control theory model objects

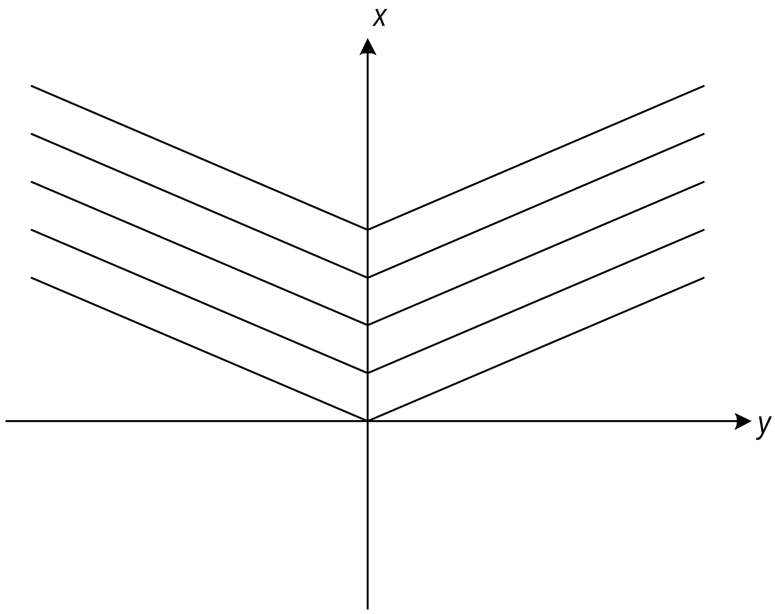

Figure 1: The graph of $x(t) = |t| + c$

with variable structure or with sliding motions that are discontinuous. Another motivation to consider discontinuous right-hand sides is of a mathematical nature. Namely, if the right-hand side is continuous but complicated, it can be useful to approximate it by a simple discontinuous function, for example, by a piecewise constant or piecewise linear function [18].

Thus the consideration of differential equations with discontinuous right-hand sides requires a generalization of the concept of solution. In the case where the right-hand side of equation $\dot{x} = f(t,x)$ is continuous in x and discontinuous only in t, it is usually possible to generalize the concept of solution using only a mathematical argument (continuity is the requirement of solution in the above example). In the case where the right-hand side of the equation is discontinuous in x, such simple mathematical arguments are often insufficient. Then the solution is defined by means of a limiting process taking into account the physical meaning of a given problem. Also, the generalization of the concept of solution must be equivalent to the solution of a differential equation with a continuous right-hand side.

So, it is very important to consider the "Differential Inclusions"

$$\dot{x}(t) \in F(t, x(t)),$$

where F is a set-valued map which associates with any point $(t, x) \in R \times R^n$ a set $F(t, x) \subset R^n$. Differential inclusions serve as models for many dynamical systems. Obviously, any process described by an ordinary differential equation

$$\dot{x}(t) = f(t, x(t))$$

can be described by a differential inclusion with the right-hand side

$$F(t, x(t)) = \{f(t, x(t))\}$$

as well.

Differential inclusions provide a good mathematical tool for studying differential equations

$$\dot{x}(t) = f(t, x(t)), \qquad x(0) = x_0$$

with discontinuous right-hand side, by embedding $f(t, x)$ into the set-valued map $F(t, x)$ which, as a set-valued map, enjoys enough regularity to have trajectories $(x(t))$ closely related to the trajectories of the original differential equation. Differential equation

$$\dot{x}(t) = f(t, x(t))$$

with discontinuous f is a rather unpleasant object from mathematical point of view. In particular, it is impossible to prove existence theorems. However, if solutions of the differential equation with discontinuous right-hand side are regarded to be solutions to the differential inclusion

$$\dot{x}(t) \in \bigcap_{\varepsilon > 0} \overline{\text{conv}}\ f(x + \varepsilon B)$$

(in the case of Krasovskij), then it is possible to develop rigorous mathematical theory of discontinuous system.

One of the most important examples of differential inclusions (discontinuous differential equations) comes from control theory. Consider a control system of the form

$$\dot{x} = g(t, x(t), u(t)), \quad u(t) \in U,\ x(t_0) = x_0,$$

where $u : [0,t] \to R^n$ is a Lebesgue measurable function. Given $u(\cdot)$, the solution to the control system is the solution to the differential equation in the Caratheodory sense when

$$f(t, x(t)) = g(t, x(t), u(t)).$$

However, control theorists are also interested in feedback control, when u is to be a function of x, $u = u(x)$, then we need to solve the feedback equation

$$\dot{x}(t) = g(t, x(t), u(x(t)) = f(t, x(t)),$$

where f does not satisfy in general the classical or Caratheodory solution concepts because u is discontinuous. This leads to consider more general solution concepts. Of course, to give a precise meaning to the notion of generalized solution, we need to assign a rule to associate with the discontinuous function f.

In order to define generalized solutions, two main approaches can be followed. The first approach is to associate a differential inclusion to the differential equation and define the generalized solutions as solutions of the associated differential inclusion. Filippov and Krasovskij solutions follow this method. The second approach consists in finding approximate solutions by means of an algorithm and taking as generalized solutions the uniform limits of the approximate solutions. Hermes and Euler solutions are constructed in this way.

Regarding the solution, in order to obtain the equivalence between a (single-valued) differential equation and the corresponding integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds$$

we had to assume, by definition, that the solution $x(\cdot)$, besides continuous, had to be absolutely continuous, which is the weakest acceptable kind of solution.

In this thesis, we introduce the four generalized solutions to the discontinuous differential equation: Krasovksij, Filippov, Hermes and Euler. We study further the relation between these solutions. Comparison between the solutions was studied. We study also some properties of the Filippov operator by using the concept of the support function and generalized some results related to a Filippov operator from one dimension space to a higher dimension space.

This thesis is composed of four chapters. In Chapter One, we gather several known definitions and theorems which are used in several books needed throughout the thesis. We give very brief introduction about: set-valued analysis, convex sets, the support function and we conclude the chapter by recalling the definition of an absolutely continuous function along with some of its properties. Chapter Two is divided into two parts. In the first part, we show that the Filippov operator is an upper-semicontinuous set-valued map. Then we use the support function to prove more results about the Filippov operator. We end this part by giving a new characterization that shows under what conditions $\mathcal{F}F(x) = F(x)$. In the second part of chapter two, we generalize two results from one-dimension to an n-dimensional space. The first one characterizes the range of the Filippov operator and the second one is the continuity of the Filippov map. Chapter Three is devoted to the study of almost everywhere singleton-valued Filippovs in n dimensional space. We generalize work by Biles and Spraker [2] that was proven in one-dimensional space, and which was based on three technical lemmas. We generalize the three lemmas by proving each lemma with new techniques. We use the Caratheodory theorem, the finite intersection property and the support function for the first, second and third lemma, respectively. Chapter Four is devoted to discontinuous differential equations. We first introduce the three generalized solutions: Filippov, Krasovskij and Hermes and the relation between them by following the work of Hajek [12]. We provide many examples and diagrams to demonstrate clearly the idea of each solution. Then we introduce the fourth generalized solution, Euler solution, which was introduced in “Nonsmooth Analysis and Control Theory” [6] by Professor Peter Wolenski and his colleagues in 1998. We show by an example that the Euler solution is not a generalization of the classical solution and also by another example that there is no relation between the Euler and Filippov solutions. We find a place to fit the Euler solution in the scheme diagram of the generalized solutions by finding a direct relation between Euler and Hermes solutions in the autonomous case. We give a table which summarizes the properties of the four generalized solutions. We conclude chapter four by a Corollary that gives (under certain conditions) all the four solutions are equal.

Chapter 1

Preliminaries and Background Notes

In this chapter, we give some basic definitions and theorems which are going to be used throughout the thesis.

1.1 Some Basic Notations

Throughout this dissertation we denote the set of real numbers by R and the usual n-dimensional space of vectors $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in R$, $i = 1, \ldots, n$ by R^n. The inner product of two vectors x and y in R^n is expressed by $\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n$. The norm of a vector $x \in R^n$ is defined by $|x| = \langle x, x \rangle^{1/2}$. We denote the unit ball in R^n by B_n:

$$B_n = \{x \in R^n : |x| \leq 1\}.$$

The open unit ball in R^n is defined by

$$\mathcal{B}_n^0 = \{x \in R^n : |x| < 1\}.$$

Let $A \subset R^n$. We let $\overline{A}$, A^0 and ∂A be closure, interior and boundary of A, respectively, $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$ and $\overline{B}_r(x_0) = \{x \in R^n : |x - x_0| \leq r\} = \overline{B_r(x_0)}$. For $A, B \subset R^n$ and $\lambda \in R$, we define

$$A + \lambda B = \{a + \lambda b : a \in A \text{ and } b \in B\}$$

and $x + B = \{x\} + B$. So, for example, $B_r(x_0) = x_0 + rB_1(0)$ and $A + B_r(0) = \{x \in R^n : d(x, A) < r\}$ where $d(\cdot, A) : R^n \to R$ the distance function defined by $d_x(A) = d(x, A) = \inf\{|x - a| : a \in A\}, \quad x \in R^n$.

The closure $\mathrm{cl}\, A$ and the interior $\mathrm{int}\, A$ of A are defined by the formulas:

$$\begin{aligned}\mathrm{cl}\, A &= \bigcap_{\varepsilon>0}(A+\varepsilon B_n)\\ \mathrm{int}\, A &= \{a \in A : \exists\, \varepsilon > 0,\ a+\varepsilon B_n \subset A\}.\end{aligned}$$

1.2 Set-Valued Analysis

Let A be a nonempty closed subset of R^n where $d_A(x) = \inf_{y\in A} \|y - x\|$ is the distance from the point x to A. Given two nonempty closed sets A and B, consider

$$e_H(A/B) = \sup\{d_B(x) : x \in A\},$$

called the excess of A over B; geometrically, $e_H(A/B) \leq \varepsilon$ means $A \subset B + B_\varepsilon(0)$ and $B \subset A + B(0, \varepsilon)$.

The *Hausdorff-distance* d_H between A and B is the symmetrization of the above concept:

$$d_H(A, B) = \max\{e_H(A/B), e_H(B/A)\}.$$

One checks immediately that $d_H(A, B) \in R^+ \cup \{+\infty\}$, but d_H is a finite-valued function when restricted to bounded closed sets. Also

$$\begin{aligned}d_H(A, B) &= 0 \Leftrightarrow A = B.\\ d_H(A, B) &= d_H(B, A).\\ d_H(A, C) &\leq d_H(A, B) + d_H(B, C).\end{aligned}$$

In other words, d_H does define a distance on the family of nonempty compact subsets of R^n.

A mapping F that assigns to each $x \in X \subset R^n$ a subset of R^n is called a multi-valued, or set-valued mapping, or more simply a multifunction; we use sometimes the notation $X \ni x \to F(x) \subset R^n$. The set of all subsets of R^n is denoted by $\rho(R^n))$, and so a multifunction F can be viewed as an ordinary function from X to $\rho(R^n))$.

The domain of F is the set of $x \in X$ such that $F(x) \neq \Phi$. Its image (or range) $F(X)$ and graph F are the unions of the sets $F(x) \subset R^n$ and $\{x\} \times F(x) \subset X \times R^n$ respectively. A selection of F is a particular function

$$f : \text{dom } F \to R^n \quad \text{with} \quad f(x) \in F(x) \quad \text{for all} \quad x.$$

Example 1.2 *Define the set-valued map $F : [0, \infty) \to \rho(R)$ by*

$$F(t) = \begin{cases} [0, \infty) & \text{if } t = 0 \\ [0, 1/t] & \text{if } t > 0. \end{cases}$$

Then $d_H(F(t), F(0)) = +\infty$.

The multifunciton F is said to be bounded-valued, closed-valued, convex-valued etc. when the sets $F(x)$ are bounded, closed, convex, etc.

If its graph is a closed set, we say that the multifunction is closed. We say that the multifunction F is locally bounded near x when:

For some neighborhood N of x and bounded set $B \subset R^n$,

$$N \subset \text{ dom } F \quad \text{and} \quad F(N) \subset B.$$

If F is locally bounded near every x in a set A, we say that F is locally bounded on A.

1.2.1 Upper Semi-Continuous Function

Recall the definition of the upper semi-continuous.

Definition 1.1 *We say that F is an upper semi-continuous (u.s.c.) at $x_0 \in R^n$ if for any open N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subset N$.*

We say that F is upper semi-continuous if it is so at every $x_0 \in R^n$.

1.3 Convex Sets

A set $A \subset R^n$ is said to be convex if $\lambda x + (1-\lambda)y \in A$ whenever $x \in A,\ y \in A$, and $\lambda \in [0,1]$. By definition it follows that an intersection of any number of convex sets is a convex set, and if $A \subset R^n,\ B \subset R^n$ are convex, α and β are real numbers, then the set $\alpha A + \beta B$ is convex. If A is convex, then $\operatorname{int} A$ and $\operatorname{cl} A$ are also convex.

Convex hulls. Let $A \subset R^n$. The intersection of all convex sets containing A is called the convex hull of A and is denoted by $\operatorname{conv} A$. The closed convex hull of the set A is the intersection of all closed convex sets containing A. It will be denoted by $\overline{\operatorname{conv}}\, A$.

A vector sum

$$\lambda_1 x_1 + \cdots + \lambda_n x_n$$

is called a convex combination of $x_1, \ldots, x_n$ if $\lambda_i \geq 0,\ \ i = 1, \ldots, n$, and $\lambda_1 + \cdots + \lambda_n = 1$. Obviously, if $x_1, \ldots, x_n$ are vectors from A, then any convex combination of $x_1, \ldots, x_n$ belongs to $\operatorname{conv} A$. The following inverse statement is very important.

Theorem 1.1 (Caratheodory). *Let $A \subset R^n$. For any $x \in \operatorname{conv} A$, there exist $x_1, \ldots, x_m \in A$ such that $m \leq n+1$ and*

$$x = \lambda_1 x_1 + \cdots + \lambda_m x_m$$

where $\lambda_i \geq 0,\ i = 1, \ldots, m$ and $\lambda_1 + \cdots + \lambda_m = 1$.

Remark 1.1 *In other words, any point $x \in \operatorname{conv} A$ can be expressed as a convex combination of at most $n+1$ points from A.*

An immediate corollary from the above theorem is

Corollary 1.1 *The convex hull of a compact set is a compact set.*

One more result that comes directly from Caratheodory's theorem is

Theorem 1.2 *If A is bounded [resp. compact], then $\operatorname{conv} S$ is bounded [resp. compact].*

The above theorem does allow us to conclude:

$$\text{If } A \text{ is bounded } \Rightarrow \overline{\text{conv}}\, A = \text{cl}(\text{conv}\, A) = \text{conv}(\text{cl}\, A).$$

We conclude this section by two lemmas mentioned in [12] that will be used in Chapter 4.

Lemma 1.1 *Let $t \to A_t$ be a measurable set-valued mapping whose values A_t are compact, convex subset of R^n, all contained in a common ball. Then $\int_0^1 A_t$ is compact and convex, where $\int_0^1 A_t = \left\{ \int_0^1 a_t : \ a_t \ \text{ is a measurable selection for } A_t \right\}$ (a_t is a measurable selection of A_t provided $t \to a_t$ is measurable and $a_t \in A_t$ almost everywhere).*

Lemma 1.2 *For $t \in [0, 1]$, let $t \to S_t$ be a set-valued mapping whose values are all contained in a common ball of R^n. If $x : [0, 1] \to R^n$ has*

$$x(t) - x(s) \in \int_s^t Sr$$

(for all $t > s$ in $[0, 1]$, then $x(\cdot)$ is absolutely continuous and satisfies

$$\dot{x}(t) \in \overline{\text{conv}}\ S_t \quad a.e.$$

in particular,

$$x(t) = x(0) + \int_0^t \dot{x}(r)dr \in x(0) + \int_0^t \overline{\text{conv}}\ S_r \quad \textit{for all } t \in [0, 1].$$

1.4 The Support Function

We proceed now to study one of the most important concepts in convex analysis which allows us to completely characterize the compact convex set. This concept is called the support function which is derived from another important concept in convex analysis which is the separation theorem which says that any two convex sets without common points can be separated by a hyperplane. Let $A \subset R^n$ and $x \in R^n$. The projection of x into A is the set defined by

$$\pi(x, A) = \{a \in A : |x - a| = d(x, A)\}.$$

Lemma 1.3 *If $x \in R^n$ and $A \subset R^n$ is a closed convex set, then $\pi(x, A)$ consists of a single point.*

Theorem 1.3 *Let $A \subset R^n$ be a convex set and let $x_0 \notin cl\, A$. Then there exists a vector $x^* \neq 0$ and a positive number ε such that*

$$\langle x, x^* \rangle \leq \langle x_0, x^* \rangle - \varepsilon \quad \textit{for all } x \in A.$$

Theorem 1.4 *Let $A \subset R^n$ be a nonempty closed convex set, and let $x_0 \notin A$. Then there exist $x \in R^n$ such that*

$$\langle x, x_0 \rangle > \sup\{\langle x, a \rangle : a \in A\}. \tag{1.1}$$

The above theorem is often called the Hahn-Banach theorem in geometric form. On the other hand, consider the right-hand side of (1.1); it suggests a function $\sigma_A : R^n \to R$ called the support function of A:

$$\sigma_A(x) = \sup\{\langle x, a \rangle : a \in A\}$$

which will be used thoroughly in Chapter two and three. If $a \in A$, we have by definition

$$\langle x, a \rangle \leq \sigma_A(x) \quad \text{for all} \quad x \in R^n;$$

but this actually characterizes the element of A; Theorem 1.3 tells that the converse is true. Therefore the test "$a \in A$?" is equivalent to the requirement $\langle x, a \rangle \leq \sigma_A(x)$ for all $x \in R^n$. When A is bounded, its support function is finite everywhere; otherwise σ_A can take on the value $+\infty$. Furthermore, σ_A is also the support function of the closure of A, and even of the closed convex hull of A.

1.5 Absolutely Continuous Functions

An absolutely continuous function plays a fundamental role in the theory of differential equations, for although it may not be differentiable at all points, it still can be recovered by integration from its derivative. In fact, it is characterized by this property, and in some sense, is the weakest acceptable kind of solution one can seek to a discontinuous differential equation.

Definition 1.2 [1]. *A function* $x : [\alpha, \beta] \to R^n$ *is called absolutely continuous if* $\forall\ \varepsilon > 0$, $\exists\ \delta$ *such that, for any countable collection of disjoint subintervals* $[\alpha_k, \beta_k]$ *of* $[\alpha, \beta]$ *such that*

$$\sum(\beta_k - \alpha_k) < \delta,$$

we have

$$\sum |x(\beta_k) - x(\alpha_k)| < \varepsilon.$$

An absolutely continuous function is at once continuous and of bounded variation (the converse is false). Any Lipschitzean function is absolutely continuous.

As it is well known a function of bounded variation, hence a fortiori an absolutely continuous function, has a finite derivative except at most on a set of measure zero. However, for a continuous function x, even of bounded variation, having a finite derivative x' except at most on a set of measure zero, it need not be true that

$$x(\beta) - x(\alpha) = \int_{\alpha}^{\beta} x'(s)ds. \tag{1.2}$$

In fact the exceptional set A of points where the derivative need not exist, being of measure zero, has no influence on the value of the integral on the right-hand side of (1.2), in the sense that, even if the derivative were to exist on such a set, nothing would change in the integral. However the behavior of x' (hence of x) on a set of measure zero can very well have an influence on the left-hand side of (1.2). For it not to occur we should have that our function x maps subsets of measure zero of (α, β) onto subsets of measure zero of R^n. This is not true in general for continuous functions of bounded variation.

1.5.1 Some Properties of an Absolutely Continuous Functions

Let f be a function from the interval $[a, b]$ to R^n.

1. If the function f is absolutely continuous on $[a, b]$, then it is of bounded variation on $[a, b]$.

2. If f is absolutely continuous, then f has a derivative almost everywhere.

3. A function f is an indefinite integral if and only if it is absolutely continuous.

4. Every absolutely continuous function is the indefinite integral of its derivative.

5. If f satisfies the Lipschitz condition, then it is absolutely continuous.

6. f is absolutely continuous if $|f'|$ is bounded.

Chapter 2

Some Properties of Filippov's Operator

In this chapter, we introduce the definition of the Filippov operator and we show that it is an upper-semicontinuous set-valued map. We use the concept of the support function to give more results about Filippov operator. We also generalize some results to a higher dimension space.

2.1 Introduction

Let $f : R^n \to R^n$ be given. The Filippov of f is defined as follows:

$$\mathcal{F}[f](x) = \bigcap_{\varepsilon>0} \bigcap_{\substack{z \\ m(z)=0}} \overline{\text{conv}}\ f(x + \varepsilon B \setminus Z),$$

where m denotes the Lebesgue measure, $\overline{\text{conv}}\ A$ represents the closure of the convex hull of the set A and $x+\varepsilon B$ represents the open ball of radius ε about the point x. The Filippov is used in defining a generalized solution of the ordinary differential equation $x' = f(t, x)$, particularly in the case of f discontinuous in x (as we will see in Chapter 4).

We denote $\mathcal{F}[f](x)$ by $F(x)$, where $F(x) : R^n \to \rho(R^n)$. Here, we treat $F(x)$ as a multivalued map, mapping real-valued measurable function into set valued functions, and investigate the properties of $F(x)$. Such results add to our understanding of this operation. We first consider choosing an appropriate domain for F. Certainly, there are a number of possibilities but we require that the domain be restricted to f's which are useful for additional equations with discontinuous right-hand side. L^∞ is the most appropriate domain because it satisfies the classical local existence theorem for Filippov solution in the case of $x' = f(t,x)$. We choose for the codomain the set $\mathcal{B} = \{F : R^n \to \rho(R^n) | F$ is upper semi-continuous, F is a convex compact subset of $R^n\}$. $\mathcal{B}$ can be made into a metric space by defining the Hausdorff distance on it.

We shall make use of the following definition.

Definition 2.1 [15]. *Let $F : R^n \to \rho(R^n)$. Then the Filippov of F is defined by*

$$\mathcal{F}[F](x) = \bigcap_{\varepsilon>0} \bigcap_{\substack{z \\ m(z)=0}} \overline{\text{conv}} \bigcup_{y \in B(x,\varepsilon) \setminus Z} \{F(y)\}.$$

The purpose of this definition is to extend the Filippov so that it can be applied to set-valued functions.

2.2 The Characterization of the Upper Semi-Continuous $\boldsymbol{F}(x)$

Recall the definition of the upper-semi-continuous map:

Definition 2.2 *We say that F is an upper semi-continuous (u.s.c.) at $x_0 \in R^n$ if for any open N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subset N$.*

We say that F is upper semi-continuous if it is so at every $x_0 \in R^n$. The following proposition gives two equivalent characterizations of upper semi-continuous multifunctions.

Proposition 2.1 *The property that a multifunction $F : R^n \to \rho(R^n)$ with closed values is upper semicontinuous is equivalent to each of the following:*

(1) $F^{-1}(A)$ is closed in R^n whenever $A \subset \rho(R^n)$ is closed;

(2) If $\{x_i\}$ and $\{v_i\}$ are sequences such that $x_i \to x_0$, $v_i \to v_0$, $v_i \in F(x_i)$, then $v_0 \in F(x_0)$.

Example 2.1 [9]. *The map $F : R \to \rho(R)$ defined by*

$$F(x) = \begin{cases} \{0\} & \text{for } x < 0 \\ \{0,1\} & \text{for } x = 0 \\ \{1\} & \text{for } x > 0 \end{cases}$$

is an upper semi-continuous map.

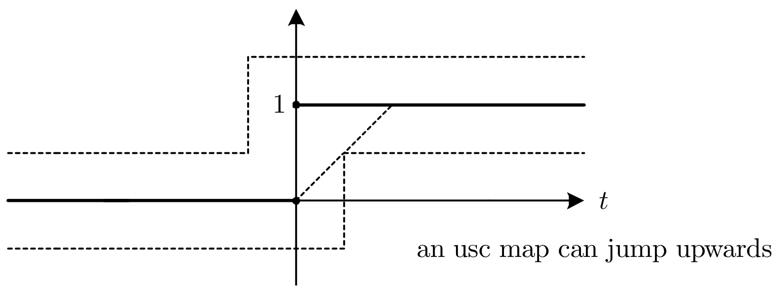

Figure 2: The graph of $F(x)$

Later it will often be essential to have convex values; here, this can simply be achieved by letting $F(0) = [0, 1]$, i.e. by filling in the gap at the point of discontinuity.

Example 2.2 *The map $F : R \to \rho(R)$ defined by*

$$F(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ [-1,1] & \text{if } x \neq 0, \end{cases}$$

is not an usc map.

Let us show that now the map $F(x)$, the Filippov operator, indeed is an usc map.

Lemma 2.1 *Let f be a single-valued map $f : R^n \to R^n$ such that $f \in L^\infty$. Then the map $F(x) : R^n \to \rho(R^n)$ given by*

$$F(x) = \bigcap_{\varepsilon>0} \bigcap_{\substack{z \\ m(z)=0}} \overline{\text{conv}}\ f(x + \varepsilon B \setminus Z)$$

is upper semi-continuous.

Proof. Let us show that if $x_i \to x_0, v_i \to v_0$ where $v_i \in F(x_i)$, then $v_0 \in F(x_0)$. Fix $\varepsilon > 0$ and the subset $Z \subset R^n$ where $m(Z) = 0$. Now, $F(x_i) = \bigcap_{\varepsilon>0} \bigcap_{\substack{z \\ m(z)=0}} \overline{\text{conv}}\ f(x_i + \varepsilon B \setminus Z)$, $\exists\ i_0 \in N$ such that for $i \geq i_0, x_i \in x_0 + \varepsilon B$. Let $\varepsilon_i = |x_i - x_0|$ so $v_i \in \overline{\text{conv}}\ f(x_i + \varepsilon_i B \setminus Z) \subseteq \overline{\text{conv}}\ f(x_0 + \varepsilon B \setminus Z)$ and since $v_i \to v_0$ therefore $v_0 \in \overline{\text{conv}}\ f(x_0 + \varepsilon B \setminus Z)$. ∎

From this result we conclude:

Proposition 2.2 $\mathcal{F}[L^\infty] \subseteq \mathcal{B}$.

Proof. From Lemma 2.1, $\mathcal{F}[f]$ is an upper semi-continuous map. Moreover, $\mathcal{F}[f](x)$ is a compact convex subset of $R^n\ \forall\ x \in R^n$ so $\mathcal{F}[f] \subseteq \mathcal{B}\ \forall\ f \in L^\infty$. ∎

Definition 2.1 helps us to prove the following important fact about the Filippov of F.

Lemma 2.2 *If $F(x)$ is upper semi-continuous, then $\mathcal{F}[F(x)] \subset F(x)\ \ \forall\ x \in R^n$.*

Proof. $F(x)$ is usc, so $\forall\ x_0 \in R^n$ and for any open N containing $F(x_0)$ $\exists$ a neighborhood M of x_0 such that $F(M) \subset N$. Consider $M = x_0 + \varepsilon B$

$$
\begin{aligned}
F(x_0) &= \bigcap_{\varepsilon>0} \bigcap_{\substack{z \\ m(z)=0}} \overline{\text{conv}}\ f(x_0 + \varepsilon B \setminus Z) \subset N \\
\mathcal{F}F(x_0) &= \bigcap_{\varepsilon>0} \bigcap_{\substack{z \\ m(z)=0}} \overline{\text{conv}} \bigcup_{y \in x_0 + \varepsilon B \setminus z} \{F(y)\}
\end{aligned}
$$

where $\forall\ y \in x_0 + \varepsilon B,\ \ F(y) \subset N$ as $\varepsilon \to 0$ (decreases) $\mathcal{F}F(x_0) \subset F(x_0)$. ∎

We will use the support function to prove more results about the Filippov $F(x)$.

Lemma 2.3 *If F is upper semi-continuous, then $\mathcal{F}[F]$ is upper semi-continuous.*

Proof. Let $F(x)$ be upper semi-continuous at $x_0 \in R^n$, so given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $\forall\ y \in x_0 + \delta B,\ F(y) \subseteq F(x_0) + \varepsilon B$ is a statement equivalent to

$$H(y, p) \leq H(x_0, p) + \varepsilon \|p\| \ \text{ where } \ p \in R^n\,.$$

The map F is usc so $\mathcal{F}F(y) \subset F(y)$ and $\mathcal{F}F(x_0) \subset F(x_0)$. So $\mathcal{F}F(y) \subseteq \mathcal{F}F(x_0) + \varepsilon B \;\; \forall \; y \in x_0 + \varepsilon B$. Hence

$$\underset{y \in \mathcal{F}F(y)}{H(y,p)} \leq \underset{\mathcal{F}F(x_0)}{H(x_0,p)} + \varepsilon \|p\| \; \forall \; y \in x_0 + \varepsilon B \;\; \forall \; p \in R^n.$$

This concludes the upper semi-continuity of $\mathcal{F}F(x)$ at x_0. ■

The following is an immediate corollary of Lemma 2.3.

Corollary 2.1 $\mathcal{F}[\mathcal{B}] \subseteq \mathcal{B}$.

Proof. Straightforward ■

The following proposition shows that the support function of an upper semi-continuous map is also an upper semi-continuous function.

Proposition 2.3 *If $F(x)$ is upper semi-continuous, then the map $x \to H_F(x,p)$ is upper semi-continuous $\forall \; x \in R^n, \; \forall \; p \in R^n$.*

Proof. Suppose that $F(x)$ is an usc at x_0, then $\forall \; \varepsilon > 0, \; \exists \; \delta > 0$ such that

$$F(y) \subseteq F(x_0) + \varepsilon B \quad \forall \; y \in x_0 + \delta B$$

that implies

$$\begin{aligned}
\sup_{v \in F(y)} \langle v,p \rangle \quad &\leq \quad \sup_{v \in F(x_0)+\varepsilon B} \langle v,p \rangle \\
&\Rightarrow \; \sup_{v \in F(y)} \langle v,p \rangle \leq \sup_{v \in F(x_0)} \langle v,p \rangle + \sup_{v \in \varepsilon B} \langle v,p \rangle \\
&\Rightarrow \; \sup_{v \in F(y)} \langle v,p \rangle \leq \sup_{v \in F(x_0)} \langle v,p \rangle + \varepsilon \|p\| \\
&\Rightarrow \; H(y,p) \leq H(x_0,p) + \varepsilon \|p\|
\end{aligned}$$

or "equivalently", $\limsup_{y \to x_0} H(y,p) \leq H(x_0,p)$. So H is usc at x_0. ■

Proposition 2.4 $\mathcal{F}F(x) \subset F(x) \Leftrightarrow \underset{\mathcal{F}F(x)}{H}(x,p) \le \underset{F(x)}{H}(x,p)$.

Proof.

$$
\begin{aligned}
\mathcal{F}F(x) \subset F(x) &\Leftrightarrow v \in F(x)\ \forall\ v \in \mathcal{F}F(x) \\
&\Leftrightarrow \underset{F(x)}{H}(x,p) \ge \langle p, v\rangle \quad \forall\ v \in \mathcal{F}F(x) \quad \forall\ p \in R^n \\
&\Leftrightarrow \underset{F(x)}{H}(x,p) \ge \sup_{v\in\mathcal{F}F(x)}\langle p, v\rangle \\
&\Leftrightarrow \underset{F(x)}{H}(x,p) \ge \underset{\mathcal{F}F(x)}{H}(x,p)
\end{aligned}
$$

■

In the literature [3] and [15]. The following question was investigated: Under what conditions $\mathcal{F}F(x) = F(x)$. Here, a new characterization of this equality is given by using the support function.

Theorem 2.1 *$\mathcal{F}F(x) = F(x)$ for $x \in R^n$ if*

$$m\{y \in x + \varepsilon B : H(y,p) > H(x,p) - \varepsilon\} > 0, \quad \forall\ \varepsilon > 0, \quad \forall\ p \in R^n.$$

Proof. Let $m\{y \in x + \varepsilon B : H(y,p) > H(x,p) - \varepsilon\} > 0\ \ \forall\ \varepsilon > 0, \quad \forall\ p \in R^n$.

$$
\begin{aligned}
H(y,p) &> H(x,p) - \varepsilon \\
\sup_{v\in F(y)}\langle p, v\rangle &> \sup_{v\in F(x)}\langle v, p\rangle - \varepsilon \\
\sup_{\substack{v\in\{\cup F(y)\} \\ y\in x+\varepsilon B}}\langle p, v\rangle &> \sup_{v\in F(x)}\langle p, v\rangle - \varepsilon \\
\sup_{\substack{v\in\overline{\text{conv}}\ \{\cup F(y)\} \\ y\in x+\varepsilon B}}\langle p, v\rangle &> \sup_{v\in F(x)}\langle p, v\rangle - \varepsilon \\
\sup_{\substack{v\in\bigcap_z \overline{\text{conv}}\ \{\cup F(y)\} \\ m(z)=0\quad y\in x+\varepsilon B\setminus Z}}\langle p, v\rangle &> \sup_{v\in F(x)}\langle p, v\rangle - \varepsilon
\end{aligned}
$$

as ε decreases

$$\sup_{v\in\bigcap_{\varepsilon>0}\bigcap_{\substack{z \\ m(z)=0}}\overline{\text{conv}}\ \{\cup F(y)\}_{y\in x+\varepsilon B\setminus Z}}\langle p, v\rangle > \sup_{v\in F(x)}\langle p, v\rangle.$$

So,

$$F(x) \subset \bigcap_{\varepsilon>0} \bigcup_{\substack{z \\ m(z)=0}} \overline{\text{conv}} \, \{\cup F(y)\}_{y \in x+\varepsilon B \setminus Z} = \mathcal{F}F(x)$$

and since $\mathcal{F}F(x) \subset F(x) \ \ \forall \, x \in R^n$, we conclude that $\mathcal{F}F(x) = F(x)$.

■

2.3 More Generalized Results About the Filippov $\mathcal{F}$

In [3], a nice characterization of the range of the Filippov operator was proved for a one-dimension only. Here, we generalize this result to a higher dimension space.

Theorem 2.2 *Let $F : R^n \to \rho(R^n)$. Then there exists $f \in L^\infty(R^n)$ such that $\mathcal{F}[f] = F$ if and only if F satisfies the following conditions:*

(I) F is upper semi-continuous.

(II) There exists $r > 0$ such that $F(x) \subset rB \ \forall \ x \in R^n$.

(III) $\mathcal{F}[F] = F$.

Proof. ($\Leftarrow$) Assume that F satisfies the three conditions, the existence of $f \in L^\infty$ such that $\mathcal{F}[f] = F$ follows from the main result in [15].

($\Rightarrow$). Suppose there exists $f \in L^\infty(R^n)$ such that $\mathcal{F}[f] = F$. From Proposition 2.1, F is an upper semi-continuous map and $\forall \ x \in R^n$ $F(x)$ is a compact subset of R^n, so (I) and (II) are satisfied. F satisfies (III) by equation 7 in [15]. ■

Example 2.3 *Consider the map $F : R \to \rho(R^2)$ defined by $F(x) = \{(x, y) : y \in R\}$. $F(x)$ is not an usc map. To show that, take $N = \left\{(x, y) : |y| < \dfrac{1}{|x|}\right\}$. It is an open neighborhood around $F(0)$, but for every $x \neq 0$, $F(x) \not\subset N$. So, F is not the Filippov of an L^∞ function.*

Example 2.4 *Define* $F : R \to \rho(R)$ *by*

$$F(x) = \begin{cases} \{0\} & \text{if } x \neq 0 \\ [0,1] & \text{if } x = 0\,. \end{cases}$$

$\mathcal{F}[F](x) = \{0\} \neq F(x)$. *So* $F(x)$ *is not the Filippov of an* L^∞ *function.*

Corollary 2.2 $\mathcal{F}$ *is not onto* $\mathcal{B}$.

In [3], Biles and Sparker proved the continuity of the one-dimensional Filippov operator $d_H(\mathcal{F}F, \mathcal{F}G) \leq d_H(F, G)$. Actually, it is more than that, it is Lipschitz of rank one. Here, we give an example showing that this result is no more valid in a higher dimension space.

Example 2.5 *Define* $F, G : R^2 \to \rho(R^2)$ *by*

$$F(x) = \begin{cases} x_1^2 + x_2^2 \leq 1 & \text{if } (x_1, x_2) = (0,0) \\ (0,0) & \text{if } (x_1, x_2) \neq (0,0) \end{cases}$$

and $G(x)$ *by the constant function where we map each* $(x_1, x_2) \in R^n \times R^n$ *to the rectangle defined by*

$$G(x) : (x_1, x_2) \to \{2 \leq x_1 \leq 3, \quad -1 \leq x_2 \leq 1\}.$$

Clearly, $\mathcal{F}F(x) = \{(0,0)\}$ *and* $\mathcal{F}G(x) = G(x)$. $d_H(\mathcal{F}F, \mathcal{F}G) = \sqrt{10}$ *while* $d_H(F, G) = 3$.

Let us investigate now that under what condition the continuity of the Filippov from $\mathcal{B}$ to $\mathcal{B}$ is valid in R^n. It turns out to be valid under the following assumption:

Theorem 2.3 *Let* F *and* $G \in \mathcal{B}$. *Let* $d_H(F, G) = r$ *where* $r > 0$. *If* $F \subset \mathcal{F}G + rB$, *then we have* $d_H(\mathcal{F}F, \mathcal{F}G) \leq d_H(F, G)$.

Proof. We have $d_H(F, G) < r$ which is equivalent to $F \subset G + rB$. F and G are usc maps $\mathcal{F}F \subset F$ and $\mathcal{F}G \subset G$. We know that for two subsets A and C from R^n if

$A \subset C$, then $A + rB \subset C + rB$ we have $\mathcal{F}F(x) \subset F(x) \subset G + rB$. Therefore

$$\sup_{v\in\mathcal{F}F(x)}\langle v, p\rangle \leq \sup_{v\in F(x)}\langle v, p\rangle \leq \sup_{v\in G(x)}\langle v, p\rangle + r\|p\|, \qquad p \in R^n.$$

Also, since $\mathcal{F}F(x) \subset F(x) \subset \mathcal{F}G + rB$,

$$\sup_{v\in\mathcal{F}F(x)}\langle v, p\rangle \leq \sup_{v\in F(x)}\langle v, p\rangle \leq \sup_{v\in\mathcal{F}G(x)}\langle v, p\rangle + r\|p\|, \qquad p \in R^n,$$

i.e., $\mathcal{F}F(x) \subset F(x) \subset \mathcal{F}G(x) + rB \subset G(x) + rB$ which concludes

$$d_H(\mathcal{F}F(x), \mathcal{F}G(x)) \leq d_H(F(x), G(x)).$$

■

Chapter 3

A Study of Almost Everywhere Singleton-Valued Filippov's

In this chapter, we give some results generalization to an almost everywhere singleton-valued Filippov map.

3.1 Introduction

In [2], Biles and Spraker investigated the following question: If the Filippov maps of f and g (where f and $g : R \to R$ are Lebesgue measurable) are the same, then are f and g also equal? Here, we are going to generalize their work and results to an n-dimensional Lebesgue measurable function.

One direction is clear. It is immediate that if f and g are measurable and $f(x) = g(x)$ a.e., then $\mathcal{F}[f](x) = \mathcal{F}[g](x)$ for all x, since the Filippov operator ignores a set of measure zero.

On the other hand, the converse is not so obvious. Suppose $A \subset [0, 1]$ is measurable so that both A and A^c intersect every interval with positive measure. Such a set is known to exist. Define $f, g : R^n \to R$ by $f(x) = \chi_A(x)$ and $g(x) = \chi_{A^c}(x)$ where $\lambda_A(x)$ and $\lambda_{A^c}(x)$ denote the characteristic functions of A and $R^n \setminus A$ respectively. Then for each $x \in R^n$, $\mathcal{F}[f](x) = \mathcal{F}[g](x) = [0, 1]$, however, f and g do not agree anywhere. Thus in general, there is no obvious relationship to characterize the relationship between f and g when $\mathcal{F}[f] = \mathcal{F}[g]$. However, Biles and Spraker [2] showed that under several sets of assumptions, a characterization does exist. Specifically, a characterization is obtained from the relationship between f and g in the case in which their Filippov maps are singleton almost everywhere. We also prove necessary and sufficient conditions for a function f whose Filippov map is a singleton almost everywhere. Related results are also given.

3.2 Technical Lemmas

This section consists of three key lemmas and their proofs, which are used to obtain the main results in the last section. Throughout $f, g : R^n \to R^n$ are given Lebesgue measurable functions.

Lemma 3.1 *Let $f : R^n \to R^n$ be given. If there exists a set $A \subseteq R^n$ of full measure such that $f|_A$ is continuous, then for each $x \in A$ we have $\mathcal{F}[f](x) = \{f(x)\}$.*

Proof. Claim. For each $x \in A$ we have $f(x) \in \mathcal{F}[f](x)$.

Proof of Claim. Fix $\varepsilon > 0$ and $Z \subset \mathbb{R}^n$ such that $m(Z) = 0$. Since A has full measure, therefore for each $n \in N$, $A \cap (B(x, 1/n) \setminus Z)$ has full measure in $B(x, 1/n)$. Thus for any n, there exists $x_n \in A \cap (B(x, 1/n) \setminus Z)$. It is clear that $x_n \to x$ and the continuity of f relative to A yields that $f(x_n) \to f(x)$. So $f(x) \in \operatorname{cl} f(B(x, \varepsilon) \setminus Z) \subseteq \overline{conv} f(B(x, \varepsilon) \setminus Z)$. The above statement holds for each $\varepsilon > 0$ and set $Z \subset R^n$ where $m(Z) = 0$. Thus, we have $f(x) \in \bigcap_{\varepsilon > 0} \bigcap_{\substack{z \\ m(z)=0}} \overline{conv} f(B(x, \varepsilon) \setminus Z) = \mathcal{F}[f](x)$ so the claim is established.

Assume $\exists\, \overline{x} \in A$ such that $\mathcal{F}[f](\overline{x}) \neq \{f(\overline{x})\}$. Thus, $\exists\, y \in R^n$ such that $y \neq f(\overline{x})$ and $y \in \mathcal{F}[f](\overline{x})$, notice that $\mathcal{F}[f](\overline{x}) \neq \Phi$; so $d(y, f(\overline{x})) > \delta > 0$.

Now, for each $\varepsilon > 0$, and $Z \subset R^n$ with $m(Z) = 0$, we have $y \in \overline{\text{conv}} f(B(\overline{x}, \varepsilon) \setminus Z)$. Set $Z_0 = R^n \setminus A$. Then $\forall\ \varepsilon > 0$, $y \in \overline{\text{conv}} f(B(\overline{x}, \varepsilon) \setminus Z_0)$. Thus $\forall\ \varepsilon > 0$, there exist $\overline{y}_\varepsilon$ and y^*_ε such that $d(\overline{y}_\varepsilon, f(\overline{x})) \leq d(y, f(\overline{x})) \leq d(y^*_\varepsilon, f(\overline{x}))$ and y^*_ε, $\overline{y}_\varepsilon \in f(B(\overline{x}, \varepsilon) \setminus Z_0)$ by Caratheodory theorem (Theorem 1.1). For each $\varepsilon > 0$, let $\omega_\varepsilon \in B(\overline{x}, \varepsilon) \setminus Z_0 = B(\overline{x}, \varepsilon) \cap A$ such that $f(\omega_\varepsilon) = y^*_\varepsilon$. Set $\beta = d(y, f(\overline{x}))$, so we have shown $\exists\ \beta > 0$ such that $\forall\ \ \varepsilon > 0\ \ \exists\ \omega_\varepsilon \in B(\overline{x}, \varepsilon) \cap A$ with $d(f(\omega_\varepsilon), f(\overline{x})) = d(y^*_\varepsilon, f(\overline{x})) \geq d(y, f(\overline{x})) = \beta$.

This contradicts the continuity of $f|_A$ at $\overline{x}$. ■

Lemma 3.2 *Let $f : R^n \to R^n$ be given. Then $\mathcal{F}[f]$ is continuous when restricted to the set A on which it is singleton.*

Proof. Assume the contrary. So $\exists$ some $\overline{x} \in A$ at which $\mathcal{F}[f]$ is discontinuous. So $\exists\, \beta > 0$ such that $\forall\, \varepsilon > 0$, $\exists\, y_\varepsilon \in A$ such that $d(\overline{x}, y_\varepsilon) < \varepsilon$. But $d(\mathcal{F}[f](\overline{x}), \mathcal{F}[f](y_\varepsilon)) \geq \beta$. Choose $\varepsilon > 0$ and a corresponding $y_\varepsilon \in A$. We know that $\forall\, \gamma > 0$ and $\forall\, Z \subset R^n$ with $m(Z) = 0$, that $\mathcal{F}[f](y_\varepsilon) \in \overline{\text{conv}} f(B(y_\varepsilon, \gamma) \setminus Z)$. Choose $\gamma > 0$ such that

$$B(y_\varepsilon, \gamma) \subseteq B(\overline{x}, \varepsilon).$$

We then have for each $Z \subset R^n$ of measure zero that $B(y_\varepsilon, \gamma) \setminus Z \subseteq B(\overline{x}, \varepsilon) \setminus Z$ and $f(B(y_\varepsilon, \gamma) \setminus Z) \subseteq f(B(\overline{x}, \varepsilon) \setminus Z)$. Thus we have

$$\mathcal{F}[f](y_\varepsilon) \in \overline{\text{conv}}\ f(B(y_\varepsilon, \gamma) \setminus Z) \subseteq \overline{\text{conv}} f(B(\overline{x}, \varepsilon) \setminus Z).$$

So, we have shown that $\forall\, \varepsilon > 0$ and $Z \subset R^n$ with $m(Z) = 0$

$$\overline{\text{conv}}\ f(B(\overline{x}, \varepsilon) \setminus Z) \cap B(\mathcal{F}[f](y_\varepsilon), \beta) \neq \Phi.$$

Let z_ε be a point on the line segment $[\mathcal{F}[f](\overline{x}), \mathcal{F}[f](y_\varepsilon)]$ such that $|\mathcal{F}[f](\overline{x}) - z_\varepsilon| = \beta > 0$. Consider the intersection of the ball $\overline{B}(z_\varepsilon, \beta)$ with $\overline{\text{conv}}\ f(B(\overline{x}, \varepsilon) \setminus Z)$. Let $S_\varepsilon = \overline{B}(z_\varepsilon, \beta) \cap \left\{ \bigcap_{\substack{z \\ m(z)=0}} \overline{\text{conv}}\ f(B(\overline{x}), \varepsilon) \setminus Z) \right\} \neq \Phi$. Each S_ε is a non-empty compact set and as $\varepsilon \to 0$, $\bigcap_{\varepsilon>0} S_\varepsilon \neq \Phi$ (by Finite-Intersection Property). Let $\overline{z} \in \bigcap_{\varepsilon>0} S_\varepsilon$. Then $[\overline{z}, \mathcal{F}[f](\overline{x})] \subseteq \mathcal{F}[f](\overline{x})$ which is a contradiction since $\mathcal{F}[f](\overline{x})$ is singleton. ■

Lemma 3.3 *Let $f : R^n \to R^n$ be given. If $\mathcal{F}[f]$ is a singleton almost everywhere, then $\mathcal{F}[f](x) = \{f(x)\}$ almost everywhere.*

Proof. Suppose not. Then there exists $A \subseteq R^n$ such that $m(A) > 0$, $\mathcal{F}[f](x)$ is a singleton for each $x \in A$ and $\mathcal{F}[f](x) \neq f(x)$.

Claim 1. There exists $\delta > 0$ such that

$$m\{x \in A : f(x) \notin B(\mathcal{F}[f](x), \delta)\} > 0.$$

Proof of Claim 1. Assume the opposite. Then for $\delta = \frac{1}{n}$,

$$m\left\{x \in A : f(x) \notin B\left(\mathcal{F}[f](x), \frac{1}{n}\right)\right\} = 0.$$

But $A = \bigcup_{n=1}^{\infty}\left\{x \in A : f(x) \notin B\left(\mathcal{F}[f](x), \frac{1}{n}\right)\right\}$, a countable union of sets of measure zero, which implies $m(A) = 0$, a contradiction.

Now let y be a point of density of $B_\delta = \{x \in A : f(x) \notin B(\mathcal{F}[f](x), \delta)\}$ such that $y \in B_\delta$.

Claim 2. There exists $\overline{\varepsilon} > 0$ such that

$$\mathcal{F}[f](x) \in B\left(\mathcal{F}[f](y), \frac{\delta}{2}\right) \qquad \forall\, x \in B(y, \overline{\varepsilon}).$$

Proof of Claim 2. Define $g : C \to R^n$ by $g(x) = \mathcal{F}[f](x) - \mathcal{F}[f](y)$ where C is the set on which $\mathcal{F}[f]$ is a Singleton. By Lemma 2, $\mathcal{F}[f]$ is continuous restricted to the set C, hence g is continuous. The fact that $g(y) = 0$ and the continuity of g restricted to C implies that $\exists\, \overline{\varepsilon} > 0$ such that $\|\mathcal{F}[f](x) - \mathcal{F}[f](y)\| < \frac{\delta}{2}$ $\forall\, x \in (y, \overline{\varepsilon})$. Observe that, for $x \in B(y, \overline{\varepsilon})$, $f(x) \notin B(\mathcal{F}[f](x), \delta)$ which implies that $f(x) \notin B\left(\mathcal{F}[f](y), \frac{\delta}{2}\right)$, hence

$$\left\{x \in A : f(x) \notin B\left(\mathcal{F}[f](y), \frac{\delta}{2}\right)\right\} \supseteq \{x \in A : f(x) \notin B(\mathcal{F}[f](x), \delta)\}.$$

Claim 3. For all $\varepsilon > 0$, $m\left[B(y, \varepsilon) \cap \left\{x \in A : f(x) \notin B\left(\mathcal{F}[f](y), \frac{\delta}{2}\right)\right\}\right] > 0$.

Proof of Claim 3. Suppose the opposite. Then, there exists $\hat{\varepsilon} > 0$ such that $m\left[B\ (y, \hat{\varepsilon}) \cap \left\{x \in A : f(x) \notin B\left(\mathcal{F}[f](y), \frac{\delta}{2}\right)\right\}\right] = 0$. Choose $\varepsilon = \min\{\hat{\varepsilon}, \overline{\varepsilon}\}$ (where

$\bar{\varepsilon}$ is chosen as in Claim 2). Then,

$$\begin{aligned} &B(y,\hat{\varepsilon}) \cap \left\{x \in A : f(x) \notin B\left(\mathcal{F}[f](y), \frac{\delta}{2}\right)\right\} \supseteq \\ &B(y,\varepsilon) \cap \left\{x \in A : f(x) \notin B\left(\mathcal{F}[f](y), \frac{\delta}{2}\right)\right\} \supseteq \\ &B(y,\varepsilon) \cap \{x \in A : f(x) \notin B(\mathcal{F}[f](x), \delta)\} \end{aligned}$$

by the previous claim. This last set has positive measure (by Claim 1), which yields a contradiction.

So, $\forall\, \varepsilon > 0,\ \ \forall\, Z \subset R^n$ with $m(Z) = 0$, choose

$$x_0 \in (B(y,\varepsilon) \setminus Z) \cap \left\{x \in A : f(x) \notin B\left(\mathcal{F}[f](y), \frac{\delta}{2}\right)\right\}.$$

Then $\overline{\text{conv}}\ f(B(y,\varepsilon) \setminus Z) \supseteq [f(x_0), \mathcal{F}[f](y)]$, which is a contradiction. ■

3.3 The Main Results

All the difficult mathematical work was contained in the previous section in proving the lemmas, and specifically generalizing them from dimension one to n. Consequences of these lemmas are presented now. First, theorems that characterize f depending on the properties of its Filippov map will be presented.

Theorem 3.1 *Let $f : R^n \to R^n$ be given. For each $x \in R^n, \mathcal{F}[f](x) = \{f(x)\}$ if and only if f is continuous.*

Proof. ($\Rightarrow$): By Lemma 3.2, $\mathcal{F}[f]$ is continuous. Thus f, being the same function, has the same property.

($\Leftarrow$): This follows from Lemma 3.1, letting $A = R^n$. ■

Theorem 3.2 *Let $f : R^n \to R^n$ be given. For each $x \in R^n$, $\mathcal{F}[f](x)$ is a singleton if and only if f agrees a.e. with a continuous function $g : R^n \to R^n$.*

Proof. (⇒): We need to construct a $g : R^n \to R^n$ with two properties:

(a) g is continuous, and

(b) $f = g$ is almost everywhere.

Define g to be the function that satisfies $\mathcal{F}[f](x) = \{g(x)\}$ for each $x \in R^n$. The (a) requirement follows from Lemma 3.2, since $\mathcal{F}[f](x)$ is a singleton for each $x \in R^n$. The (b) requirement follows from Lemma 3. It tells us that $\mathcal{F}[f](x) = \{f(x)\}$ a.e. and thus $f(x) = g(x)$ a.e.

(⇐): Let $g : R^n \to R^n$ represent a function that is continuous and satisfies $g(x) = f(x)$ a.e. It follows easily that $\mathcal{F}[g](x) = \mathcal{F}[f](x)$ for all $x \in R^n$. By Lemma 3.1, $\mathcal{F}[g](x) = \{g(x)\}$ for all $x \in R^n$. Thus, for any $x \in R^n$, $\mathcal{F}[f](x) = \mathcal{F}[g](x) = \{g(x)\}$, a singleton. ■

Theorem 3.3 *Let $f : R^n \to R^n$ be given. Then the following three properties are equivalent:*

(i) $\mathcal{F}[f](x)$ is a singleton almost everywhere.

(ii) $\mathcal{F}[f](x) = \{f(x)\}$ is almost everywhere.

(iii) There exists a set $A \subseteq R^n$ of full measure such that $f|_A$ is continuous.

Proof. (i) ⇒ (ii): This is precisely Lemma 3.3.

(ii) ⇒ (iii): $\mathcal{F}[f](x)$ is a singleton a.e., hence, by Lemma 3.2, $\mathcal{F}[f]$ is continuous when restricted to a set of full meausre, call it S_1. By hypothesis, $\mathcal{F}[f](x) = \{f(x)\}$ on a set of full measure, call it S_2. We now let $A = S_1 \cap S_2$, and note that $f|_A$ is continuous.

(iii) ⇒ (ii): Follows easily from Lemma 3.1.

(ii) ⇒ (i): Trivial, since, for each $x \in R^n$, $\{f(x)\}$ is a singleton. ■

Theorem 3.4 *Let $f, g : R^n \to R^n$ be given such that both $\mathcal{F}[f](x)$ and $\mathcal{F}[g](x)$ are singleton a.e. Then, the following are equivalent:*

(i) For each $x \in R^n$, $\mathcal{F}[f](x) = \mathcal{F}[g](x)$.

(ii) $\mathcal{F}[f](x) = \mathcal{F}[g](x)$ almost everywhere.

(ii) $f(x) = g(x)$ almost everywhere.

Proof. (i) $\Rightarrow$ (ii): Tirivial.

(ii) $\Rightarrow$ (iii): Since $\mathcal{F}[f](x)$ is a singleton a.e., we have by Lemma 3.3 that $\mathcal{F}[f](x) = \{f(x)\}$ a.e. Similalry, $\mathcal{F}[g](x) = \{g(x)\}$ a.e. Thus, by hypothesis, $\{f(x)\} = \{g(x)\}$ a.e., that is $f(x) = g(x)$ a.e.

(iii) $\Rightarrow$ (i): Clear, since the construction of the Filippov "ignores" sets of measure zero. ■

Chapter 4

Discontinuous Differential Equations

In this chapter, we introduce the four generalized solutions to the discontinuous differential equations: Filippov, Krasovskij, Hermes and Euler. Some relation between the solutions are given. The Euler solution is compared with the other three solutions.

4.1 Introduction

We know that if f is a continuous function in some (t, x) domain D, then the differential equation

$$x' = f(t, x) \quad \text{with } x(t_0) = x_0 \tag{4.1}$$

is equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds. \tag{4.2}$$

That is to say, if φ is a solution to (4.1) on some interval I for which $\varphi(t_0) = x_0$, then $x = \varphi(t)$ will satisfy (4.2) on I and conversely. It is also known that equation (4.2) makes sense for many discontinuous functions f. Since the continuity of f guaranteed that a solution of (4.1) was of class C^1. Thus, if a continuously differentiable solution of (4.1) is not demanded, the continuity restriction of f can be relaxed. Suppose the function f is continuous in the variable x but discontinuous in the variable t. Then one can extend the notion of the differential equation (4.1) by defining (4.1) to be the following problem: "To find an absolutely continuous function φ defined on a real t

interval I such that

(i) $(t, \varphi(t)) \in D \qquad (t \in I)$.

(ii) $\varphi'(t) = f(t, \varphi(t))$ for all $t \in I$, except on a set of Lebesgue-measure zero".

If such an interval I and function φ exist, then φ is said to be a solution of (4.1) in the extended sense on I or (Caratheodory sense). Absolute continuity of the solution guarantees the existence of φ' almost everywhere on I except on a set of Lebesgue measure zero, so that (ii) makes sense. If f is continuous on D, and φ is a solution of (4.1) in the Caratheodory sense, then from (ii) φ' is continuous on I, and therefore the more general solution of the equation (4.1), and of solution φ, reduces to the ordinary definition of (4.1) when f is continuous on D. As regards the existence of a solution of (4.1), Caratheodory has proved the following quite general theorem under the assumption that f be bounded by a Lebesgue-Integrable function of t.

We now present the known existence theorem for solution of Caratheodory differential equation.

Theorem 4.1 [8] (Caratheodory). *Let f be a vector-valued function on R^n defined on the region $D = \{|t - t_0| \leq a$ and $|x - x_0| \leq b\}$, and suppose f is measurable in t for each fixed x, continuous in x for each fixed t. If there exists a Lebesgue-integrable function $g(t)$ on the interval $|t-t_0| \leq a$ such that $|f(t, x)| \leq g(t)$ where $(t, x) \in D$, then there exists a solution φ of (4.1) in the extended sense on some interval $|t - t_0| \leq a$ satisfying $\varphi(t_0) = x_0$.*

The above solution (Caratheodory solution denoted by $\mathcal{C}$) is not always applicable in the case that the function f is discontinuous in both variables t and x, so we need more general notion of solution to the discontinuous differential equation with a discontinuous right-hand side).

In order to define generalized solutions, two main approaches can be followed. The first approach consists in defining approximate solutions by means of an algorithm and taking as generalized solutions the uniform limits of the approximate solutions.

Hermes and Euler solutions are constructed in this way. The second approach is to associate a differential inclusion to the differential equation and define the generalized solutions as solutions of the associated differential inclusion. Filippov and Krasovskij solutions follow this method.

We are now going to list the definition of each solution along with the main solution existence theorem of each one.

Here, $f : R \times R^n \to R^n$ (or $f : R^n \to R^n$ in the autonomous case) is a locally bounded and does not grow too fast with respect to x, say $|f(t,x)| \leq \gamma|x| + c$ (linear growth role) where γ and c are positive constant. Also, we assume that f is measurable in t ($t \to f(t,x)$) measurable for each fixed x.

4.2 The Four Generalized Solutions

4.2.1 Krasoviskij and Filippov Solutions

Before we go to the definitions, a little preparation is useful. Assume given a set-valued mapping G, from points $(t,x) \in R \times R^n$ to subsets $G(t,x) \subset R^n$. We define operators K and F

$$KG(t,x) = \bigcap_{\varepsilon>0} \overline{\text{conv}}\ G(t, x+\varepsilon B) \tag{4.3}$$

$$FG(t,x) = \bigcap_{\varepsilon>0} \bigcap_{\substack{z \\ m(z)=0}} \overline{\text{conv}}\ G(t, x+\varepsilon B \setminus Z), \tag{4.4}$$

where $\overline{\text{conv}}\ G$ represents the closure of the convex hull of the set G, $x+\varepsilon B$ represents the open ball of radius ε about the point x and m denotes the Lebesgue measure.

Definition 4.1 (Krasoviskij). *Let $x : J \to R^n$ (J is an interval in R) be absolutely continuous on each compact subinterval of J. Then x is called a Krasovskij solution (or $\mathcal{K}$-solution) of (4.1) iff*

$$\dot{x}(t) \in Kf(t,x) \ \ a.e. \ \ in \ \ J.$$

From the definition, it is clear that

(i) $\forall(t,x) \in R \times R^n$, $f(t,x) \in Kf(t,x)$ provides that $\mathcal{C} \subset \mathcal{K}$, any Caratheodory solution is a Krasovskij solution.

(ii) The map $Kf(t,x)$ is an upper semicontinuous with compact convex values (Chapter two).

(iii) Whenever f is continuous at (t,x), $Kf(t,x) = \{f(t,x)\}$.

Certainly, any solution to the differential equation (4.1) is a solution to the differential inclusion $\dot{x}(t) \in Kf(t,x)$. We stress the point that whenever f is continuous at $x(t)$, then a solution to the differential inclusion satisfies the equation $\dot{x}(t) = f(t,x(t))$.

In order to obtain this result, we do not need property (i) at points when f is not continuous. We can look for smaller set-valued map (Filippov) which still satisfies properties (ii) and (iii).

Definition 4.2 *(Filippov). An absolutely continuous function $x : J \to R^n$ is said to be a Filippov solution (or $\mathcal{F}$-solution) of (4.1) if it is a solution to the differential inclusion*

$$\dot{x}(t) \in Ff(t,x) \quad a.e. \ in \ J.$$

We can notice from the definition of Filippov the following:

(i) The map $\mathcal{F}[f](t,x)$ is an upper semi-continuous with a nonempty convex compact values (Chapter 2).

(ii) Whenever f is continuous at (t,x), $\mathcal{F}[f](t,x) = \{f(t,x)\}$.

(iii) $f(t,x)$ belongs to $\mathcal{F}[f](t,x)$ at almost every (t,x).

Since $Ff \subset Kf$ which then yields to $\mathcal{F} \subset \mathcal{K}$, so every Filippov solution is a Krasovskij solution.

The idea behind the concepts of Krasovskij and Filippov solutions is that the value of a solution at a certain point should be determined by the behavior of its derivative in the nearby points. Moreover, the definition of Filippov solution suggests that possible misbehavior of the derivative on null measure sets could be ignored [12].

Before we proceed to the main solution existence theorem, let us consider an example demonstrating the idea of the operations of Krasovskij and Filippov.

Example 4.1 *Compute the Krasovskij and Filippov of $f(x)$ where*

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Solution

$$\begin{aligned} Kf(0) &= \bigcap_{\varepsilon>0} \overline{\text{conv}}\ f(0+\varepsilon B) = \bigcap_{\varepsilon>0} \overline{\text{conv}}\ f(-\varepsilon,\varepsilon) \\ &= \bigcap_{\varepsilon>0} \overline{\text{conv}}\ \{0,1\} = [0,1] \\ \mathcal{F}[f](0) &= \bigcap_{\varepsilon>0} \bigcap_{\substack{z \\ m(z)=0}} \overline{\text{conv}}\ f(0+\varepsilon B \setminus \{0\}) \\ &= \bigcap_{\varepsilon>0} \bigcap_{\substack{z \\ m(z)=0}} \overline{\text{conv}}\ f(-\varepsilon,\varepsilon) \setminus \{0\}) = \{1\}. \end{aligned}$$

Theorem 4.2 (Existence Theorem) [1, 6, 18]. *Let $G : [0,T]\times R^n \to R^n$ be a set-valued map with closed convex values. Assume that*

1. *The set-valued map $x \to G(t,x)$ is upper semi-continuous for almost all $t \in [0,T]$.*

2. *For any $x \in R^n$, there exists a measurable function $t \to f(t,x)$ satisfying $f(t,x) \in G(t,x)$.*

3. *There exists a function $g(t) \in L_1([0,T],R^n)$ such that $|f(t,x)| \leq g(t)$, $t \in [0,T]$.*

Then for any $x_0 \in R^n$, there exists a solution to the differential inclusion $\dot{x} \in G(t,x)$, $t \in [0,T]$ with $x(0) = x_0$.

At this moment, we have the following generalized solutions scheme

$$\begin{array}{ccc} \mathcal{F} & \subset & \mathcal{K} \\ & & \cup \\ & & \mathcal{C} \end{array}$$

Diagram

4.2.2 Comparison Between Filippov and Caratheodory Solutions

We saw a direct relationship between Caratheodory-Krasovskij solution from one side and Filippov-Krasoviskij solutions from another side. But what about Caratheodory and Filippov solutions? The answer will be very clear from the following two examples:

Example 4.2 *Consider $f : R \to R$ defined by*

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

and suppose that $x_0 = 0$. Then $x = 0$ is a Filippov solution but not a Caratheodory solution.

So we conclude from the above example that it is possible for Filippov solution to fail to be a Caratheodory solution.

Example 4.3 *Define $f : R \to R$ by*

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases}$$

and let $x_0 = 0$. Then $x = 0$ is a Caratheodory solution but not a Filippov solution.

Thus the Filippov solution is not a generalization of a Caratheodory solution. It is even not a generalization of the classical (Newton) solution (denoted by $\mathcal{N}$), as it appears in the following example.

Example 4.4 *Consider $\dot{x} = f(x)$, where $f(x)$ is defined by*

$$f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

and let $x_0 = 0$. In a classical sense (Newton), $x(t) = 0$ is a solution while $x(t) = 0$ is not a solution in the Filippov sense.

The natural question to be asked here is when or under what conditions these two solutions, Caratheodory and Filippov, are the same? It turns out to be the same if the function $f(t, x)$ is measurable in the variable t and continuous with respect to the variable x for arbitrary fixed t and the reason is that "because of the continuity of $f(t, x)$ with respect to x, the set $\mathcal{F}[f](t, x)$ consists of a single point which coincides with $f(t, x)$" [10].

So, with the continuity of $f(t, x)$ with respect to the variable x we have

$$\mathcal{F} \quad = \quad \mathcal{K} \quad = \quad \mathcal{C}$$

Diagram

The following fact, due to Hajek [12] presents a simple condition for $\mathcal{F} = \mathcal{K}$. This is useful since a lot is known about $\mathcal{F}$-solutions (Chapters 2 and 3) while the information on $\mathcal{K}$-solutions is rather meager.

Lemma 4.1 *Consider $\dot{x} = f(x)$ (autonomous) under the following assumption of f: there exists a disjoint decomposition*

$$R^n = UM_i \text{ with } M_i \subset \overline{Int\ M_i} \tag{4.5}$$

and continuous $f_i : R^n \to R^n$ such that $f = f_i$ on M_i. Then each $\mathcal{K}$-solution is a $\mathcal{F}$-solution (so $\mathcal{F} = \mathcal{K}$).

Proof. It suffices to show $Kf \subset Ff$, or that $f(x + \varepsilon B) \subset \overline{f((x + \varepsilon B) \setminus Z)}$ for each $x \in R^n$, $\varepsilon > 0$, null set Z. Take any $y \in x + \varepsilon B$, find k so that $y \in M_k$ (thus

$f(y) = f_k(y))$. From (4.5) there exist $y_j \to y$ in $\text{Int}\, M_k$, obviously we may even take $y_i \in \text{Int}\ M_k \setminus Z$ and, of course, $y_i \in x + \varepsilon B$. By continuity, $f_k(y_i) \to f_k(y)$, so that $f(y)$ is in the closure of $f((x + \varepsilon B) \setminus Z)$ as asserted. ■

With the consideration of the above lemma, we have the following scheme

$$\mathcal{F} = \mathcal{K}$$

$$\cup$$

$$\mathcal{C}$$

Diagram

Let us move now to the third generalized solution which is called, according to [12], the Hermes solution.

4.2.3 Hermes Solution

Based on the vague requirement that solutions of feedback problems should be stable under error of measurement, say $p(t)$ which enters into the equation through "feeding" $f(x(t) + p(t))$ instead of $f(x(t))$ was considered in Hermes's work according to [12]. This led Hajek [12] to define the so-called Hermes solution as the following:

Definition 4.3 (Hermes). *Let $x : J \to R^n$ (J is an interval in R) be absolutely continuous on each compact subinterval of J. Then x is called a Hermes solution (or $\mathcal{H}$-solution) of (4.1) if and only if there exist measurable functions $p_k : J \to R^n$ and $\mathcal{C}$-solutions x_k of $\dot{y} = f(t, y + p_k(t))$ such that*

$$p_k \to 0, \quad x_k \to x$$

uniformly on each compact subinterval of J.

The inner perturbation $p_k(\cdot)$ of the above definition, as in $\dot{x} = f(x + p_k(t))$, may be contrasted with outer perturbations $q_k(\cdot)$ in $\dot{x} = f(x) + q_k(t)$. On the one hand, for continuous f, every small inner perturbation obviously is a small outer one,

$$\dot{x} = f(x + p_k(t)) = f(x) + q_k(t),$$
$$q_k = f(x + p_k) - f(x).$$

Even without continuity, the converse holds in the limit: if $\dot{x} = f(x) + q_k(t)$ and we set $y = x - \int^t q_k$, then

$$\dot{y} = f(y + p_k) \text{ and } p_k \to 0, \quad y \to x \text{ for } p_k = \int^t q_k \qquad [12].$$

Of the generalized solutions, as we will see, the most closely related solutions are Krasovskij and Hermes, at least, both have to do with different species of inner perturbation εB and $p_k(t)$.

Notice that Hermes solution is a generalization of Caratheodory solution, $p_k = 0$ leads to $\mathcal{C} \subset \mathcal{H}$.

4.2.4 Existence For Hermes Solution

Theorem 4.3 [12]. *Let f be locally bounded and measurable in t. Then, for any initial data t_0 and x_0, there is a Hermes solution $x(\cdot)$ of $\dot{x} = f(t, x)$ with $x(t_0) = x_0$, defined at least on some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.*

Proof. In the usual manner, we need only consider $t_0 = 0,\ x_0 = 0$; treat only a right neighborhood $[0, \varepsilon)$ of $t_0 = 0$; and assume that f is bounded globally, by some constant φ; otherwise one replaces f by a bounded function which coincides with f inside an appropriate neighborhood.

For each $\delta = 1, 1/2, 1/3, \ldots$ construct the analogue of the Euler polygonal arc, a function $y(\cdot)$ as follows. For $j = 0, 1, 2, \ldots$, we set $t_j = j\delta,\ y_0 = 0$, and then

$$y(t) = y_j + \int_{t_j}^t f(s, y_j)ds \quad \text{in} \quad [t_j, t_{j+1}],$$

where $y_{j+1} = y(t_{j+1})$. Then $y(\cdot)$ has a Lipschitz constant φ, and

$$\dot{y}(t) = f(t, y(t) + p(t)) \quad \text{a.e.}$$

where

$$|p(t)| = |y_j - y(t)| \le \varphi|t - t_j| \le \varphi\delta.$$

Thus $y(\cdot)$ is a $\mathcal{C}$-solution of the perturbed equation with inner perturbations $p \to 0$ uniformly as $\delta \to 0^+$. Since all $y(0) = 0$ and φ is a common Lipschitz constant, the theorem of Arzela and Ascoli applies. Thus some subsequence of the $y = y_\delta$ converges uniformly, by definition, the limit is a $\mathcal{H}$-solution of $\dot{x} = f(t,x)$. ∎

Remark 4.1 *Local boundedness of f and the measurability in the variable t ensure the existence of the three solutions: Hermes, Filippov and Krasovskij for any initial condition. For the autonomous case $\dot{x} = f(x)$ local boundedness of f is enough for the three solutions to exist.*

Example 4.5 *Consider the differential equation*

$$\dot{x} = \begin{cases} 1 & \text{if } x \le 0 \\ -1 & \text{if } x > 0 \end{cases}$$

with $x(0) = 0$ on $[0,1]$.

Following the construction of the Hermes solution in the above theorem, the solution is $x(t) = 0$ although it fails to satisfy the equation at any t.

Corollary 4.1 *For the autonomous differential equation $\dot{x} = f(x)$. Let f be locally bounded. Then there are Filippov, Hermes and Krasovskij solutions for any initial data.*

4.2.5 Comparison Between Hermes and Krasovskij Solution

Hajek in [12] established a very strong relationship between Krasovskij and Hermes solution in conclusion of the closure theorem for a Krasovskij solution. Here, we present the slight modification of this theorem followed by the relation between Hermes and Krasovskij.

Theorem 4.4 (Closure Theorem for $\mathcal{K}$-Solutions). *Let $x_k(\cdot)$ be Krasovskij solutions of*

$$\dot{y} = f(t, y + p_k(t, y)) + q_k(t, y)$$

on $[0, 1]$, with $p_k \to 0$ and $q_k \to 0$ uniformly; assuming that f is locally bounded and measurable in t.

(I) If $x_k(\cdot)$ converges uniformly, then the limit function is a Krasovskij solution of $\dot{x} = f(t, x)$.

(II) Unless $x_k(0) \to \infty$, some subsequence of the $x_k(\cdot)$ does converge uniformly, at least on some $[0, \varepsilon]$ (with $\varepsilon > 0$ depending only on f and $\liminf |x_k(0)|$).

Proof. First assume $x_k \to x$ uniformly; choose $\varepsilon > 0, \delta > 0$ arbitrarily. Then for large indices,

$$|p_k(\cdot)| < \varepsilon,\ \ |q_k(\cdot)| < \delta \text{ and } |x_k(\cdot) - x(\cdot)| < \varepsilon$$

on $[0, 1]$, so that, a.e.

$$\dot{x}_k(t) \in \overline{\text{conv}}\ (f(t, x_k(t) + \varepsilon B) + \delta B) \subset \overline{\text{conv}}\ (f(t, x(t) + 2\varepsilon B) + \delta B).$$

It follows that, for any $t > s$ in $[0, 1]$,

$$x_k(t) - x_k(s) \in \int_s^t \overline{\text{conv}}\ (f(r, x(r) + 2\varepsilon B) + \delta B) dr.$$

The assumptions of Lemma 1.1 are satisfied; hence, on taking limits as $k \to \infty$,

$$x(t) - x(s) \in \int_s^t \overline{\text{conv}}\ (\cdots).$$

By Lemma 1.2, $x(\cdot)$ is absolutely continuous, and

$$\dot{x}(t) \in \overline{\text{conv}}\ (f(t, x(t) + 2\varepsilon B) + \delta B) \text{ a.e.}$$

Take limits over a sequence $\delta \to 0$ to obtain

$$\dot{x}(t) \in \overline{\text{conv}}\ f(t, x(t) + 2\varepsilon B) \text{ a.e.}$$

and then over a sequence $\varepsilon \to 0$ to verify

$$\dot{x}(t) \in Kf(t, x(t)) \quad \text{a.e.}$$

Thus, indeed $x(\cdot)$ is a $\mathcal{K}$-solution. For the proof of the second assertion, consult [12]. ■

Corollary 4.2 *For $\dot{x} = f(t, x)$, with f is locally bounded and measurable in t, each Hermes solution is a Krasovskij solution.*

Proof. Take any $\mathcal{H}$-solution $x(\cdot)$, and the appropriate $\mathcal{C}$-solutions $x_k \to x$ uniformly of $\dot{y} = f(t, y + p_k(t))$ ($p_k \to 0$ uniformly). Trivially, the $x_k(\cdot)$ are Krasovskij solutions; thus by the above theorem, $x = \lim x_k$ is a $\mathcal{K}$-solution of our equation. ■

Example 4.6 *Consider the bounded function f defined by*

$$f(x) = \begin{cases} 1 & \textit{if } x = 0 \\ -1 & \textit{if } x \neq 0 \end{cases}$$

with $x(0) = 0$ on $[0, 1]$. The Hermes solution $x(t) = 0$ is also a Krasovskij solution, where $0 \in Kf(0) = [-1, 1]$.

$x(t) = 0$ is not a Filippov solution since $0 \notin \mathcal{F}[f](0) = \{-1\}$.

At this stage we have the following general containments for the generalized solutions

$$\begin{array}{ccccc} \mathcal{F} & \subset & \mathcal{K} & \supset & \mathcal{H} \\ & & \cup & & \\ & & \mathcal{C} & & \end{array}$$

Diagram

In [12], Hajek proved the converse of the last theorem for the autonomous case:

Theorem 4.5 *Consider the autonomous system, $\dot{x} = f(x)$ where f is locally bounded. Then every Krasovskij solution is a Hermes solution.*

Proof. See [12]. ■

By combining Theorems 4.4 and 4.5, we have

Corollary 4.3 *In the autonomous differential equation $\dot{x} = f(x)$ where f is locally bounded:*

(i) Hermes and Krasovskij solutions coincide.

(ii) Every Filippov solution is a Hermes solution.

By considering the autonomous case, we have the following diagram

$$\begin{array}{ccccc} \mathcal{F} & \subset & \mathcal{K} & = & \mathcal{H} \\ & & \bigcup & & \\ & & \mathcal{C} & & \end{array}$$

Diagram

Another relation between the generalized solutions can be gained by considering the condition of Lemma 4.1 where we assume here a local boundedness in f.

Corollary 4.4 *Consider $\dot{x} = f(x)$ where f is locally bounded and there exists a disjoint decomposition*

$$R^n = UM_i \quad \textit{with } M_i \subset \overline{\textit{Int } M_i}$$

and continuous $f_i : R^n \to R^n$ such that $f = f_i$ on M_i. Then $\mathcal{K} = \mathcal{F} = \mathcal{H}$.

Let us move now to the fourth generalized solution to a discontinuous differential equation, which is called Euler solution, which was first introduced in "Nonsmooth Analysis and Control Theory" [7] by Professor Peter Wolenski and his colleagues (1997) after almost two decades from the publication of Hajek's paper (1978). Our aim here is to compare the Euler solution with the other generalized solutions and to find a place for Euler solution to fit in the diagram scheme for the generalized solutions.

4.2.6 Euler Solution

Many of us will have seen methods of calculating solutions of ordinary differential equations; how would we study in concrete terms the calculation of trajectories of the differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.,} \quad t \in [a, b]. \tag{4.6}$$

The most straightforward approach to calculating a trajectory is to first find a *selection* f of F; i.e., a function f such that $f(t, x) \in F(t, x)$ for all (t, x). Then, we consider the differential equation $\dot{x} = f(t, x)$; any solution will presumably satisfy (4.6).

The problem with this approach lies in finding selections f with the regularity properties (e.g., continuity) required by the usual theory of differential equations. This selection issue is an interesting and well-studied one, but not one that we intend nor need to dwell upon. Instead, we will consider a generalized concept of solution to $\dot{x} = f(t, x)$, one which requires particular regularity of f.

Let us now consider the so-called Cauchy or *initial-value problem*

$$\dot{x}(t) = f(t, x(t)) \;\; x(a) = x_0, \tag{4.7}$$

where f is simply any function from $[a, b] \times \mathbb{R}^n$ to $\mathbb{R}^n$. How would we begin to calculate numerically a solution (4.7)? Recalling the classical Euler iterative scheme from ordinary differential equations, we suspect that a reasonable answer is obtained by discretizing in time. So let

$$\pi = \{t_0, t_1, \ldots, t_{N-1}, t_n\}$$

be a partition of $[a, b]$, where $t_0 = a$ and $t_N = b$. (We do not require uniform partitions: thus the interval lengths $t_i - t_{i-1}$ may differ.)

We proceed by considering, on the interval $[t_0, t_1]$, the differential equation with *constant* right-hand side

$$\dot{x}(t) = f(t_0, x_0), \quad x(t_0) = x_0.$$

Of course this has a unique solution $x(t)$ on $[t_0, t_1]$, since the right side is constant; we define $x_1 : x(t_1)$. Next we iterate, by considering on $[t_1, t_2]$ the initial-value problem

$$\dot{x}(t) = f(t_1, x_1), \quad x(t_1) = x_1.$$

The next so-called *node* of the scheme is $x_2 = x(t_2)$. We proceed in this manner until an arc x_π (which is in fact piecewise affine) has been defined on all of $[a, b]$. We use the notation x_π to emphasize the role played by the particular partition π in determining x_π, which has been called in the past (and in our present) the *Euler polygonal arc* corresponding to the partition π, or similar words to that effect.

The *diameter* (or *mesh size*) μ_π of the partition π is given by

$$\mu_\pi : \max\{t_i - t_{i-1} : 1 \leq i \leq N\}.$$

An *Euler solution* to the initial-value problem (4.7) means any arc x which is the uniform limit of Euler polygonal arcs x_{π_j}, corresponding as above to some sequence π_j such that $\pi_j \to 0$, where this connotes convergence of the diameters $\mu_{\pi_j} \to 0$ (evidently, the corresponding number N_j of partition points in π_j must then go to infinity). We will also say that an arc x on $[a, b]$ is an *Euler arc* for f when x is an Euler solution on $[a, b]$ as above to the initial-value problem (4.7) for the "right" initial condition, namely $x_0 = x(a)$.

The following examples help understanding more the concept of the Euler solution.

Example 4.7 *Define* $f : R \to R$ *by*

$$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ -1 & \text{if } x > 0. \end{cases}$$

Let $x_0 = 0, \quad a = 0$ *and* $b = 1$. *Then* $x(t) = 0$ *is the only Euler solution, although it fails to satisfy* $\dot{x}(t) = f(t, x(t))$ *at any* t.

Example 4.8 *Let* f *be defined as follows*

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ -1 & \text{otherwise,} \end{cases}$$

where $x_0 = 0, \quad a = 0$ *and* $b = 1$. *If we consider only a uniform partition of* $[0, 1]$, *then* $x(t) = 0$ *is the only solution to the problem, otherwise we have another solution to the initial-value problem which is* $x(t) = -t$.

We come now to the main existence theorem of an Euler solution.

Theorem 4.6 [7]. *Suppose that for positive constants* γ *and* c *and for all* $(t, x) \in [a, b] \times \mathbb{R}^n$, *we have the linear growth condition*

$$\|f(t, x)\| \leq \gamma \|x\| + c,$$

where f *is otherwise arbitrary. Then:*

(a) At least one Euler solution x *to the initial-value problem (4.7) exists on* $[a, b]$, *and any Euler solution is Lipschitz.*

(b) Any Euler arc x *for* f *on* $[a, b]$ *satisfies*

$$\|x(t) - x(a)\| \leq (t - a)e^{\gamma(t-a)}(c + \gamma)\|x(a)\|), \quad a \leq t \leq b.$$

(c) If f *is continuous, then any Euler arc* x *for* f *on* $[a, b]$ *is continuously differentiable on* (a, b) *and satisfies* $\dot{x}(t) = f(t, x(t)) \;\; \forall\, t \in (a, b)$.

Before we prove the theorem, the following example demonstrates the importance of the linear growth condition.

Example 4.9 *Consider the classical single-valued differential equation*

$$x' = \frac{-1}{2} x^{3/2} \quad \text{with} \quad x(0) = 1 \text{ on } [0, 2],$$

which has solution $x(t) = \dfrac{1}{(t-1)^2}$. *Obviously the solution "blows up" at* $t = 1$ *and so cannot be defined on all of* $[0, 2]$.

Proof. Let $\pi := \{t_0, t_1, \ldots, t_N\}$ be a partition of $[a, b]$, and let x_π be the corresponding Euler polygonal arc, with the nodes of x_π being denoted $x_0, x_1, \ldots, x_N$ as usual. On the interval (t_i, t_{i+1}) we have

$$\|\dot{x}_\pi(t)\| = \|f(t_i, x_i)\| \leq \gamma \|x_i\| + c,$$

whence

$$\begin{aligned}\|x_{i+1}-x_0\| &\le \|x_{i+1}-x_i\|+\|x_i-x_0\| \\ &\le (t_{i+1}-t_i)(\gamma\|x_i\|+c)+\|x_i-x_0\| \\ &\le [(t_{i+1}-t_i)\gamma+1]\|x_i-x_0\|+(t_{i+1}-t_i)(\gamma\|x_0\|+c).\end{aligned}$$

We now require the following exercise in induction:

Exercise 1 *Let $r_0, r_1, \ldots, r_N$ be nonnegative numbers satisfying*

$$r_{i+1} \le (1+\delta_i)r_i + \Delta_i, \quad i = 0, 1, \ldots, N-1,$$

where $\delta_i \ge 0$ and $\Delta_i \ge 0$, $r_0 = 0$. Then

$$r_N \le \left(\exp\left(\sum_{i=0}^{N-1}\delta_i\right)\right)\sum_{i=0}^{N-1}\Delta_i.$$

We apply this result to derive, for $i = 1, 2, \ldots, N$,

$$\|x_i - x_0\| \le M,$$

where

$$M := (b-a)e^{\gamma(b-a)}(\gamma\|x_0\|+c).$$

Thus all the nodes x_i lie in the ball $\overline{B}(x_0; M)$; by convexity this is true of all the values $x_\pi(t)$, $a \le t \le b$. Since the derivative along any linear portion of x_π is determined by the values of f at the nodes, we obtain as well the following uniform bound on $[a, b]$:

$$\|\dot{x}_\pi\|_\infty \le k = \gamma M + c.$$

Therefore x_π is Lipschitz of rank k on $[a, b]$.

Now let π_j be a sequence of partitions such that $\pi_j \to 0$; i.e., such that μ_{π_j} goes to zero, and (necessarily) $N_j \to \infty$. Then the corresponding polygonal arcs x_{π_j} on $[a, b]$ all satisfy

$$x_{\pi_j}(a) = x_0, \quad \|x_{\pi_j} - x_0\|_\infty \le M, \quad \|\dot{x}_{\pi_j}\|_\infty \le k.$$

It follows that the family $\{x_{\pi_j}\}$ is equicontinuous and uniformly bounded; then by the well-known theorem of Arzela and Ascoli, some subsequence of it converges uniformly to a continuous function x. the limiting function inherits the Lipschitz rank k on $[a, b]$, and in consequence is absolutely continuous (i.e., x is an arc). Thus by definition x is an Euler solution of the initial-value problem (4.7) on $[a, b]$, and assertion (a) of the theorem is proved.

The inequality in part(b) of the theorem is inherited by x from the sequence of polygonal arcs generating it (we identify t with b). There remains to prove part (c) of the theorem.

To this end, let x_{π_j} denote a sequence of polygonal arcs for problem (4.7) converging uniformly to an Euler solution x. As shown above, the arcs x_{π_j} all lie in a certain ball $\overline{B}(x_0; M)$ and they all satisfy a Lipschitz condition of the same rank k. Since a continuous function on $\mathbb{R}^n$ is uniformly continuous on compact sets, for any $\varepsilon > 0$, we can find $\delta > 0$ such that

$$t, \widetilde{t} \in [a, b], \quad x, \widetilde{x} \in \overline{B}(x_0; M), \qquad |t - \widetilde{t}| < \delta,$$
$$\|x - \widetilde{x}\| < \delta \Rightarrow \|f(t, x) - f(\widetilde{t}, \widetilde{x})\| < \varepsilon.$$

Now let j be large enough so that the partition diameter μ_{π_j} satisfies $\mu_{x_j} < \delta$ and $k\mu_{\pi_j} < \delta$. For any point t which is not one of the finitely many points at which $x_{\pi_j}(t)$ is a node, we have $\dot{x}_{\pi_j}(t) = f(\widetilde{t}, x_{\pi_j}(\widetilde{t}))$ for some $\widetilde{t}$ within $\mu_{\pi_j} < \delta$ of t. Thus, since

$$\|x_{\pi_j}(t) - x_{\pi_j}(\widetilde{t})\| \leq k\mu_{\pi_j} < \delta,$$

we deduce

$$\|\dot{x}_{\pi_j}(t) - f(t, x_{\pi_j}(t))\| = \|f(t, x_{\pi_j}(t)) - f(\widetilde{t}, x_{\pi_j}(\widetilde{t}))\| < \varepsilon.$$

It follows that for any t in $[a, b]$, we have

$$\begin{aligned}&\left\| x_{\pi_j}(t) - x_{\pi_j}(a) - \int_a^t f(t, x_{\pi_j}(\tau))d\tau \right\| \\ &\qquad = \left\| \int_0^t \{\dot{x}_{\pi_j}(\tau) - f(\tau, x_{\pi_j}(\tau))\} d\tau \right\| < \varepsilon(t - a) \leq (b - a).\end{aligned}$$

Letting $j \to \infty$, we obtain from this

$$\left\| x(t) - x_0 - \int_a^t f(\tau, x(\tau))d\tau \right\| \leq \varepsilon(b - a).$$

Since ε is arbitrary, it follows that

$$x(t) = x_0 + \int_a^t f(\tau, x(\tau)) d\tau,$$

which implies (since the integrand is continuous) that x is C^1 and $\dot{x}(t) = f(t, x(t))$ for all $t \in (a, b)$. ■

Table 4.1 summarizes the properties of the four generalized solutions:

Table 4.1

Properties of the Generalized Solutions

Property \ Solutions	$\mathcal{E}$	$\mathcal{H}$	$\mathcal{F}$	$\mathcal{K}$
Condition of f	Admits the linear growth condition	Local boundedness and measurability in t	Local boundedness and measurability in t	Local boundedness and measurability in t
Existence of solution	Always exist	Always exist	Always exist	Always exist
Nature of solution	Absolutely continuous & locally Lipschitz	Absolutely continuous & locally Lipschitz	Absolutely continuous & locally Lipschitz	Absolutely continuous & locally Lipschitz
Solution satisfaction	It might fail to satify the differential equation	It might fail to satify the differential equation	Satisfies the differential inclusion almost everywhere	Satisfies the differential inclusion almost everywhere

4.2.7 Comparison Between Euler and the Other Generalized Solutions

Our aim here is to study the relation between Euler solution with the other types of solutions and fit $\mathcal{E}$ in the scheme diagram of the generalized solutions. Not surprisingly, Euler solution is not a generalization of the classical solution.

Example 4.10 *Consider the differential equation*

$$\dot{x} = f(x) = \frac{3}{2}x^{1/3} \text{ with } x(0) = 0$$

on the interval $[0, 1]$*. The equation has three distinct classical solutions:*

$$x(t) = t^{3/2}, x(t) = -t^{3/2} \text{ and } x(t) = 0$$

while the only Euler solution is $x(t) = 0$*.*

The following example carries no good news about the relation between Euler and Filippov solutions.

Example 4.11 *Let* $f(x)$ *be the function defined by*

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ -1 & \text{if } x \neq 0 \end{cases}$$

on the interval $[0, 1]$ *with* $x(0) = 0$*. A simple calculation shows that* $x(t) = 0$ *is an Euler solution, while* $x(t) = 0$ *is not a Filippov solution.*

The construction of Hermes solution in the autonomous case, $\dot{x} = f(x)$, is identical to the way that Euler solution was constructed: they are both the uniform limits of Euler polygonal arc (Euler arc) by using the so-called Arzela-Ascoli theorem. That leads us to the following fact.

Lemma 4.2 *Let* f *be locally bounded (admit the linear growth condition) for the differential equation* $\dot{x} = f(x)$*. Then Euler and Hermes are identical* $\mathcal{E} = \mathcal{H}$*.*

Example 4.12 *Let f be the function*

$$f(x) = \begin{cases} 1 & \textit{if } x \leq 0 \\ -1 & \textit{if } x > 0 \end{cases}$$

Construct an Euler (Hermes) solution to the differential equation

$$\dot{x} = f(x) \text{ on } [0,1] \text{ with } x(0) = 0.$$

Solution *For simplicity, let us consider a uniform partition to the interval $[0,1]$. Let $\delta = \pi = 1$.*

Euler*:*

$$\begin{aligned} x_0 &= t_0 = 0 \\ \dot{x}(t) &= f(0,0) = 1. \end{aligned}$$

So, $x(t) = t + c$ by using the initial condition $x(0) = 0$, then $x(t) = t$ on $[0,1]$.

Hermes*:*

$$y(t) = y_0 + \int_0^t f(0)ds = 0 + \int_0^t 1ds = t \text{ on } [0,1]$$

where $y_0 = 0$.

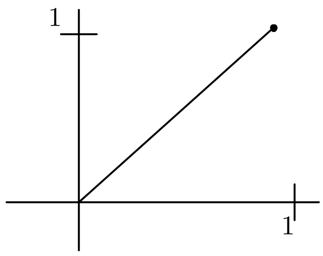

Figure 3: The graph of $x(t)$ on $[0,1]$ when $\delta = 1$

Let $\pi = \delta = 1/2$, on $[0, 1/2]$ with $x(0) = 0$.

Euler*: $\dot{x}(t) = f(0,0) = 1$ so $x(t) = t$, and on $[1/2, 1]$,*

$$\begin{aligned} \dot{x} &= f(t_1, x_1) \\ \dot{x} &= f(1/2, 1/2) = -1 \\ x(t) &= -t + x_0 \text{ with the initial condition} \\ x(t) &= -t + 1 \end{aligned}$$

Hermes*: On* $[0, 1/2]$*, we have* $j = t_0 = y_0 = 0$

$$y(t) = 0 + \int_{t_0}^{t} f(y_0)ds = 0 + \int_0^t 1ds = t$$

and on $[1/2, 1]$*,* $t_1 = 1/2,\ t_2 = 1,\ j = 1, \quad y_1 = 1/2$ *and* $y_2 = 0$

$$\begin{aligned} y(t) &= y_1 + \int_{t_1}^{t} f(y_1)ds + \frac{1}{2} + \int_{1/2}^{t} (-1)ds \\ &= \frac{1}{2} - t + \frac{1}{2} = -t + 1, \end{aligned}$$

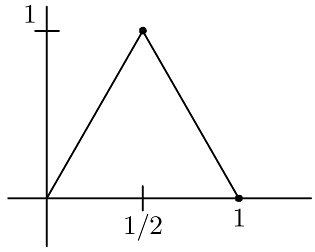

Figure 4: The graph of $x(t)$ on $[0, 1]$ when $\delta = 1/2$

and if we continue in this manner, we reach the "saw-toothed" functions

$$x_\delta(t) = x_\pi(t) = \begin{cases} t - \dfrac{2i}{k} & on\ \dfrac{2i}{k} \le t \le \dfrac{2i+1}{k} \\ \dfrac{2i+2}{k} - t & on\ \dfrac{2i+1}{k} \le t \le \dfrac{2i+2}{k} \end{cases} \qquad i = 0, 1, 2, \ldots.$$

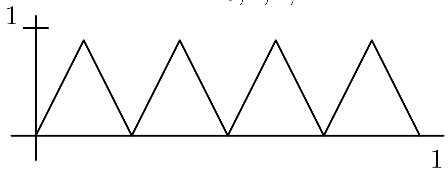

Figure 5: The graph of $x_\delta(t)$

as $k \to \infty, \quad x_k(t) \to x(t) = 0$ *on* $[0, 1]$*. Notice that the solution fails to satisfy the differential equation.*

More relation can be observed by combining Theorem 4.5 and Lemma 4.2.

Corollary 4.5 *In the autonomous differential equation, $\dot{x} = f(x)$, where f is locally bounded (admits a linear growth condition). Then Krasovskij, Hermes and Euler coincide $\mathcal{K} = \mathcal{H} = \mathcal{E}$.*

By considering the above corollary, we have the following scheme.

$$\begin{array}{ccccc} \mathcal{F} \subset \mathcal{K} & = & \mathcal{H} & = & \mathcal{E} \\ & & \bigcup & & \\ & & \mathcal{C} & & \end{array}$$

Diagram

We conclude the chapter by the following results:

Corollary 4.4 *Consider $\dot{x} = f(x)$ where f has the following assumption: there exits a disjoint decomposition*

$$R^n = UM_i \text{ with } M_i \subset \overline{Int\ M_i}$$

and continuous $f_i : R^n \to R^n$ such that $f = f_i$ on M_i. Then all the four generalized solutions: Krasovskij, Filippov, Hermes, and Euler are equal

$$\mathcal{K} = \mathcal{F} = \mathcal{H} = \mathcal{E}.$$

In view of Corollary 4.4, we have the following scheme:

$$\begin{array}{c} \mathcal{K} = \mathcal{F} = \mathcal{H} = \mathcal{E} \\ \cup \\ \mathcal{C} \end{array}$$

Diagram

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Vita

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